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ON THE ZERO MODULE OF RATIONAL MATRIX FUNCTIONS

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1. Introduction

Module theoretic methods have been introduced into system theory by Kalman [1] and have since proved to be central to the theory of linear systems. Their greatest impact has been initially in the analysis and solution of the realization problem and the study of feedback in later stages. In particular the state module of a transfer function is determined, up to isomorphism, by its pole structure. Surprisingly, as rightly pointed out by Wyman and Sain [2], no attempt has been made in analysing the zero structure of rational matrix functions from a module theoretic point of view. Possibly the closest in spirit, though highly indirect, is the geometric control theory analysis using the quotient of the maximal (A,B)-invariant subspace in Ker C by the maximal reachability subspace in Ker C using a properly defined state feedback map, where (A,B,C) is any canonical realization of the transfer function.

This situation has been remedied to a large extent by the recent paper of Wyman and Sain [2] which is the main motivation for this short note.

The definition of the zero module given by Wyman and Sain applies just as well to any rational, in particular polynomial, matrix function without any assumption of coprimeness. Thus for any pxm rational matrix function G, the base field F being arbitrary, we define the zero module by

\[ Z(G) = \frac{G^{-1}(FP[A]) + FM[A]}{Ker \ G + FM[A]} \quad (1.1) \]

Of course in analogy with the scalar case one expects the zero information to be included in the numerator of any coprime factorization of G. In fact if

\[ G = T^{-1}U = NB^{-1} \quad (1.2) \]

are respectively left and right coprime factorizations of G then it is easily checked that \( Z(G) = Z(U) = Z(N) \). In this connection the work of Pugh and Shelton [3] is also relevant.

If instead of representations of the form (1.2) of a transfer function G we consider Rosenbrock [4] type representations of the form

\[ G = VT^{-1}U + W \quad (1.3) \]

with V,T,U and W polynomial matrices of which T is assumed nonsingular, then the polar information, assuming left coprimeness of T and U and right coprimeness of T and V, is determined by T, and a natural question is the representation of the zero module in terms of the data T,U,V and W. Following Rosenbrock we define the polynomial system matrix associated with (1.3) to be the polynomial matrix

\[ P = \begin{pmatrix} T & U \\ -V & W \end{pmatrix} \quad (1.4) \]

In the next section we state the main results concerning the relation between the zero structure of G and the polynomial system matrix.

2. The Polynomial System Matrix as a Numerator

There are some indications that the polynomial system matrix (1.4) associated with the representation (1.3) of a transfer function G behaves like a numerator in a coprime factorization of G. Especially suggestive is a comparison of corollaries 3.12 and 8.6 in Emre and Hautus [5] which give a characterization of the maximal (A,B)-invariant subspace in Ker C in terms of the representations \( G = T^{-1}U \) and (1.3). In fact the analysis given in Fuhrmann and Willems [6] can be extended to this case to yield the following.

Theorem 2.1: Let P be the polynomial system matrix associated with representation (1.3) of a transfer function G,

(i) If \( P = E_P E_0 \) is a factorization of P with \( E_0 \) nonsingular then \( V = P_1 E_0 K \) is an \( (A,B) \)-invariant subspace in Ker C.

(ii) \( V K_T \) is an \( (A,B) \)-invariant subspace in Ker C if and only if \( V = P_1 E_0 \) where \( P = P E_0 \) is a factorization with \( E_0 \) nonsingular.

Proof of Theorem 2.1: Let P be the polynomial system matrix associated with representation (1.3) of a transfer function G,

\[ Z(G) = \frac{pr_2(p^{-1}(F^{sp}[A]) + FM[A])}{pr_2(Ker P) + FM[A]} \quad (2.1) \]

(iii) If in addition V and T are right coprime then \( Z(V) = Z(G) \), the isomorphism being the one induced by \( pr_2 \).

We note first the following.

Lemma 2.3: Let \( (U) \in \mathbb{F}^{s+m}(\{1\}) \), then \( (U) \in Ker P \) if and only if \( v \in Ker G \) and \( u = -T^{-1}v \).

Proof of Theorem 2.3. We break the proof into several steps.
(a) By Lemma 2.3 Ker \(G = \text{pr}_2 \text{Ker} \ P\) and so \(\text{Ker} \ P + \text{F}^m[\lambda] = \text{pr}_2(\text{Ker} \ P + \text{F}^m[\lambda]).\)

(b) We show \(G^{-1}(p\text{F}[\lambda]) \subset \text{pr}_2(F^{-1}(\text{F}^s\text{P}[\lambda])).\) Indeed if \(h_2 \in G^{-1}(\text{F}[\lambda])\) then \(Gh_2 = p_2 \in \text{F}[\lambda].\) Define \(h_1\) by \(h_1 = -T^{-1}Uh_2\) then \(P(h_1) = (0) \in \text{F}^s\text{P}[\lambda]\) which proves the inclusion. In particular we obtain the inclusion \(G^{-1}(\text{F}[\lambda]) + \text{F}^m[\lambda] \subset \text{pr}_2(\text{F}^{-1}(\text{F}^s\text{P}[\lambda]) + \text{F}^m[\lambda]).\)

(c) Using left coprimeness of \(T\) and \(U\) we will show \((2.3)\) By the left coprimeness of \(T\) and \(U\) we have \(\text{Ker} \ G = \text{pr}_2 \text{Ker} \ P\) and so that \((2.3)\) follows. Since \(\text{pr}_2 \text{Ker} \ P = \text{F}^m[\lambda]\) it follows that \((2.4)\) holds.

(d) If \(h_2 \in \text{pr}_2(F^{-1}(\text{F}^s\text{P}[\lambda]))\) there exists \(h_1\) such that \(h_2 \notin \text{F}^m[\lambda]\) and \(h_2 \notin \text{pr}_2(\text{Ker} \ P + \text{F}^m[\lambda]).\) Conversely if \(h_2 \in \text{pr}_2(\text{Ker} \ P + \text{F}^m[\lambda])\) then there exist \(h_1\) and polynomials \(g\) such that \(P(h_2) = (0)\) or \(h_2 \notin \text{pr}_2(\text{Ker} \ P + \text{F}^m[\lambda]).\) We will show that \(h_2 \notin \text{pr}_2(\text{Ker} \ P + \text{F}^m[\lambda]).\) When \(h_2 \notin \text{pr}_2(\text{Ker} \ P + \text{F}^m[\lambda])\) and \(h_2 \in \text{pr}_2(\text{Ker} \ P + \text{F}^m[\lambda])\) then we have \(h_2 \in \text{pr}_2(\text{Ker} \ P + \text{F}^m[\lambda]).\)

(e) In an analogous way one proves \(\text{pr}_2(F^{-1}(\text{F}^s\text{P}[\lambda])) + \text{F}^m[\lambda] = \text{pr}_2(F^{-1}(\text{F}^s\text{P}[\lambda]) + \text{F}^m[\lambda]).\)

(f) Equalities \(2.5\) and \(2.6\) and part (i) imply that the induced map \(\text{pr}_2 : \mathcal{Z}(P) \to \mathcal{Z}(G)\) is surjective.

(g) As a final step we show that if \(V\) and \(T\) are right coprime then the map \(\text{pr}_2\) is also injective and so an isomorphism. To this end assume \(h_1 \in \text{pr}_2(\text{Ker} \ P + \text{F}^m[\lambda])\) and \(h_2 \in \text{pr}_2(\text{Ker} \ P + \text{F}^m[\lambda]).\) We will show that \(h_1 \in \text{Ker} \ P + \text{F}^m[\lambda].\) Our assum-