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ON DISTRIBUTION SPACES BASED ON
JACOBI POLYNOMIALS

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S.J.L. van Eijndhoven and J. de Graaf
SUMMARY

The Jacobi operator

$$A_{\alpha, \beta} = - (1-x^2) \frac{d^2}{dx^2} - (\beta - \alpha - (\alpha + \beta + 2)x) \frac{d}{dx}, \quad \alpha > -1, \beta > -1,$$

is self-adjoint and positive in the Hilbert space

$$X_{\alpha, \beta} = L_2([-1,1], (1-x)^{\alpha}(1+x)^{\beta}dx).$$

We study the analyticity domain $D^\omega(A_{\alpha, \beta}^\nu)$ and the entireness domain $D^\omega(\exp(A_{\alpha, \beta}^\nu))$ of the $\nu$-th power, $\nu \geq \frac{1}{2}$, of the operator $A_{\alpha, \beta}$ in $X_{\alpha, \beta}$.

It is shown that for fixed $\nu$ the analyticity and entireness domains do not depend on $\alpha$ and $\beta$. Moreover, for each fixed $\nu$ the mentioned domains are characterized as spaces of analytic functions of suitable well-described (growth-) classes.

Next the spaces $D^\omega(A_{\alpha, \beta}^\nu)$ and $D^\omega(\exp(A_{\alpha, \beta}^\nu))$ are considered as test spaces for distribution theories. They are very special examples of general functional analytic constructions as given by De Graaf and Van Eijndhoven.

The distribution spaces (dual spaces of $D^\omega(A_{\alpha, \beta}^\nu)$ and $D^\omega(\exp(A_{\alpha, \beta}^\nu))$) are described in detail and many natural examples of continuous linear functionals and (extendible) continuous linear mappings are given. These examples are based on simple geometric and analytic considerations. Further, expansions of distributions in Jacobi polynomials and Jacobi functions of the second kind are studied and sharp estimates on the expansion coefficients are produced.

For $\nu = \frac{1}{2}$ the relation with (ultra-) hyperfunction theory is discussed.

Finally, some large groups of analytic functions are represented as (extendible) linear operators on the mentioned spaces.

AMS Classifications 46F05, 46F10, 46F15, 47D05, 33A65.
PRELIMINARIES

In the past decennia generalized functions have been introduced in many different ways. We mention Schwartz's distribution theory in which generalized functions are regarded as continuous linear functionals on some locally convex space of good functions, Jones' theory on generalised functions, in which generalized functions are weak limits of regular sequences of good functions, and the theory on hyperfunctions in which generalized functions are 'boundary values' of analytic functions defined in a region of the complex plane. Probably the easiest way to introduce generalized functions is by means of formal series expansions with respect to some orthonormal basis in a Hilbert space. Here we shall clarify this method a bit more.

Let $(v_n)_{n=0}^\infty$ be an orthonormal basis in a Hilbert space $X$. The Riesz-Fischer theorem says that each $f$ in $X$ is represented in $\ell_2$ by the $\ell_2$-sequence $((f,v_n))_{n=0}^\infty$, and $\|f - \sum_{n=0}^N (f,v_n)v_n\| \to 0$ as $N \to \infty$. Now consider a vector space $E$ the elements of which are formal series $\sum_{n=0}^\infty a_n v_n$ where the sequences $(a_n)_{n=0}^\infty$ belong to a class which contains $\ell_2$. A candidate for the dual of $E$ can be obtained by considering sequences $(b_n)_{n=0}^\infty$ for which $\sum_{n=0}^\infty |a_n b_n| < \infty$ for all $(a_n)_{n=0}^\infty$ in $E$. Let $D$ denote the vector space of the related series $\sum_{n=0}^\infty b_n v_n$. Then it is clear that $D \subset X \subset E$. Thus we get a functional analytic analogue of the so-called pansion theory [7] developed by Korevaar.

We note that Korevaar looks at the Hilbert space $L_2(\mathbb{R})$ with the Hermite functions as an orthonormal basis.
2.

Analyticity spaces and trajectory spaces

A simplified version of the theory [6] on generalized functions developed by De Graaf can be described as follows. The space $E$ consists of all formal series $\sum_{n=0}^{\infty} a_n v_n$ where the sequences $(a_n)_{n=0}^{\infty}$ satisfy

\[(0.1) \quad \forall t > 0 : \sup_{n \in \mathbb{N}} (|a_n| e^{-\lambda_n t}) < \infty.\]

Here $(\lambda_n)_{n=0}^{\infty}$ is a fixed sequence of positive numbers with the property that $0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ...$ and $\sum_{n=0}^{\infty} e^{-\lambda_n t} < \infty$ for all $t > 0$. The set \{$(\lambda_n)_{n \in \mathbb{N}} \cup \{0\}$\} is the spectrum of the positive self-adjoint operator $H$ loosely defined by

\[(0.2) \quad Hv_n = \lambda_n v_n, \quad n = 0,1,2,\ldots.\]

Let $(a_n)_{n=0}^{\infty}$ be a sequence with property (0.1). Then to $(a_n)_{n=0}^{\infty}$ we let correspond the mapping $F: (0,\infty) \rightarrow X$ defined by

$$F(t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} a_n v_n.$$  

It is clear that the mapping $F$ satisfies $F(t+\tau) = e^{-\tau H} F(t)$, $t,\tau > 0$.

Conversely, if a mapping $G: (0,\infty) \rightarrow X$ satisfies $G(t+\tau) = e^{-\tau H} G(t)$ for all $t,\tau > 0$, then there exists a sequence $(a_n)_{n=0}^{\infty}$ which obeys condition (0.1) and for all $t > 0$, $G(t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} a_n v_n$. So the set of all sequences $(a_n)_{n=0}^{\infty}$ determined by condition (0.1) can be identified with the set of all mappings $F$ from $(0,\infty)$ into $X$ with the mentioned property.

Since each positive self-adjoint operator $A$ generates a one parameter semigroup of so-called smoothing operators, the previous considerations lead to the following more general definition.
(0.4) **Definition**

Let $A$ be a positive self-adjoint operator in $X$. Then the space $T_{X,A}$, called the trajectory space, consists of all mappings $F: (0,\infty) \to X$ with the property that

$$
\forall t > 0 \forall \tau > 0: F(t+\tau) = e^{-\tau A} F(t).
$$

Heuristically, the space $T_{X,A}$ can be regarded as the space of 'initial conditions' $u(0)$ to the evolution equation $\frac{du}{dt} = -Au$, which give rise to a solution $t \mapsto u(t), \, t > 0$, through $X$.

The topology for the space $T_{X,A}$ is generated by the seminorms

$$
F \mapsto \|F(t)\|_X, \quad t > 0, \quad F \in T_{X,A}.
$$

With these seminorms $T_{X,A}$ becomes a Frechet space. Moreover, $T_{X,A}$ is Montel iff the operators $e^{-tA}, \, t > 0$, are compact; $T_{X,A}$ is nuclear iff the operators $e^{-tA}, \, t > 0$, are Hilbert-Schmidt. (Observe that $T_{X,H}$ is nuclear.)

Let $f \in X, \, f = \sum_{n=0}^{\infty} (f,v_n)v_n$, satisfy $(f,v_n) = O(e^{-\lambda_n t})$ for some $t > 0$. Then for every $F \in T_{X,H}, \, F = \sum_{n=0}^{\infty} a_n v_n$, we have

$$
\sum_{n=0}^{N} |a_n(f,v_n)| \leq \left( \sum_{n=0}^{N} |a_n|^{2} e^{-2 \lambda_n t} \right)^{\frac{1}{2}} \left( \sum_{n=0}^{N} |(f,v_n)|^{2} e^{2 \lambda_n \tau} \right)^{\frac{1}{2}}.
$$

If we take $0 < \tau < t$ the right-hand side converges if $N \to \infty$. So the series $\sum_{n=0}^{\infty} a_n (f,v_n)$ is absolutely convergent. We remark that the order estimate on the sequence $((f,v_n))_{n=0}^{\infty}$ implies that $f$ is contained in the domain of each unbounded self-adjoint operator $e^{tH}, \, 0 < \tau < t$. This leads to the general definition of analyticity space.
4.

(0.5) **Definition**

\[ S_{x,A} := \bigcup_{t>0} e^{-tA}(X) = \{ e^{-tA}w \mid t > 0, w \in X \} . \]

It is clear that \( S_{x,A} \) is a dense subspace of \( X \). Since \( S_{x,A} \) consists of precisely all analytic vectors of \( A \), we call the spaces of type \( S_{x,A} \) analyticity spaces.

The natural topology for the space \( S_{x,A} \) is the inductive limit topology induced by the spaces \( e^{-tA}(X) \) with Hilbert space topology generated by the inner product \( (f,g)_t := (e^{tA}f, e^{tA}g) \), \( f, g \in e^{-tA}(X) \). This inductive limit is not strict. The construction of seminorms which generate the inductive limit topology has led to a thorough description of several topological features of \( S_{x,A} \). For example, the sequence \( (f_n)_{n \in \mathbb{N}} \) is a null sequence in \( S_{x,A} \) iff there exists \( t > 0 \) such that \( (e^{tA}f_n)_{n \in \mathbb{N}} \) is a null sequence in \( X \). Further, we note that \( S_{x,A} \) is complete, bornological and barreled.

On \( S_{x,A} \times T_{x,A} \) we introduce the pairing \( \langle \cdot, \cdot \rangle \) by

(0.6) \[ \langle f, G \rangle = (e^{\tau A}f, G(\tau))_X . \]

The definition (0.6) makes sense for \( \tau > 0 \) sufficiently small and it does not depend on the choice of \( \tau \). Through this pairing, all strongly continuous functionals on \( S_{x,A} \) are represented by the members of \( T_{x,A} \). Conversely, all continuous functionals on \( T_{x,A} \) are represented by the members of \( S_{x,A} \). The spaces \( S_{x,A} \) and \( T_{x,A} \) are in duality.
Entireness spaces and ultra-trajectory spaces

Another distribution theory which is a considerable generalization of the theory of tempered distributions, has been developed by Van Eijndhoven, see [2]. In order to introduce this theory along the lines of formal series expansions we start with a fixed sequence of positive numbers \((\mu_n)_{n=0}^{\infty}\) which are ordered, \(0 < \mu_0 \leq \mu_1 \leq \ldots \) and \(\mu_n \to \infty, n \to \infty\). Further, there has to be \(t_0 > 0\) such that \(\sum_{n=0}^{\infty} e^{-\mu_n t_0} < \infty\). Now for the space \(E\) we take the vector space of all formal series \(\sum_{n=0}^{\infty} \frac{a}{n!} n!\) where the sequences \((a_n)_{n=0}^{\infty}\) satisfy

\[
(0.7.a) \quad \exists_{t > 0}: \sup_{n \in \mathbb{N} \cup \{0\}} \left| a_n \right| e^{-\mu_n t} < \infty
\]

or equivalently

\[
(0.7.b) \quad \exists_{t > 0} \forall s \geq t: \sum_{n=0}^{\infty} \left| a_n \right|^2 e^{-2\mu_n s} < \infty.
\]

To a sequence \((a_n)_{n=0}^{\infty}\) with the property (0.7.b) we associate a mapping \(\Phi\) from \([t, \infty)\) into \(X\) as follows:

\[
\Phi(s) = \sum_{n=0}^{\infty} e^{-\mu_n s} a_n n!, \quad s \geq t.
\]

Note that

\[
\Phi(s_1 + s_2) = e^{-s_1 H} \Phi(s_2), \quad s_1 \geq 0, \ s_2 \geq t.
\]

On the other hand, let \(\Psi: [t, \infty) \to X\) satisfy

\[
\Psi(s_1 + s_2) = e^{-s_1 H} \Psi(s_2), \quad s_1 \geq 0, \ s_2 \geq \tilde{t}.
\]

Then there exists a sequence \((b_n)_{n=0}^{\infty}\), which satisfies (0.7.b) and
6.

\[ \forall(s) = \sum_{n=0}^{\infty} e^{-\|n\| s} b_n n_n, \quad s \geq t. \]

We thus arrive at the following general definition.

(0.8) Definition

Let \( A \) be a positive self-adjoint operator in \( X \). Let \( X_t, t > 0 \), denote the space of all mappings \( \Phi \) from \( [t, \infty) \) into \( X \) satisfying

\[ \Phi(s_1 + s_2) = e^{-s_1 A} \Phi(s_2), \quad s_1 \geq 0, \quad s_2 \geq t. \]

Then the space \( \sigma(X,A) \) is defined to be the inductive limit

\[ \sigma(X,A) = \bigcup_{t>0} X_t. \]

(We note that \( X_t \subset X \) for all \( t, 0 < t < T \).)

The space \( X_t \) is a Hilbert space with inner product \( \langle \phi, \psi \rangle(t) = \langle \phi(t), \psi(t) \rangle_X \).

We note that \( X_t \) is a copy of \( X \).

The space \( \sigma(X,A) \) is called the ultra-trajectory space. Inspired by [6] explicit seminorms have been constructed which generate a locally convex topology equivalent to the inductive limit topology. It has been proved that \( \sigma(X,A) \) is complete, bornological and barreled. Further, \( \sigma(X,A) \) is Montel iff \( e^{-tA} \) is compact for some \( t > 0 \); \( \sigma(X,A) \) is nuclear iff \( e^{-tA} \) is Hilbert-Schmidt for some \( t > 0 \).

Suppose that the Fourier coefficients \( (f,v_n), n \in \mathbb{N} \cup \{0\}, \) of an element \( f \in X \) satisfy \( \forall t > 0: \langle f, v_n \rangle = 0(e^{-\|n\| t}) \) or, equivalently, \( \forall t > 0: \sum_{n=0}^{\infty} |\langle f, v_n \rangle|^2 e^{-\|n\| t} < \infty \). Then for any sequence \( (a_n)_{n=0}^{\infty} \) satisfying (0.7) the series \( \sum_{n=0}^{\infty} a_n (f, v_n) \) converges absolutely. So the following definition seems suitable in this context.
(0.9) Definition
Let $A$ be a positive self-adjoint operator in $X$. Then the space $\tau(X,A)$ is defined by

$$\tau(X,A) = \bigcap_{t>0} e^{-tA}(X) = D((e^A)^\omega).$$

The space $\tau(X,A)$ is called the entireness space, because it contains all entire vectors of $A$. We note that $w$ is an entire vector of $A$ iff

$$\forall a>0 \exists b>0 \forall n \in \mathbb{N}: \|A^nw\| \leq n! a^n b.$$ 

With the seminorms

$$w \mapsto \|e^{tA}w\|_X, \quad w \in \tau(X,A), \quad t > 0,$$

$\tau(X,A)$ becomes a Frechet space.

The pairing between the spaces $\tau(X,A)$ and $\sigma(X,A)$ is defined as follows. Let $w \in \tau(X,A)$ and let $\phi \in \sigma(X,A)$. Then

$$(0.10) \quad <w,\phi> = (e^{sA}w,\phi(s))_X$$

where $s > 0$ has to be taken so large that $\phi \in X_s$. We note that (0.10) does not depend on the choice of $s$. With this pairing $\sigma(X,A)$ is a representation of the strong dual of $\tau(X,A)$ and, conversely, $\tau(X,A)$ is a representation of the strong dual of $\sigma(X,A)$.

From a topological point of view the spaces $\tau(X,A)$ and $T_{X,A}$, and $\sigma(X,A)$ and $S_{X,A}$ have similar properties. So the theory [2] can be considered as a kind of reverse of the theory [6]. For instance, a sequence $(\phi_n)_{n \in \mathbb{N}}$ in $\sigma(X,A)$ is a null sequence iff there exists $s > 0$ such that $(\phi_n)_{n \in \mathbb{N}} \subset X_s$ and

$$\|\phi_n(s)\| \to 0$$

as $n \to \infty$. Finally we mention the following inclusion scheme

$$\tau(X,A) \subset S_{X,A} \subset X \subset T_{X,A} \subset \sigma(X,A).$$
SOME NOTATIONS

In this paper we consider the Hilbert spaces

\[ X_{\alpha, \beta} = L^2([-1,1], (1-x)^\alpha (1+x)^\beta \, dx) \]

and the positive self-adjoint operator \( A_{\alpha, \beta} \) in \( X_{\alpha, \beta} \)

\[ A_{\alpha, \beta} = -(1-x^2) \frac{d^2}{dx^2} - ((\beta - \alpha) - (\alpha + \beta + 2)x) \frac{d}{dx} \]

where we take \( \alpha \) and \( \beta \) larger than \(-1\). The operator \( A_{\alpha, \beta} \) has a discrete spectrum: \( \{ n(n+\alpha+\beta+1) \mid n \in \mathbb{N} \cup \{0\} \} \). Its normalized eigenvectors are the normalized Jacobi polynomials \( R_n^{(\alpha, \beta)} \),

\[ R_n^{(\alpha, \beta)} = \left[ \frac{\alpha + \beta + 2n + 1}{2^{\alpha + \beta + 1} \Gamma(n+1)\Gamma(n+\alpha+\beta+1)} \right]^\frac{1}{2} p_n^{(\alpha, \beta)}(x) \]

where

\[ p_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{n! 2^n (1-x)^\alpha (1+x)^\beta} \left( \frac{d}{dx} \right)^n [(1-x)^\alpha (1+x)^\beta] \]

(cf. [8], p. 209).

In this report we shall work out the following program:

- Classification of the space \( S_{X_{\alpha, \beta}, (A_{\alpha, \beta})^\nu} \) and \( \tau(X_{\alpha, \beta}, (A_{\alpha, \beta})^\nu) \) where we take \( \nu \geq \frac{1}{2} \) and \( \alpha, \beta > -1 \). We get the following result: For all pairs \( (\alpha, \beta) \) and \( (\gamma, \delta) \) and for all \( \nu \geq \frac{1}{2} \)

\[ S_{X_{\alpha, \beta}, (A_{\alpha, \beta})^\nu} = S_{X_{\alpha, \beta}, (A_{\alpha, \beta})^\nu} \]

and, also,

\[ \tau(X_{\alpha, \beta}, (A_{\alpha, \beta})^\nu) = \tau(X_{\alpha, \beta}, (A_{\alpha, \beta})^\nu) \]

With the transformation \( x = \cos \theta \), we get
$Y_{\alpha,\beta} = L_2([0,\pi],(1 - \cos \theta)^\alpha(1 + \cos \theta)^\beta \sin \theta \, d\theta)$

and

$C_{\alpha,\beta} = -\frac{d^2}{d\theta^2} + (\beta - \alpha) \frac{1}{\sin \theta} \frac{d}{d\theta} - (\alpha + \beta + 1) \cot \theta \frac{d}{d\theta}.$

It is clear that

$C_{\alpha,\beta}(R_n^{(\alpha,\beta)}(\cos \theta)) = n(n+\alpha+\beta+1)R_n^{(\alpha,\beta)}(\cos \theta).$

Moreover, for all $\nu \geq \frac{1}{2},$

$S_{\nu} Y_{\alpha,\beta}(C_{\alpha,\beta}) = S_{\nu} Y_{\delta}^*,(C_{\delta}^*)^\nu$

and

$\tau(\nu Y_{\alpha,\beta}(C_{\alpha,\beta})^\nu) = \tau(\nu Y_{\delta}^*,(C_{\delta}^*)^\nu).$

- Characterization of the spaces $S_{Y_{-\frac{1}{2}},-\frac{1}{2}}(C_{-\frac{1}{2},-\frac{1}{2}})$ and $\tau(Y_{-\frac{1}{2},-\frac{1}{2}}(C_{-\frac{1}{2},-\frac{1}{2}}), for each $\nu \geq \frac{1}{2},$ in classical analytic terms. Here we employ the relations

$R_n^{(-\frac{1}{2},-\frac{1}{2})}(\cos \theta) = \sqrt{\frac{2}{\pi}} \cos n\theta, \ n \in \mathbb{N}$

and

$R_0^{(-\frac{1}{2},-\frac{1}{2})}(\cos \theta) = \frac{1}{\sqrt{\pi}}.$

By means of this characterization we can also describe the spaces $S_{X_{-\frac{1}{2},-\frac{1}{2}}(A_{-\frac{1}{2},-\frac{1}{2}})}$ and $\tau(X_{-\frac{1}{2},-\frac{1}{2}}(A_{-\frac{1}{2},-\frac{1}{2}}), \nu \geq \frac{1}{2}.$

- Relations between the spaces $T_{X_{\alpha,\beta}}(A_{\alpha,\beta})^\frac{1}{2}$ and $\sigma(X_{\alpha,\beta},(A_{\alpha,\beta})^\frac{1}{2})$ with some classes of hyperfunctions, respectively ultra-hyperfunctions.
SURVEY OF RESULTS

This University Report is a contribution to our research project of making a link between our general functional analytic theory on analyticity spaces, trajectory spaces, entireness spaces and ultra-trajectory spaces on one hand and "classical" analysis and distribution theory on the other hand. Similar results have been achieved in the papers [1], [3], [4] and [5] where test spaces have been introduced in which the Hermite functions and the Laguerre functions serve as bases in the way pointed out in the preliminaries. In the present paper the main emphasis is on the test spaces in which the Jacobi polynomials establish bases. So the elements of these test function spaces are determined by sequences of expansion coefficients of certain growth orders.

We give a detailed classification of such test spaces and characterize them as classes of analytic functions which satisfy specific growth conditions. This characterization enables to describe a variety of continuous linear functionals and continuous linear operators on these spaces in classical analytic terms. As a further result we find conditions on the asymptotics of the coefficients in the Jacobi series expansion of an analytic (entire) function which belongs to such a growth class. Finally, we show that the duals of some of our test spaces can be represented by spaces of (ultra-) hyperfunctions. Each (ultra-) hyperfunction can be expanded in so-called Jacobi functions of the second kind. We describe the convergence of these expansions in terms of complex analysis.

Now we go into more detail.

In Chapter I we study the general classification problem which can be stated as follows. Let there be given a separable Hilbert space X and a positive self-adjoint operator A in X. Look for pairs (Y,B) such that $S_{X,A} = S_{Y,B}$ and...
\( \tau(X, A) = \tau(Y, B) \), both set theoretically and topologically. For instance, if \( \omega \) is a continuous bijection on \( S_{X, A} \), respectively \( \tau(X, A) \), then we may take \( Y = X^{\omega} \) and \( B = \omega^{-1} A \omega \). Here \( X^{\omega} \) denotes the completion of \( S_{X, A} (\tau(X, A)) \) with respect to the norm \( \| \cdot \|_{X} = \| \omega \|_{X} \). Special attention is given to the case \( X = \ell_{2} \) and \( A = \Lambda \) with \( \Lambda \) a diagonal operator in \( \ell_{2} \). Also, we mention the case that \( B \) is a perturbation of \( A \). This has been investigated in an earlier paper [4] and quoted in Chapter 1 for completeness. Further, we pay attention to the interrelation between the spaces \( T_{X, A} \) and \( T_{Y, B} \) which both are representations of the dual \( S'_{X, A} \) if \( S_{X, A} = S_{Y, B} \).

In Chapter 2 we apply the abstract results on classification of analyticity spaces and entireness spaces to the following concrete case.

Let \( X_{\alpha, \beta} \) be the Hilbert space \( L_{2}([-1, 1], (1-x)^{\alpha}(1+x)^{\beta} dx) \) and let \( A_{\alpha, \beta} \) be the positive self-adjoint operator

\[
A_{\alpha, \beta} = - (1-x^2) \frac{d^2}{dx^2} - (\beta-\alpha) \frac{d}{dx} + (\alpha+\beta+2)x \frac{d}{dx}
\]

as introduced in the Preliminaries. We show that for all pairs \( (\alpha, \beta), (\gamma, \delta) \) with \( \alpha, \beta, \gamma, \delta > -1 \) and for all \( \nu \geq \frac{1}{2} \)

\[
S_{X_{\alpha, \beta}, (A_{\alpha, \beta})^{\nu}} = S_{X_{\gamma, \delta}, (A_{\gamma, \delta})^{\nu}}
\]

and

\[
\tau(X_{\alpha, \beta}, (A_{\alpha, \beta})^{\nu}) = \tau(X_{\gamma, \delta}, (A_{\gamma, \delta})^{\nu})
\]

The coordinate transformation \( x = \cos \theta \) transforms the Hilbert space \( X_{\alpha, \beta} \) in

\[
Y_{\alpha, \beta} = L_{2}([0, \pi], (1 - \cos \theta)^{\alpha}(1 + \cos \theta)^{\beta} \sin \theta d\theta)
\]

and the operator \( A_{\alpha, \beta} \) becomes...
\[ C_{\alpha, \beta} = -\frac{d^2}{d\theta^2} + (\beta - \alpha) \frac{1}{\sin \theta} \frac{d}{d\theta} - (\alpha + \beta + 1) \cot \theta \frac{d}{d\theta}. \]

We have for all pairs \((\alpha, \beta), (\gamma, \delta)\), \(\alpha, \beta, \gamma, \delta > -1\), and all \(\nu \geq \frac{1}{2}\)

\[ S_{Y_{\alpha, \beta}, (C_{\alpha, \beta})} \tau = S_{Y_{\gamma, \delta}, (C_{\gamma, \delta})} \tau \]

and

\[ \tau(Y_{\alpha, \beta}^{\nu}, (C_{\alpha, \beta})^{\nu}) = \tau(Y_{\gamma, \delta}^{\nu}, (C_{\gamma, \delta})^{\nu}). \]

In order to apply the general theorems of Chapter 1, we derive rather subtle estimates for the matrix entries of the operators \(\frac{d}{dx}\) and \(x \frac{d}{dx}\) with respect to the Jacobi polynomial bases. (See the Appendix to this paper.)

Chapter 3 deals with the characterization problem. We look for a description in terms of classical analysis, of the spaces

\[ S_{X_{\alpha, \beta}, (A_{\alpha, \beta})} \tau, \tau(X_{\alpha, \beta}, (A_{\alpha, \beta})^{\nu}), \]

\[ S_{Y_{\alpha, \beta}, (C_{\alpha, \beta})} \tau \text{ and } \tau(Y_{\alpha, \beta}^{\nu}, (C_{\alpha, \beta})^{\nu}). \]

To this end, we start with a characterization of the spaces

\[ \mathcal{S}_{L^2([-\pi, \pi])}, \left(-\frac{d^2}{d\theta^2}\right)^\nu \text{ and } \tau\left(L^2([-\pi, \pi]), \left(-\frac{d^2}{d\theta^2}\right)^\nu\right), \nu \geq \frac{1}{2}. \]

These spaces can be presented as classes of \(2\pi\)-periodic analytic functions of well-defined growth behaviour dependent on \(\nu\). Since

\[ S_{Y_{\alpha, \beta}, (C_{\alpha, \beta})}^{\nu} = S_{Y_{-\frac{1}{2}, -\frac{1}{2}}, (C_{-\frac{1}{2}, -\frac{1}{2}})}^{\nu} = S_{L^2([0, \pi]), \left(-\frac{d^2}{d\theta^2}\right)^\nu}, \alpha, \beta > -1, \]

the space \(S_{Y_{\alpha, \beta}, (C_{\alpha, \beta})}^{\nu}\) is the subspace of \(S_{L^2([-\pi, \pi]), \left(-\frac{d^2}{d\theta^2}\right)^\nu}\) which consists of all even functions in \(S_{L^2([-\pi, \pi]), \left(-\frac{d^2}{d\theta^2}\right)^\nu}\). Similarly, \(\tau(Y_{\alpha, \beta}^{\nu}, (C_{\alpha, \beta})^{\nu})\) contains precisely all even functions in \(\tau\left(L^2([-\pi, \pi]), \left(-\frac{d^2}{d\theta^2}\right)^\nu\right)\). The
characterization of the spaces $S_{X,\alpha,\nu}(A,\beta)^{p}$ and $\tau(X,\alpha,\beta)(A,\beta)^{p}$ is then obtained by means of the conformal mapping $w = \cos z$.

Important consequences of this characterization are theorems on approximation of analytic (entire) functions by means of (normalized) Jacobi polynomials. We mention the following.

- Let $f$ be an entire function satisfying

$$|f(w)| \leq A \exp\left(B \left(\log |w|\right)^{2\nu-1}\right), \quad |w| \geq 1,$$

where $A, B > 0$ and $\nu > \frac{1}{2}$. Let $\alpha, \beta > -1$. Then

$$f = \sum_{n=0}^{\infty} a^n (\alpha, \beta) R_n (\alpha, \beta)$$

where the coefficients $a^n (\alpha, \beta)$ satisfy

$$\sup_{n \in \mathbb{N}} |a^n (\alpha, \beta)| \exp(n^{\nu}(t-\varepsilon)) < \infty$$

for all $0 < \varepsilon < t$ with $t = (2\nu)^{-2\nu} \left(\frac{2\nu-1}{B}\right)^{2\nu-1}$. The series

$$\exp\left(-B \left(\log |w|\right)^{2\nu-1}\right) \sum_{n=0}^{\infty} a^n (\alpha, \beta) R_n (\alpha, \beta)(w)$$

converges uniformly on $w$.

- Let $\alpha, \beta > -1$, and let $(b^n (\alpha, \beta))_{n \in \mathbb{N} \cup \{0\}}$ be a sequence for which

$$\sup_{n \in \mathbb{N}} \left|b^n (\alpha, \beta) \exp(n^{\nu}t)\right| < \infty \text{ for some } \nu > \frac{1}{2} \text{ and some } t > 0.$$ Then the function $f$ defined by $f(w) = \sum_{n=0}^{\infty} b^n (\alpha, \beta) R_n (\alpha, \beta)(w)$ is entire analytic and

$$|f(w)| \leq A_f \exp\left(B_f \left(\log |w|\right)^{2\nu-1}\right), \quad |w| \geq 1.$$
Here $A_f > 0$, and for $B_f$ any number larger than $(2\nu-1)\left(\frac{(2\nu)^{-2\nu}}{t}\right)^{\frac{1}{2\nu-1}}$ can be taken.

Let $f$ be an analytic function in an open neighbourhood of the interval $[-1,1]$. Let $\alpha, \beta > -1$. Then $f = \sum_{n=0}^{\infty} a_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}$ where the coefficients $a_n^{(\alpha,\beta)}$ satisfy $\sup_n |a_n^{(\alpha,\beta)}| \exp(nt) < \infty$ for some $t > 0$ depending on $\alpha$ and $\beta$. The series converges uniformly on a sufficiently small neighbourhood of $[-1,1]$. (Cf. [9], where sharper results have been obtained.)

Let $\alpha, \beta > -1$ and let $(b_n^{(\alpha,\beta)})_{n=0}^{\infty} \in \mathbb{C}^{\mathbb{N}\cup\{0\}}$. Then $\sum_{n=0}^{\infty} b_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}$ represents an entire function iff for all $t > 0$, $\sup_{n \in \mathbb{N}} |b_n^{(\alpha,\beta)}| \exp(nt) < \infty$.

Since

$$\delta_z = \sum_{n=0}^{\infty} R_n^{(\alpha,\beta)}(z)R_n^{(\alpha,\beta)}(A_{\alpha,\beta}^z) \in \mathcal{H}_{\alpha,\beta}(A_{\alpha,\beta}^z)$$

for all $\nu > \frac{1}{2}$, we get:

$$\forall t > 0 \, \forall \omega, \xi \in B > (2\nu)^{-2\nu}\left(\frac{2\nu-1}{t}\right)^{\frac{1}{2\nu-1}} \exists A_{B,\omega} > 0$$

$$\left| \sum_{n=0}^{\infty} \exp(-(n(n+\alpha+\beta+1))^\nu t) R_n^{(\alpha,\beta)}(z) R_n^{(\alpha,\beta)}(\omega) \right| \leq$$

$$\leq A_{B,\omega} \exp\left(B\log(\max(1,|z|))\right)\left(\frac{2\nu}{t}\right)^{\frac{1}{2\nu-1}}.$$

With the aid of these classical analytic descriptions and with elementary geometrical considerations we introduce natural classes of continuous linear functionals and continuous (extendible) linear mappings on all $S$- and $\tau$-spaces mentioned above.
In Chapters 4 and 5 we limit ourselves to the case \( v = \frac{1}{2} \). The duals

\[
S'_{L_2([-\pi, \pi])}, \left(-\frac{d^2}{d\theta^2}\right)^{\frac{1}{2}}, \quad S'_{Y_{a,\beta},(C_{a,\beta})^{\frac{1}{2}}} \quad \text{and} \quad S'_{X_{a,\beta},(A_{a,\beta})^{\frac{1}{2}}}
\]

are linked to three classes of hyperfunctions. The representation of the considered linear functionals as hyperfunctions is by means of contour integrals.

In Chapter 4 we introduce a natural Frechet topology on each of these hyperfunction spaces. As a consequence we get the following classical result (see [9], p. 250): A function \( \theta \) which is analytic on the region \( \mathbb{C} \setminus [-1,1] \) with \( \theta(\infty) = 0 \), can be expanded in a series of associated Jacobi functions, \( Q_n^{(\alpha,\beta)} \) defined by

\[
Q_n^{(\alpha,\beta)}(\omega) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{1}{x - \omega} R_n^{(\alpha,\beta)}(x) dx.
\]

The coefficients \( a_n^{(\alpha,\beta)} \) in this series satisfy:

\[
\forall t > 0 \sup_{n \in \mathbb{N}} |a_n^{(\alpha,\beta)} e^{-nt}| < \infty
\]

and the series \( \sum_{n=0}^{\infty} a_n^{(\alpha,\beta)} Q_n^{(\alpha,\beta)} \) converges uniformly outside each open neighbourhood of \([-1,1]\). Results in this direction have also been obtained in [10].

In Chapter 5 the duals \( \tau' \left( L_2([-\pi, \pi]), \left(-\frac{d^2}{d\theta^2}\right)^{\frac{1}{2}} \right), \tau' \left( Y_{a,\beta},(C_{a,\beta})^{\frac{1}{2}} \right) \) and \( \tau' \left( X_{a,\beta},(A_{a,\beta})^{\frac{1}{2}} \right) \) are treated in the same way as the corresponding \( S' \)-spaces in the previous chapter. These duals can be represented by classes of so-called ultra-hyperfunctions. Each space of ultra-hyperfunctions can be regarded as an inductive limit of Banach spaces. It follows that the spaces \( \sigma \left( L_2([-\pi, \pi]), \left(-\frac{d^2}{d\theta^2}\right)^{\frac{1}{2}} \right), \sigma \left( Y_{a,\beta},(C_{a,\beta})^{\frac{1}{2}} \right) \) and \( \sigma \left( X_{a,\beta},(A_{a,\beta})^{\frac{1}{2}} \right) \) are homeomorphic to certain ultra-hyperfunction spaces. Finally, classical results are obtained concerning the expansion in associated Jacobi function series of functions which are analytic at infinity.
In Chapter 6 we deal with two large continuous groups of analytic functions. These groups can be represented as groups of continuous linear mappings on (ultra-) hyperfunction spaces.
1. CLASSIFICATIONS OF ANALYTICITY SPACES AND ENTIRENESS SPACES

In this chapter, we discuss two methods to classify analyticity spaces and entireness spaces. First, we consider the case of a general positive self-adjoint operator in a Hilbert space. Next, we turn to a more concrete situation.

Let $X$ denote a Hilbert space and let $A$ be a positive self-adjoint operator in $X$. In our paper [5] we have proved the following classification theorems, in which a central role is played by perturbations.

(1.1) Theorem

Let $P$ be a linear operator in $X$ with $D(P) \supset S_{X,A^\nu}$ where we take $\nu > 0$ fixed. Let the following conditions be satisfied:

(i) There exists a Hilbert space $Y$ such that the operator $\exp(-tA^\nu)$ maps $X$ into $Y$ for all $t > 0$.

(ii) The operator $A + P$ defined on $S_{X,A^\nu}$ can be extended to a positive self-adjoint operator in $Y$ (denoted by $A + P$ also).

(iii) There exists a monotone non-increasing function $\varphi$ on the open interval $(0,1)$ such that

$$\forall r, 0 < r < 1: \| \exp(rA^\nu)P \| \leq \varphi(r).$$

Then $S_{X,A^\nu} \subset S_{Y,(A+P)^\nu}$.

Remark. In fact we proved that under the conditions of Theorem (1.1) the following result is valid: $\forall t > 0$ $\exists \tau > 0$ $\exists \tau > 0$ such that the operator $\exp(\tau A^\nu)\exp(\tau(A+P)^\nu)\exp(-tA^\nu)$ is bounded on $X$.

For entireness spaces the following similar theorem is valid:
(1.2) **Theorem**

Let \( P \) be a linear operator in \( X \) such that \( D(P) \supset \exp(-tA^V)(X) \) for some \( t > 0 \) and some \( v > 0 \) fixed. Let the following conditions be satisfied:

(i) There exists a Hilbert space \( Y \) and there exists \( t > 0 \) such that the operator \( \exp(-tA^v) \) maps \( X \) into \( Y \).

(ii) The operator \( A + P \) defined on \( \tau(X,A^v) \) is extendable to a positive and self-adjoint operator in \( Y \).

(iii) There exist positive constants \( d \) and \( r_0 \) and there exists \( q, \) \( 0 < q < \frac{1}{v} \) such that

\[
\forall \tau > r_0 : \| \exp(\tau A^v)PA^{-1} \exp(-\tau A^v) \| < dr^q .
\]

Then \( \tau(X,A^v) \subset \tau(Y,(A+P)^v) \).

**Remark.** In fact we proved that under the conditions of Theorem (1.2) the following result is valid: \( \forall t > 0 \) \( \forall \tau > 0 \) \( \exists t > 0 \) such that the operator \( \exp(\tau A^v)\exp(\tau(A+P)^v)\exp(-\tau A^v) \) is bounded on \( X \).

Next we consider another way of classifying analyticity spaces and entireness spaces. We prove that each continuous bijection \( \omega \) on \( S_{X,A} \) \( (\tau(X,A)) \) gives rise to a Hilbert space \( X^\omega \) and a positive self-adjoint operator \( A^\omega \) in \( X^\omega \) such that \( S_{X,A} = S_{X^\omega,A^\omega} \) \( (\tau(X,A) = \tau(X^\omega,A^\omega)) \).

We denote the inner product of \( X \) by \( \langle \cdot, \cdot \rangle \). Let \( \omega \) be a continuous bijection on the analyticity space \( S_{X,A} \). Since also its inverse mapping \( \omega^{-1} \) is continuous on \( S_{X,A} \) the sesquilinear form

\[
(f,g)_\omega = (\omega f, \omega g) , \quad f,g \in S_{X,A}
\]
turns $S_{X,A}$ into a pre-Hilbert space. By $\mathcal{H}_W$ we denote the Hilbert completion of $S_{X,A}$ with respect to $(\cdot, \cdot)_W$. Then we have

(1.3) Theorem
The operator $\omega^{-1}AW$ with domain $S_{X,A}$ is positive and essentially self-adjoint in $\mathcal{H}_W$. If $A_W^\omega$ denotes its unique positive self-adjoint extension in $\mathcal{H}_W$, then

$S_{X,A} = S_{X,W,A_W^\omega}$

Proof
Let $f = \omega^{-1}g \in S_{X,A}$. Since $g \in S_{X,A}$ there exist $a, b > 0$ such that

$$\|A^n g\| \leq ab^n n! , \quad n \in \mathbb{N}.$$  

Hence

$$\| (\omega^{-1}A) f \|_W = \| \omega (\omega^{-1}A^n \omega) \omega^{-1} g \| \leq ab^n n!$$  

for all $n \in \mathbb{N}$. Thus it follows that the analyticity domain of the operator $\omega^{-1}A_W$ in $\mathcal{H}_W$ contains $S_{X,A}$. Since $S_{X,A}$ is dense in $\mathcal{H}_W$, the analyticity domain of $\omega^{-1}A_W$ is dense in $\mathcal{H}_W$. So following [11], Theorem 8.31, $\omega^{-1}A_W$ is essentially self-adjoint in $\mathcal{H}_W$. It is clear that $\omega^{-1}A_W$ is positive on $S_{X,A} \subset \mathcal{H}_W$. Hence the unique self-adjoint extension $A_W$ is positive. Since $S_{X,W,A_W}^\omega$ contains precisely all analytic vectors of $A_W$, we obtain $S_{X,A} \subset S_{X,W,A_W}^\omega$. On $S_{X,A}$ we have for all $t > 0$

$$\omega e^{-tA} f = e^{-tA} \omega f.$$  

Now observe that the linear mapping $\omega$ on $S_{X,A}$ can be extended to a bounded linear operator from $\mathcal{H}_W$ onto $\mathcal{H}$. We denote this extension also by $\omega$. It follows that for all $x \in \mathcal{H}_W$

$$e^{-tA} x = \omega^{-1} e^{-tA} \omega x.$$  

The assertion $S_{X,A} \supseteq S_{x,A}^{\omega}$ is then obtained from the observation that $\omega x \in X$ for all $x \in X$ and that $\omega^{-1}$ is continuous on $S_{X,A}$. \[ \square \]

Since $S_{X,A} = S_{x,A}^{\omega}$, the elements $f \in S_{X,A}$ can be paired both with the elements $G \in T_{X,A}$

\[ \langle f,G \rangle = (e^A_f,G(t)) , \quad t > 0 \text{ sufficiently small} , \]

and with the elements $H \in T_{x,A}^{\omega}$

\[ \langle f,H \rangle = (e^A_f,H(p))_\omega , \quad p > 0 \text{ sufficiently small}. \]

Let $\ell$ be a continuous linear functional on $S_{X,A}$. Then there exists $T(\ell) \in T_{X,A}$ such that $\ell(f) = \langle f,T(\ell) \rangle$, $f \in S_{X,A}$, and also $T_{\omega}(\ell) \in T_{x,A}^{\omega}$ such that $\ell(f) = \langle f,T_{\omega}(\ell) \rangle$ $f \in S_{X,A}$. So the antilinear mappings $T: \ell \mapsto T(\ell)$ and $T_{\omega}: \ell \mapsto T_{\omega}(\ell)$ are isomorphisms from $S_{X,A}$ onto $T_{X,A}$ and from $S_{X,A}$ onto $T_{x,A}^{\omega}$, respectively.

We shall investigate the relationship between $T(\ell)$ and $T_{\omega}(\ell)$. To this end, we define the mapping $j$ on $T_{X,A}$ by

\[ j(F): t \mapsto \omega^{-1} F(t) , \quad t > 0 , \quad F \in T_{X,A} . \]

It is clear that $(j(F))(t) \in S_{X,A} \subset X^{\omega}$ for all $t > 0$, and also that for all $t, \tau > 0$

\[ (j(F))(t+\tau) = \omega^{-1} F(t+\tau) = (\omega^{-1} e^{-\tau A^\omega})^{\omega^{-1}} F(t) = e^{-\tau A^\omega} (j(F))(t) . \]

Hence $j(F) \in T_{X,A}^{\omega}$. Further it follows that $j$ is a continuous linear mapping and that $j$ is invertible with

\[ j^{-1}(H): t \mapsto \omega H(t) , \quad H \in T_{x,A}^{\omega} . \]
Now let \( \ell \in S_{X,A}^1 \). Then for all \( f \in S_{X,A} \),

\[
\ell(f) = ((\omega^{-1})' \ell)(\omega f) = \langle \omega f, T((\omega^{-1})' \ell) \rangle \\
= (e^{TA} \omega f, (T((\omega^{-1})' \ell))'(T)) \\
= (e^{TA} \omega f \omega (j(T((\omega^{-1})' \ell))'(T))) \\
= \langle \ell, j(T((\omega^{-1})' \ell)) \rangle_{\omega}. 
\]

We thus obtain

\[
\ell_{\omega}(\ell) = j(T((\omega^{-1})' \ell)) .
\]

In the same way

\[
\ell(T) = f^{-1}(\ell_{\omega}(\omega' \ell)) .
\]

It leads to the following result

\[
\ell_{\omega} = j \circ \ell \circ (\omega^{-1})' ; \quad \ell = f^{-1} \circ \ell_{\omega} \circ \omega'.
\]

which expresses the relation between the representations \( \ell \) and \( \ell_{\omega} \).

We now repeat the above considerations for the entireness space \( \tau(X,A) \).

Let \( \nu \) be a continuous bijection on the Frechet space \( \tau(X,A) \). Then \( \nu^{-1} \) is also continuous on \( \tau(X,A) \). With the inner product

\[
(f, g)_\nu = (\nu f, \nu g)
\]

\( \tau(X,A) \) becomes a pre-Hilbert space. Let \( X' \) denote the Hilbert space completion of \( \tau(X,A) \) with respect to \((\cdot,\cdot)_\nu\). It is clear that the operator \( \nu^{-1}A\nu \) with domain \( \tau(X,A) \subset X' \) is symmetric and positive. Similarly to Theorem (1.3) we have
(1.4) Theorem

The operator $\mathcal{V}^{-1} A \mathcal{V}$ is essentially self-adjoint and positive. Let $A^\mathcal{V}$ denote its unique self-adjoint extension in $X^\mathcal{V}$. Then

$$\tau(X,A) = \tau(X^\mathcal{V},A^\mathcal{V}).$$

Proof

By standard arguments it can be proved that $\tau(X,A)$ is contained in the analyticity domain of the operator $\mathcal{V}^{-1} A \mathcal{V}$. Since $\tau(X,A)$ is dense in $X^\mathcal{V}$, it follows that $\mathcal{V}^{-1} A \mathcal{V}$ is essentially self-adjoint. The operator $\mathcal{V}$ defined on $\tau(X,A)$, can be extended to a bounded operator from $X^\mathcal{V}$ into $X$. We denote this extension also by $\mathcal{V}$. Conversely, for each $g \in X$ the linear functional $\ell_g : f \mapsto (\mathcal{V}f,g)$ is continuous on $X^\mathcal{V}$. So there exists $h \in X^\mathcal{V}$ such that

$$(\mathcal{V}f,g) = (f,h)_{\mathcal{V}}.$$ 

Hence $g = \mathcal{V}h$. Therefore we write $h = \mathcal{V}^{-1} g$, and

$$g \in X \leftrightarrow \mathcal{V}^{-1} g \in X^\mathcal{V}.$$ 

We thus obtain

$$\mathcal{V}^{-1}(e^{-tA}(X)) = \mathcal{V}^{-1} e^{-tA} \mathcal{V}(X^\mathcal{V}),$$

and finally

$$\tau(X,A) = \mathcal{V}^{-1} \left( \bigcap_{t>0} e^{-tA}(X) \right) = \mathcal{V}^{-1} \left( \bigcap_{t>0} e^{-tA}(X) \right) =$$

$$= \bigcap_{t>0} (\mathcal{V}^{-1} e^{-tA}(X^\mathcal{V})) = \bigcap_{t>0} e^{-tA^\mathcal{V}}(X^\mathcal{V}) = \tau(X^\mathcal{V},A^\mathcal{V}).$$

We omit the investigation of the relationship between $\sigma(X,A)$ and $\sigma(X^\mathcal{V},A^\mathcal{V})$. 


Next we apply Theorems (1.3) and (1.4) to the following concrete case.

Let $H$ be a positive self-adjoint operator in the separable Hilbert space $X$, and let $H$ have a discrete spectrum. Then in $X$ there exists a complete orthonormal basis $(e_n)_{n=0}^{\infty}$ and there are positive numbers $\lambda_k$, $k \in \mathbb{N} \cup \{0\}$, such that $He_n = \lambda_n e_n$. The spaces $S_{X,H}$ and $T_{X,H}$ are nuclear iff $\sum_{k=0}^{\infty} e^{-\lambda_k t} < \infty$ for all $t > 0$. The spaces $\tau(X,H)$ and $\sigma(X,H)$ are nuclear iff there exists $t > 0$ such that $\sum_{k=0}^{\infty} e^{-\lambda_k t} < \infty$.

We define the unitary operator $U$ from $X$ into $\ell_2$ by

$$Uw = ((\omega, e_n))_{n=0}^{\infty}, \quad w \in X.$$ 

Then

$$U(S_{X,H}) = S_{\ell_2, \Lambda}, \quad U(\tau(X,H)) = \tau(\ell_2, \Lambda), \quad U(T_{X,H}) = T_{\ell_2, \Lambda}$$

and

$$U(\sigma(X,H)) = \sigma(\ell_2, \Lambda).$$

Here $\Lambda$ denotes the diagonal operator with matrix

$$\Lambda_{kl} = \lambda_k \delta_{kl}, \quad k, l \in \mathbb{N} \cup \{0\}.$$ 

So, instead of the spaces $S_{X,H}$ and $\tau(X,H)$, we can study the spaces $S_{\ell_2, \Lambda}$ and $\tau(\ell_2, \Lambda)$ as well. For instance, in $S_{\ell_2, \Lambda}$ there exists a natural identification between continuous linear mappings on this space and infinite matrices. In the sequel we make no distinction between a linear operator in $\ell_2$ and its corresponding matrix.

Let $\Lambda$ be a diagonal matrix with $\Lambda_{kk} = \lambda_k$, $k \in \mathbb{N} \cup \{0\}$, where $0 < \lambda_0 \leq \lambda_1 \leq \ldots$ and where $\sum_{k=0}^{\infty} e^{-\lambda_k t} < \infty$ for all $t > 0$. Also, let $D$ be a diagonal matrix with $D_{kk} > 0$ and with

$$D_{kk} = c(\lambda_k)^{\alpha}(1 + o(1))$$
for positive constants $c$ and $\alpha$. Then we have

$(1.5)$ Lemma

Let $\nu > 0$. Then $S_{\ell_2, \lambda^\nu} = S_{\ell_2, \mathcal{D}^\nu/\alpha}$ and $\tau(\ell_2, \lambda^\nu) = \tau(\ell_2, \mathcal{D}^\nu/\alpha)$.

Proof

For each $t > 0$, we define the subspaces $R_1(t)$ and $R_2(t)$ as follows:

$$R_1(t) = \{(f_j) \in \ell_2 \mid \sup_{j \in \mathbb{N} \cup \{0\}} (|f_j| \exp(\lambda_j^\nu t)) < \infty\}$$

$$R_2(t) = \{(f_j) \in \ell_2 \mid \sup_{j \in \mathbb{N} \cup \{0\}} (|f_j| \exp((\mathcal{D}_{jj})^{\nu/\alpha} t)) < \infty\}.$$  

The following relations are not hard to prove:

$$S_{\ell_2, \lambda^\nu} = \bigcup_{t>0} R_1(t), \quad \tau(\ell_2, \lambda^\nu) = \bigcap_{t>0} R_1(t),$$

$$S_{\ell_2, \mathcal{D}^\nu/\alpha} = \bigcup_{t>0} R_2(t), \quad \tau(\ell_2, \mathcal{D}^\nu/\alpha) = \bigcap_{t>0} R_2(t).$$

Let $g \in R_1(t)$. Then there exists $K_1 > 0$ such that

$$K_1 \geq |g_j| \exp(\lambda_j^\nu t) = |g_j| \exp((c\lambda_j)^\nu t c^{-\nu}) \geq |g_j| \exp((\mathcal{D}_{jj})^{\nu/\alpha} t c^{-\nu})$$

for all $j > j_0$ with $j_0 \in \mathbb{N}$ so large that

$$(\mathcal{D}_{jj})^{\nu/\alpha} \leq 2(c\lambda_j)^\nu, \quad j > j_0.$$  

Hence $g \in R_2(\frac{1}{t} c^{-\nu})$, and $R_1(t) \subset R_2(\frac{1}{t} c^{-\nu})$.

Conversely, let $h \in R_2(t)$. Then there exists $K_2 > 0$ such that
Moreover, there exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$
\[ D_{jj} \geq \frac{1}{2} c \lambda_j^a. \]
So for $j > j_0$ we obtain
\[ |h_j| \exp((D_{jj})^{\nu/2a} t) \geq |h_j| \exp(\lambda_j^a (\frac{1}{2} c \nu) t). \]
Thus we find that $h \in R_1((\frac{1}{2} c \nu) t)$ and $R_2(t) \subset R_1((\frac{1}{2} c \nu) t)$. Now our assertion is proved by taking intersections or unions.

In the remaining part of this chapter we consider the diagonal matrix $A$ with $\lambda_j = j$. Observe that for all $\nu, t > 0$, $\sum_{j=0}^{\infty} e^{-j \nu t} < \infty$. It implies that the analyticity space $S(\ell_2, \Lambda^\nu)$ and the entireness space $\tau(\ell_2, \Lambda^\nu)$ are both nuclear.

Let $K = (K_{mn})$ be a linear operator in $\ell_2$. We assume that the entries $K_{mn}$ satisfy the following conditions:

(1.6) \[ K_{mn} = 0 \quad \text{for } m > n \]
\[ \exists C > 0 \forall m, n : |K_{mm} - K_{nn}| > C \]
\[ \exists D > 0 \exists \gamma > 0 \forall m, 0 \leq m < n \forall n : |K_{mn}| \leq D n^\gamma. \]

It is clear that the numbers $K_{nn}$, $n \in \mathbb{N} \cup \{0\}$, are eigenvalues of $K$ with eigenvectors $u_{(n)}$, say, $u_{(n)}^{(n)} = 0$ for $j > n$. If we take $u_{(n)}^{(n)} = 1$, then the $u_{(n)}^{(m)}$, $m = 0, 1, \ldots, n$, have to satisfy
\[ (K_{mm} - K_{nn})u_{m}^{(n)} + K_{m, m+1} u_{m+1}^{(n)} + \ldots + K_{mn} u_{n}^{(n)} = 0 \]
for $m = 0, 1, \ldots, n-1$. Next, we define the linear operator $S = (S_{mn})$ by
Then algebraically we have the relation $KS = SM$, where $M$ denotes the diagonal matrix $M_k = K_{kk}$, $k \in \mathbb{N} \cup \{0\}$. We note that $M$ is injective on finite sequences.

In the remaining part of this chapter we take $\mu > 1$ fixed. We want to prove that $S$ is a continuous bijection on $S_{\ell_2^n, \lambda^\mu}$ and on $\tau(\ell_2^n, \lambda^\mu)$. To this end, first note that

$$S_{mn} = \begin{cases} 0 & \text{if } m > n \\ u_m(n) & \text{if } 0 \leq m \leq n . \end{cases}$$

For $t > 0$ we put

$$\sigma_{mn}^t = e^{-t(n^\mu - m^\mu)} S_{mn} .$$

Then for $\sigma_{mn}^t$ the following recurrent relation is valid

$$\sigma_{mn}^t = \frac{1}{K_{mm} - K_{nn}} \sum_{k=1}^{n-m} K_{m,n+k} \sigma_{m+k,n}^t .$$

We take a fixed $N_t \in \mathbb{N}$ so large that for all $n > N_t$

$$\frac{D}{C} n^{\gamma+1} \exp((-\rho-1)t \ n^{\rho-1}) < 1$$

where we set $\rho = \min(\mu,2)$. It leads to the following result.

(1.7) Lemma

$$\forall n > N_t \ \forall m, 0 \leq m \leq n : |\sigma_{mn}^t| \leq \exp(-t(n^\mu - (m-n))) .$$
Proof

Let \( n > N_t \). Since \( \sigma_{nn}^t = 1 \), the inequality is valid for \( m = n \). Hence, with 'backward induction' it follows that

\[
\exp(\tau(n-m))\left|\sigma_{mn}^t\right| \leq
\]

\[
\leq \frac{D}{K_{nn} - K_{nn}} \sum_{k=1}^{n-m} (m+k)^\gamma \exp(-\tau((m+k)^\mu - m^\mu)) \cdot
\]

\[
\cdot \exp(\tau^\mu(n-m)) \cdot \exp(-\tau(m+k)^\mu(n-m-k)) \leq
\]

\[
\leq \frac{D}{C} n^\gamma \sum_{k=1}^{n-m} \exp(-\tau((m+k)^\mu - m^\mu)) \leq
\]

\[
\leq \frac{D}{C} n^\gamma \sum_{k=1}^{n-m} \exp(-\tau k(\rho-1)(m+k)^{\rho-2}) \leq
\]

\[
\leq \frac{D}{C} n^\gamma \sum_{k=1}^{n-m} \exp(-\tau k\rho^{-1}k)\]

\[
\leq \frac{D}{C} n^\gamma+1 \exp(-\tau\rho^{-1}) < 1. \]

If we put

\[
L_t := \max_{0 \leq s \leq N_t, 0 \leq m \leq N_t} \left|\sigma_{mn}^t\right|
\]

Then Lemma (1.7) gives

(1.8) Lemma

\[
\forall \tau > 0 \exists C_t > 0: \sup_{n, m \in \mathbb{N} \cup \{0\}} \exp(-\tau(n^\mu - m^\mu))|S_{mn}| \leq C_t
\]
The same techniques as used above apply in the proof of the following lemma:

(1.9) Lemma

Let \( V = (V_{mn}) \) satisfy

\[
V_{mn} = \begin{cases} 
0 & \text{if } m > n \\
1 & \text{if } m = n 
\end{cases}
\]

and, also,

\[
\forall t > 0 \exists B_t > 0: \sup_{n, m \in \mathbb{N} \cup \{0\}} |\exp(-t(n^\mu - m^\nu))V_{mn}| < B_t.
\]

Then \( V \) is invertible and \( V^{-1} \) satisfies

\[
(V^{-1})_{mn} = 0 \quad \text{if } m > n,
\]

\[
(V^{-1})_{mn} = 1 \quad \text{if } m = n,
\]

and

\[
\forall t > 0 \forall \varepsilon > 0: \sup_{n, m} \left( |(V^{-1})_{mn}| \exp(-n^\mu(t + \varepsilon) + m^\mu t) \right) < \infty.
\]

Proof

Let \( (W_{mn}) \) denote the inverse matrix of \( (V_{mn}) \) (which exists algebraically!). Then we have

\[
[\begin{array}{l}
\omega_{mn} = 1 \\
\omega_{mn} = 0 \quad \text{if } m > n \\
\omega_{mn} = - \sum_{k=m+1}^{n} V_{mk} \omega_{nk} \quad \text{if } 0 \leq m < n
\end{array}]
\]

Now we put

\[
\omega^t_{mn} = \exp(-t(n^\mu - m^\nu))\omega_{mn} \quad \text{and} \quad \varphi^t_{mn} = \exp(-t(n^\mu - m^\nu))V_{mn}.
\]
We want to estimate $\omega_{mn}^t$. To this end note first that

$$\omega_{mn}^t = - \sum_{k=m+1}^{n} q_{mk}^t \omega_{kn}^t, \quad 0 \leq m < n.$$ 

Hence by assumption

$$|\omega_{mn}^t| \leq B_t \sum_{k=m+1}^{n} |\omega_{kn}^t|$$

where we may as well suppose that $B_t > 1$.

We assert that $|\omega_{mn}^t| \leq 2^{n-m} (B_t)^{n-m}, \quad 0 \leq m \leq n$. To show this we use backward induction:

- $|\omega_{mn}^t| = 1 < 2$

- $|\omega_{mn}^t| \leq \frac{B_t}{2^{n-m+1}} \sum_{k=m+1}^{n} B_t^{n-k} 2^{-k} 2^{n-k+1} = \sum_{k=m+1}^{n} B_t^{n-k} 2^{-k} 2^{n-k+1} \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$

(1.10) Corollary

Let $\mathcal{V}$ be as in Lemma (1.9). Then

$$\forall t > 0 \quad \forall \epsilon > 0: \quad \| e^{tR} \mathcal{V} e^{-(t+\epsilon)R} \| < \infty$$

and

$$\forall t > 0 \quad \forall \epsilon > 0: \quad \| e^{tR} \mathcal{V}^{-1} e^{-(t+\epsilon)R} \| < \infty.$$ 

Here $\| \cdot \|$ denotes the Hilbert-Schmidt norm, i.e. the norm of $\ell_2 \otimes \ell_2$. 

Proof
Let \( t > 0 \) and let \( \varepsilon > 0 \). Let \( N, M \in \mathbb{N} \). Then we estimate as follows:

\[
\left( \sum_{m,n=0}^{M,N} |V_{mn}| \exp(-(t+\varepsilon)n^\mu + tm^\mu) \right)^{\frac{1}{2}} \leq \sup_{n,m \in \mathbb{N} \cup \{0\}} (|V_{mn}| \exp(-(t+\frac{1}{2}\varepsilon)(n^\mu - m^\mu)) \left( \sum_{m,n=0}^{\infty} \exp(-\varepsilon(n^\mu + m^\mu)) \right)^{\frac{1}{2}}).
\]

So, by Lemma (1.9), \( \| e^{t\Lambda^\mu} V e^{-(t+\varepsilon)\Lambda^\mu} \| < \infty \).

Also, we have

\[
\left( \sum_{m,n=0}^{M,N} |(V^{-1})_{mn}| \exp(-(t+\varepsilon)n^\mu + (t+\frac{1}{2}\varepsilon)m^\mu) \cdot \left( \sum_{m,n=0}^{\infty} \exp(-\frac{1}{2}\varepsilon n^\mu - \varepsilon m^\mu) \right)^{\frac{1}{2}} \right) \leq \sup_{n,m \in \mathbb{N}} (|V_{mn}| \exp(-(t+\varepsilon)n^\mu + (t+\frac{1}{2}\varepsilon)m^\mu)) \cdot \left( \sum_{m,n=0}^{\infty} \exp(-\frac{1}{2}\varepsilon n^\mu - \varepsilon m^\mu) \right)^{\frac{1}{2}}).
\]

Hence, \( \| e^{t\Lambda^\mu} V^{-1} e^{-(t+\varepsilon)\Lambda^\mu} \| < \infty \).

If we apply the previous results to the diagonalizing operator \( S \), we get

(1.11) Theorem
The linear operator \( S \) and the inverse \( S^{-1} \) satisfy

\[ \forall t>0 \forall \varepsilon>0; \quad \| e^{t\Lambda^\mu} S e^{-(t+\varepsilon)\Lambda^\mu} \| < \infty \]

and

\[ \forall t>0 \forall \varepsilon>0; \quad \| e^{t\Lambda^\mu} S^{-1} e^{-(t+\varepsilon)\Lambda^\mu} \| < \infty . \]
We note that both $S$ and $S^{-1}$ map $e^{-(t+c)\lambda^u}$ into $e^{-t\lambda^u}$ for any $\varepsilon > 0$ and any $t > 0$.

From the characterization of continuous linear mappings on analyticity spaces and the entireness spaces (cf. [6c] and [2]) it follows that

(1.12) Corollary

I. The operator $S$ is a continuous bijection on $S_{\ell_2,\lambda^u}$ with continuous inverse $S^{-1}$.

II. The operator $S$ is a continuous bijection on $\tau(\ell_2,\lambda^u)$ with continuous inverse $S^{-1}$.

Now we assume in addition that the matrix entries $K_{nn}$, $n \in \mathbb{N} \cup \{0\}$ of the matrix $(K_{nn})$ introduced in (1.6), satisfy

$$K_{nn} = cn^\alpha(1 + o(1))$$

for some positive constant $c$. If we define the diagonal operator $M$ by

$$M_k = K_{kk}, \quad k \in \mathbb{N} \cup \{0\},$$

then $S_{\ell_2,\lambda^u} = S_{\ell_2,\lambda^u/a}$ by Lemma (1.5). Moreover, the operator $S$ is continuous on $S_{\ell_2,\lambda^u/a}$ and has a continuous inverse. We have the relation $K = SMS^{-1}$.

In the first part of this section the Hilbert space $(\ell_2)^{S^{-1}}$ has been introduced. It follows that $K$ is a positive self-adjoint operator in $(\ell_2)^{S^{-1}}$, which satisfies $K^{\mu/a} = SM^{\mu/a}S^{-1}$. It leads to the following classification result.

(1.13) Theorem

I. $S_{\ell_2,\lambda^u} = S_{(\ell_2)^{S^{-1}},K^{\mu/a}}$. 
II. \[ \tau(\ell_2, \Lambda^\mu) = \tau((\ell_2)^{S^{-1}}, k'^{\mu/\alpha}). \]

**Proof**

Cf. Corollary (1.12), Lemmas (1.5) and (1.6). \(\square\)
2. THE CLASSIFICATION OF ANALYTICITY SPACES AND ENTIRENESS SPACES GENERATED BY THE JACOBI POLYNOMIALS

The two classification methods for analyticity and entireness spaces discussed in the previous chapter, lead to the classification of the spaces \( S_{X_{\alpha,\beta},(A_{\alpha,\beta})^\nu} \) and \( \tau(X_{\alpha,\beta},(A_{\alpha,\beta})^\nu) \) with \( \alpha, \beta > -1 \) and \( \nu \geq \frac{1}{2} \). Here \( X_{\alpha,\beta} \) denotes the Hilbert space \( L^2([-1,1],(1-x)^\alpha(1+x)^\beta \, dx) \) and \( A_{\alpha,\beta} \) the positive self-adjoint operator

\[
- (1-x^2) \frac{d^2}{dx^2} + (\alpha+\beta+2)x \frac{d}{dx} - (\beta-\alpha) \frac{d}{dx}.
\]

It is well known that the Jacobi polynomials \( p_n^{(\alpha,\beta)} \), \( n \in \mathbb{N} \cup \{0\} \), are the eigenfunctions of \( A_{\alpha,\beta} \) with eigenvalues \( n(n+\alpha+\beta+1) \). (For the definition of \( p_n^{(\alpha,\beta)} \) we refer to [8], p. 208.) Instead of \( p_n^{(\alpha,\beta)} \) we rather consider the normalized Jacobi polynomials

\[
R_n^{(\alpha,\beta)} = \left\{ \sqrt{\frac{2n+\alpha+\beta+1}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}} \right\}^{\frac{1}{2}} p_n^{(\alpha,\beta)}
\]

which constitute an orthonormal basis in \( X_{\alpha,\beta} \).

The normalized Chebyshev polynomials \( T_n = R_n^{\frac{1}{2},-\frac{1}{2}} \) establish a special class of Jacobi polynomials. They satisfy

\[
(2.1) \quad T_n(\cos \theta) = \sqrt{\frac{2}{\pi}} \cos n\theta, \quad n \geq 1, \quad \theta \in [0,\pi],
\]

\[
T_0(\cos \theta) = \sqrt{\frac{1}{\pi}}, \quad \theta \in [0,\pi].
\]

We note that \( X_{-\frac{1}{2},-\frac{1}{2}} = L^2([-1,1],(1-x^2)^{-\frac{1}{2}} \, dx) \) and, also, that \( A_{-\frac{1}{2},-\frac{1}{2}} = - (1-x^2) \frac{d^2}{dx^2} + x \frac{d}{dx} \). The eigenvalues of \( A_{-\frac{1}{2},-\frac{1}{2}} \) are the numbers \( n^2 \), \( n \in \mathbb{N} \cup \{0\} \).
If we consider the transformation \( x = \cos \theta \), then the operator \( A_{a,b} \) transforms into

\[
C_{a,b} = -\frac{d^2}{d\theta^2} + (\beta - \alpha) \frac{1}{\sin \theta} \frac{d}{d\theta} - (\alpha + \beta + 1) \cot \theta \frac{d}{d\theta}.
\]

So, with the aid of (2.1) it follows that the matrix of \( A_{a,b} \) with respect to the orthonormal basis \( (T_n)_{n=0}^\infty \) satisfies

\[
\begin{cases}
(A_{a,b} T_n, T_m)_{X_{-\frac{1}{2},-\frac{1}{2}}} = 0 & \text{if } n > m \\
(A_{a,b} T_n, T_n)_{X_{-\frac{1}{2},-\frac{1}{2}}} = n(n+a+b+1) \\
|\langle A_{a,b} T_n, T_m \rangle_{X_{-\frac{1}{2},-\frac{1}{2}}}| \leq Dn & \text{if } 0 \leq m < n
\end{cases}
\]

for some positive constant \( D \). So we can apply the theory of the previous chapter. It yields an operator \( S_{a,b} \) with the property

\[
A_{a,b} = S_{a,b} \Lambda_{a,b} (S_{a,b})^{-1}
\]

where \( \Lambda_{a,b} \) denotes the diagonal operator \( \Lambda_{a,b} T_n = n(n+a+b+1)T_n \).

Moreover, for all \( \mu > 1 \) the operator \( S_{a,b} \) is a continuous bijection on the analyticity space \( S_{X_{-\frac{1}{2},-\frac{1}{2}}, (A_{-\frac{1}{2},-\frac{1}{2}})^{\mu/2}} \) as well as on the entireness space \( \tau(X_{-\frac{1}{2},-\frac{1}{2}}, (A_{-\frac{1}{2},-\frac{1}{2}})^{\mu/2}) \) with continuous inverse \( S_{a,b}^{-1} \).

We note that we have proved in chapter 1 the following sharper result:

\[
\forall t > 0 \forall \epsilon > 0 : \| \exp(t(A_{\frac{1}{2},-\frac{1}{2}})^{\mu/2}) S_{a,b} \exp(-(t+\epsilon)(A_{-\frac{1}{2},-\frac{1}{2}})^{\mu/2}) \| < \infty
\]

\[
\forall t > 0 \forall \epsilon > 0 : \| \exp(t(A_{-\frac{1}{2},-\frac{1}{2}})^{\mu/2}) S_{a,b}^{-1} \exp(-(t+\epsilon)(A_{-\frac{1}{2},-\frac{1}{2}})^{\mu/2}) \| < \infty
\]

for all \( \mu > 1 \).
We find that

$$\hat{p}_n^{(\alpha, \beta)} = T_n + (S_{\alpha, \beta})_{n-1, n} T_{n-1} + \ldots + (S_{\alpha, \beta})_{1, n} T_1 + (S_{\alpha, \beta})_{0, n} T_0$$

is an eigenfunction of $A_{\alpha, \beta}$. In order to arrive at the spaces $S_{X_{\alpha, \beta}, (A_{\alpha, \beta})^{\mu/2}}$ and $\tau(X_{\alpha, \beta}, (A_{\alpha, \beta})^{\mu/2})$ we have to normalize the polynomials $\hat{p}_n^{(\alpha, \beta)}$ with respect to the norm of $X_{\alpha, \beta}$. To put it differently, we have to compute the constants $d_n^{(\alpha, \beta)}$ which satisfy

$$\hat{p}_n^{(\alpha, \beta)} = d_n^{(\alpha, \beta)} \hat{p}_n^{(\alpha, \beta)}.$$ 

Following [8], p. 210, the following relations are valid for all $n \in \mathbb{N}$:

$$T_n(x) = 2^{n-1} \sqrt{\frac{2}{\pi}} x^n + \ldots x^{n-1} + \ldots$$

and

$$p_n^{(\alpha, \beta)} = \frac{1}{2^n} \left(\frac{2n+\alpha+\beta}{n}\right) x^n + \ldots x^{n-1} + \ldots.$$ 

So we obtain

$$T_n(x) = \sqrt{\frac{2}{\pi}} \left(\frac{2^{2n-1}}{(2n+\alpha+\beta)}\right) p_n^{(\alpha, \beta)}(x) + \ldots \frac{p_n^{(\alpha, \beta)}(x)}{n!} + \ldots.$$ 

Therefore

$$\hat{p}_n^{(\alpha, \beta)} = T_n + (S_{\alpha, \beta})_{n-1, n} T_{n-1} + \ldots$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{2^{2n-1}}{(2n+\alpha+\beta)}\right) p_n^{(\alpha, \beta)} = \sqrt{\frac{2}{\pi}} \left(\frac{2^{2n-1}}{(2n+\alpha+\beta)}\right) \frac{1}{o_n^{(\alpha, \beta)}} R_n^{(\alpha, \beta)}$$

with

$$o_n^{(\alpha, \beta)} = \left(\frac{2n + \alpha + \beta + 1}{2^{\alpha+\beta+1}} \frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right)^{\frac{1}{2}}.$$
With the aid of Lemma (a.11) and the result (a.13.ii) of the Appendix it follows that there are constants \( q_{\alpha, \beta} \) and \( p_{\alpha, \beta} > 0 \) such that

\[
|d_n^{(\alpha, \beta)}| \leq q_{\alpha, \beta}(n+1)^{-1}
\]

and

\[
|d_n^{(\alpha, \beta)}| \leq p_{\alpha, \beta}(n+1).
\]

So the diagonal operator \( D_{\alpha, \beta} \) with entries \( d_n^{(\alpha, \beta)} \) and its inverse \( D_{\alpha, \beta}^{-1} \) satisfy the following

\[
(2.5) \quad \forall \varepsilon > 0: \quad \|[D_{\alpha, \beta} \exp (-\varepsilon (A_{-\frac{1}{2}, -\frac{1}{2}})^{1/2})]|| < \infty
\]

\[
(2.5) \quad \forall \varepsilon > 0: \quad \|[D_{\alpha, \beta}^{-1} \exp (-\varepsilon (A_{-\frac{1}{2}, -\frac{1}{2}})^{1/2})]|| < \infty
\]

for each \( \mu > 1 \).

So if we put \( W_{\alpha, \beta} = S_{\alpha, \beta} D_{\alpha, \beta} \), then \( W_{\alpha, \beta} \) is a continuous bijection both on S_{X_{-\frac{1}{2}, -\frac{1}{2}}, A_{-\frac{1}{2}, -\frac{1}{2}}} and on \( \tau(X_{-\frac{1}{2}, -\frac{1}{2}}, A_{-\frac{1}{2}, -\frac{1}{2}}) \) with continuous inverse \( W_{\alpha, \beta}^{-1} \).

Further we obtain from (2.3) and (2.5):

\[
(2.6) \quad \text{Lemma}
\]

I. \( \forall t > 0 \forall \varepsilon > 0: \quad \|[\exp(t(A_{-\frac{1}{2}, -\frac{1}{2}})^{1/2}) W_{\alpha, \beta} \exp(-t+\varepsilon)(A_{-\frac{1}{2}, -\frac{1}{2}})^{1/2})]|| < \infty \)

II. \( \forall t > 0 \forall \varepsilon > 0: \quad \|[\exp(t(A_{-\frac{1}{2}, -\frac{1}{2}})^{1/2}) W_{\alpha, \beta}^{-1} \exp(-t+\varepsilon)(A_{-\frac{1}{2}, -\frac{1}{2}})^{1/2})]|| < \infty \)

with \( \mu > 1 \).

Since \( W_{\alpha, \beta} T_{\alpha, \beta} = R_{\alpha, \beta}^{(\alpha, \beta)} \), \( n \in \mathbb{N} \cup \{0\} \), and since \( A_{\alpha, \beta} = W_{\alpha, \beta} \wedge_{\alpha, \beta} \) \( W_{\alpha, \beta}^{-1} \) we obtain from Theorem (1.3) and Theorem 1.4)
(2.7) **Theorem**

Let $\mu > 1$. Then for all $\alpha, \beta > -1$

$$S_{X_{\alpha, \beta}, (A_{\alpha, \beta})^{\mu/2}} = S_{L_2([-1, 1], (1-x^2)^{-\frac{1}{2}} dx), \left(- (1-x^2) \frac{d^2}{dx^2} + x \frac{d}{dx}\right)^{\mu/2}}$$

and

$$\tau(X_{\alpha, \beta}, (A_{\alpha, \beta})^{\mu/2}) = \tau(L_2([-1, 1], (1-x^2)^{-\frac{1}{2}} dx), \left(- (1-x^2) \frac{d^2}{dx^2} + x \frac{d}{dx}\right)^{\mu/2})$$

If we apply the transformation $x = \cos \theta$, we get

(2.8) **Corollary**

Let $\mu > 1$. Then for all $\alpha, \beta > -1$

$$S_{Y_{\alpha, \beta}, (C_{\alpha, \beta})^{\mu/2}} = S_{L_2([0, \pi], d\theta), \left(- \frac{d}{d\theta} \right)^{\mu/2}}$$

and

$$\tau(Y_{\alpha, \beta}, (C_{\alpha, \beta})^{\mu/2}) = \tau(L_2([0, \pi], d\theta), \left(- \frac{d}{d\theta} \right)^{\mu/2})$$

Here $Y_{\alpha, \beta}$ denotes the Hilbert space $L_2([0, \pi], (1 - \cos \theta)^\alpha (1 + \sin \theta)^\beta d\theta)$.

The classification method, based on the use of continuous bijections is not refined enough to obtain the classification of the spaces $S_{X_{\alpha, \beta}, (A_{\alpha, \beta})^{\mu/2}}$ and $\tau(X_{\alpha, \beta}, (A_{\alpha, \beta})^{\mu/2})$. For this case we use the other classification method, based on perturbations. With Theorems (1.1) and (1.2) we will obtain the results of Theorem 2.7 for $\mu = 1$. We note that the spaces $T_{X_{\alpha, \beta}, (A_{\alpha, \beta})^{\frac{1}{2}}}$ and $\sigma(X_{\alpha, \beta}, (A_{\alpha, \beta})^{\frac{1}{2}})$ will be related to the spaces of (ultra-) hyperfunctions in the Chapters 6 and 7.

In the Appendix we have shown that the matrix elements of the operators $D = \frac{d}{dx}$ and $xD$ satisfy the following estimates:
(2.9) \[ |(D_n^{(\alpha, \beta)}, R_k^{(\alpha, \beta)})_{\alpha, \beta}| \leq \begin{cases} 0 & \text{if } k \geq n \\ G_{\alpha, \beta} \frac{n^{3/2}(n-k)}{(k+1)^{1/2}} & \text{if } 0 \leq k < n \end{cases} \]

where \( G_{\alpha, \beta} \) is some positive constant;  

(2.10) \[ |((xD)_n^{(\alpha, \beta)}, R_k^{(\alpha, \beta)})_{\alpha, \beta}| \leq \begin{cases} 0 & \text{if } k > n \\ n & \text{if } k = n \\ H_{\alpha, \beta} \frac{(\alpha+1)^{3/2}}{(k+1)^{1/2}} & \text{if } 0 \leq k < n \end{cases} \]

for some positive constant \( H_{\alpha, \beta} \).

For each \( \alpha, \beta > -1 \) we put \( \tilde{A}_{\alpha, \beta} = A_{\alpha, \beta} + I \), and further

\[
 p_{\gamma, \delta}^{\alpha, \beta} = 
 ((\delta-\gamma) - (\beta-\alpha))D - ((\delta+\gamma) - (\beta+\alpha))(xD) .
\]

Then we have the relation

\[ \tilde{A}_{\gamma, \delta} = \tilde{A}_{\alpha, \beta} + p_{\gamma, \delta}^{\alpha, \beta} . \]

With the aid of the estimates (2.9) and (2.10) we prove the following auxiliary results.

(2.11) Lemma

Let \( \alpha, \beta > -1 \) and let \( \frac{1}{2} \leq \nu \leq 1 \). Then there exists a positive constant \( e_{\alpha, \beta} \) such that for all \( r > 0 \)

\[ \| \exp(r(\tilde{A}_{\alpha, \beta})^\nu)D(\tilde{A}_{\alpha, \beta})^{-1}\exp(-r(\tilde{A}_{\alpha, \beta})^\nu) \|_{\alpha, \beta} \leq e_{\alpha, \beta} r^{-3/2} . \]

Proof

For each \( r > 0 \) we define the weighted shift operators \( \tilde{W}_n^{(\alpha, \beta)}(r) \), \( n \in \mathbb{N} \),
First we shall prove that \( W_n^{(\alpha, \beta)}(r) \) is a bounded linear operator on \( X_{\alpha, \beta} \) for all \( r > 0 \). Therefore, note first that

\[
((n+\ell)(n+\ell+\alpha+\beta+1)+1)^{\nu} - (\ell(\ell+\alpha+\beta+1)+1)^{\nu} = \nu\left[ ((n+\ell)(n+\ell+\alpha+\beta+1)+1) - (\ell(\ell+\alpha+\beta+1)+1) \right] \xi(n, \ell)^{\nu-1}
\]

for some number \( \xi(n, \ell) \in [(\ell+\alpha+\beta+1)+1, (n+\ell)(n+\ell+\alpha+\beta+1)+1] \).

We get the following estimation

\[
((n+\ell)(n+\ell+\alpha+\beta+1)+1)^{\nu} - (\ell(\ell+\alpha+\beta+1)+1)^{\nu} \geq \nu n(n+\ell+\alpha+\beta+1)((n+\ell)(n+\ell+\alpha+\beta+1)+1)^{\nu-1} \geq \nu n.
\]

In addition, for all \( n \in \mathbb{N} \) we have by the estimate (2.9)

\[
\sup_{\ell \in \mathbb{N} \cup \{0\}} \left\{ \| (R_n^{(\alpha, \beta)}, R_\ell^{(\alpha, \beta)}) \|_{\alpha, \beta} \frac{1}{(n+\ell)(n+\ell+\alpha+\beta+1)+1} \right\} \leq \frac{G_{\alpha, \beta}}{(n+\ell)^{3/2}} \frac{n}{(\ell+1)^{1/2}} \frac{1}{(n+\ell)(n+\ell+\alpha+\beta+1)+1} \leq G_{\alpha, \beta} n^{1/2}.
\]

So the norm of the operator \( W_n^{(\alpha, \beta)}(r) \) is smaller than \( G_{\alpha, \beta} n^{1/2} e^{-\nu r n} \), for each \( n \in \mathbb{N} \). Hence the series \( \sum_{n=1}^{\infty} \| W_n^{(\alpha, \beta)}(r) \| \) converges. Thus we obtain the following estimation.
\begin{align*}
\| \exp(r(\tilde{A}_{\alpha,\beta})^\nu)D(\tilde{A}_{\alpha,\beta})^{-1} \exp(-r(\tilde{A}_{\alpha,\beta})^\nu) \| & \leq \sum_{n=1}^{\infty} \| \nu_n^{(\alpha,\beta)}(r) \| \\
& \leq C_{\alpha,\beta} \sum_{n=1}^{\infty} n^{1/2} \exp(-\nu r^n) \leq e_{\alpha,\beta} r^{3/2}
\end{align*}
for some positive constant $e_{\alpha,\beta}$.

Comparing the bounds for the matrix elements of the operator $xD$ and of the operator $D$, the reader will see that the same method as used in the proof of Lemma (2.11) applies in the proof of the following result.

(2.12) **Lemma**

Let $\alpha, \beta > -1$ and let $\frac{1}{4} \leq \nu \leq 1$. Then there exists a positive constant $f_{\alpha,\beta}$ such that for all $r > 0$

\[ \| \exp(r(\tilde{A}_{\alpha,\beta})^\nu)(xD)\tilde{A}_{\alpha,\beta}^{-1}\exp(-r(\tilde{A}_{\alpha,\beta})^\nu) \| \leq f_{\alpha,\beta} r^{-1} + 1. \]

**Remark.** From the preceding lemmas it follows that $D$ and $xD$ are well-defined continuous linear mappings on the analyticity spaces $S_{X,\alpha,\beta}$, $(\tilde{A}_{\alpha,\beta})^\nu$ and on the entireness spaces $\tau(X,\alpha,\beta, (\tilde{A}_{\alpha,\beta})^\nu)$ with $\frac{1}{4} \leq \nu \leq 1$ and $\alpha, \beta > -1$.

**Remark.** The Lemmas (2.11) and (2.12) also hold for $\nu > 1$. However, for simplicity in the estimation we have given the proof in case of $\frac{1}{4} \leq \nu \leq 1$ only.

We want to apply Theorem (1.1) and Theorem (1.2) to the perturbations $p_{\alpha,\beta}^{\gamma,\delta}$. Therefore we need the following lemma.

(2.13) **Lemma**

Let $\alpha, \beta, \gamma, \delta > -1$ and let $\nu > 0$. Then for every $t > 0$ the linear operator $\exp(-t(\tilde{A}_{\alpha,\beta})^\nu)$ maps $X_{\alpha,\beta}$ continuously into $X_{\gamma,\delta}$. 
Proof

By [8], p. 217, there exist $q > 0$ and $c > 0$ such that

$$\max_{-1 \leq x \leq 1} |P_n^{(\alpha, \beta)}(x)| \leq c(n+1)^q.$$ 

So with the aid of (a.13.i) we derive

$$\max_{-1 \leq x \leq 1} |R_n^{(\alpha, \beta)}(x)| \leq c_0(n+1)^q \leq c \alpha_2(n+1)^q + \frac{1}{2}.$$ 

Now let $\ell \in \mathbb{N}$ be so large that $\ell > q + \frac{3}{2}$. Then for each $f \in X_\alpha, \beta$

$$\int_{-1}^{1} |((\tilde{\alpha}, \beta)^{-\ell} f)(x)|^2 (1-x)^{\gamma}(1+x)^{\delta} \, dx \leq$$

$$\leq \int_{-1}^{1} (1-x)^{[\gamma] + 1} (1+x)^{[\delta] + 1} |((\tilde{\alpha}, \beta)^{-\ell} f)(x)|^2 (1-x)^{-1+\gamma-[\gamma]}(1+x)^{-1+\delta-[\delta]} \, dx \leq$$

$$\leq 2^{[\gamma] + [\delta] + 2} \max_{-1 \leq x \leq 1} |((\tilde{\alpha}, \beta)^{-\ell} f)(x)|^2 \int_{-1}^{1} (1-x)^{-1+\gamma-[\gamma]}(1+x)^{-1+\delta-[\delta]} \, dx \leq$$

$$\leq h_{\gamma, \delta} \max_{-1 \leq x \leq 1} |((\tilde{\alpha}, \beta)^{-\ell} f)(x)|^2$$

for some constant $h_{\gamma, \delta} > 0$ which does not depend on $f \in X_\alpha, \beta$.

Consider the following estimation

$$\max_{-1 \leq x \leq 1} |((\tilde{\alpha}, \beta)^{-\ell} f)(x)|^2 =$$

$$= \max_{-1 \leq x \leq 1} \left| \sum_{n=0}^{\infty} (f, R_n^{(\alpha, \beta)})_{\alpha, \beta} \left( \frac{1}{n(n+\alpha+\beta+1)+1} \right) \frac{\ell}{R_n^{(\alpha, \beta)}(x)} \right|^2 \leq$$

$$\leq \left( \sum_{n=0}^{\infty} |(f, R_n^{(\alpha, \beta)})_{\alpha, \beta} | \left( \frac{1}{n(n+\alpha+\beta+1)+1} \right) \ell \max_{-1 \leq x \leq 1} |R_n^{(\alpha, \beta)}(x)| \right)^2 \leq$$
Thus we have found that there exists a constant $\alpha_{\gamma, \delta} > 0$ independent on the choice of $f$ such that

$$\| (a_{\alpha, \beta})^{-\frac{1}{2}} \|_{\gamma, \delta} \leq \alpha_{\alpha, \beta} \| f \|_{\alpha, \beta}.$$ 

So $(a_{\alpha, \beta})^{-\frac{1}{2}}$ maps $X_{\alpha, \beta}$ continuously into $X_{\gamma, \delta}$.

Finally we note that $\exp(-t(a_{\gamma, \delta}))$ can be written as

$$\exp(-t(a_{\gamma, \delta})) = (a_{\alpha, \beta})^{-\frac{1}{2}} \{ (a_{\alpha, \beta})^{\frac{1}{2}} \exp(-t(a_{\alpha, \beta})) \}.$$ 

Since the operator between $\{ \}$ maps $X_{\alpha, \beta}$ continuously into itself the proof is complete.

(2.14) Theorem

Let $\alpha, \beta, \gamma, \delta > -1$ and let $\frac{1}{2} \leq \nu \leq 1$. Then

$$S_{X_{\alpha, \beta}}(a_{\alpha, \beta})^{\nu} = S_{X_{\gamma, \delta}}(a_{\gamma, \delta})^{\nu}$$

and, also

$$\tau(X_{\alpha, \beta}, (a_{\alpha, \beta})^{\nu}) = (X_{\gamma, \delta}, (a_{\gamma, \delta})^{\nu}).$$

Proof

Following Lemma (1.5) it is sufficient to prove that $S_{X_{\alpha, \beta}}(a_{\alpha, \beta})^{\nu} = S_{X_{\gamma, \delta}}(a_{\gamma, \delta})^{\nu}$ and that $\tau(X_{\alpha, \beta}, (a_{\alpha, \beta})^{\nu}) = \tau(X_{\gamma, \delta}, (a_{\gamma, \delta})^{\nu})$. By the Lemmas (2.1), (2.12) and (2.13) the following assertions are valid:

- The perturbation $p_{\alpha, \beta}^{\gamma, \delta} = (((\delta - \gamma) - (\beta - \alpha))D - ((\delta + \gamma) - (\beta + \alpha))XD$ satisfies
\[ \exists d > 0 \forall r > 0: \| \exp(r(\tilde{A}_{\alpha,\beta})^\nu)p_{\alpha,\beta}(\tilde{A}_{\alpha,\beta})^{-1} \exp(-r(\tilde{A}_{\alpha,\beta})^\nu) \| \leq d(r^{-3/2} + r^{-1/2}) \]

- \( D(p_{\alpha,\beta}, \delta) \supset S_{X_{\alpha,\beta}}(\tilde{A}_{\alpha,\beta})^\nu \) and \( \tilde{A}_{\gamma,\delta} = \tilde{A}_{\alpha,\beta} + p_{\alpha,\beta}^\nu \) is a positive self-adjoint operator in \( X_{\gamma,\delta} \).

- For all \( t > 0 \), \( \exp(-t(\tilde{A}_{\alpha,\beta})^\nu) \) maps \( X_{\alpha,\beta} \) continuously in \( X_{\gamma,\delta} \).

Then by Theorem (1.1), \( S_{X_{\alpha,\beta}}(\tilde{A}_{\alpha,\beta})^\nu \subset S_{X_{\gamma,\delta}}(\tilde{A}_{\gamma,\delta})^\nu \), and by Theorem (1.2), \( \tau(X_{\alpha,\beta}, (\tilde{A}_{\alpha,\beta})^\nu) \subset \tau(X_{\gamma,\delta}, (\tilde{A}_{\gamma,\delta})^\nu) \).

If we interchange the roles of \( (\alpha,\beta) \) and \( (\gamma,\delta) \), the wanted result is obtained.

Application of the transformation \( x = \cos \theta \) yields

(2.15) Corollary

Let \( \alpha, \beta, \gamma, \delta > -1 \) and let \( \frac{1}{2} \leq \nu \leq 1 \). Then

\[ S_{X_{\alpha,\beta}}(C_{\alpha,\beta})^\nu = S_{X_{\gamma,\delta}}(C_{\gamma,\delta})^\nu \]

and

\[ \tau(X_{\alpha,\beta}, (C_{\alpha,\beta})^\nu) = \tau(X_{\gamma,\delta}, (C_{\gamma,\delta})^\nu) \]

For \( \alpha = \beta = -\frac{1}{2} \), we have seen that \( R_{-\frac{1}{2}, -\frac{1}{2}}(\cos \theta) = \sqrt{\frac{1}{\pi}} \) and \( R_{-\frac{1}{2}, -\frac{1}{2}}(\cos \theta) = \sqrt{\frac{2}{\pi}} \cos n\theta, n \in \mathbb{N} \). So it becomes rather easy to characterize the spaces \( S_{Y_{-\frac{1}{2}, -\frac{1}{2}}}((C_{-\frac{1}{2}, -\frac{1}{2}})^\nu) \) and \( \tau(Y_{-\frac{1}{2}, -\frac{1}{2}}, (C_{-\frac{1}{2}, -\frac{1}{2}})^\nu) \) in classical analytic terms. Then Theorem (2.7) and Theorem (2.14) yield the same analytic description for the other values of \( \alpha, \beta > -1 \).

Further, by means of the conformal mapping \( w = \cos z \), we get the characterization of the spaces \( S_{X_{\alpha,\beta}}(A_{\alpha,\beta})^\nu \) and \( \tau(X_{\alpha,\beta}, (A_{\alpha,\beta})^\nu) \). This program will be carried out in the next chapter.
3. THE CHARACTERIZATION OF THE ANALYTICITY SPACES AND ENTIRENESS SPACES BASED ON THE JACOBI POLYNOMIALS, WITH APPLICATIONS TO CLASSICAL ANALYSIS

In this chapter we intend to derive characterizations in classical analytic terms of the elements in the analyticity spaces \( \mathcal{S}_{\alpha,\beta}^0(\mathcal{C}_{\alpha,\beta}^\nu) \) and in the entireness spaces \( \tau(\mathcal{Y}_{\alpha,\beta},(\mathcal{C}_{\alpha,\beta}^\nu)) \) where we consider \( \alpha,\beta > -1 \) and \( \nu \geq \frac{1}{2} \).

Following the results obtained in the previous chapter we only have to consider one pair \( \mathcal{Y}_{\alpha,\beta},\mathcal{C}_{\alpha,\beta} \). It is rather natural therefore to study the case \( \alpha = \beta = -\frac{1}{2} \), i.e., the so-called Chebyshev case.

For convenience we put \( \mathcal{Y} = \mathcal{Y}_{-\frac{1}{2},-\frac{1}{2}} = L_2([0,\pi]) \) and \( \mathcal{C} = \mathcal{C}_{-\frac{1}{2},-\frac{1}{2}} = -\frac{d^2}{d\theta^2} \).

The normalized eigenfunctions \( c_n \) of the operator \( \mathcal{C} \) given by

\[
c_n(\theta) = \sqrt{\frac{2}{\pi}} \cos n\theta,
\]

\[
c_0(\theta) = \sqrt{\frac{1}{\pi}}, \quad \theta \in [0,\pi], \quad n \in \mathbb{N},
\]

establish an orthonormal basis in \( \mathcal{Y} \). We want to determine the analytic behaviour of the series

\[
z \mapsto \sum_{n=0}^{\infty} a_n c_n(z)
\]

for all coefficients \( (a_n)_{n=0}^{\infty} \) satisfying the following order estimate

\[
\exists t > 0: a_n = O(\exp(-t n^{2\nu})), \quad n \in \mathbb{N} \cup \{0\}, \quad \nu \geq \frac{1}{2}.
\]

Since \( \cos nz = \frac{e^{inz} + e^{-inz}}{z} \) it is necessary and sufficient to characterize the analytic behaviour of the type of series

\[
z \mapsto \sum_{n=-\infty}^{\infty} b_n e^{inz}
\]

where \( b_n = O(\exp(-|n|^{2\nu} t)) \) for some \( t > 0 \).
We observe that the functions \( e_n : \theta \rightarrow \frac{1}{\sqrt{2\pi}} e^{in\theta}, \theta \in [-\pi, \pi], n \in \mathbb{Z} \), establish an orthonormal basis for the Hilbert space \( L_2([-\pi, \pi]) \). They are the eigenfunctions of the positive self-adjoint operator \( \Delta = -\frac{d^2}{d\theta^2} \) in \( L_2([-\pi, \pi]) \) with eigenvalues \( n^2, n \in \mathbb{Z} \). It is clear that the analyticity space \( S_{Y, C^\nu} \) consists of all even functions in \( S_{L_2([-\pi, \pi]), \Delta^\nu} \) and, similarly, that the entireness space \( \tau(Y, C^\nu) \) consists of all even functions in \( \tau(L_2([-\pi, \pi]), \Delta^\nu) \).

The program of this chapter is the following. In a separate section we characterize the elements of \( S_{L_2([-\pi, \pi]), \Delta^\nu} \) and \( \tau(L_2([-\pi, \pi]), \Delta^\nu), \nu \geq \frac{1}{2} \). These spaces correspond to classes of \( 2\pi \)-periodic, analytic functions which mostly satisfy certain growth conditions. In a rather natural way we define norms in these classes of analytic functions. This leads to an alternative description of sequential convergence in the corresponding analyticity spaces and the corresponding entireness spaces.

As we have seen all results carry over to the spaces \( S_{Y, C^\nu} \) and \( \tau(Y, C^\nu), \nu \geq \frac{1}{2} \). We thus obtain vector spaces of even, \( 2\pi \)-periodic, analytic functions with their natural norms.

We next employ the conformal mapping \( w = \cos z \) to the function classes associated to \( S_{Y, C^\nu} \) and \( \tau(Y, C^\nu) \). The thus obtained classes of analytic functions lead to a characterization of the spaces \( S_{X, (A_{\alpha, \beta})^\nu} \) and \( \tau(X, (A_{\alpha, \beta})^\nu) \) for \( \nu \geq \frac{1}{2} \) and \( \alpha, \beta > -1 \). Moreover we derive a classical description of sequential convergence in these spaces. As a result we are able to connect the growth behaviour of analytic (entire) functions with the growth of their expansion coefficients with respect to bases of Jacobi polynomials \( (R_n^{(\alpha, \beta)}) \).
3.1. The characterization of $S_{L^2([-\pi, \pi]), \Delta^v}$ and $\tau(L^2([-\pi, \pi]), \Delta^v)$

First we discuss the case $v = \frac{1}{2}$. To this end, consider the following lemma:

(3.1.1) Lemma

Let $(a_n)_{n=-\infty}^{\infty} \in \mathbb{C}^Z$. Then for the expression

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{inz}$$

the following assertions are valid.

I. $f$ is an analytic function on an open strip $|\text{Im } z| < t$, $t > 0$, and it tends to boundary functions $x \mapsto f(x \pm it)$ in $L^2([-\pi, \pi])$-sense iff

$$\left( a_n e^{tn} \right)_{n=-\infty}^{\infty} \in \ell^2.$$

II. $f$ is an analytic function on an open strip $|\text{Im } z| < t_0$, $t_0 > 0$, iff for all $t$ with $0 < t < t_0$ the sequence $(a_n e^{tn})_{n=-\infty}^{\infty}$ tends in $\ell^2$-sense to the sequence $(a_n e^{nt})_{n=-\infty}^{\infty}$ (or equivalently, $(a_n e^{nt})_{n=-\infty}^{\infty}$ converges uniformly on a strips $t \pm t$ or $y \sim -t$) the sequence $(a_n e^{-ny})_{n=-\infty}^{\infty}$ tends in $\ell^2$-sense to the sequence $(a_n e^{-nt})_{n=-\infty}^{\infty}$ (or $(a_n e^{-nt})_{n=-\infty}^{\infty}$).

Proof

I $\Rightarrow$ $f(x+iy) = \sum_{n=-\infty}^{\infty} a_n e^{-ny} e^{inx}$ converges uniformly on all strips $\{x+iy \mid |y| \leq \tau < t\}$. For $y \pm t$ (or $y \pm -t$) the sequence $(a_n e^{-ny})_{n=-\infty}^{\infty}$ tends in $\ell^2$-sense to the sequence $(a_n e^{-nt})_{n=-\infty}^{\infty}$ (or $(a_n e^{nt})_{n=-\infty}^{\infty}$).

I $\Rightarrow$ Both $(a_n e^{nt})_{n=-\infty}^{\infty}$ and $(a_n e^{-nt})_{n=-\infty}^{\infty}$ have to be $\ell^2$-sequences.

Hence $(a_n e^{n|y|})_{n=-\infty}^{\infty}$ is an $\ell^2$-sequence.

I $\Rightarrow$ Similar to I $\Rightarrow$.

II $\Rightarrow$ For all $y \in (-t, t)$ we have $\int_{-\pi}^{\pi} |f(x+iy)|^2 \, dx < \infty$. Therefore

$$\left( a_n e^{n|y|} \right)_{n=-\infty}^{\infty} \in \ell^2$$

and hence $(a_n e^{n|y|}) \in \ell^\infty$ since the interval $(-t, t)$ is open.
We introduce the following notation: \( S_t \) denotes the symmetric open strip of width \( 2t \) around the real axis, i.e.

\[
S_t = \{ z \in \mathbb{C} \mid |\text{Im} \, z| < t \}.
\]

(3.1.2) **Definition**

Let \( t > 0 \). Then the class \( A(S_t, 2\pi\text{-per}) \) consists of \( 2\pi \)-periodic functions for which there exists \( \varepsilon > 0 \) such that \( f \) is analytic on \( S_{t+\varepsilon} \).

The class \( A(\mathbb{C}, 2\pi\text{-per}) \) consists of all \( 2\pi \)-periodic entire analytic functions.

(3.1.3) **Theorem**

Let \( t > 0 \). Then \( f \in L_2([-\pi, \pi]) \) can be extended to \( f \in A(S_t, 2\pi\text{-per}) \) iff

\[
f \in \exp(-t\Delta^\frac{1}{2})(S_L^2([-\pi, \pi]), \Delta^\frac{1}{2}).
\]

Proof

Immediately from Lemma 3.1.1.

We note that

\[
\exp(-t\Delta^\frac{1}{2})(S_L^2([-\pi, \pi]), \Delta^\frac{1}{2}) = \bigcup_{\tau > t} \exp(-\tau\Delta^\frac{1}{2})(L_2([-\pi, \pi])).
\]

We obtain the following characterization:

(3.14) **Theorem**

I. \( f \in S_L^2([-\pi, \pi]), \Delta^\frac{1}{2} \) iff there exists \( t > 0 \) such that \( f \in A(S_t, 2\pi\text{-per}) \).

II. \( f \in \tau(L_2([-\pi, \pi]), \Delta^\frac{1}{2}) \) iff \( f \in A(\mathbb{C}, 2\pi\text{-per}) \).

Next we shall show that the 'functional analytic' topologies of the spaces \( S_L^2([-\pi, \pi]), \Delta^\frac{1}{2} \) and \( \tau(L_2([-\pi, \pi]), \Delta^\frac{1}{2}) \) are equivalent to topologies which are related to the analytic properties of the functions in these spaces.
To this end, let \( L^2_{2,t}([\pi,\pi]) \), \( t > 0 \), denote the Hilbert space \\
\( e^{-t\Delta^{\frac{1}{2}}}L^2([\pi,\pi]) \) with inner product \((f,g)_{t,\frac{1}{2}} = (e^{t\Delta^{\frac{1}{2}}}f, e^{t\Delta^{\frac{1}{2}}}g)_{L^2} \) where \((,\)\)\( _{L^2} \) denotes the inner product of \( L^2([\pi,\pi]) \). We denote the norm in \( L^2_{2,t}([\pi,\pi]) \) \\
by \( \| \cdot \|_{t,\frac{1}{2}} \).

In the space \( A(S_t,2\pi\text{-per}) \) we define the following norm \\
\[ p_t(f) = \sup_{|\text{Im } z| \leq t} |f(z)|, \quad f \in A(S_t,2\pi\text{-per}). \]

Note that with the usual identifications \\
\[ S_{L^2_2([-\pi,\pi])}^{\Delta^{\frac{1}{2}}} = \bigcup_{t>0} L^2_{2,t}([\pi,\pi]) = \bigcup_{t>0} A(S_t,2\pi\text{-per}), \]

and \\
\[ \tau(L^2([\pi,\pi]),\Delta^{\frac{1}{2}}) = \bigcap_{t>0} L^2_{2,t}([\pi,\pi]) = \bigcap_{t>0} A(S_t,2\pi\text{-per}), \]

as sets.

Now let \( f \in L^2_{2,t}([\pi,\pi]) \) for some fixed \( t > 0 \). From the characterization in \\
Lemma (1.3.1) it follows that \( f \in A(S_t,2\pi\text{-per}) \) for all \( t, 0 < \tau < t \). Let \( \tau \), \\
with \( 0 < \tau < t \) be fixed, and consider the following estimation \\
\[ p_\tau(f) = \sup_{|\text{Im } z| \leq \tau} \left| \sum_{n=-\infty}^{\infty} (f,e_n) e_n(z) \right| \leq \]

\[ \leq \left( \sum_{n=-\infty}^{\infty} e^{2n|\tau|} (f,e_n)^2 \right)^{\frac{1}{2}} \sup_{|\text{Im } z| \leq \tau} \left( \frac{1}{2\pi} \right) \left( \sum_{n=-\infty}^{\infty} e^{-2|n|\tau} e^{inz} \right)^2 \leq \]

\[ \leq \| f \|_{t,\frac{1}{2}} \sqrt{\frac{1}{2\pi}} \left( \sum_{n=-\infty}^{\infty} e^{-2|n|(t-\tau)} \right)^{\frac{1}{2}}. \]

Thus we obtain with \\
\[ a_{t,\tau} = \sqrt{\frac{1}{\pi}} (1 - e^{-2(t-\tau)})^{-\frac{1}{2}} \]

the inequality \\
\[ (\ast) \quad p_\tau(f) \leq a_{t,\tau} \| f \|_{t,\frac{1}{2}}. \]
Conversely, let \( f \in A(\overline{S}_t, 2\pi \text{-per}) \) for some \( t > 0 \). Then by Lemma (1.3.1) we have \( f \in \mathbb{L}_{2,t}([\pi, \pi]) \). Write \( f = \sum_{n=-\infty}^{\infty} a_n e_n \). Then \( a_n \) is given by

\[
a_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx, \quad n \in \mathbb{Z}.
\]

We have the following relations

\[
a_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x+it) e^{-in(x+it)} \, dx = \frac{e^{-int}}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x+it) e^{-inx} \, dx.
\]

So for all \( n \in \mathbb{N} \)

\[
|a_n| \leq e^{-|n|t} \sqrt{2\pi} \mathcal{P}_t(f).
\]

Thus we obtain for all \( \tau, 0 < \tau < t \),

\[
(\ast \ast) \quad \|f\|_{\tau, \frac{1}{2}} = \left( \sum_{n=-\infty}^{\infty} e^{2|n|\tau |a_n|^2} \right)^{\frac{1}{2}} \leq \sqrt{2\pi} \mathcal{P}_t(f) \left( \sum_{n=-\infty}^{\infty} e^{-2|n|(t-\tau)} \right)^{\frac{1}{2}} = 2\pi a_{t, \tau} \mathcal{P}_t(f).
\]

The inequalities (\ast) and (\ast \ast) yield the following result.

(3.1.5) Theorem

I. Let \( t > 0 \). Let \( (f_n)_{n\in\mathbb{N}} \) be a sequence in \( e^{-t\Delta^{\frac{1}{2}}} \mathcal{S}_{L_2([-\pi, \pi]), \Delta^{\frac{1}{2}}} \). The sequence \( (e^{t\Delta^{\frac{1}{2}}} f_n)_{n\in\mathbb{N}} \) is a null sequence in \( \mathcal{S}_{L_2([-\pi, \pi]), \Delta^{\frac{1}{2}}} \) iff there exists \( \epsilon > 0 \) such that \( \mathcal{P}_{t+\epsilon}(f_n) \to 0 \), i.e., \( f_n(z) \to 0 \) uniformly on \( |\text{Im } z| \leq t+\epsilon \).
II. The sequence \((g_n)_{n \in \mathbb{N}}\) is a null sequence in \(\tau(L_2([-\pi, \pi]), \Delta^\perp)\) iff for all \(t > 0\), \(p_t(g_n) \to 0\), i.e., \(g_n(z) \to 0\) uniformly on each strip \(|\text{Im } z| \leq t|\).

Remark. The sequence \((f_n)_{n \in \mathbb{N}}\) is a null sequence in \(S^\perp \Delta^\perp\) iff \((f_n) \subset A(S^\perp, 2\pi\text{-per})\) for some \(t > 0\) and \(p_t(f_n) \to 0\), i.e., \(f_n(z) \to 0\) uniformly on \(|\text{Im } z| \leq t|\).

In the remaining part of this section we investigate the case \(\nu > \frac{1}{2}\). The next two lemmas are of main importance.

(3.1.6) Lemma

Let \((a_n)_{n=-\infty}^{\infty} \subset \mathbb{C}^\perp\). Then for the expression

\[ f(z) = \sum_{n=-\infty}^{\infty} a_n e^{inz}, \quad z = x + iy \in \mathbb{C}, \]

the following assertions are valid:

I. Let \(\nu \geq 1\). Then \(f\) is a \(2\pi\text{-periodic}\) analytic function of growth behaviour

\[ (\ast) \quad \int_{-\pi}^{\pi} |f(x+iy)|^2 dx \leq A^2 \exp \left(2B|y|^{\frac{2\nu}{\nu-1}}\right), \quad A \geq 0, \quad B > 0 \]

iff

\[ (\ast\ast) \quad (a_n \exp \left|t|n|^{\frac{2\nu}{\nu-1}}\right)_{n=-\infty}^{\infty} \in \ell_\infty \quad \text{for} \quad t = (2\nu)^{-2\nu} \left(\frac{2\nu - 1}{B}\right)^{2\nu-1}. \]

II. Let \(\nu > \frac{1}{2}\). If \((a_n \exp(t|n|^{\frac{2\nu}{\nu-1}}))_{n=-\infty}^{\infty} \in \ell_2\) for \(t = (2\nu)^{-2\nu} \left(\frac{2\nu - 1}{B}\right)^{2\nu-1}\),

then \(f\) is a \(2\pi\text{-periodic}\) function of growth behaviour \((\ast)\). Moreover, if \(f\) satisfies the growth condition \((\ast)\), then the sequence \((a_n)_{n=-\infty}^{\infty}\) satisfies \((\ast\ast)\).
Proof

I $\Rightarrow$ Consider the following estimation

$$A^2 \geq \exp \left( -2B |y|^{2\nu-1} \right) \int_{-\pi}^{\pi} |f(x+iy)|^2 \, dx =$$

$$= \frac{1}{2\pi} \exp \left( -2B |y|^{1/\nu - 2} \right) \sum_{n=-\infty}^{\infty} e^{-2ny} |a_n|^2 =$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left\{ \exp \left( -2B |y|^{2\nu - 2} \right) - 2ny - 2(2\nu - 2) \left( \frac{2\nu - 1}{B} \right)^{2\nu - 1} |n|^{2\nu} \right\} \cdot$$

$$\cdot \exp \left( 2(2\nu - 2) \left( \frac{2\nu - 1}{B} \right)^{2\nu - 1} |n|^{2\nu} |a_n|^2 \right).$$

If we now take

$$y_k = -\left( \frac{2\nu - 1}{2\nu B} \right)^{2\nu - 1} \sgn(k) |k|^{2\nu - 1}, \quad k \in \mathbb{Z},$$

then in the k-th term the factor between curly brackets equals 1. Hence

$$\exp \left( 2(2\nu - 2) \left( \frac{2\nu - 1}{B} \right)^{2\nu - 1} |k|^{2\nu} |a_k|^2 \right) \leq 2\pi A^2,$$

for all $k \in \mathbb{Z}$. This proves that for $t = (2\nu - 2) \left( \frac{2\nu - 1}{B} \right)^{2\nu - 1}$ the sequence

$$(a_n \exp(t|n|^{2\nu}))_{n=-\infty}^{\infty}$$

belongs to $\ell_\infty$.

I $\Leftarrow$ The analyticity of $f$ is trivial. We estimate as follows:

$$\int_{-\pi}^{\pi} |f(x+iy)|^2 \, dx = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left( |a_n|^2 e^{2t|n|^{2\nu}} - e^{-2t|n|^{2\nu}} |2ny| \right) \leq$$
The proof is complete if we can show that the series between \{\cdot\} is uniformly bounded for all $y \in \mathbb{R}$. Then just take

$$B = \left(\frac{2v-1}{2v}\right)^{\frac{1}{2v-1}}.$$ 

In order to show this uniform boundedness, consider the function

$$x \mapsto \exp(-2t|x|^{2v} - 2xy), \quad x \in \mathbb{R},$$

for fixed $y$. This function attains its maximum $\exp 2t(2v-1)\left(\frac{|y|}{2vt}\right)^{2v-1}$ at the point $x = -\left|\frac{y}{2vt}\right|^{2v-1} \text{sign}(y)$. Left to this point the function increases, right to this point it decreases. Then we get the estimate

$$\sum_{n=-\infty}^{\infty} e^{-2t|n|^{2v} - 2ny} \leq \exp \left(2t(2v-1)\left(\frac{|y|}{2vt}\right)^{2v-1}\right) + 2 \int_{-\infty}^{\infty} e^{-2t|x|^{2v} + 2xy} dx.$$ 

Finally, standard asymptotic techniques yield

$$\sum_{n=-\infty}^{\infty} e^{-2t|n|^{2v} - 2ny} \leq C(1 + |y|) \exp \left(2t(2v-1)\left(\frac{|y|}{2vt}\right)^{2v-1}\right).$$

So for $v \geq 1$ the series \{\cdot\} is uniformly bounded in $y$.

II. In order to prove the first part of the statement we estimate as follows:
\[ \int_{-\pi}^{\pi} |f(x+iy)|^2 dx = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} |a_n|^2 e^{2t|n|^2 \nu} \{e^{-2t|n|^2 \nu - 2n\tau}\}. \]

For \( t = (2\nu)^{-2\nu} \left( \frac{2\nu - 1}{B} \right)^{2\nu - 1} \) the factor between \( \{ \cdot \} \) is bounded by \( e^{2B|y|^{2\nu - 1}} \). For the second part of the statement see the proof of I \( \Rightarrow \). \( \square \)

(3.1.7) \textbf{Lemma}

Let \( (a_n)_{n=-\infty}^{\infty} \in \mathbb{C}^Z \). Then for the expression

\[ f(z) = \sum_{n=-\infty}^{\infty} a_n e^{inz} \]

the following assertions are valid for all \( \nu > \frac{1}{2} \).

I. If \( (a_n \exp(|n|^{2\nu} t))_{n=-\infty}^{\infty} \in \ell_2 \), then \( f \) is entire analytic and

\[
\sup_{z=(x+iy) \in \mathbb{C}} \left( \exp \left( -B|y|^{2\nu - 1} \right) |f(x+iy)| \right) \leq \leq a^{(\nu)}_t, b \left( \sum_{n=-\infty}^{\infty} |a_n|^2 \exp(2|n|^{2\nu} t) \right)^{\frac{1}{2}},
\]

where

\[ B > (2\nu - 1) \left( \frac{1}{(2\nu - 2\nu^2)^{2\nu - 1}} \right) \]

and where

\[ a^{(\nu)}_{t, b} = \frac{1}{\sqrt{2\pi}} \left( \sum_{n=-\infty}^{\infty} e^{-2(t-\tau)|n|^{2\nu}} \right)^{\frac{1}{2}} \]

with \( t = (2\nu)^{-2\nu} \left( \frac{2\nu - 1}{B} \right)^{2\nu - 1} < \tau \).
II. If \( f \) is an entire function which satisfies

\[
sup_{z=(x+iy) \in \mathbb{C}} \left( \exp \left( -B |y|^{2\nu-1} \right) |f(x+iy)| \right) < \infty
\]

for some \( B > 0 \), then for \( t = (2\nu)^{-\frac{\nu}{B}} \left( \frac{2\nu - 1}{B} \right)^{2\nu-1} \) we have

\[
|a_n| \exp(|n|^{2\nu} t) \leq \sqrt{2\pi} \sup_{z=x+iy} \left( \exp \left( -B |y|^{2\nu-1} \right) |f(x+iy)| \right).
\]

**Proof**

1. Suppose \( (a_n \exp(|n|^{2\nu} t) \}_{n=-\infty}^{\infty} \in \ell_2 \). Then it is clear that \( f \) is an entire analytic function. We estimate as follows:

\[
|f(x+iy)| = \frac{1}{\sqrt{2\pi}} \left| \sum_{n=-\infty}^{\infty} a_n \exp(t|n|^{\nu}) (\exp(-t|n|^{2\nu} + \text{in}(x+iy))) \right| \leq \frac{1}{\sqrt{2\pi}} \left( \sum_{n=-\infty}^{\infty} |a_n|^2 \exp(2t|n|^{2\nu}) \right)^{\frac{1}{2}} \left( \sum_{n=-\infty}^{\infty} \exp(-2(t-\tau)|n|^{2\nu}) \right)^{\frac{1}{2}} \cdot \exp(-2t|n|^{2\nu} - 2ny)^{\frac{1}{2}}
\]

where we take \( \tau = (2\nu)^{-\frac{\nu}{B}} \left( \frac{2\nu - 1}{B} \right)^{2\nu-1} \) with \( B > (2\nu-1) \left( \frac{2\nu}{t} \right)^{\frac{1}{2\nu-1}} \) fixed. As in Lemma (1.3.6) we have for all \( n \in \mathbb{Z} \)

\[
\exp(-2t|n|^{2\nu} - 2ny) \leq \exp \left( 2B |y|^{\nu} \right), \quad y \in \mathbb{R}.
\]

Thus we obtain for all \( x,y \in \mathbb{R} \)

\[
|f(x+iy)| \leq a^{(\nu)}_{t,B} \left( \sum_{n=-\infty}^{\infty} |a_n|^2 \exp(2t|n|^{2\nu}) \right)^{\frac{1}{2}} \exp \left( B |y|^{2\nu-1} \right).
\]
II. Suppose $f$ is entire analytic and for some $B > 0$

$$\sup_{x+iy \in \mathbb{C}} \left( \exp\left(-\frac{2\nu}{2\nu-1}B|y| \right) |f(x+iy)| \right) < \infty.$$ 

The coefficients $a_n$, $n \in \mathbb{Z}$, are given by

$$a_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x+iy)e^{-in(x+iy)} \, dx.$$ 

Now let $n \in \mathbb{Z}$, and choose

$$y_n = -\left(\frac{2\nu - 1}{2\nu B}\right)^{2\nu-1}|n|^{2\nu-1} \text{sign}(n).$$

Then

$$|a_n| \leq \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \left( |f(x+iy_n)| \exp\left(-\frac{2\nu}{2\nu-1}B|y_n| \right) \exp\left(B|y_n|^{2\nu-1} + ny_n \right) \right) \, dx$$

$$\leq \frac{2\pi}{\sqrt{2\pi}} \exp(-t|n|^{2\nu}) \sup_{(x+iy) \in \mathbb{C}} \exp\left(-\frac{2\nu}{2\nu-1}B|y| \right) |f(x+iy)|$$

where $t = (2\nu)^{-2\nu}\left(\frac{2\nu - 1}{B}\right)^{2\nu-1}.$

We obtain

$$|a_n| \leq \sqrt{2\pi} \exp(-t|n|^{2\nu}) \sup_{(x+iy) \in \mathbb{C}} \left( \exp\left(-\frac{2\nu}{2\nu-1}B|y| \right) |f(x+iy)| \right).$$

We now introduce the following classes of entire functions.

(3.1.8) Definition

Let $\mu > 1$, and let $B > 0$. Then the class

$$A(\mathbb{C}; 2\pi\text{-per,} \mu, B)$$
consists of all $2\pi$-periodic, entire analytic functions $f$ satisfying

$$\sup_{x \in \mathbb{R}} |f(x+iy)| \leq A_f \exp(B_f |y|^\nu)$$

for some constants $A_f > 0$ and $0 < B_f < B$ independent of $y \in \mathbb{R}$.

By the previous lemmas we obtain the following characterization.

(3.1.9) Theorem

Let $\nu > \frac{1}{2}$ and let $B > 0$. Put $t = (2\nu)^{-\nu}\left(\frac{2\nu - 1}{B}\right)^{2\nu - 1}$. Then $f \in L_2([-\pi,\pi])$ can be extended to

$$f \in A(\mathcal{C}; 2\pi \text{-per}, \frac{2\nu}{2\nu - 1}, B) \quad \text{iff} \quad f \in \exp(-t \Delta^\nu)(S_{L_2([-\pi,\pi]), \Delta^\nu}).$$

Proof

$\Rightarrow$ Let $f \in \exp(-t \Delta^\nu)(S_{L_2([-\pi,\pi]), \Delta^\nu})$. Then $f = \sum_{n=-\infty}^{\infty} a_n e_n$ where

$$(a_n \exp(t + \tau) |n|^{2\nu})_{n=-\infty}^{\infty} \in \ell_2$$

for some fixed $\tau > 0$.

By Lemma (3.1.7) we get

$$\sup_{(x+iy) \in \mathcal{C}} \exp\left(-\widetilde{B} |y|^{\frac{2\nu - 1}{2\nu - 1}}\right) |f(x+iy)| < \infty$$

where $\widetilde{B}$ satisfies

$$(2\nu - 1)\left(\frac{(2\nu)^{-\nu}}{t + \tau}\right)^{2\nu - 1} < \widetilde{B} < (2 - 1)\left(\frac{(2\nu)^{-\nu}}{t}\right)^{2\nu - 1} = B.$$

Hence $f \in A(\mathcal{C}, 2\pi \text{-per}, \frac{2\nu}{2\nu - 1}, B)$.

$\Leftarrow$ Let $f \in A(\mathcal{C}, 2\pi \text{-per}, \frac{2\nu}{2\nu - 1}, B)$. There exist constants $A_f > 0$ and $0 < B_f < B$ such that
By Lemma (3.1.7) the sequence \((a_n)_{n=-\infty}^{\infty}\) defined by \(a_n = (f,e_n), n \in \mathbb{Z}\), satisfies \((a_n \exp(\eta |n^{2\nu}|))_{n=-\infty}^{\infty} \in L_{\infty}\), where
\[
\tilde{\eta} = (2\nu - 2\nu - 1)\left(\frac{2\nu - 1}{B_f}\right)^{2\nu - 1} > t.
\]

So for all \(\tau\) with \(\tilde{\tau} > \tau > t\), \(f \in \exp(-\tau \Delta^{\nu})(\mathbb{L}_2([-\pi,\pi]))\) and hence
\[
f \in \exp(-\tau \Delta^{\nu})(\mathbb{S}_{\mathbb{L}_2([-\pi,\pi]),\Delta^{\nu}}).
\]

Since
\[
\mathbb{S}_{\mathbb{L}_2([-\pi,\pi]),\Delta^{\nu}} = \bigcup_{\tau > 0} \exp(-\tau \Delta^{\nu})(\mathbb{S}_{\mathbb{L}_2([-\pi,\pi]),\Delta^{\nu}})
\]
and since
\[
\tau(\mathbb{L}_2([-\pi,\pi]),\Delta^{\nu}) = \bigcap_{\tau > 0} \exp(-\tau \Delta^{\nu})(\mathbb{S}_{\mathbb{L}_2([-\pi,\pi]),\Delta^{\nu}}),
\]
as a consequence of the previous theorem we derive

(3.1.10) Theorem

Let \(\nu > \frac{1}{2}\). Then the following characterizations are valid:

I. \(f \in \mathbb{S}_{\mathbb{L}_2([-\pi,\pi]),\Delta^{\nu}}\) iff there exists a constant \(B > 0\) such that the entire analytic extension of \(f\) is an element of \(A(\mathbb{C},2\pi\text{-per},\frac{2\nu}{2\nu - 1},B)\).

II. \(f \in \tau(\mathbb{L}_2([-\pi,\pi]),\Delta^{\nu})\) iff for all \(\varepsilon > 0\) the entire analytic extension of \(f\) is an element of \(A(\mathbb{C},2\pi\text{-per},\frac{2\nu}{2\nu - 1},\varepsilon)\).

(I.e. \(\forall \varepsilon > 0 \exists A_\varepsilon > 0 \forall y \in \mathbb{R}: \sup_{x \in \mathbb{R}} |f(x+iy)| \leq A_\varepsilon \exp\left(\varepsilon |y|^{\frac{2\nu}{2\nu - 1}}\right)\).
We also derive a classical analytic description of sequential convergence. In $A(\xi, 2\pi\text{-per}, \mu, B)$ we define the following norm

$$q_B^{(\mu)}(f) = \sup_{z=x+iy \in \mathcal{C}} (\exp(-B|y|^\mu)|f(x+iy)|).$$

We note that $q_B^{(\mu)}$ is finite for all $f \in A(\xi, 2\pi\text{-per}, \mu, B)$. Let $L_2^{(\nu)}([-\pi, \pi])$, $t > 0$, $\nu > \frac{1}{2}$, denote the Hilbert space $e^{-t\Delta^\nu} (L_2([-\pi, \pi]))$ with inner product

$$(f, g)_{t, \nu} = (e^{t\Delta^\nu} f, e^{t\Delta^\nu} g)_{L_2}.$$ 

We denote the norm in $L_2^{(\nu)}([-\pi, \pi])$ by $\|\cdot\|_{t, \nu}$.

Let $\nu > \frac{1}{2}$ and let $t > 0$. Put

$$B_{t, \nu} = (2\nu-1)^{-1} \left(\frac{2\nu-2\nu}{t} \frac{2\nu-1}{t}\right).$$

Let $f \in L_2^{(\nu)}([-\pi, \pi])$. Then $f \in A(\xi, 2\pi\text{-per}, \frac{2\nu}{2\nu-1}, B)$ for all $B > B_{t, \nu}$. By Lemma (3.1.7) we have

$$(*) \quad \forall_{B > B_{t, \nu}} \exists_{a(\nu)}: q_B^{(\nu)}(f) \leq a(B) \|f\|_{t, \nu}.$$ 

Conversely, let $\nu > \frac{1}{2}$ and let $B > 0$. Put $t_{\nu, B} = (2\nu-2\nu(2\nu-1)2\nu-1)^{2\nu-1}$. Let $f \in A(\xi, 2\pi\text{-per}, \frac{2\nu}{2\nu-1}, B)$. Then by Lemma (1.3.7)

$$\sup_{n \in \mathbb{Z}} (\exp(t_{\nu, B}|n|^{2\nu})|f(e_n)|) \leq \sqrt{2\pi} q_B^{(\nu)}(f).$$

Thus we obtain
The inequalities (*) and (**) give rise to the following descriptions:

\((3.1.11) \text{ Theorem}\)

Let \( \nu > \frac{1}{2} \).

I. Let \( t > 0 \). Let \( (f_n)_{n \in \mathbb{N}} \) be a sequence in \( e^{-t \Delta^\nu} S_{L_2([-\pi, \pi]), \Delta^\nu} \). Then

\[ (e^{t \Delta^\nu} f_n)_{n \in \mathbb{N}} \text{ is a null sequence in } S_{L_2([-\pi, \pi]), \Delta^\nu} \text{ iff } q_B (f_n) \to 0 \]

for some \( B \) with \( 0 < B < B_t \), i.e.

\[ \sup_{(x+iy) \in \mathbb{C}, v} \left( \exp \left( \frac{2\nu}{2\nu - 1} |x| \right) |f_n(x+iy)| \right) \to 0 \text{ as } n \to \infty. \]

II. The sequence \( (g_n)_{n \in \mathbb{N}} \) is a null sequence in \( \tau(L_2([-\pi, \pi]), \Delta^\nu) \) iff for all \( \epsilon > 0 \), \( q_\epsilon (f_n) \to 0 \) as \( n \to \infty \), i.e.

\[ \forall \epsilon > 0 : \sup_{(x+iy) \in \mathbb{C}, v} \left( \exp \left( \frac{2\nu}{2\nu - 1} |x| \right) |f_n(x+iy)| \right) \to 0, \quad n \to \infty. \]

Proof

I. If \( (e^{t \Delta^\nu} f_n)_{n \in \mathbb{N}} \) is a null sequence in \( S_{L_2([-\pi, \pi]), \Delta^\nu} \), then there exists \( \epsilon > 0 \) such that \( \|f_n\|_{t+\epsilon, \nu} \to 0 \) as \( n \to \infty \). So \( q_B (f_n) \to 0 \) as \( n \to \infty \), where \( B \) is any number larger than \( B_{t+\epsilon, \nu} \). Since \( B_{t+\epsilon, \nu} > B_{t, \nu} \), we can choose \( B < B_{t, \nu} \).

Conversely, let the sequence \( (f_n)_{n \in \mathbb{N}} \) satisfy \( q_B (f_n) \to 0 \) as \( n \to \infty \) for some \( B \) with \( 0 < B < B_{t, \nu} \). Following (**), \( \|f_n\|_{s, \nu} \to 0 \) for all \( s \) with \( 0 < s < t_{\nu, B} \). Since \( t < t_{\nu, B} \), there exists \( \epsilon > 0 \) such that \( \|f_n\|_{t+\epsilon, \nu} \to 0 \) as \( n \to \infty \).
II. Similar to the proof of I.

Remark. Let \( \nu > \frac{1}{2} \). The sequence \( (f_n)_{n \in \mathbb{N}} \) is a null sequence in \( S_{L^2([-\pi, \pi]), \Delta^\nu} \) iff there exists \( B > 0 \) such that \( (f_n)_{n \in \mathbb{N}} \subseteq A(\mathcal{C}, 2\pi\text{-per}, \frac{2
u}{2\nu-1}, B) \) and

\[
\sup_{(x+iy) \in \mathcal{C}} \left( \exp \left(-B|y|^{\frac{2\nu}{2\nu-1}}\right)|f_n(x+iy)| \right) \to 0 \quad \text{as} \ n \to \infty.
\]

Remark

- Examples of continuous linear functionals on \( S_{L^2([-\pi, \pi]), \Delta^\frac{1}{2}} \) are the evaluation functionals \( \delta^{(k)}_x : f \mapsto (-1)^k f^{(k)}(x), \ x \in \mathbb{R}, \ k \in \mathbb{N} \cup \{0\} \).

Cf. Theorem 3.1.5.

- Examples of continuous linear functionals on \( \tau(L^2([-\pi, \pi]), \Delta^\frac{1}{2}) \) are the evaluation functionals \( \delta^{(k)}_z : f \mapsto (-1)^k f^{(k)}(z), \ z \in \mathbb{C}, \ k \in \mathbb{N} \cup \{0\} \).

A wider class of continuous linear functionals is obtained as follows. Consider a Borel measure \( \mu \) of bounded support on the vertical strip \([-\pi, \pi] \times i\mathbb{R}\) and define \( \ell_\mu \) by

\[
\ell_\mu(f) = \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} f(x+iy)d\mu(x,y).
\]

Cf. Theorem 3.1.5.

- Examples of continuous linear functionals on \( S_{L^2([-\pi, \pi]), \Delta^\nu}, \nu > \frac{1}{2} \), are \( \delta^{(k)}_z, \ z \in \mathbb{C}, \ k \in \mathbb{N} \cup \{0\} \). A wider class is obtained as follows. Consider Borel measures \( \mu \) with the property

\[
\forall B > 0 : \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \exp \left(-B|y|^{\frac{2\nu}{2\nu-1}}\right)d\mu(x,y) < \infty.
\]

Then define \( \ell_\mu \) by

\[
\ell_\mu(f) = \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} f(x+iy)d\mu(x,y).
\]
Examples of continuous linear functionals in $\tau(L_2([-\pi,\pi]),\Delta^\nu)$ are $\delta_z^{(k)}$, $z \in \mathbb{C}$, $k \in \mathbb{N} \cup \{0\}$. A wider class consists of all linear functionals $\ell_\mu$:

$$\ell_\mu(f) = \int_\mathbb{R} \int_{-\infty}^{\infty} f(x+iy)d\mu(x,y)$$

where $\mu$ is a Borel measure satisfying

$$\exists B > 0: \int_\mathbb{R} \int_{-\infty}^{\infty} \exp\left(Bl\frac{2v}{2v-1}\right) d|\mu(x,y)| < \infty.$$

3.2. The characterization of the spaces

$$S = L_2([-1,1],(1-x^2)^{-\frac{1}{2}}dx), \left(-(1-x^2)\frac{d^2}{dx^2} + x \frac{d}{dx}\right)^\nu$$

and

$$\tau(L_2([-1,1],(1-x^2)^{-\frac{1}{2}}dx), \left(-(1-x^2)\frac{d^2}{dx^2} + x \frac{d}{dx}\right)^\nu),$$

and related classical analytic results

In the previous section we have described the spaces $S$ and $\tau$ in full detail. As observed in the introduction to this chapter, the spaces $S$ and $\tau$ can be regarded as the even part of $S$ and $\tau$. It is therefore natural to introduce the following classes of analytic functions.

(3.2.1) Definition

I. Let $t > 0$. Then the class $A(S_t,2\pi$-per,even) consists of all $2\pi$-periodic and even functions $f$ for which there exists $\varepsilon > 0$ such that $f$ is analytic on $|\text{Im } z| < t+\varepsilon$. 
II. The class $A(\mathcal{C}, 2\pi \text{-per}, \text{even})$ consists of all $2\pi$-periodic and even entire analytic functions.

III. Let $\mu > 1$ and let $B > 0$. Then the class $A(\mathcal{C}, 2\pi \text{-per}, \text{even}; \mu, B)$ consists of all $2\pi$-periodic and even entire analytic functions $f$ satisfying

$$\sup_{x \in \mathbb{R}} |f(x+iy)| \leq A_f \exp(B_f |y|^\mu)$$

for some constants $A_f > 0$ and $0 < B_f < B$ independent of $y \in \mathbb{R}$.

Employing the notation $Y = L^2([0, \pi])$ and $C = -\frac{d^2}{d\theta^2}$, we immediately derive from Theorem (3.1.3) and Theorem (3.1.9)

(3.2.2) Theorem

I. Let $f \in Y$ can be extended to $f \in A(S_t, 2\pi \text{-per}, \text{even})$ iff

$$f \in e^{-tC^\frac{1}{2}}(S_{Y,C^\frac{1}{2}}).$$

II. Let $\nu > \frac{1}{2}$ and let $B > 0$. Put $t = (2\nu)^{-2\nu}\left(\frac{2\nu - 1}{B}\right)^{2\nu-1}$. Then $f \in Y$ can be extended to $f \in A(\mathcal{C}, 2\pi \text{-per}, \text{even}; 2\nu^{-1}, B)$ iff $f \in e^{-tC^\nu}(S_{Y,C^\nu})$.

Thus we obtain the following characterization of the spaces $S_{Y,C^\nu}$ and $\tau(Y,C^\nu)$ for all $\nu \geq \frac{1}{2}$.

(3.2.3) Theorem

I. $f \in S_{Y,C^\frac{1}{2}}$ iff $f$ has an analytic extension which belongs to $A(S_t, 2\pi \text{-per}, \text{even})$ for some $t > 0$.

II. $f \in \tau(Y,C^\frac{1}{2})$ iff $f$ has an analytic extension which belongs to $A(\mathcal{C}, 2\pi \text{-per}, \text{even})$. 
III. Let \( f \in S_{Y, C^\nu} \), \( \nu > \frac{1}{2} \), iff there exists \( B > 0 \) such that the analytic extension of \( f \) belongs to \( A(\mathbb{C}, 2\pi\text{-per, even}; \frac{2\nu}{2\nu-1}, B) \).

IV. Let \( f \in \tau(Y, C^\nu) \)), \( \nu > \frac{1}{2} \), iff \( f \) has an analytic extension which belongs to each \( A(\mathbb{C}, 2\pi\text{-per, even}; \frac{2\nu}{2\nu-1}, \epsilon) \), \( \epsilon > 0 \).

In the sequel, the elements of \( S_{Y, C^\nu} \) and \( \tau(Y, C^\nu) \) and their analytic extensions will be identified. It is clear that also the analytic descriptions of sequential convergence derived for the spaces \( S_{L_2([-\pi, \pi]), \Delta^\nu} \) and \( \tau(L_2([-\pi, \pi]), \Delta^\nu) \) carry over to the spaces \( S_{Y, C^\nu} \) and \( \tau(Y, C^\nu) \).

(3.2.4) Theorem

I. Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence in \( A(\mathbb{C}, 2\pi\text{-per, even}) \).

\[
\text{The sequence } \{e^{tC^\frac{1}{2}}(S_{Y, C^\frac{1}{2}}) = A(\mathbb{S}_t, 2\pi\text{-per, even}) \}\text{ is a null sequence in } S_{Y, C^\frac{1}{2}} \text{ iff}
\]

\[
\sup_{|\text{Im } z| \leq t + \epsilon} (|f_n(z)|) \to 0 \text{ as } n \to \infty \text{ for some } \epsilon > 0.
\]

II. The sequence \( \{\varphi_n\}_{n \in \mathbb{N}} \) is a null sequence in \( \tau(Y, C^\frac{1}{2}) \) iff for all \( t > 0 \), \( \sup_{|\text{Im } z| \leq t} (|\varphi_n(z)|) \to 0 \text{ as } n \to \infty. \)

Let \( \nu > \frac{1}{2} \).

III. Let \( \{g_n\}_{n \in \mathbb{N}} \) be a sequence in \( A(\mathbb{C}, 2\pi\text{-per, even}; \frac{2\nu}{2\nu-1}, B) \).

\[
e^{-tC^\nu}(S_{Y, C^\nu}) = A(\mathbb{C}, 2\pi\text{-per, even}; \frac{2\nu}{2\nu-1}, B)
\]

with \( B = (2\nu-1)\left(\frac{2\nu}{t}\right)^{\frac{1}{2\nu-1}} \).
The sequence \((e^{tC^\nu}g_n)_{n \in \mathbb{N}}\) is a null sequence in \(S_{Y,C^\nu}\) iff
\[
\sup_{(x+iy) \in \mathbb{C}} \left( \exp\left(-B_{\epsilon}|y|^{2\nu-1}\right)|g_n(x+iy)| \right) \to 0 \quad \text{as } n \to \infty
\]
for some \(B_{\epsilon}\) with \(0 < B_{\epsilon} < B\).

IV. The sequence \((\psi_n)_{n \in \mathbb{N}}\) is a null sequence in \(\tau(Y,C^\nu)\) iff
\[
\sup_{(x+iy) \in \mathbb{C}} \left( \exp\left(-\epsilon|y|^{2\nu-1}\right)|\psi_n(x+iy)| \right) \to 0 \quad \text{as } n \to \infty
\]
for all \(\epsilon > 0\).

Proof

Cf. Theorem (3.1.5) and Theorem (3.1.11).

For examples of continuous linear functionals on the spaces \(S_{Y,C^\nu}\) and \(\tau(Y,C^\nu)\), \(\nu \geq \frac{1}{2}\), see the end of the previous section.

As noted before, the coordinate transformation \(x = \cos \theta\) gives rise to a unitary mapping from \(Y = L_2([0,\pi])\) onto \(X = L_2([-1,1],(1-x^2)^{-\frac{1}{2}} dx)\). Now put
\[
A = A_{-\frac{1}{2},-\frac{1}{2}} = - (1-x^2) \frac{d^2}{dx^2} + x \frac{d}{dx}.
\]

It follows that an element \(h \in X\) is in \(S_{X,A^\nu}(\tau(X,A^\nu))\) iff the function \(\tilde{h}: \theta \mapsto h(\cos \theta)\) is an element of \(S_{Y,C^\nu}(\tau(Y,C^\nu))\). So in order to characterize the spaces \(S_{X,A^\nu}(\tau(X,A^\nu))\) by classes of analytic functions, we have to consider the conformal mapping \(w = \cos z\) from the strip \(0 \leq \text{Re } z \leq \pi\) in the complex \(z\)-plane onto the complex \(w\)-plane. By this conformal mapping the rectangle \(\{z \in \mathbb{C} \mid 0 < \text{Re } z < \pi, \text{Im } z < t\}\) is carried into the region in the complex \(w\)-plane given by \(G_t = E_t \setminus (L_t^- \cup L_t^+),\) where
If we consider an even, $2\pi$-periodic function $f$ which is analytic on the strip $|\text{Im } z| < t$, then $w \mapsto f(\arccos w)$ is a function which is analytic on the region $G_t$. However, since $f$ is even and $2\pi$-periodic, the function is continuous on $L^+_t$ and $L^-_t$. Therefore $w \mapsto f(\arccos w)$ can be extended to an analytic function on $E_t$. Consequently, the conformal mapping $w = \cos z$ transforms every even, $2\pi$-periodic, entire analytic function of $z$ into an entire analytic function of $w$.

The above observations will eventually lead to the wanted characterization of the spaces $S_{X,A^v}$ and $\tau(X,A^v)$, $v \geq \frac{1}{2}$, in classical analytic terms.

First we define the following classes of analytic functions.
(3.2.5) Definition

I. For each $t > 0$, the class $A(E_t)$ consists of all functions $f$ for which there exists $\varepsilon > 0$ such that $f$ is analytic on $E_{t+\varepsilon}$.

II. The class $A(\mathbb{C})$ consists of all entire analytic functions.

III. Let $K > 0$ and let $\rho > 1$. Then the class $A(\mathbb{C};\rho;B)$ consists of all entire analytic functions $h$ for which there exist constants $A_h > 0$ and $0 < B_h < B$ such that

$$|h(w)| \leq A_h |w|^{\rho-1}B_h (\log(\max(1,|w|))^{\rho-1}.$$ 

The following important result is valid.

(3.2.6) Lemma

The conformal mapping $w = \cos z$ yields a bijection

I. From $A(S_t,2\pi\text{-per,even})$ onto $A(E_t)$.

II. From $A(\mathbb{C},2\pi\text{-per,even})$ onto $A(\mathbb{C})$.

III. From $A(\mathbb{C},2\pi\text{-per,even};\mu,B)$ onto $A(\mathbb{C};\mu,B)$.

Proof

Taking into consideration the mentioned properties of the transformation $w = \cos z$, the proofs of I and II are left out. So we only prove part III. To this end, let $f \in A(\mathbb{C},2\pi\text{-per,even};\mu,B)$. By assumption, there exist $A_f > 0$ and $0 < B_f < B$ such that

$$\forall x \in \mathbb{R} \forall y \in \mathbb{R} : |f(x+iy)| \leq A_f \exp(B_f |y|^\mu).$$

Put $z = x+iy$, and $w = u+iv$. We have $w = \cos z$, and therefore $u = \cos x \cosh y$, $v = \sin x \sinh y$, where $\sinh^2 y \leq u^2 + v^2 < \cosh^2 y$. Here we only consider $|w| = R = \sqrt{u^2 + v^2} \geq 1$. 

We obtain the following straightforward estimation:

\[ |f(\arccos w)| = |f(z)| \leq A_f \exp(B_f |y|^\mu) \leq \]
\[ \leq A_f \exp(B_f (\arcsinh R)^\mu) \leq \]
\[ \leq A_f \exp(B_f (\log(R + \sqrt{1 + R^2}))^\mu) \leq \]
\[ \leq A_f \exp \left( B_f (\log R)^\mu \left( 1 + \frac{\log(1 + \sqrt{1 + 1/R^2})}{\log R} \right)^\mu \right). \]

Let \( \varepsilon > 0 \) be so small that \( B_f + \varepsilon < B \). Further, let \( R_0 > 1 \) be so large that for all \( R > R_0 \) the factor between \( \{ \cdot \} \) is smaller than \( 1 + \frac{\varepsilon}{B_f} \). Then for \( R < R_0 \) we have

\[ |f(\arccos w)| \leq A_f \exp(B_f (\log(R_0 + \sqrt{1 + R_0^2}))^\mu) \]

and for \( |w| \geq R_0 \) we have

\[ |f(\arccos w)| \leq A_f \exp (B_f (\log |w|)^\mu (1 + \frac{\varepsilon}{B_f})) \]
\[ = A_f \exp ((B_f + \varepsilon)(\log |w|)^\mu) . \]

So for all \( w \in \mathbb{C} \) the following estimate is valid

\[ |f(\arccos w)| \leq \tilde{A}_f |w|^\mu (\log(\max(1, |w|)))^{\mu-1} . \]

with

\[ \tilde{A}_f = A_f (1 + \exp(B_f (\log(R_0 + \sqrt{1 + R_0^2}))^\mu) \quad \text{and} \quad \tilde{B}_f = B_f + \varepsilon < B . \]

Thus we find that the function \( w \mapsto f(\arccos w) \) is an element of \( A(\mathbb{C}; \mu, B) \).

Conversely, let \( h \in A(\mathbb{C}; \mu, B) \). Then \( z \mapsto h(\cos z) \) is \( 2\pi \)-periodic, even and entire analytic. By assumption there exist \( A_h > 0 \) and \( 0 < B_h < B \) such that
\[ |h(\cos z)| = |h(w)| \leq A_h \exp(B_h(\log |w|)^\mu), \quad |w| \geq 1. \]

Since \( |w| \leq \cosh y \), we get
\[ |h(\cos z)| \leq A_h \exp(B_h(\log \cosh y)^\mu). \]

Now for all \( y \in \mathbb{R} \) we have
\[ \cosh y = \frac{1}{2}(e^y + e^{-y}) \leq e^{|y|} \]

and hence
\[ |h(\cos z)| \leq A_h \exp(B_h|y|^\mu). \]

It follows that
\[ (z \mapsto h(\cos z)) \in A(\mathbb{C},2\pi\text{-}per,\text{even};\mu,B). \]

From Theorem (3.2.2) and the above lemma we obtain the following results.

(3.2.7) Theorem

I. Let \( t > 0 \). Then \( h \in X \) can be extended to \( h \in A(E_t) \) iff \( h \in e^{-tA^\frac{1}{2}}(S_{X,A^\frac{1}{2}}) \).

II. Let \( v > \frac{1}{4} \) and let \( B > 0 \). Put \( t = (2v)^{-2}\left(\frac{2v - 1}{B}\right)^{2v - 1} \). Then \( h \in X \) can be extended to \( h \in A(\mathbb{C}; \frac{2v}{2v - 1}, B) \) iff \( h \in e^{-tA^v}(S_{X,A^v}) \).

(3.2.8) Theorem

I. \( h \in S_{X,A^\frac{1}{4}} \) iff there exists \( t > 0 \) such that \( h \) has an analytic extension which is a member of \( A(E_t) \).

II. \( h \in \tau(X,A) \) if \( h \) has an entire analytic extension.

III. \( h \in S_{X,A^v}, v > \frac{1}{4}, \) iff \( h \) has an entire analytic extension which is a member of \( A(\mathbb{C}; \frac{2v}{2v - 1}, B) \) for some \( B > 0 \).
IV. \( h \in \tau(X,A^\nu) \), \( \nu > \frac{1}{2} \), iff \( h \) has an entire analytic extension which is a member of each \( A(c; \frac{2\nu}{2\nu-1}, \varepsilon) \), \( \varepsilon > 0 \).

It is natural to identify the elements of \( S(X,A^\nu) \), \( \nu \geq \frac{1}{2} \), and its corresponding analytic extensions to the complex plane.

As a consequence of the classification results, which have been proved in the previous chapter, we have

(3.2.9) Corollary

Let \( \alpha, \beta > -1 \). Then

\[
S_{X,\alpha,\beta,}^{(A_{\alpha,\beta})^\frac{1}{2}} = \bigcup_{t>0} A(\overline{E}_t), \quad \tau(X,\alpha,\beta,)^{(A_{\alpha,\beta})^\frac{1}{2}} = \mathbb{C},
\]

\[
S_{X,\alpha,\beta,}^{(A_{\alpha,\beta})^\nu} = \bigcup_{B>0} A(c; \frac{2\nu}{2\nu-1}, B),
\]

\[
\tau(X,\alpha,\beta,)^{(A_{\alpha,\beta})^\nu} = \bigcap_{B>0} A(c; \frac{2\nu}{2\nu-1}, B),
\]

where \( \nu > \frac{1}{2} \).

As a natural consequence of the classical description of sequential convergence in the spaces \( S_{X,\nu} \) and \( \tau(Y,C^\nu) \), \( \nu \geq \frac{1}{2} \), the sequential convergence in the spaces \( S_{X,\nu} \) and \( \tau(X,A^\nu) \) can be described in classical analytic terms.

(3.2.10) Theorem

I. Let \( (h_n)_{n \in \mathbb{N}} \) be a sequence in \( e^{-tA^\frac{1}{2}} S_{X,A^\nu} = A(\overline{E}_t) \). The sequence \( (e^{tA^\frac{1}{2}} h_n)_{n \in \mathbb{N}} \) is a null sequence in \( S_{X,A^\nu} \) iff there exists \( \varepsilon > 0 \) such that

\[
\sup_{w \in E_{t+\varepsilon}} (|h_n(w)|) \to 0, \quad n \to \infty.
\]
II. The sequence \( (\theta_n)_{n \in \mathbb{N}} \) is a null sequence in \( \tau(X, A^{1/2}) \) iff for all \( t > 0 \),
\[
\sup_{w \in \mathbb{C}} (|\theta_n(w)|) \to 0 \quad \text{as} \quad n \to \infty.
\]

Let \( \nu > \frac{1}{2} \).

III. Let \( (h_n)_{n \in \mathbb{N}} \) be a sequence in \( e^{-tA^{\nu}}(S, X, A^{\nu}) = A(\mathbb{C}; \frac{2\nu}{2\nu-1}, B) \)
with
\[
B = (2\nu-1) \left( \frac{(2\nu)^{-2\nu}}{t} \right)^{\frac{2\nu}{2\nu-1}}.
\]
The sequence \( (e^{tA^{\nu}}g_n)_{n \in \mathbb{N}} \) is a null sequence in \( S, X, A^{\nu} \) iff
\[
\sup_{w \in \mathbb{C}, |w| \geq 1} \left( \exp \left( -B \varepsilon (\log |w|)^{\frac{2\nu}{2\nu-1}} \right) |g_n(w)| \right) \to 0 \quad \text{as} \quad n \to \infty
\]
for some \( B \varepsilon \) with \( 0 < B \varepsilon < B \).

IV. The sequence \( (\theta_n)_{n \in \mathbb{N}} \) is a null sequence in \( \tau(X, A^{\nu}) \) iff for all \( \varepsilon > 0 \)
\[
\sup_{w \in \mathbb{C}, |w| \geq 1} \left( \exp \left( -\varepsilon (\log |w|)^{\frac{2\nu}{2\nu-1}} \right) |\theta_n(w)| \right) \to 0 \quad \text{as} \quad n \to \infty
\]
and \( \theta_n(w) \to 0 \) for all \( w \in \mathbb{C} \), with \( |w| < 1 \).

We recall that the Chebyshev polynomials \( T_n = R_n(-\frac{1}{2}, -\frac{1}{2}) \) are the normalized eigenfunctions in \( X \) of the operator \( A = A^{\frac{1}{2}, -\frac{1}{2}} \). So for \( f \in e^{-tA^{\frac{1}{2}}} (S, X, A^{\frac{1}{2}}) \) it follows that the sequence
\[
\left( \sum_{n=0}^{N} (f, T_n)_{X, A^{\frac{1}{2}}} T_n \right)_{N \in \mathbb{N}}
\]
converges to \( f \) in the sense that \( \left( e^{tA^{\frac{1}{2}}}(f - \sum_{n=0}^{N} (f, T_n) T_n) \right) \) is a null sequence in \( S, X, A^{\frac{1}{2}} \). So by the previous theorem
So the series \( \sum_{n=0}^{\infty} (f,T_n)_{T_n}(w) \) converges uniformly on \( \overline{E}_t \) to \( f \).

Let \( \alpha, \beta > -1 \). Since \( S_{X,A_1^{\frac{1}{2}}} = S_{X_{\alpha, \beta}^t(A_{\alpha, \beta})^{\frac{1}{2}}} \), \( f \) can be written as

\[
\begin{align*}
f &= \sum_{n=0}^{\infty} (f,R_n^{(\alpha, \beta)})_{\alpha, \beta} R_n^{(\alpha, \beta)}.
\end{align*}
\]

Here the series converges in \( S_{X,A_1^{\frac{1}{2}}} \). It follows that there exists \( \tau > 0 \) such that the sequence

\[
\left( e^{\tau A^{\frac{1}{2}}} \left( f - \sum_{n=0}^{N} (f,R_n^{(\alpha, \beta)})_{\alpha, \beta} R_n^{(\alpha, \beta)} \right) \right)_{N \in \mathbb{N}}
\]

is a null sequence in \( S_{X,A_1^{\frac{1}{2}}} \). Hence by the previous theorem the series

\[
\sum_{n=0}^{\infty} (f,R_n^{(\alpha, \beta)})_{\alpha, \beta} R_n^{(\alpha, \beta)}
\]

converges uniformly to \( f \) on \( \overline{E}_t \). (In fact the series converges uniformly on \( \overline{E}_t \), see [9], p. 243.)

(3.2.11) **Theorem**

I. Let \( \alpha, \beta > -1 \), and let \( f \in A(\overline{E}_t) = e^{-\tau A^{\frac{1}{2}}}(S_{X,A_1^{\frac{1}{2}}}) \). Then \( f \) can be written as

\[
\begin{align*}
f &= \sum_{n=0}^{\infty} (f,R_n^{(\alpha, \beta)})_{\alpha, \beta} R_n^{(\alpha, \beta)}
\end{align*}
\]

where \( (f,R_n^{(\alpha, \beta)})_{\alpha, \beta} = O(e^{-sn}) \) for some \( s > 0 \). Here the series converges uniformly on a region \( \overline{E}_t \) with \( \tau \) dependent on \( \alpha, \beta, t \). If \( \alpha = \beta = -\frac{1}{2} \), we can take \( \overline{E}_t = \overline{E}_t \).

II. Let \( \alpha, \beta > -1 \), and let \( \theta \in A(\overline{E}) = \tau(X,A_1^{\frac{1}{2}}) \). Then \( \theta \) can be written as

\[
\begin{align*}
\theta &= \sum_{n=0}^{\infty} (\theta,R_n^{(\alpha, \beta)})_{\alpha, \beta} R_n^{(\alpha, \beta)}
\end{align*}
\]
with $(\Theta_n^{(\alpha, \beta)}_{\infty})_{\alpha, \beta} = O(e^{-ns})$ for all $s > 0$. Here the series converges uniformly on each region $E_t$, $t > 0$.

Next, we shall prove similar results in case $\nu > \frac{1}{2}$.

To this end, we first note that the linear mapping $\omega_{\alpha, \beta}$ introduced in Lemma (2.6), is a continuous bijection on $S_{X, A^\nu}$. Since

$$\forall t > 0 \forall \epsilon > 0: \| e^{(t-\epsilon)A^\nu} \omega_{\alpha, \beta} e^{-tA^\nu} \|_X < \infty$$

and, also,

$$\forall t > 0 \forall \epsilon > 0: \| e^{(t-\epsilon)A^\nu} (\omega_{\alpha, \beta})^{-1} e^{-tA^\nu} \|_X < \infty,$$

it even follows that for all $t > 0$

$$(\omega_{\alpha, \beta})^{\pm 1} (e^{-tA^\nu} (S_{X, A^\nu})) = e^{-tA^\nu} (S_{X, A^\nu}).$$

Further, since $A_{\alpha, \beta} = \omega_{\alpha, \beta} A_{\alpha, \beta}^{-1}$, we also have

$$(\omega_{\alpha, \beta}) (e^{-tA^\nu} (S_{X, A^\nu})) = e^{-t(A_{\alpha, \beta})^\nu} (S_{X_{\alpha, \beta}, (A_{\alpha, \beta})^\nu}).$$

Let $f \in e^{-tA^\nu} (S_{X, A^\nu})$. Then the sequence

$$\left( e^{-tA^\nu} \left( \sum_{n=0}^{N} (f, T_n^X) X_n - f \right) \right)_{N \in \mathbb{N}}$$

is a null sequence in $S_{X, A^\nu}$. So by Theorem (3.2.10)

$$\sup_{\omega \in \mathcal{C},} \left( \exp \left( -B(\log |\omega|) \frac{2\nu}{2\nu - 1} \right) |f(\omega) - \sum_{n=0}^{N} (f, T_n^X) T_n(\omega)| \right) + 0$$

for $|\omega| \geq 1$. 

as $N \to \infty$, and

$$f(\omega) = \sum_{n=0}^{\infty} (f, T_n) T_n(\omega) \quad \text{for all } \omega \in \mathcal{E}, \ |\omega| < 1.$$  

Here $B = (2v-1)(2v-2v)2v-1$. Now let $g = (\omega, \beta)^{-1} f$. Then $g \in e^{-tA^\nu} (S_{X,A^\nu})$. So $\left( e^{tA^\nu} \left( g - \sum_{n=0}^{N} (g, T_n) X T_n \right) \right)_{N \in \mathbb{N}}$ is a null sequence in $S_{X,A^\nu}$. Since $\omega_{\alpha, \beta} T_n = R_n(\alpha, \beta), n \in \mathbb{N} \cup \{0\}$, $f$ is represented by

$$f = \omega_{\alpha, \beta} g = \sum_{n=0}^{\infty} (g, T_n) X R_n(\alpha, \beta).$$

Now we observe that there exists $\varepsilon > 0$ such that the sequence

$$\left( e^{(t+\varepsilon)A^\nu} \left( g - \sum_{n=0}^{N} (g, T_n) X T_n \right) \right)_{N \in \mathbb{N}}$$

is a null sequence in $S_{X,A^\nu}$. Further, the following relation is valid for all $N \in \mathbb{N}$

$$e^{tA^\nu} \sum_{n=0}^{N} (g, T_n) X R_n(\alpha, \beta) =$$

$$= \left( e^{tA^\nu} \omega_{\alpha, \beta} e^{-(t+\varepsilon)A^\nu} \right) \left( e^{(t+\varepsilon)A^\nu} \sum_{n=0}^{N} (g, T_n) X T_n \right).$$

Hence $\left( e^{tA^\nu} \left( f - \sum_{n=0}^{N} (g, T_n) X R_n(\alpha, \beta) \right) \right)_{N \in \mathbb{N}}$ is a null sequence in $S_{X,A^\nu}$. It thus follows that pointwise

$$f(\omega) = \sum_{n=0}^{\infty} (g, T_n) X R_n(\alpha, \beta)(\omega)$$
and moreover, with \( B = (2 \nu - 1) \left( \frac{2^\nu}{t} \right)^{2^\nu - 1} \),
\[
\sup_{w \in \mathbb{C}, |w| \geq 1} \left( \exp \left( -B \left( \log |w| \right)^{2^\nu - 1} \right) \right) |f(w) - \sum_{n=0}^{N} (g_{n}, t_{n}) \mathcal{R}_{n}^{(\alpha, \beta)}(w)| > 0
\]
as \( N \to \infty \). Summarizing, we obtain the following classical analytic type of result.

(3.2.12) **Theorem**

Let \( \nu > \frac{1}{2} \) and let \( B > 0 \). Put \( t = (2 \nu - 2)^{2^\nu - 1} \). Let \( \alpha, \beta > -1 \), and let \( f \in A(\mathbb{C}; \frac{2^\nu}{2^\nu - 1}, B) \). Then there exist coefficients \( a_{n}^{(\alpha, \beta)}, n \in \mathbb{N} \cup \{0\} \), satisfying \( a_{n}^{(\alpha, \beta)} = O(e^{-n^2 \nu (t+\epsilon)}) \) for some \( \epsilon > 0 \), such that
\[
f = \sum_{n=0}^{\infty} a_{n}^{(\alpha, \beta)} \mathcal{R}_{n}^{(\alpha, \beta)}(a, \beta)
\]
pointwise on \( \mathbb{C} \). Moreover,
\[
\sup_{w \in \mathbb{C}, |w| \geq 1} \left( \exp \left( -B \left( \log |w| \right)^{2^\nu - 1} \right) \right) |f(w) - \sum_{n=0}^{N} a_{n}^{(\alpha, \beta)} \mathcal{R}_{n}^{(\alpha, \beta)}(a, \beta)(w)| \to 0
\]
as \( N \to \infty \).

Conversely, let \( f \in X_{\alpha}^{\nu} \), \( f = \sum_{n=0}^{\infty} b_{n}^{(\alpha, \beta)} \mathcal{R}_{n}^{(\alpha, \beta)}(a, \beta) \) where the coefficients \( b_{n}^{(\alpha, \beta)} \) satisfy \( b_{n}^{(\alpha, \beta)} = O(e^{-n^2 \nu (t+\epsilon)}) \) for some \( \epsilon > 0 \). Then \( f \) has an analytic extension which is a member of \( A(\mathbb{C}; \frac{2^\nu}{2^\nu - 1}, B) \).

**Remark.** Examples of continuous linear functionals on the spaces \( X_{\alpha}^{\nu} \) and \( \tau(X, A^{\nu}) \), \( \nu \geq \frac{1}{2} \), can be obtained from measures in the complex plane along similar lines as at the end of section 3.1. As special examples we mention
the evaluation functionals \( \delta_{x}^{(k)}, \ x \in [-1, 1], \ k \in \mathbb{N} \cup \{0\}, \) on \( S_{X, A^{\perp}} \) and
\( \delta_{z}^{(k)}, \ z \in C, \ k \in \mathbb{N} \cup \{0\}, \) on \( \tau(X, A^{\perp}), S_{X, A^{\perp}} \) and \( \tau(X, A^\nu), \nu > \frac{1}{2}. \)

Remark. The characterizations given in Theorem (3.2.3) and Theorem (3.2.8)
lead to a great number of continuous linear mappings on the spaces \( S_{Y, C^\nu}, \)
\( \tau(Y, C^\nu), S_{X, A^\nu} \) and \( \tau(X, A^\nu), \nu \geq \frac{1}{2}. \) We list some examples here.

\(- \quad S_{Y, C^\frac{1}{2}}:\)

\( T_{nn}, \ defined \ by \ (T_{nn} f)(x) = f(x+n\pi), \ n \in \mathbb{Z}; \)

\( Z_n, \ defined \ by \ (Z_n f)(x) = f(nx), \ n \in \mathbb{Z}; \)

\( M_\phi, \ defined \ by \ (M_\phi f)(x) = \phi(x)f(x), \ \text{with} \ \phi \in S_{Y, C^\frac{1}{2}}; \)

\( \sin x \frac{d}{dx} ; \ \frac{1}{\sin x} \frac{d}{dx}. \)

We note that the commutant of \( \frac{1}{\sin x} \frac{d}{dx} \) and \( \cos x \) equals \(-1.\)

\(- \quad \tau(Y, C^\frac{1}{2}):\)

\( T_{nn}, \ n \in \mathbb{Z}; M_\phi, \ \phi \in \tau(Y, C^\frac{1}{2}); \sin x \frac{d}{dx}; T_{ib}, \ b \in \mathbb{R}, \ defined \ by \)

\( (T_{ib} f)(z) = f(z+ib); Z_n; \frac{1}{\sin x} \frac{d}{dx}. \)

\(- \quad S_{Y, C^\nu} \ and \ \tau(Y, C^\nu) \ with \ \nu > \frac{1}{2}:\)

\( T_q, \ \text{with} \ q = n\pi + ib, \ n \in \mathbb{Z}, \ b \in \mathbb{R}, \ defined \ by \ (T_q f)(z) = f(z+q); \)

\( M_\phi, \ \phi \in S_{Y, C^\nu} (\tau(Y, C^\nu)); Z_n, \frac{1}{\sin x} \frac{d}{dx}; \sin x \frac{d}{dx}. \)

\(- \quad S_{X, A^\frac{1}{2}}:\)

\( M_\phi, \ \phi \in S_{X, A^\frac{1}{2}}; \frac{d}{dx}; Z_\lambda, \ \lambda \in \mathbb{R}, \ |\lambda| \leq 1, \ defined \ by \ (Z_\lambda f)(x) = f(\lambda x). \)
\[ \tau(X, A^{\frac{1}{2}}) : \]

\[ M_{\varphi}, \varphi \in \tau(X, A^{\frac{1}{2}}); \quad \frac{d}{dx} ; \quad Z_{\lambda}, \lambda \in \mathbb{C}; \quad T_{a}, a \in \mathbb{C}, \text{ defined by } (T_{a} f)(w) = f(w+a); \]

\[ K_{h}, h \in \tau(X, A^{\frac{1}{2}}) \text{ defined by } (K_{h} f)(w) = f(h(w)). \]

- \( S_{X, A^{\psi}} \) and \( \tau(X, A^{\psi}), \psi > \frac{1}{2} : \)

\[ M_{\varphi}, \varphi \in S_{X, A^{\psi}} (\in \tau(X, A^{\psi})); \quad \frac{d}{dx} ; \quad Z_{\lambda}, \lambda \in \mathbb{C}; \quad T_{a}, a \in \mathbb{C}. \]
4. HYPERFUNCTIONS AND TRAJECTORY SPACES

In order to investigate the relations between hyperfunction theory and some special types of analyticity spaces and trajectory spaces, we start with the following simple example.

Let \( S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \) denote the unit circle in the complex plane. In \( S^1 \) we consider the usual Lebesgue measure. The functions \( p_n : z \mapsto \frac{1}{\sqrt{2\pi}} z^n \), \( n \in \mathbb{N} \cup \{ 0 \} \), establish an orthonormal basis for a Hilbert subspace \( H \) of \( L_2(S^1) \); \( H \) is called the Hardy-Lebesgue space. In \( H \) we define the positive self-adjoint operator \( N \) as follows:

\[
D(N) = \left\{ f \in H \mid \sum_{n=0}^{\infty} n^2 |(f, p_n)|^2 < \infty \right\}
\]

and

\[
Nf = \sum_{n=0}^{\infty} n(f, p_n)p_n, \quad f \in D(N).
\]

(4.1) Theorem

\( f \in S_{H,N} \) iff there exists \( r > 1 \) such that \( f \) can be extended to an analytic function on \( \{ z \in \mathbb{C} \mid |z| < r \} \).

Proof

Let \( f \in S_{H,N} \). Then \( f = \sum_{n=0}^{\infty} \frac{e^{-nt}}{\sqrt{2\pi}} a_n p_n \) for some \( t > 0 \) and some \( \ell_2 \)-sequence \( (a_n)_{n=0}^{\infty} \). The series \( \sum_{n=0}^{\infty} \frac{e^{-nt}}{\sqrt{2\pi}} a_n z^n \) converges uniformly on each disc \( \{ z \in \mathbb{C} \mid |z| \leq \rho \} \) with \( \rho < e^t \). Hence the function \( z \mapsto \sum_{n=0}^{\infty} \frac{e^{-nt}}{\sqrt{2\pi}} a_n z^n \), \( |z| < e^t \), is analytic.

Conversely, if \( g \) is an analytic function on \( \{ z \in \mathbb{C} \mid |z| < r \} \) for some \( r > 1 \), then \( g \) can be written as \( g(z) = \sum_{n=0}^{\infty} b_n z^n \). This series converges uniformly on
\[ |z| \leq \rho < r. \] By Cauchy's criterion it follows that \( b_n = O(e^{-nt}) \) for \( t > 0. \)

\[ 1 < e^t < r. \] So \( g \in S_{H,N} \)

Let \( f \in S_{H,N}. \) Then there exist \( \tau > 0 \) and \( g \in S_{H,N} \) such that \( f = e^{-\tau N} g. \) We have

\[
(e^{-\tau N} g)(z) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{-n\tau}(g, p_n) z^n = g(e^{-\tau} z)
\]

and, equivalently, \((e^{\tau N} f)(z) = f(e^\tau z)\) for \(|z| < r \) with \( r > 1 \) sufficiently small.

Next, we shall represent the trajectory space \( T_{H,N} \) as a space of analytic functions. Each element \( F \) of \( T_{H,N} \) corresponds to a formal series \( \sum_{n=0}^{\infty} \xi_n p_n \), where the sequence \( (\xi_n)_{n=0}^{\infty} \) satisfies the order estimate \( \forall t > 0: \xi_n = O(e^{nt}). \)

We interpret these formal series by means of the notion of trajectory.

Here, however, there is another possibility. Let \( z \in \mathbb{C} \) with \(|z| \leq r < 1. \) Then

\[
\left| \sum_{n=M}^{N} \xi_n \frac{z^n}{\sqrt{2\pi}} \right| \leq \frac{1}{\sqrt{2\pi}} \sum_{n=M}^{N} |\xi_n| r^n \leq \frac{1}{\sqrt{2\pi}} \sup_{n \in \mathbb{N} \setminus \{0\}} \left( |\xi_n| \left( \frac{2r}{n+1} \right)^n \right) \sum_{n=M}^{N} \left( \frac{r+1}{2} \right)^n.
\]

So the series \( \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \xi_n \frac{z^n}{\sqrt{2\pi}} \) converges uniformly on \( \{z \in \mathbb{C} \mid |z| \leq r\}. \)

Hence the function \( z \mapsto \sum_{n=0}^{\infty} \xi_n \frac{z^n}{\sqrt{2\pi}} \) is analytic on \( \{z \in \mathbb{C} \mid |z| < 1\}. \)

(4.2) **Theorem**

\[ F = \sum_{n=0}^{\infty} \xi_n p_n \in T_{H,N} \iff \text{the function } z \mapsto \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \xi_n z^n \text{ is analytic on } \{z \in \mathbb{C} \mid |z| < 1\}. \]

**Proof**

It only remains to prove the implication \( \Rightarrow \). So let \( h \) be an analytic function on \( \{z \in \mathbb{C} \mid |z| < 1\}. \) Then \( h(z) = \sum_{n=0}^{\infty} \theta_n z^n, \) where the series converges
uniformly on each disc \( \{ z \in \mathbb{C} \mid |z| \leq \rho \} \), \( 0 < \rho < 1 \). Hence \( \theta_n = O(r^n) \) for every \( r > 1 \).

Let \( F \in \mathcal{T}_{H,N} \), \( F(t) = \sum_{n=0}^{\infty} e^{-nt} \xi_n p_n \). Then

\[
F(z, t) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{-nt} \xi_n z^n = \hat{F}(e^{-t} z)
\]

where \( \hat{F} \) denotes the analytic function corresponding to \( F \).

The function \( \tilde{F} : w \to \frac{i}{w} \hat{F}(\frac{1}{w}) \) is analytic on \( \{ w \in \mathbb{C} \mid |w| > 1 \} \) and \( \tilde{F}(w) \to 0 \) as \( w \to \infty \). It is clear that each function \( \Phi \) which is analytic on \( \{ w \in \mathbb{C} \mid |w| > 1 \} \) and for which \( \Phi(w) \to 0 \) as \( w \to \infty \), can be written as \( \Phi(w) = \frac{i}{w} \hat{G}(\frac{1}{w}) \) for a unique \( G \in \mathcal{T}_{H,N} \).

(4.3) Theorem

Let \( f \in \mathcal{S}_{H,N} \) and let \( G \in \mathcal{T}_{H,N} \). Choose \( r > 1 \) so small that \( f \) is analytic on \( \{ z \in \mathbb{C} \mid |z| < r \} \). Then

\[
\langle f, G \rangle = \int_{|w| = \rho} f(w) \tilde{G}(w) dw , \quad 1 < \rho < r ,
\]

where the path of integration is clockwise oriented.

Proof

\[
\langle f, G \rangle = \langle e^{\tau N} f, G(\tau) \rangle = \int_{\mathcal{S}^1} f(e^{\tau} z) \hat{G}(e^{-\tau} z) ds = \]

\[
= \frac{1}{i} \int_{-\pi}^{\pi} f(e^{\tau + i\varphi}) \hat{G}(e^{-\tau - i\varphi}) e^{\tau - i\varphi} d\varphi = \]

\[
= \int_{|w| = e^{\tau}} \overline{f(w)} \tilde{G}(w) dw . \]

(4.4) Definition

The space $H(S^1,+)$ of hyperfunctions contains all ("boundary values of") analytic functions $\Phi$ on $\{w \in \mathbb{C} \mid |w| > 1\}$ with the property that $\Phi(w) \to 0$ as $w \to \infty$.

Let $w_0 \in \mathbb{C}$ with $|w_0| > 1$. Then the function $\zeta_{w_0}$ defined by

$$\zeta_{w_0}: z \mapsto \frac{1}{2\pi i} \frac{1}{w_0 - z}$$

can be regarded as an element of $S_{H,N}$. By Theorem (4.3) it follows that for each $G \in T_{H,N}$

$$\langle \zeta_{w_0}, G \rangle = \frac{1}{2\pi i} \oint_{|w| = r} \frac{1}{w - w_0} \tilde{G}(w) dw = \tilde{G}(w_0)$$

where $1 < r < |w_0|$. So the linear mapping

$$G \mapsto \tilde{G}: w \mapsto \langle \zeta_{w_0}, G \rangle$$

is a bijection from $T_{H,N}$ onto the class $H(S^1,+)$.

We have discussed now the connection between a specific trajectory space and a specific class of hyperfunctions. In a similar way, we shall relate the trajectory spaces $T_{L_2([-\pi,\pi]), A^\frac{1}{2}}$, $T_{\gamma, C^\frac{1}{2}}$ and $T_{X, A^\frac{1}{2}}$ (cf. Chapter 3) to suitable spaces of hyperfunctions in the remaining part of this chapter.

In the Hilbert space $L_2([-\pi,\pi])$ an orthonormal basis is established by the functions $e_n: x \mapsto \frac{1}{\sqrt{2\pi}} e^{inx}$, $n \in \mathbb{Z}$. As in Chapter 3 the positive self-adjoint operator $\Delta$ is defined by $\Delta e_n = n^2 e_n$, $n \in \mathbb{Z}$. We have already characterized the space $S_{L_2([-\pi,\pi]), A^\frac{1}{2}}$. It consists of all $2\pi$-periodic functions $f$ which are analytic on a strip around the real axis, where the width of the strip depends on $f$. 
Let \( \mathcal{f} \in \mathcal{L}_2([-\pi, \pi]), \Delta^\frac{1}{2} \). Then for sufficiently small \( \rho > 0 \)

\[
f(z) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} (f(e_n)e^{in\rho}e^{inz} + \frac{i}{2}(f,e_0)), \quad |\text{Im } z| < \rho.
\]

We define

\[
f_+(z) = \frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} (f,e_n)e^{inz} - \frac{i}{2}(f,e_0) \right), \quad |\text{Im } z| < \rho
\]

and

\[
f_-(z) = \frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} (f,e_n)e^{-inz} - \frac{i}{2}(f,e_0) \right), \quad |\text{Im } z| < \rho.
\]

It follows that \( f = f_+ + f_- \). Moreover, \( f_+ \) extends to a \( 2\pi \)-periodic function which is analytic on \( \text{Im } z > -\rho \), and \( f_- \) to a \( 2\pi \)-periodic function which is analytic on \( \text{Im } z < \rho \). Further,

\[
(e^{-it\Delta^\frac{1}{2}}f_+)(z) = f_+(z+it), \quad \text{Im}(z+it) > -\rho,
\]

\[
(e^{-it\Delta^\frac{1}{2}}f_-)(z) = f_-(z-it), \quad \text{Im}(z-it) < \rho.
\]

Now let \( 0 < t < \rho \), and let \( x \in \mathbb{R} \). Then

\[
(e^{it\Delta^\frac{1}{2}}f)(x) = f_+(x-it) + f_-(x+it) = f(x-it) + f(x+it) - f_+(x+it) - f_-(x-it).
\]

In addition we have

\[
f_+(x+it) = \frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} (f,e_n)e^{inx}e^{inz} - \frac{i}{2}(f,e_0) \right) =
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) \left( \sum_{n=0}^{\infty} e^{-in(\xi-x-it)} - \frac{i}{2} \right) d\xi =
\]
\[ f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) \left( \frac{1}{1 - e^{-i(\xi-x-\xi)}} - \frac{1}{1 - e^{-i(\xi-x+\xi)}} \right) d\xi \]

and, similarly,

\[ f_-(x-it) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) \left( \frac{1}{1 - e^{-i(\xi-x-\xi)}} - \frac{1}{1 - e^{-i(\xi-x+\xi)}} \right) d\xi. \]

So as a side result we get the following expression for \( e^{t\Delta t} f \)

\[ (e^{t\Delta t} f)(x) = f(x-it) + f(x+it) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) \frac{\sinh t}{\cosh t - \cos(x-\xi)} d\xi. \]

(Note that \[ 2i \frac{\sinh t}{\cosh t - \cos(x-\xi)} = \cot \frac{1}{2}(\xi-x+it) - \cot \frac{1}{2}(\xi-x-it). \])

The elements of \( T_{L^2([-\pi, \pi]), \Lambda^1} \) can be decomposed in a similar way. So for \( F \in T_{L^2([-\pi, \pi]), \Lambda^1}, F = \sum_{n=-\infty}^{\infty} \xi_n e_n \), we put

\[ F_+ = \sum_{n=0}^{\infty} \xi_n e_n - \frac{1}{2} \xi_0 e_0 \]

and

\[ F_- = \sum_{n=0}^{\infty} \xi_{-n} e_{-n} - \frac{1}{2} \xi_0 e_0. \]

Since \( \forall t > 0: \xi_n = O(e^{nt}), \) it is clear that the function

\[ \hat{F}_+(z) = \frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} \xi_n e^{inz} - \frac{1}{2} \xi_0 \right) \]

is a 2\pi-periodic function which is analytic for \( \text{Im} z > 0. \) We have

\[ \hat{F}_+(x,t) = \hat{F}_+(x+it), \quad t > 0, \ x \in \mathbb{R}. \]

Similarly,
\[ \hat{F}_-(z) = \frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} \xi_{-n} e^{-inz} - \frac{i}{2} \xi_0 \right) \]
is a $2\pi$-periodic function which is analytic on $\text{Im} \ z < 0$. Here we have

\[ F_-(x,t) = \hat{F}_-(x-it), \quad t > 0, \ x \in \mathbb{R}. \]

Now let $f \in S_{L^2([-\pi, \pi])}, \Delta_{\mathbb{R}}^\frac{1}{2}$ and let $G \in T_{L^2([-\pi, \pi])}, \Delta_{\mathbb{R}}^\frac{1}{2}$. Then for $t$ sufficiently small

\[ \int_{-\pi}^{\pi} f(x-it) \hat{G}_+(x+it) \, dx = \]

\[ = <f+\frac{1}{2}(f,e_0)e_0,G_+> + (e^{-t\Delta_{\mathbb{R}}^\frac{1}{2}}(f,-\frac{1}{2}(f,e_0)e_0),G_+(t))_{L^2}. \]

Since $G_+(t)$ is orthogonal to $e^{-t\Delta_{\mathbb{R}}^\frac{1}{2}}(f,-\frac{1}{2}(f,e_0)e_0)$, the second summand is zero.

In the same way we derive

\[ \int_{-\pi}^{\pi} f(x+it) \hat{G}_-(x-it) \, dx = <f-\frac{1}{2}(f,e_0)e_0,G_->. \]

Thus the following result is obtained:

\[ <f,G> = <f+\frac{1}{2}(f,e_0)e_0,G_+> + <f-\frac{1}{2}(f,e_0)e_0,G_-> = \]

\[ = \int_{-\pi}^{\pi} f(x-it) \hat{G}_+(x+it) \, dx + \int_{-\pi}^{\pi} f(x+it) \hat{G}_-(x-it) \, dx = \]

\[ = \oint_{C_\mathbb{L}} f(z) H_G(z) \, dz. \]

Here the path of integration $C_\mathbb{L}$ consists of the two directed line segments, 
\{ $z \in \mathbb{C} \mid \text{Im} \ z = -t$, $\pi \geq \text{Re} \ z \geq -\pi$ \} and \{ $z \in \mathbb{C} \mid \text{Im} \ z = t$, $-\pi \leq \text{Re} \ z \leq \pi$ \}. 

The function $H_G$ is defined by

$$H_G(z) = \begin{cases} \hat{G}_+(z), & \text{Im } z > 0, \\ -\hat{G}_-(z), & \text{Im } z < 0. \end{cases}$$

Now let $0 < t < \text{Im } z$. Then with $G(t) = \sum_{n=-\infty}^{\infty} e^{-nt} \varepsilon_n e^{\pi i n t}$, $t > 0$,

$$H_G(z) = \hat{G}_+(z) = \left( \sum_{n=0}^{\infty} \varepsilon_n e^{-nt} e^{\pi i n (z-it)} - \frac{1}{4\pi} \right) =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{4\pi} \left( \sum_{n=0}^{\infty} e^{\pi i n x} e^{-\pi i (z-it)} \right) dx =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} G(x,t) e^{\frac{1}{4\pi} x} \left( \sum_{n=0}^{\infty} e^{\pi i n x} e^{-\pi i z} - \frac{1}{4} \right) dx =$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} G(x,t) e^{\frac{1}{4\pi} x} (\cot \frac{1}{4}(x-z)) dx.$$

In the same way we obtain for $0 < t < -\text{Im } z$

$$H_G(z) = \frac{1}{4\pi} \int_{-\pi}^{\pi} G(x,t) e^{\frac{1}{4\pi} x} (\cot \frac{1}{4}(x-z)) dx.$$

This leads to the following result.

(4.5) Theorem

Put $\gamma_z : x \mapsto \frac{i}{4\pi} \cot(\frac{1}{4}(x-z))$, $x \in \mathbb{R}$, $z \in \mathbb{C}$. Then the analytic function $H_G$ associated to the trajectory $G \in L^2(\mathbb{R}, \mathbb{C})$ is represented by

$$H_G(z) = \langle \gamma_z, G \rangle, \quad \text{Im } z \neq 0.$$
Proof

We note that for $z \in \mathbb{C}$, $\text{Im } z \neq 0$, the function $\eta_z$ is an element of $S_{L_2([\pi,\pi]),\Delta^1}$. Hence

$$\langle \eta_z, G \rangle = \int_{-\pi}^{\pi} G(x,t)(e^{i\Delta^1 x})_z(x)dx = H_G(z),$$

where $0 < t < |\text{Im } z|$. □

(4.6) Corollary

Let $G \in T_{L_2([\pi,\pi]),\Delta^1}$, let $f \in S_{L_2([\pi,\pi]),\Delta^1}$ and let $t > 0$ sufficiently small. Then

$$\langle f, G \rangle = \int_{\mathbb{C}} f(z) \overline{\langle \eta_z, G \rangle} \, dz.$$ 

Let $\phi$ be a $2\pi$-periodic function which is analytic on $\text{Im } z \neq 0$ and for which

$$\lim_{\text{Im } z \to -\infty} \phi(z)$$

exists and equals $\lim_{\text{Im } z \to -\infty} (-\phi(z))$. Then $\phi$ can be written as

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} \xi_n e^{inz} - i\xi_0 \right), \quad \text{Im } z > 0,$$

$$\phi(z) = -\frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} \xi_{-n} e^{-inz} - i\xi_0 \right), \quad \text{Im } z < 0,$$

where $\xi_n = 0(e^{|n|t})$ for all $t > 0$. It follows that $\phi = H_G$ for $G = \sum_{n=-\infty}^{\infty} \xi_n e_n$.

Now we are in a position to introduce a class of periodic hyperfunctions.

(4.6) Definition

The space of $2\pi$-periodic hyperfunctions $H(\mathbb{R}, 2\pi$-per) consists of all $2\pi$-periodic functions $\phi$ which are analytic on $\mathbb{C} \setminus \mathbb{R}$ and for which

$$\lim_{\text{Im } z \to +\infty} \phi(z) = -\lim_{\text{Im } z \to -\infty} \phi(z)$$

exists.
(4.7) Theorem

\[ \phi \in H(\mathbb{R}, 2\pi\text{-per}) \text{ iff there exists } G \in T_{L^2([-\pi, \pi])}, \Delta^1 \text{ such that } \phi = H_G. \]

Remark. Let \( \phi = H_G \in H(\mathbb{R}, 2\pi\text{-per}) \). Since \( G(x,t) = \hat{\phi}_+(x+it) + \hat{\phi}_-(x-it) = \phi(x+it) - \phi(x-it) \), it follows that \( \lim_{\varepsilon \to 0} (\phi(x+i\varepsilon) - \phi(x-i\varepsilon)) \) exists in 'distributional' sense. If this limit equals an \( L_1 \)-function, say \( h \), then

\[ \int_{-\pi}^{\pi} \phi(z) f(z) dz = \int_{-\pi}^{\pi} h(x) f(x) dx, \quad f \in L^1([-\pi, \pi]), \Delta^1. \]

Conversely, by Theorem (4.7), to every \( L_1 \)-function \( h \), there corresponds a hyperfunction \( \phi \) such that \( h(x) = \lim_{\varepsilon \to 0} (\phi(x+i\varepsilon) - \phi(x-i\varepsilon)) \) because \( h \) can be regarded as an element of \( T_{L^2([-\pi, \pi])}, \Delta^1 \).

Our next aim is to describe a class of hyperfunctions which is related to the trajectory space \( T \). Here \( Y = L^2([0, \pi]) \) and \( C \) denotes the positive self-adjoint operator \( \left( -\frac{d^2}{dx^2} \right) \) where the domain of \(-\frac{d^2}{dx^2}\) consists of the functions \( f \) in \( H^2([-\pi, \pi]) \) with \( f'(0) = f'(\pi) = 0 \). The functions \( c_n \)

\[ c_n : x \mapsto \sqrt{\frac{2}{\pi}} \cos nx, \quad n \in \mathbb{N}, \]

and \( c_0 : x \mapsto \sqrt{\frac{1}{\pi}} \) establish an orthonormal basis in \( Y \) and they satisfy

\[ C^{\frac{1}{2}} c_n = nc_n, \quad n \in \mathbb{N} \cup \{0\}. \]

So the space \( T_{Y, c^{\frac{1}{2}}} \) consists of all trajectories \( G \in T_{L^2([-\pi, \pi]), \Delta^1} \) such that

\[ G(x,t) = \sum_{n=1}^{\infty} \frac{e^{-nt}}{\xi_n} \frac{\sin x}{\sqrt{2\pi}} + \int_{\mathbb{R}} e^{-nt} \frac{\xi_n}{\sqrt{2\pi}} e^{-in\xi_0} d\xi, \]

where \( \xi_n = 0(e^{nt}) \) for all \( t > 0 \). Hence \( G(x,t) = G(-x,t) \). We insert this relation into the integral expression for \( H_G \) where we take \( 0 < t < \text{Im } z \),
\[ H_G(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(x, t) \left( \sum_{n=0}^{\infty} e^{inx} e^{-in(z+it)} - \frac{1}{2} \right) dx = \]

\[ = \frac{1}{2\pi} \int_{0}^{\pi} G(x, t) \left( \sum_{n=0}^{\infty} e^{nt} e^{-inz} (e^{inx} + e^{-inx}) - 1 \right) dx = \]

\[ = \frac{i}{4\pi} \int_{0}^{\pi} G(x, t) e^{tC} (\cot \frac{1}{2}(x-z) - \cot \frac{1}{2}(x+z)) dx = \]

\[ = \int_{0}^{\pi} G(x, t) (e^{tC} \theta_z(x)) dx, \]

where

\[ \theta_z : x \mapsto \frac{1}{2\pi i} \frac{\sin z}{\cos x - \sin z}, \quad \text{Im } z \neq 0. \]

In the same way, for all \( z \in \mathbb{C} \) with \( \text{Im } z < -t < 0 \)

\[ H_G(z) = \int_{0}^{\pi} G(x, t) (e^{tC} \theta_z)(x) dx. \]

We note that \( H_G \) is a 2\pi-periodic, odd function which is analytic on \( \text{Im } z = 0 \).

Moreover

\[ \lim_{\text{Im } z \to -\infty} H_G(z) = - \lim_{\text{Im } z \to \infty} H_G(z). \]

(4.8) Theorem

Let \( G \in \mathcal{T}, G(x, t) = \sum_{n=0}^{\infty} e^{-nt} \xi_n e_n(x) \). Then \( H_G \) defined by

\[ H_G(z) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \xi_n e^{inz} + \frac{\xi_0}{2\sqrt{\pi}} , \quad \text{Im } z > 0 , \]
88.

\[ H_G(z) = -\frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \xi_n e^{-inz} - \frac{\xi_0}{2\sqrt{\pi}}, \quad \text{Im } z < 0, \]

is represented by

\[ H_G(z) = \langle \delta_z, G \rangle_Y = \int_0^\pi G(x,t)(e^{tC_1^1} G(x)dx. \]

Moreover, for each \( f \in S_{Y,C_1^1} \) and each \( t > 0 \) sufficiently small

\[ \langle f, G \rangle = \int_{K_t} f(z) H_G(z)dz, \]

where

\[ K_t = \{ z \in \mathbb{C} \mid 0 \leq \text{Re } z \leq \pi, \text{Im } z = t \} \cup \{ z \in \mathbb{C} \mid \pi \leq \text{Re } z \geq 0, \text{Im } z = -t \} \]

with clockwise orientation.

**Proof**

Cf. Theorem (4.5).

Let \( \phi \) be an odd, \( 2\pi \)-periodic function which is analytic on \( \mathbb{C} \setminus \mathbb{R} \) and also satisfies \( \lim_{\text{Im } z \to -\infty} \phi(z) = -\lim_{\text{Im } z \to \infty} \phi(z) \). Then it is not hard to see that there exists a unique \( G \in T_{Y,C_1^1} \) such that \( \phi = H_G \). It thus makes sense to introduce the following class of hyperfunctions.

(4.10) **Definition**

The space of \( 2\pi \)-periodic hyperfunctions \( H(\mathbb{R}, 2\pi\text{-per}, \text{odd}) \) contains all hyperfunctions in \( H(\mathbb{R}, 2\pi\text{-per}) \) which are odd.

(4.11) **Theorem**

\( \forall \in H(\mathbb{R}, 2\pi\text{-per}, \text{odd}) \) iff there exists \( G \in T_{Y,C_1^1} \) such that \( \psi = H_G \).
(4.12) Corollary

Let \( \ell \) be a continuous linear functional on \( S_{Y,C^1} \). Then for all \( f \in S_{Y,C^1} \)

\[
\ell(f) = \int \frac{f(z) \ell(\theta_z)}{K_t} \, dz
\]

or equivalently

\[
\ell(f) = -\int \frac{f(w) \ell(\theta_w)}{K_t} \, dw.
\]

Here \( t > 0 \) has to be chosen small enough (dependent on \( f \in S_{Y,C^1} \)).

Proof

There exists \( G \in T_{Y,C^1} \) such that \( \ell(f) = \langle f, G \rangle_Y \), \( f \in S_{Y,C^1} \). Hence

\[
H_G(z) = \langle \theta^{-1}_z, G \rangle_Y = \ell(\theta_z).
\]

By Theorem (4.8) the assertion follows.

Next we transform the class of hyperfunctions \( H(\mathbb{R}, 2\pi\text{-per, odd}) \) into the class of hyperfunctions \( H([-1,1]) \). Heuristically, the elements of \( H([-1,1]) \) can be considered as 'jumps' \( \Omega(x+i0) - \Omega(x-i0), x \in [-1,1] \), of analytic functions \( \Omega \) on \( \mathbb{C} \setminus [-1,1] \).

To this end, we note first that each element \( f \in Y \) can be written as

\[
f = \sum_{n=0}^{\infty} a_n c_n
\]

where \( (a_n)_{n=0}^{\infty} \in \ell_2 \). Since the Chebyshev polynomials \( T_n = R_n^{(-\frac{1}{2},-\frac{1}{2})} \), \( n \in \mathbb{N} \cup \{0\} \), establish an orthonormal basis for the Hilbert space

\( X = L_2([-1,1],(1-\xi^2)^{-\frac{1}{2}} \, d\xi) \), and since \( T_n(\cos x) = c_n(x) \), the transformation \( \xi = \cos x \) gives rise to a unitary operator \( \mathcal{U} \) from \( Y \) onto \( X \). The operator \( \mathcal{U} \) satisfies \( \mathcal{U}c^* = A \) with \( A = \left( \frac{d^2}{d\xi^2} + \xi \frac{d}{d\xi} \right)^{\frac{1}{2}}. \) So \( \mathcal{U} \) is a continuous
linear bijection from $S_{Y,C^1}$ onto $S_{X,A^1}$ with continuous inverse $U^*$. Hence, any continuous linear functional $m$ on $S_{X,A^1}$ can be written as $m = \mathcal{L} \circ U^*$, i.e. $m(f) = \mathcal{L}(U^*f)$, $\mathcal{L} \in S_{X,A^1}$, $f \in S_{Y,C^1}$.

Now let $\mathcal{L}$ be a continuous linear functional on $S_{Y,C^1}$. As follows from Theorem 4.8, the function $z \mapsto \overline{\mathcal{L}(\theta z)}$, $\text{Im } z \neq 0$, can be written as

$$\overline{\mathcal{L}(\theta z)} = \sin z \theta(\cos z),$$

where the function $\theta$ is analytic on $C \setminus [-1,1]$.

Further, since $\frac{\overline{\mathcal{L}(\theta z)}}{\sin z} \to 0$ as $|\text{Im } z| \to \infty$, it follows that $\theta(w) \to 0$ as $w \to \infty$.

Conversely, for a function $\Omega$ which is analytic on $C \setminus [-1,1]$ and which satisfies $\Omega(w) \to 0$ as $w \to \infty$, the function $z \mapsto \sin z \Omega(\cos z)$ belongs to the class $H(\mathbb{R},2\pi\text{-per,odd})$. All this leads to the following definition.

\begin{equation}
\text{(4.13) Definition}
\end{equation}

The space of hyperfunctions $H([-1,1])$ consists of all functions $\theta$ which are analytic on $C \setminus [-1,1]$ and satisfy $\theta(w) \to 0$ as $w \to \infty$.

We now turn to the investigation of the relation between the above defined space of hyperfunctions $H([-1,1])$ and the dual space $S_{Y,C^1}'$. Each linear functional $m$ on $S_{X,A^1}$ can be written as $m = \mathcal{L} \circ U^*$ where $\mathcal{L} \in S_{Y,C^1}'$. It follows that

$$\frac{1}{\sin z} \overline{\mathcal{L}(\theta z)} = \frac{1}{\sin z} \overline{m(U\theta z)} = -\overline{m(\tau_{\cos z})},$$

where $\tau_w$ is defined by

$$\tau_w(y) = \frac{1}{2\pi i} \frac{1}{w - y}.$$

Observe that $\frac{1}{\sin z} (U\theta z)(y) = \tau_{\cos z}(y)$. Thus we obtain
(4.14) **Theorem**

I. For each continuous linear functional $m$ on $S_{X,A^{1/2}}$ there exists

$\Omega \in H([-1,1])$, given by $\Omega: w \mapsto m(\tau_w)$, such that

$$\bar{m}(f) = \oint_{\partial E_t} f(\bar{w}) \Omega(w) \, dw, \quad f \in S_{X,A^{1/2}},$$

where $t > 0$ has to be chosen sufficiently small dependent on $f$.

II. Let $\theta \in H([-1,1])$. Then the linear functional $n$ defined by

$$n(f) = \oint_{\partial E_t} f(\bar{w}) \theta(w) \, dw, \quad f \in S_{X,A^{1/2}},$$

is continuous.

**Remark.** $\partial E_t$ denotes the clockwise oriented contour

$$\left\{ z = x+iy \mid \left( \frac{x}{\cosh t} \right)^2 + \left( \frac{y}{\sinh t} \right)^2 = 1 \right\}.$$

**Proof**

I. If $m \in S'_{X,A^{1/2}}$, then $w \mapsto \bar{m}(\tau_w)$ is a hyperfunction in $H([-1,1])$. Further, for $f \in S_{X,A^{1/2}}$ and $t > 0$ sufficiently small we get by Corollary (4.12)

$$\oint_{\partial E_t} f(\bar{w}) \bar{m}(\tau_w) \, dw = \oint_{K_t} f(\cos z)(m \circ U)(\theta_z) \, dz =$$

$$= (m \circ U)(U^*f) = \bar{m}(f).$$

II. Since $\sin z \theta(\cos z)$ is a member of $H(\mathbb{R}, 2\pi$-per, odd), there exists

$G \in T_{Y, C^{1/2}}$ such that $H_G(z) = -\sin z \theta(\cos z)$, $z \in G \setminus \mathbb{R}$. Hence for $f \in S_{X,A^{1/2}}$ and $t > 0$ sufficiently small,
In Chapter 2 we have proved that $S_{X,A}^{1/2} = S_{X,\alpha,\beta}^{1/2}$ for all $\alpha, \beta > -1$. So for each $m \in S'$ and for each $\alpha, \beta$ with $\alpha, \beta > -1$, there exists $G(\alpha, \beta) \in T_{X,\alpha,\beta}^{1/2}$ such that

\[ m(f) = \left< f, G(\alpha, \beta) \right>_{X,\alpha,\beta}, \quad f \in S_{X,A}^{1/2}. \]

Then Theorem (4.14) leads to

(4.15) Corollary

Let $\alpha, \beta > -1$. Then $\theta \in H([-1,1])$ iff there exists $G(\alpha, \beta) \in T_{X,\alpha,\beta}^{1/2}$ such that $\theta(w) = \left< \tau, G(\alpha, \beta) \right>_{X,\alpha,\beta}$. Furthermore, 

\[ \left< f, G(\alpha, \beta) \right>_{X,\alpha,\beta} = \int_{\mathbb{R}} f(w) \theta(w) dw, \quad f \in S_{X,A}^{1/2}. \]

A natural way to attach to an integrable function $\varphi$ on $[-1,1]$ a hyperfunction $\Lambda_{\varphi}$ is the following:

\[ \Lambda_{\varphi}(w) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{1}{w - y} \varphi(y) dy. \]

The Legendre case $\alpha = \beta = 0$ seems best affiliated to the correspondence $\varphi \leftrightarrow \Lambda_{\varphi}$, because $X_{0,0} = L^2([-1,1], d\xi)$. 
Remark. If $G \in L^1([-1,1]) \subset T_{X_{G_0,0}}(A^{1/2})$, then $y \mapsto \frac{1}{w-y} G(y)$ is integrable on $[-1,1]$ for $w \notin [-1,1]$. By Corollary (4.15) we have

$$
\Lambda_G(w) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{1}{w-y} G(y) dy
$$

is an element of $H([-1,1])$, and for all $f \in S_{X,A^{1/2}}$

$$
\int_{-1}^{1} \overline{f(y)} G(y) dy = \int_{\mathbb{R}^2} \overline{f(\omega)} \Lambda_G(w) dw.
$$

Next we turn to some topological considerations. Let $S_t^+$ denote the exterior strip $\{z \in \mathbb{C} \mid \text{Im } z > t\}$ and $\overline{S_t^+}$ its closure. Then in $H(\mathbb{R}, 2\pi\text{-per, odd})$ we introduce the following seminorms:

$$
\rho_t(\phi) = \sup_{z \in \overline{S_t^+}} |\phi(z)|, \quad \phi \in H(\mathbb{R}, 2\pi\text{-per, odd}).
$$

We note that $\rho_t(\phi)$ is well-defined because $\phi(u+iv)$ tends to a constant as $v \to \pm \infty$ uniformly for $u \in \mathbb{R}$.

As proved, each $\phi \in H(\mathbb{R}, 2\pi\text{-per, odd})$ equals some $H_G : z \mapsto \frac{1}{2\pi} G(z)$ where $G \in T_{Y,C^{1/2}}$. We shall show that for each $t > 0$ there exist positive constants $k_t^{(1)}$ and $k_t^{(2)}$ such that

$$
k_t^{(1)} \|G(2t)\|_Y \leq \rho_t(H_G) \leq k_t^{(2)} \|G(t)\|_Y.
$$

Let $G \in T_{Y,C^{1/2}}$, $G(x,t) = \sum_{n=0}^{\infty} e^{-nt} \xi_n c_n(x)$. By definition

$$
H_G(z) = \sum_{n=1}^{\infty} \xi_n \frac{\text{inz}}{\sqrt{2\pi}} + \frac{\xi_0}{2\sqrt{\pi}}, \quad \text{Im } z > 0
$$
Let \( t > 0 \). Then for \( \text{Im } z \geq t \)

\[
|H_G(z)| = \left| \sum_{n=1}^{\infty} \xi_n e^{-i\text{nt}} \frac{e^{\text{in}(z-\frac{1}{2}it)}}{\sqrt{2\pi}} + \frac{\xi_0}{2\sqrt{\pi}} \right| \leq \\
\leq \frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} |\xi_n|^2 e^{-\text{nt}} \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} e^{-\text{nt}} \right)^{\frac{1}{2}} \leq \\
\leq \frac{1}{\sqrt{2\pi}} (1-e^{-t})^{-\frac{1}{2}} \|G(\frac{1}{2}t)\|_{\mathcal{Y}}.
\]

Similarly, for \( \text{Im } z \leq -t \)

\[
|H_G(z)| \leq \frac{1}{\sqrt{2\pi}} (1-e^{-t})^{-\frac{1}{2}} \|G(\frac{1}{2}t)\|_{\mathcal{Y}}.
\]

Thus it follows that

\[
\rho_t(H_G) \leq \frac{1}{\sqrt{2\pi}} (1-e^{-t})^{-\frac{1}{2}} \|G(\frac{1}{2}t)\|_{\mathcal{Y}}.
\]

On the other hand, for \( n \geq 1 \)

\[
|\xi_n| = |\langle c_n, G \rangle| = \frac{2\sqrt{2}}{\sqrt{\pi}} \left| \int_{K_t} \cos(nz)H_G(z)dz \right|
\]

and for \( n = 0 \)

\[
|\xi_0| = |\langle c_0, G \rangle| = \frac{2}{\sqrt{\pi}} \left| \int_{K_t} H_G(z)dz \right|
\]

where \( t > 0 \) may be taken arbitrarily large.

Consider the following estimation:
If \( \frac{1}{t} \cos(nz)H_G(z)dz \leq \frac{1}{t} \int_0^\pi e^{-in(x+it)}H_G(x+it)dx + \)

\[ + \frac{1}{t} \int_0^\pi e^{in(x-it)}H_G(x-it)dx \leq n e^{nt} \rho_t(H_G). \]

Hence \( |\xi_n|e^{-nt} \leq 2\sqrt{2\pi} \rho_t(H_G), \) \( n \in \mathbb{N}. \) Similarly, \( |\xi_0| \leq 2\sqrt{\pi} \rho_t(H_G). \) Thus we obtain

\[ \|G(2t)\|_Y = \left( \sum_{n=0}^\infty |\xi_n|^2 e^{-4nt} \right)^{1/2} \leq \rho_t(H_G) 2\sqrt{2\pi} \left(1-e^{-2t}\right)^{1/2}. \]

(4.17) Theorem

Let \((G_n)_{n=0}^\infty\) be a sequence in \( T \). Then \( G_n \to 0, \ n \to \infty \), in \( T \) \( Y, C^1_{\text{iff}} \)

\[ \rho_t(H_G) \to 0, \ n \to \infty, \] for all \( t > 0. \)

Correspondingly, we define the following seminorms in \( H([-1,1]) \)

\[ \sigma_t(\Omega) = \sup_{w \in \overline{E_t^*}} |\Omega(w)| \]

where \( \overline{E_t^*} \) denotes the closure of the region in \( \mathbb{C} \) given by

\[ E_t^* = \{w \in \mathbb{C} \mid w = u+iv, \left(\frac{u}{\cosh t}\right)^2 + \left(\frac{v}{\sinh t}\right)^2 > 1\}. \]

We have the following inequality

\[ k_t^{(1)} \|F(2t)\| \leq \sigma_t(A_F) \leq k_t^{(2)} \|F(it)\|, \quad F \in T_{X, A_t^{1/2}}, \]

where \( A_F \in H([-1,1]) \) is given by \( A_F(w) = \langle w, F \rangle_X. \)

(4.18) Theorem

Let \((F_n)_{n=0}^\infty\) be a sequence in \( T_{X, A_t^{1/2}} \). Then \( F_n \to 0, \ n \to \infty, \) iff \( \sigma_t(A_{F_n}) \to 0, \ n \to \infty, \) for all \( t > 0. \)
(4.19) Corollary

Let \( \left( F_n \right)_{n=0}^{\infty} \) be a sequence in \( T_{X,\alpha,\beta; (A_{\alpha,\beta})^\frac{1}{2}} \). Then \( F_n \to 0, n \to \infty \), in \( T_{X,\alpha,\beta; (A_{\alpha,\beta})^\frac{1}{2}} \) iff \( \sigma_t (\lambda_{F_n}^{(\alpha,\beta)}) \to 0 \) as \( n \to \infty \) for all \( t > 0 \), where \( \lambda_{F_n}^{(\alpha,\beta)}(\omega) \to \frac{1}{\tau_\omega} F_n \).

Let \( \ell \) be a continuous linear functional on \( S_{X,\alpha,\beta}^\frac{1}{2} \). Following the classification theorem that we proved in Chapter 3, the functional \( \ell \) can be represented by an element \( G^{(\alpha,\beta)} \in T_{X,\alpha,\beta; (A_{\alpha,\beta})^\frac{1}{2}} \) for each \( \alpha, \beta > -1 \), i.e.

\[
\ell(f) = <f, G^{(\alpha,\beta)} >_{\alpha,\beta}, \quad f \in S_{X,\alpha,\beta}^\frac{1}{2}.
\]

On the other hand, the functional \( \ell \) can also be represented by a hyperfunction \( \theta \in \mathcal{H}([-1,1]) \) as follows

\[
\overline{\ell}(f) = \int_{\mathcal{E}_\ell} f(\omega) \theta(\omega) d\omega.
\]

Now take \( \alpha, \beta > -1 \) fixed. Then

\[
\overline{\ell}(\tau_w) = - <\tau_w, G^{(\alpha,\beta)} >_{\alpha,\beta}, \quad \omega \in \mathcal{C} \setminus [-1,1].
\]

Since

\[
G^{(\alpha,\beta)} = \sum_{n=0}^{\infty} \xi_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)},
\]

where the series converges strongly in \( T_{X,\alpha,\beta; (A_{\alpha,\beta})^\frac{1}{2}} \), we get the following result

\[
\overline{\ell}(\omega) = \sum_{n=0}^{\infty} \xi_n^{(\alpha,\beta)} <\tau_w, R_n^{(\alpha,\beta)} >_{\alpha,\beta}
\]

where the series converges absolutely for all \( \omega \in \mathcal{C} \setminus [-1,1] \). Hence
\[ \theta(w) = \sum_{n=0}^{\infty} \xi_n^{(a,\beta)} R_n^{(a,\beta)}(w) \]

where

\[ R_n^{(a,\beta)}(w) = (R_n^{(a,\beta)}, \tau_w)^{a,\beta} = \frac{1}{2\pi i} \int_{-1}^{1} \frac{1}{x-w} R_n^{(a,\beta)}(x)(1-x)^{a}(1+x)^{\beta} \, dx. \]

By Corollary (4.19) the series \( \sum_{n=0}^{\infty} \xi_n^{(a,\beta)} R_n^{(a,\beta)} \) converges uniformly on each \( \overline{E_t^x}, \, t > 0. \)

We note that

\[ R_n^{(a,\beta)}(w) = \left[ \frac{(2n+\alpha+\beta+1)}{2^{\alpha+\beta+1}} \frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \right]^{1/2} (1-w)^{a}(1+w)^{\beta} Q_n^{(a,\beta)}(w), \]

with \( Q_n^{(a,\beta)} \) the Jacobi functions of the second kind (see [8], p. 215).

The functions \( R_n^{(a,\beta)} \) are analytic outside the closed interval \([-1,1]\). We have the following theorem.

(4.20) Theorem

Let \( \theta \in H([-1,1]) \). Then for each \( a, \beta > -1 \), there exists numbers \( \xi_n^{(a,\beta)} \), \( n \in \mathbb{N} \cup \{0\} \), such that \( \forall_{t>0}: \xi_n^{(a,\beta)} = O(e^{nt}) \) and

\[ \theta(w) = \sum_{n=0}^{\infty} \xi_n^{(a,\beta)} R_n^{(a,\beta)}(w), \quad w \in \mathbb{C} \setminus [-1,1]. \]

The series converges uniformly on \( \overline{E_t^x} \) for each \( t > 0. \)

We note that it follows from the above theorem that any hyperfunction in \( H([-1,1]) \) can be expanded with respect to the Jacobi functions of the second kind in a series which is uniformly convergent outside any open neighbourhood of \([-1,1]\).
5. ULTRA-HYPERFUNCTIONS AND ULTRA-TRAJECTORY SPACES

We start with a similar simple example as in the previous chapter. So let $S^1$ be the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. The Hardy-Lebesgue space $H$ which is a subspace of $L^2(S^1)$ has an orthonormal basis constituted by the functions $p_n : z \mapsto \frac{z^n}{\sqrt{2\pi}}$. In $H$ the positive self-adjoint operator $N$ is defined by

$$D(N) = \left\{ f \in H \mid \sum_{n=0}^{\infty} n^2 |(f, p_n)|^2 < \infty \right\}$$

and

$$Nf = \sum_{n=0}^{\infty} n(f, p_n)p_n, \quad f \in D(N).$$

In the following theorem the elements of the entireness space $\tau(H, N)$ are characterized.

(5.1) Theorem

$f \in \tau(H, N)$ iff $f$ can be extended to an entire analytic function.

Proof

Let $f \in \tau(H, N)$. Then $f = \sum_{n=0}^{\infty} (f, p_n)p_n$ where $(f, p_n) = O(e^{-nt})$ for all $t > 0$. So the power series $\sum_{n=0}^{\infty} (f, p_n)z^n$ converges uniformly on each closed disc with centre $z = 0$ and radius $e^t$, $t > 0$. Consequently, $f : z \mapsto \sum_{n=0}^{\infty} (f, p_n)z^n$ is an entire function.

Conversely, if $g$ is an entire function. Then $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{C}$, where the series converges uniformly on each closed disc with centre $z = 0$ and radius $e^t$, $t > 0$. Hence the sequence $(a_n e^{nt})_{n=0}^{\infty}$ belongs to $l_\infty$ for all $t > 0$. }
We note that for all $f \in \tau(H,N)$ and all $z,w \in \mathbb{C}$

$$(e^{wn} f)(z) = f(e^w z).$$

The ultra-trajectory space $\sigma(H,N)$ can also be represented as a space of analytic functions. To see this, remind that each element $G$ of $\sigma(H,N)$ corresponds to a formal series $\sum_{n=0}^{\infty} b_n z^n$ where $b_n = O(e^{nt})$ for some $t > 0$.

Now let $z \in \mathbb{C}$ with $|z| < e^{-t}$. Then it is clear that the series $\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} b_n z^n$ converges, and $G: z \mapsto \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} b_n z^n$ is an analytic function on $|z| < e^{-t}$. By Cauchy's criterion it follows that for a function $h$, $h(z) = \sum_{n=0}^{\infty} a_n z^n$ which is analytic on the disc $|z| < e^{-t}$ for some $t > 0$, the coefficients $a_n$ satisfy $a_n = O(e^{nt})$. So we get

(5.2) Theorem

$G = \sum_{n=0}^{\infty} b_n z^n \in \sigma(H,N)$ iff the function $z \mapsto \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} b_n z^n$ is analytic on $\{z \in \mathbb{C} \mid |z| < e^{-t}\}$ for $t > 0$ sufficiently large.

Let $\hat{G}$ denote the analytic function corresponding to $G \in \sigma(H,N)$. Then for all $s > 0$, $(e^{-sn} G)(z) = \hat{G}(e^{-s} z)$. Further, the function $\tilde{G}$ defined by

$$\tilde{G}: w \mapsto i \frac{1}{w} \hat{G}(\frac{1}{w})$$

is analytic on $\{w \in \mathbb{C} \mid |w| > e^r\}$ and $\tilde{G}(w) \to 0$ as $w \to \infty$. Conversely, any function $k$ which is analytic on $\{w \in \mathbb{C} \mid |w| > e^r\}$ and for which $k(w) \to 0$ as $w \to \infty$, can be written as $k(w) = i \frac{1}{w} \hat{K}(\frac{1}{w})$ for a unique $K \in \sigma(H,N)$.

(5.3) Theorem

Let $f \in \tau(H,N)$ and let $G \in \sigma(H,N)$. Choose $r > 1$ so large that $\tilde{G}$ is analytic on $\{w \in \mathbb{C} \mid |w| > r\}$. Then
\[ \langle f, G \rangle = \int_{|w| = \rho} \overline{f(w)} \, \overline{G(w)} \, dw, \quad \rho > r \]

where the contour is a circle with radius \( \rho \) and has clockwise orientation.

\textbf{Proof}

Let \( t > 0 \) be sufficiently large. Then

\[ \langle f, G \rangle = \left( e^{tN} f, G(t) \right)_H = \]

\[ = \int_{-\pi}^{\pi} \overline{f(e^{t+i\varphi})} \, \overline{G(e^{-t+i\varphi})} \, d\varphi = \]

\[ = \frac{1}{i} \int_{-\pi}^{\pi} \frac{\overline{f(e^{-t-i\varphi})}}{\overline{G(e^{-t-i\varphi})}} \, e^{-i\varphi} \, d\varphi = \]

\[ = \oint_{|w| = e^t} \overline{f(w)} \, \overline{G(w)} \, dw. \]

\[ (5.4) \text{ Definition} \]

The class of ultra-hyperfunctions \( U(S^1,+) \) consists of all functions \( \varphi \) which are analytic on some region \( \{ z \in \mathbb{C} \mid |z| > r \} \) for \( r > 1 \) dependent on \( \varphi \). Moreover, \( \varphi(w) \) tends to zero if \( |w| \) tends to infinity.

Let \( w_0 \in \mathbb{C} \) with \( |w_0| > 1 \). Then the function \( \zeta_{w_0} \)

\[ \zeta_{w_0}(z) = \frac{1}{2\pi i} \frac{1}{\frac{1}{w_0} - \frac{1}{z}}, \quad |z| = 1 \]

satisfies

\[ e^{tN}(\zeta_{w_0}(z)) = \frac{1}{2\pi i} \frac{1}{\frac{1}{w_0} - e^t \frac{1}{z}}. \]
So $e^{tN}(t_0, t_0) \in X$ for all $t > 0$ with $e^t < |w_0|$. Now let $G \in \sigma(H, N)$. Then there exists $t > 0$ such that for all $s \geq t$, $G(s) \in X$. Now take $w_0 \in \mathbb{C}$ with $|w_0| > e^t$. Then

$$
\left( e^{tN} t_{w_0}, G(t) \right)_H = \frac{1}{2\pi i} \int_{|z-w_0|=\varepsilon} \frac{1}{e^z - w_0} G(e^{-t}z) \, dz = \frac{1}{2\pi i} \int_{|w|=e^t} \frac{1}{w-w_0} \tilde{G}(w) \, dw = \tilde{G}(w_0).$

Hence $\tilde{G}(w) = \left( e^{tN} t_{w_0}, G(t) \right)_H$ for all $w \in \mathbb{C}$ with $|w| > e^t$.

We have discussed now the connection between a specific ultra-trajectory space and a specific class of ultra-hyperfunctions. In a similar way we shall relate the ultra-trajectory spaces $\sigma(L_2([-\pi, \pi]), \Lambda^\frac{1}{2})$, $\sigma(Y, C^\frac{1}{2})$ and $\sigma(X, \Lambda^\frac{1}{2})$ which are defined in Chapter 3, to suitable spaces of ultra-hyperfunctions in the remaining part of this chapter.

The functions $e_n: x \mapsto \frac{1}{\sqrt{2\pi n}} e^{inx}$, $n \in \mathbb{Z}$, establish an orthonormal basis on the Hilbert space $L_2([-\pi, \pi])$. Again, let $\Delta$ denote the positive self-adjoint operator in $L_2([-\pi, \pi])$ defined by $\Delta e_n = \pi^2 e_n$. In Chapter 3 we have shown that the entireness space $\tau(L_2([-\pi, \pi]), \Lambda^\frac{1}{2})$ consists of all $2\pi$-periodic entire functions.

Let $f \in \tau(L_2([-\pi, \pi]), \Lambda^\frac{1}{2})$. Then for all $z \in \mathbb{C}$

$$
f(z) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} (f, e_n) e^{inz}.
$$

We define the functions $f_+$ and $f_-$ as follows:

$$
f_+(z) = \frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} (f, e_n) e^{inz} - \frac{1}{2} (f, e_0) \right), \quad z \in \mathbb{C}
$$
and

\[ f_-(z) = \frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} (f, e_{-n}) e^{-inz} - \frac{1}{2}(f, e_0) \right), \quad z \in \mathbb{C}. \]

Then \( f = f_+ + f_- \) and both \( f_+ \) and \( f_- \) are entire analytic and \( 2\pi \)-periodic.

Further, for all \( w \in \mathbb{C} \) we have

\[ (e^{w\Delta^\frac{1}{2}} f_+)(z) = f_+(z-iw) \]

and

\[ (e^{w\Delta^\frac{1}{2}} f_-)(z) = f(z+iw). \]

From Chapter 4 it follows that for all \( t > 0 \)

\[ (e^{t\Delta^\frac{1}{2}} f)(x) = f(x-it) + f(x+it) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) \frac{\sinh t}{\cosh t - \cos(x-\xi)} \, d\xi. \]

The elements of \( \sigma(L_2([-\pi, \pi]), \Lambda^\frac{1}{2}) \) are decomposed similarly. So for \( F \in \sigma(L_2([-\pi, \pi]), \Lambda^\frac{1}{2}) \), \( F = \sum_{n=-\infty}^{\infty} b_n e_n \), we put

\[ F_+ = \sum_{n=0}^{\infty} b_n e_n - \frac{1}{2} b_0 e_0 \]

and

\[ F_- = \sum_{n=0}^{\infty} b_n e_{-n} - \frac{1}{2} b_0 e_0. \]

Because \( b_n = O(e^{\left| n \right| t}) \) for \( t > 0 \) sufficiently large, the function

\[ \hat{F}_+(z) = \frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} b_n e^{inz} - \frac{1}{2} b_0 \right) \]

is \( 2\pi \)-periodic and analytic on \( \{z \in \mathbb{C} \mid \text{Im } z > t \} \). For \( t > t \) we have
\[ F_+(x, \tau) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} b_n e^{-\tau n} e^{inx} - \frac{1}{2} b_0 = \hat{F}_+(x+i\tau), \quad x \in \mathbb{R}. \]

Similarly, \( \hat{F}_- \) defined by \( \hat{F}_-(z) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} b_n e^{-inz} - \frac{1}{2} b_0 \) is \( 2\pi \)-periodic and analytic on \( \text{Im} \ z < -t \). WE have

\[ F_-(x, \tau) = \hat{F}_-(x-i\tau), \quad x \in \mathbb{R}, \tau > t. \]

Let \( f \in \tau(L^2([-\pi, \pi]), \Delta^{\frac{1}{2}}) \), and let \( G \in \sigma(L^2([-\pi, \pi]), \Delta^{\frac{1}{2}}) \). The following relation is valid for \( \tau > 0 \) large enough:

\[ \langle f, G \rangle = (\tau^{\Delta^{\frac{1}{2}}} f, G(t))_{L^2} = \int_{-\pi}^{\pi} f(x-it) \hat{G}_+(x+it) dx + \int_{-\pi}^{\pi} f(x+it) \hat{G}_-(x-it) dx. \]

Now define \( U_G \) for \( z, |\text{Im} \ z| > t \) as follows

\[ U_G(z) = \begin{cases} \hat{G}_+(z), & \text{Im} \ z > t \\ -\hat{G}_-(z), & \text{Im} \ z < t \end{cases}. \]

\( U_G \) is \( 2\pi \)-periodic and analytic. Then the integral expression for \( \langle f, G \rangle \) can be written as

\[ \langle f, G \rangle = \oint_{C_t} \overline{f(z)} U_G(z) dz, \]

where \( C_t \) consists of the two directed line segments \( L^+_t = \{ z \in \mathbb{C} \mid \text{Im} \ z = t, -\pi \leq \text{Re} \ z \leq \pi \} \) and \( L^-_t = \{ z \in \mathbb{C} \mid \text{Im} \ z = -t, \pi \leq \text{Re} \ z \geq -\pi \} \). So the integrations takes place in clockwise fashion.
Next we derive an integral expression for the analytic function $U_G$, where 

$$G = \sum_{n=-\infty}^{\infty} a_n e^{-|n|s} e_n \in \sigma(L_2([-\pi,\pi]),\Delta^\frac{1}{2}).$$

Note first that there exists $t > 0$ such that $G(s) = \sum_{n=-\infty}^{\infty} a_n e^{-|n|s} e_n \in L_2([-\pi,\pi])$ for all $s \geq t$. So for $\text{Im } z > t$

$$U_G(z) = \hat{G}_+(z) = \frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} a_n e^{-tn} \sin(z-it) - \frac{i}{tn} \right).$$

Then we derive

$$\hat{G}_+(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(x,t) \left( \sum_{n=0}^{\infty} \frac{\sin x}{n} \frac{\sin(nz+it)}{n} - \frac{i}{n} \right) dx =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} G(x,t) e^{t\Delta^\frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{\sin x}{n} \frac{\sin(nz)}{n} - \frac{i}{n} \right)} dx =$$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} G(x,t) e^{t\Delta^\frac{1}{2} \left( \cot \frac{1}{2}(x-z) \right)} dx .$$

In the same way, we obtain for $\text{Im } z < -t$

$$U_G(z) = -\hat{G}_-(z) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} G(x,t) e^{t\Delta^\frac{1}{2} \left( \cot \frac{1}{2}(x-z) \right)} dx .$$

(5.5) **Theorem**

I. The analytic function $U_G$ associated to $G \in \sigma(L_2([-\pi,\pi]),\Delta^\frac{1}{2})$ is represented by the integral expression

$$U_G(z) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} G(x,t) e^{t\Delta^\frac{1}{2} \left( \cot \frac{1}{2}(x-z) \right)} dx ,$$

where $t > 0$ must be taken so large that $G(s)$ makes sense as an element of $L_2([-\pi,\pi])$ for $s > t$, and where $\text{Im } z > t$. 
II. For all \( f \in \tau(L_2([-\pi,\pi]),\Delta^1) \), and all \( G \in \sigma(L_2([-\pi,\pi]),\Delta^1) \)

\[
\langle f, G \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z) G(z) \, dz .
\]

Remark. The functions \( n_z : x \mapsto \frac{1}{4\pi} \cot \frac{1}{2}(z-x) \), are not the elements of the entireness space \( \tau(L_2([-\pi,\pi]),\Delta^1) \). However, for \( |\text{Im} \, z| > t \), \( x \mapsto \cot \frac{1}{2}(x-z) \)

is a member of \( e^{-t\Delta^1}L_2([-\pi,\pi]) \). We have

\[
U_G(z) = (e^{t\Delta^1} n_z, G(t))_{L_2([-\pi,\pi])} .
\]

Let \( \psi \) be a \( 2\pi \)-periodic function which is analytic on \( |\text{Im} \, z| > r \) for some \( r > 0 \), and also satisfies \( \lim_{\text{Im} \, z \to \infty} \psi(z) = \lim_{\text{Im} \, z \to -\infty} (-\psi(z)) \). Then \( \psi \) can be written as

\[
\psi(z) = \frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} a_n e^{inz} - \frac{1}{2}a_0 \right), \quad \text{Im} \, z > r
\]

\[
\psi(z) = \frac{1}{\sqrt{2\pi}} \left( \sum_{n=0}^{\infty} a_n e^{inz} - \frac{1}{2}a_0 \right), \quad \text{Im} \, z < r ,
\]

where \( a_n = O(e^{n|r|}) \). So \( \psi = U_G \) with \( G = \sum_{n=-\infty}^{\infty} a_n e^n \).

Now we are in a position to define a class of periodic ultra-hyperfunctions.

(5.6) Definition

The space of \( 2\pi \)-periodic ultra-hyperfunctions \( U(\mathbb{R},2\pi\text{-per}) \) consists of all \( 2\pi \)-periodic functions \( \phi \), satisfying

- There exists \( t > 0 \) such that \( \phi \) is analytic on \( |\text{Im} \, z| > t \);

- \( \lim_{\text{Im} \, z \to \infty} \phi(z) = -\lim_{\text{Im} \, z \to -\infty} \phi(z) \) exists.
(5.7) Theorem

Let $\phi \in U(\mathbb{R}, 2\pi$-per). Then there exists a unique ultra-trajectory $G \in \sigma(L_2([\pi, \pi]), A^\frac{1}{2})$ such that $\phi = U_G$. \hfill \Box

The aim of this chapter is to describe the ultra-hyperfunctions which are related to the ultra-trajectory space $\sigma(X, A^\frac{1}{2})$ where $X = L_2([-1, 1], (1-x^2)^{-\frac{1}{2}} \, dx)$ and where $A = -(1-x^2) \frac{d^2}{dx^2} + x \frac{d}{dx}$. To this end, we first consider the ultra-trajectory space $\sigma(Y, C^\frac{1}{2})$ where $Y = L_2([0, \pi])$ and where $C$ is the positive self-adjoint operator in $Y$ defined by $C_n = nC_n$, $n \in \mathbb{N} \cup \{0\}$. Here the functions $c_n$ are given by

$$c_n(x) = \sqrt{\frac{2}{\pi}} \cos nx, \quad c_0(x) = \sqrt{\frac{1}{\pi}}, \quad n \in \mathbb{N}.$$ 

So the space $\sigma(Y, C^\frac{1}{2})$ considered as a subspace of $\sigma(L_2([-\pi, \pi]), A^\frac{1}{2})$ consists of all elements $G \in \sigma(L_2([-\pi, \pi]), A^\frac{1}{2})$ such that $G(x, t) = G(-x, t)$ for sufficiently large $t > 0$, i.e.

$$G(x, t) = \sum_{n=1}^{\infty} e^{-nt} b_n \frac{e^{inx}}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} e^{-nt} b_n \frac{e^{-inx}}{\sqrt{2\pi}} + b_0 \frac{e^{-inx}}{\sqrt{\pi}}.$$

Inserting this relation for the elements of $\sigma(Y, C^\frac{1}{2})$, we derive the following integral expression for $U_G$ by Theorem (5.5):

$$U_G(z) = \int_{0}^{\pi} G(x, t) (e^{tC^\frac{1}{2}} \theta_z(x)) \, dx, \quad |\text{Im} \, z| > t,$$

where

$$\theta_z(x) = \frac{1}{2\pi i} \frac{\sin z}{\cos x - \cos z}.$$ 

We emphasize that the value of $t$ depends on the choice of $G \in \sigma(Y, C^\frac{1}{2})$. 
(5.8) Theorem
Let \( G \in \sigma(Y, C^1) \), \( G(x,t) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} b_n e^{-nt} \cos nx + \sqrt{\frac{1}{\pi}} b_0 \). Take \( t_0 \) so large that \( G(t_0) \in Y \).
Then \( U_G \), defined by
\[
U_G(z) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} b_n e^{inz} + \frac{1}{2\sqrt{\pi}} b_0, \quad \text{Im } z > t_0
\]
\[
U_G(z) = -\frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} b_n e^{-inz} - \frac{1}{2\sqrt{\pi}} b_0, \quad \text{Im } z < -t_0
\]
is represented by
\[
U_G(z) = \int_0^\pi G(x,t_0) (e^{t_0 C_2 - ix z})(x) \, dx.
\]
In addition, for all \( f \in \tau(Y, C^1) \) and all \( t > t_0 \)
\[
< f, G > = \int_{K_t} \overline{f(z)} \, U_G(z) \, dz
\]
where \( K_t = \{ z \in C \mid 0 \leq \text{Re } z \leq \pi, \text{Im } z = t \} \cup \{ z \in C \mid \pi \geq \text{Re } z \geq 0, \text{Im } z = -t \}. \)
The integration takes place in clockwise fashion.

Straightforward arguments give the result that any \( 2\pi \)-periodic odd function \( \phi \) which is analytic on \( |\text{Im } z| > t \) for some \( t > 0 \) and for which the limit
\[
\lim_{\text{Im } z \to \infty} \phi(z)
\]
exists, equals a function \( U_G \) for some \( G \in \sigma(Y, C^1) \). Therefore it makes sense to introduce the class \( U(\mathbb{R}, 2\pi\text{-per, odd}) \) of ultra-hyperfunctions, which are in one to one correspondence with the ultra-trajectory space \( \sigma(Y, C^1) \).

(5.9) Definition
The class \( U(\mathbb{R}, 2\pi\text{-per, odd}) \) consists of all ultra-hyperfunctions \( \psi \) in \( U(\mathbb{R}, 2\pi\text{-per}) \) which are odd.
(5.10) **Theorem**

\[ \forall \in U(\mathbb{R}, 2\pi\text{-per,odd}) \text{ iff there exists } \mathcal{C} \in \sigma(Y, \mathbb{C}^1) \text{ such that } \forall = \mathcal{U}_\mathcal{C}. \]

(5.11) **Corollary**

Let \( \mathcal{L} \) be a continuous linear functional on \( \tau(Y, \mathbb{C}^1) \). Then there exists \( \forall \in U(\mathbb{R}, 2\pi\text{-per,odd}) \) such that

\[ \mathcal{L}(f) = \int_{K_t} f(z) \overline{\forall(z)} \, dz, \quad f \in \tau(Y, \mathbb{C}^1). \]

Here \( t > 0 \) has to be chosen so large that \( \forall \) is analytic for \( |\text{Im} \, z| \geq t \).

**Proof**

A linear functional \( \mathcal{L} \) on \( \tau(Y, \mathbb{C}^1) \) is continuous iff \( \mathcal{L} \) is represented by an element \( \mathcal{H} \in \sigma(Y, \mathbb{C}^1) \) by \( \mathcal{L}(f) = \langle f, \mathcal{H} \rangle \). So following Theorem (5.8), for sufficiently large \( t > 0 \),

\[ \mathcal{L}(f) = \langle f, \mathcal{H} \rangle = \int_{K_t} f(z) \overline{U_H(z)} \, dz. \]

In the previous chapter we have shown that the conformal mapping \( w = \cos z \) transforms the class of hyperfunctions \( \mathcal{H}(\mathbb{R}, 2\pi\text{-per,odd}) \) into the class of hyperfunctions \( \mathcal{H}([-1,1]) \). Here we repeat this exercition for the class of ultra-hyperfunctions \( U(\mathbb{R}, 2\pi\text{-per,odd}). \)

Let \( \phi \in U(\mathbb{R}, 2\pi\text{-per,odd}) \). Then there exists a function \( \Theta \) such that

\( \phi(z) = \sin z \, \Theta(\cos z) \). Since \( \phi \) is analytic on the region \( |\text{Im} \, z| > t \) for some sufficiently large \( t \), the function \( \Theta \) has to be analytic on the open region

\( E^*_t \) outside the ellips \( \{ z \in \mathbb{C} \mid z = x + iy, \left( \frac{x}{\cosh t} \right)^2 + \left( \frac{y}{\sinh t} \right)^2 = 1 \} \).

From the characterizations of the elements of \( U(\mathbb{R}, 2\pi\text{-per,odd}) \) it also follows that \( \lim \Theta(w) = 0 \). Conversely, if \( \Omega \) is analytic on a region \( E^*_t \) and
satisfies \( \lim_{w \to \infty} \Theta(w) = 0 \), then \( z \mapsto \sin z \Theta(\cos z) \) is an element of \( U(\mathbb{R}, 2\pi \text{-per,odd}) \). This leads to the following definition.

(5.12) Definition

The class of ultra-hyperfunctions \( U([-1,1]) \) consists of all functions \( \Theta \) which are analytic on a region \( E^*_t \) for some \( t > 0 \), and satisfy \( \lim_{w \to \infty} \Theta(w) = 0 \).

Remark. \( H([-1,1]) \) is contained in \( U([-1,1]) \).

Next we want to relate the class \( U([-1,1]) \) to the ultra-trajectory space \( \sigma(X,A^{\frac{1}{2}}) \) where, as we have seen, \( X = L^2([-1,1], (1-\xi^2)^{-\frac{1}{2}} \, d\xi) \) and where the positive self-adjoint operator \( A \) is defined by

\[
A = - (1-\xi^2) \frac{d^2}{d\xi^2} + \xi \frac{d}{d\xi}.
\]

In the previous chapter we have observed that the linear operator \( U \) originating from the transformation \( u = \cos x \) is a unitary operator from \( Y \) onto \( X \). In addition, \( UCUC^* = A \). It is clear that \( U(\tau(Y,C^{\frac{1}{2}})) = \tau(X,A^{\frac{1}{2}}) \) and also that \( U \) is extendible to \( \sigma(Y,C^{\frac{1}{2}}) \) with \( U(\sigma(Y,C^{\frac{1}{2}})) = \sigma(X,A^{\frac{1}{2}}) \).

Now let \( \Theta \in U([-1,1]) \). Then \( (z \mapsto \sin z \Theta(\cos z)) \in U(\mathbb{R}, 2\pi \text{-per,odd}) \). So there exists \( G \in \sigma(Y,C^{\frac{1}{2}}) \) such that for all \( z \), \( \text{Im } z > t \),

\[
\sin z \Theta(\cos z) = (e^{tC^{\frac{1}{2}}} \Theta, C(t))_Y = (e^{tA^{\frac{1}{2}}} U_{\theta_z}, UG(t))_X.
\]

Now observe that \( s \mapsto (UG)(s) \), \( s \geq t \), belongs to \( \sigma(X,A^{\frac{1}{2}}) \), and also that

\[
\frac{1}{\sin z} U(\theta_z) = \frac{1}{2\pi i} \frac{1}{\xi - w}, \quad w = \cos z.
\]

Put \( \tau_w : \xi \mapsto \frac{1}{2\pi i} \frac{1}{w - \xi} \), \( \xi \in [-1,1] \). Then it follows that for all \( w \in E^*_t \),
Since $\mathcal{U}^*$ maps $\sigma(X,A^\frac{1}{2})$ onto $\sigma(Y,C^\frac{1}{2})$, we obtain that for each $H \in \sigma(X,A^\frac{1}{2})$, the function
\[
\Omega: w \mapsto (e^{tA^{\frac{1}{2}}} \tau_w, H(t))_X
\]
is analytic on $E_t^*$ and satisfies $\Omega(w) \to 0$ as $w \to \infty$. Here $t > 0$ has to be chosen fixed and so large that the ultra-trajectory $H$ is well defined on $[t, \infty)$. (The region of analyticity of $\Omega$ depends on $H$.) Thus we obtain

(5.13) Theorem

$\Omega \in \mathcal{U}([-1,1])$ iff there exists $H \in \sigma(X,A^\frac{1}{2})$ such that

\[
\Omega(w) = (e^{tA^{\frac{1}{2}}} \tau_w, H(t))_X, \quad w \in E_t^*.
\]

Similarly to Corollary (5.11) we get

(5.14) Corollary

Let $m$ be a continuous linear functional on $\tau(X,A^\frac{1}{2})$. Then there exists $\Omega \in \mathcal{U}([-1,1])$ such that

\[
m(h) = \int_{\partial E_t^*} h(\bar{w}) \Omega(w) dw, \quad h \in \tau(X,A^\frac{1}{2}),
\]

where $t > 0$ must be chosen so large that $\Omega$ is analytic outside $E_t^*$. Here $\partial E_t^*$ denotes the contour

\[
\{ w = u + iv \mid \left( \frac{u}{\cosh t} \right)^2 + \left( \frac{v}{\sinh t} \right)^2 = 1 \}
\]
which is clockwise oriented.
Proof

The functional \( m \circ U \) is continuous on \( \tau(Y, C^1) \). By Corollary (5.11) there exists \( \psi \in U(\mathbb{R}, 2\pi\text{-per, odd}) \) such that

\[
\int_{K_t} (m \circ U)(f) = \int_{\mathbb{R}} f(z)(-\psi(z)) dz
\]

for \( t > 0 \) sufficiently large. Since \( \psi(z) = \sin z \Omega(\cos z) \) for a unique \( \Omega \in U([-1, 1]) \) and since \( f(z) = h(\cos z) \), for a unique \( h \in \tau(X, A^\frac{1}{2}) \), we derive

\[
m(h) = \int_{K_t} h(\cos z)(-\Omega(\cos z)) \sin z \, dz = \int_{E_t} h(w) \Omega(w) \, dw.
\]

(5.15) Corollary

Let \( H \in \sigma(X, A^\frac{1}{2}) \) and let \( g \in \tau(X, A^\frac{1}{2}) \). Then for \( t > 0 \) large enough

\[
\langle g, H \rangle = \int_{E_t^*} g(\omega) \Omega_H(\omega) \, dw
\]

where

\[
\Omega_H(\omega) = (e^{\omega A^\frac{1}{2}} \tau_w, H(\omega))_X, \quad \omega \in E_s^*, \quad s > t.
\]

(Remark. \( \tau(X, A^\frac{1}{2}) \) consists of all entire analytic functions.)

We note that it does not make sense to write

\[
\Omega_H(\omega) = \langle \omega, H \rangle
\]

since \( \tau_w \) is not an element of \( \tau(X, A^\frac{1}{2}) \).
Therefore, in contradistinction to the case of hyperfunctions \( H([-1,1]) \) we cannot write \( \Omega(w) = \mathcal{L}(\tau_w) \) for \( \ell \in \tau(X,A_{\frac{1}{2}}) \). We need a specific representation of \( \tau(X,A_{\frac{1}{2}}) \) by means of the ultra-trajectory space \( \sigma(X,A_{\frac{1}{2}}) \). We now want to show that we could have used \( \sigma(X_{\alpha,\beta},(A_{\alpha,\beta})^{\frac{1}{2}}) \) for arbitrary \( \alpha, \beta > -1 \), equally well.

In Chapter 3 we have introduced the positive self-adjoint operators \( A_{\alpha,\beta} \), \( \alpha, \beta > -1 \), in the Hilbert space \( X_{\alpha,\beta} = L_2([-1,1],(1-x)^{\alpha}(1+x)^{\beta} \, dx) \),

\[
A_{\alpha,\beta} = - (1-x^2) \frac{d^2}{dx^2} + (\alpha+\beta+2) x \frac{d}{dx} - (\beta-\alpha) \frac{d}{dx}.
\]

Observe that \( A = A_{-\frac{1}{2},-\frac{1}{2}} \). It follows from the classification theorem (1.1) that for all \( \alpha, \beta > -1 \)

\[
\tau(X_{\alpha,\beta},(A_{\alpha,\beta})^{\frac{1}{2}}) = \tau(X, A_{\frac{1}{2}}).
\]

So the strongdual \( \tau(X, A_{\frac{1}{2}}) \) admits infinitely many representations as an ultra-trajectory space, viz. \( \sigma(X_{\alpha,\beta},(A_{\alpha,\beta})^{\frac{1}{2}}), \alpha, \beta > -1 \).

Here we want to show that the way of adjoining ultra-hyperfunctions of \( U([-1,1]) \) to elements of \( \tau(X, A_{\frac{1}{2}}) \) is independent of the choice of this representation.

To this end we first note that in the proof of the classification theorem (1.1) we in fact did show that for all \( \tilde{t} > 0 \) there exist \( t > 0 \) such that \( \exp(\tilde{t}(A_{\alpha,\beta})^{\frac{1}{2}}) \exp(-tA_{\frac{1}{2}}) \) is a bounded operator from \( X \) into \( X_{\alpha,\beta} \). Further, let \( \ell \) be a continuous linear functional on \( \tau(X, A_{\frac{1}{2}}) = \tau(X_{\alpha,\beta},(A_{\alpha,\beta})^{\frac{1}{2}}) \). Then there exists \( G_{\alpha,\beta} \in \sigma(X_{\alpha,\beta},(A_{\alpha,\beta})^{\frac{1}{2}}) \) and \( G \in \sigma(X, A_{\frac{1}{2}}) \) such that

\[
\ell(f) = \langle f, G \rangle_X \quad \text{and} \quad \ell(f) = \langle f, G_{\alpha,\beta} \rangle_{X_{\alpha,\beta}}.
\]
Moreover, there exists $t_0 > 0$ such that $G$ is a well defined mapping from $[t_0, \omega)$ into $X$ and $G_{a, \beta}$ a well defined mapping from $[t_0, \omega)$ into $X_{a, \beta}$. It thus follows that for all $t > t_0$, the linear functional $\ell \circ \exp(-tA^{\frac{1}{2}})$ can be uniquely extended to a bounded linear functional on $X$ given by

$$w \mapsto (w, G(t))_X, \quad w \in X.$$  

Similarly, for all $t > t_0$, $\ell \circ \exp(-t(A_{a, \beta})^{\frac{1}{2}})$ can be extended to a bounded linear functional on $X_{a, \beta}$,

$$(\ell \circ \exp -t(A_{a, \beta})^{\frac{1}{2}})(u) = (u, G_{a, \beta}(t))_{X_{a, \beta}}, \quad u \in X_{a, \beta}.$$  

Now let $\tilde{\tau} > t_0$. Then there exists $t > t_0$ such that the operator $\exp(\tilde{\tau}(A_{a, \beta})^{\frac{1}{2}})\exp(-tA^{\frac{1}{2}})$ is bounded from $X$ into $X_{a, \beta}$. So for $w \in E_{t}^*$ we have

$$\exp(\tilde{\tau}(A_{a, \beta})^{\frac{1}{2}})\tau_w = [\exp(\tilde{\tau}(A_{a, \beta})^{\frac{1}{2}})\exp(-tA^{\frac{1}{2}})] \exp(tA^{\frac{1}{2}})\tau_w$$

and hence $\exp(\tilde{\tau}(A_{a, \beta})^{\frac{1}{2}})\tau_w \in X_{a, \beta}$ for all $w \in E_{t}^*$. Moreover,

$$(\ell \circ \exp(-\tilde{\tau}(A_{a, \beta})^{\frac{1}{2}}))\exp(\tilde{\tau}(A_{a, \beta})^{\frac{1}{2}})\tau_w = (\exp \tilde{\tau}(A_{a, \beta})^{\frac{1}{2}} \tau_w, G_{a, \beta}(\tilde{\tau}))_{X_{a, \beta}}.$$  

Thus we obtain

$$(\exp(tA^{\frac{1}{2}})\tau_w, G(t))_X = (\ell \circ \exp(-tA^{\frac{1}{2}}))(\exp(tA^{\frac{1}{2}})\tau_w) =$$

$$= (\ell \circ \exp(-\tilde{\tau}(A_{a, \beta})^{\frac{1}{2}}))(\exp \tilde{\tau}(A_{a, \beta})^{\frac{1}{2}} \exp(-tA^{\frac{1}{2}}))\exp(tA^{\frac{1}{2}})\tau_w =$$

$$= (\ell \circ \exp -\tilde{\tau}(A_{a, \beta})^{\frac{1}{2}})\exp \tilde{\tau}(A_{a, \beta})^{\frac{1}{2}} \tau_w =$$

$$= (\exp \tilde{\tau}(A_{a, \beta})^{\frac{1}{2}} \tau_w, G_{a, \beta}(\tilde{\tau}))_{X_{a, \beta}}.$$  

It leads to the following result.
(5.16) Lemma

Let \( \alpha, \beta > -1 \) and let \( G \in \sigma(X,A^\frac{1}{2}) \). Then there exists \( G_{\alpha,\beta} \in \sigma(X,\beta'(A_{\alpha,\beta}^\frac{1}{2})) \) and also \( t > 0 \) and \( \tilde{t} > 0 \) such that

\[
W \mapsto (\exp(tA_{\alpha,\beta}^\frac{1}{2})W,G(t))_X
\]

is an analytic function on \( E_t^* \). Further on \( E_t^* \) we have

\[
(\exp(tA_{\alpha,\beta}^\frac{1}{2})W,G(t))_X = (\exp(\tilde{t}(A_{\alpha,\beta}^\frac{1}{2}))W,G_{\alpha,\beta}(\tilde{t}))_X
\]

as functions of \( W \).

\[ \square \]

Summarized

To any element \( \ell \in \tau' \) (\( \tau = \tau(X,A^\frac{1}{2}) = \tau(X,\beta'(A_{\alpha,\beta}^\frac{1}{2})) \)) we associate an ultra-hyperfunction on a region \( E_t^* \) which does not depend on the specific choice of representation of \( \tau' \).

We devote the remaining part of this chapter to some topological features.

Let \( S_t^* \) denote the exterior strip \( \{z \in \mathbb{C} \mid |\text{Im } z| > t\} \), \( t > 0 \). We introduce the following subclasses of \( U(\mathbb{R},2\pi\text{-per,odd}) \):

\[ \Phi \in U(\mathbb{R},2\pi\text{-per,odd};t) \iff \Phi \text{ is a } 2\pi\text{-periodic odd function which is analytic on } S_t^* \text{ and satisfies } \sup_{z \in S_t^*} \frac{1}{\sin z} |\Phi(z)| < \infty. \]

We observe that \( U(\mathbb{R},2\pi\text{-per,odd}) = \bigcup_{t>0} U(\mathbb{R},2\pi\text{-per,odd};t) \).

In the preliminaries we have introduced \( \sigma(Y,C^\frac{1}{2}) \) as the union \( \bigcup_{t>0} Y_t \) where \( Y_t \) consists of all mappings \( H: [t,\infty) \to Y \) with the property that \( H(s_1+s_2) = e^{-s_1}H(s_2) \), \( s_1 \geq 0, s_2 \geq t \). In \( Y_t \) we define the Hilbert space norm \( H \mapsto \|H(t)\|_Y \), \( H \in Y_t \).
Now let $G \in Y_t$, $t > 0$. Then the corresponding ultra-hyperfunction $U_G$ is a member of $U(\mathbb{R}, 2\pi$-per, odd; $t + \epsilon$) for any $\epsilon > 0$. Let $\epsilon > 0$. Then for all $z \in S^*_{t+\epsilon}$ with $\text{Im } z \geq t + \epsilon$ we have

$$\left| \frac{U_G(z)}{\sin z} \right| = \left| \frac{1}{\sin z} \left( \sum_{n=1}^{\infty} (G(t), c_n) \frac{e^{(z-it)}}{\sqrt{2\pi}} + \frac{(G(t), c_0)}{2\sqrt{\pi}} \right) \right| \leq$$

$$\leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sinh t} \left( \sum_{n=0}^{\infty} |(G(t), c_n)|^2 \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} e^{-2\epsilon n} \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sinh t} \left( \frac{1}{1 - e^{-2\epsilon}} \right)^{\frac{1}{2}} \|G(t)\|_Y.$$

Similarly, for $z \in S_{t+\epsilon}$ with $\text{Im } z \leq -(t + \epsilon)$,

$$\left| \frac{U_G(z)}{\sin z} \right| \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sinh t} \left( \frac{1}{1 - e^{-2\epsilon}} \right)^{\frac{1}{2}} \|G(t)\|_Y.$$

Thus we have shown the following implication

$$G \in Y_t \Rightarrow (i) \forall \epsilon > 0: U_G \in U(\mathbb{R}, 2\pi$-$\text{per, odd}; t + \epsilon)$

and

(ii) $\forall \epsilon > 0 \exists K_{\epsilon, t}: \sup_{z \in S^*_{t+\epsilon}} \left| \frac{U_G(z)}{\sin z} \right| \leq K_{\epsilon, t} \|G(t)\|$. 

Now let $\phi \in U(\mathbb{R}, 2\pi$-$\text{per, odd}; t)$. Then for $n \geq 1$ we put

$$a_n = 2 \sqrt{\frac{2}{\pi}} \int_{K_t} \cos(nz) \phi(z) dz,$$

and, for $n = 0$,

$$a_0 = \frac{2}{\sqrt{\pi}} \int_{K_t} \phi(z) dz.$$
(cf. Theorem (5.8)). Consider the following straightforward estimations

\[ |a_n| \leq \frac{2}{\pi} \sup_{\zeta \in K_t} |\sin \zeta \cos(n\zeta)| \sup_{\zeta \in S^*_t} \left| \frac{\Omega(\zeta)}{\sin \zeta} \right| \left( \int_{K_t} |d\zeta| \right) \leq \]

\[ \leq 2\sqrt{2\pi} e^{nt} \sinh t \sup_{\zeta \in S^*_t} \left| \frac{\Omega(\zeta)}{\sin \zeta} \right| \]

and

\[ |a_0| \leq 4\sqrt{\pi} \sup_{\zeta \in S^*_t} \left| \frac{\Omega(\zeta)}{\sin \zeta} \right| . \]

It follows that \( \sum_{n=0}^{\infty} |a_n| e^{-2n(t+\epsilon)} < \infty \) for all \( \epsilon > 0 \). So if we define \( F_\Omega \) by

\[ F_\Omega(s) = \sum_{n=0}^{\infty} a_n e^{-ns} c_n, \quad s > t, \]

then \( F_\Omega \in Y_{t+\epsilon} \) for any \( \epsilon > 0 \). Moreover, we have shown that

\[ \|F_\Omega(t+\epsilon)\| \leq N_{t, \epsilon} \sup_{\zeta \in S^*_t} \left| \frac{\Omega(\zeta)}{\sin \zeta} \right| . \]

It leads to the following result.

(5.17) Lemma

I. Let \( T > 0 \). Then \( F \in U Y_t \) iff \( U_F \in U U(\mathbb{R}, 2\pi\text{-per, odd}; t) \).

II. Let \( F \in Y_t, \ t > 0 \). Then

\[ \forall \epsilon > 0 \exists K_{\epsilon, t} : \sup_{\zeta \in S^*_{t+\epsilon}} \left| U_F(\zeta) \right| / \sin \zeta \leq K_{\epsilon, t} \|F(t)\|_{Y} . \]

III. Let \( U_F \in U(\mathbb{R}, 2\pi\text{-per, odd}; t) \). Then

\[ \forall \epsilon > 0 \exists N_{\epsilon, t} : \|F(t+\epsilon)\|_{Y} \leq N_{\epsilon, t} \sup_{\zeta \in S^*_t} \left| U_F(\zeta) \right| / \sin \zeta . \]
It is natural to introduce the subclasses \( U([-1,1];t) \) of \( U([-1,1]) \) as follows:

\[
\Omega \in U([-1,1];t) \text{ iff } \Omega \text{ is an analytic function on } E_t^* \text{ for which } \\
\lim_{w \to \infty} \Omega(w) = 0 \text{ and also } \sup_{w \in E_t^*} |\Omega(w)| < \infty.
\]

Again we note that \( U([-1,1]) = \bigcup_{t>0} U([-1,1];t) \). Further, for each \( \theta \in U([-1,1];t) \) the function \( z \mapsto \sin z \theta(\cos z) \) belongs to \( U(\mathbb{R}, 2\pi\text{-per, odd};t) \).

Let \( X_t \) denote the Hilbert space which consists of all mappings \( H \) from \([t,\infty)\) into \( X \) which satisfy \( H(s_1+s_2) = e^{-s_1A_{1/2}}H(s_2), s_1 \geq 0, s_2 \geq t \). Then the unitary operator \( U \) originating from the transformation \( u = \cos x \) maps \( Y_t \) onto \( X_t \). Thus we obtain

\[(5.18) \text{ Lemma}\]

I. Let \( T > 0 \). Then \( F \in \bigcup_{t>T} X_t \) iff \( \Omega_F \in U([-1,1];T) \).

II. Let \( F \in X_t, t > 0 \). Then

\[
\forall \varepsilon > 0 \exists K_{\varepsilon, t} > 0: \sup_{w \in E_t^*} |\Omega_F(w)| \leq K_{\varepsilon, t} \|F(t)\|_X.
\]

III. Let \( \Omega_F \in U([-1,1];t) \). Then

\[
\forall \varepsilon > 0 \exists N_{\varepsilon, t} > 0: \|F(t+\varepsilon)\|_X \leq N_{\varepsilon, t} \sup_{w \in E_t^*} |\Omega_F(w)|.
\]

Here \( \Omega_F \) denotes the unique ultra-hyperfunction corresponding to an element \( F \) of \( \sigma(X,A_{1/2}) \).

Now we obtain the following classical result (cf. Theorem (4.18)).
Let \( \theta \) be a function which is analytic on \( E^*_r, r > 0 \), and which vanishes at infinity. Then for each \( \alpha, \beta > -1 \), the function \( \theta \) can be expanded in a series

\[
\theta = \sum_{n=0}^{\infty} \xi_n^{(\alpha, \beta)} R_n^{(\alpha, \beta)}(a, \beta)
\]

which converges uniformly on a region \( E^*_r \). The coefficients \( \xi_n^{(\alpha, \beta)} \) satisfy \( \xi_n^{(\alpha, \beta)} = O(e^{nt}) \) for some \( t > 0 \) dependent on \( \alpha, \beta \). If \( \alpha = \beta = -\frac{1}{4} \) the series converges uniformly on each region \( E^*_t, t > r \), and \( \xi_n^{(-\frac{1}{4}, -\frac{1}{4})} = O(e^{nt}) \).

**Proof**

By Corollary (5.15) and Theorem (5.13), the linear functional \( \ell \),

\[
\ell(f) = \int_{\beta E} f(w) \theta(w) dw, \quad f \in \tau(X, A^{\frac{1}{2}}), \quad t > r,
\]

is continuous on \( \tau(X, A^{\frac{1}{2}}) \). By Lemma (5.16) for each \( \alpha, \beta > -1 \) there exists \( t_{\alpha, \beta} > 0 \) such that the linear functional \( \ell \circ \exp(-t_{\alpha, \beta}(A_{\alpha, \beta})^{\frac{1}{2}}) \) can be extended to a continuous linear functional on \( X_{\alpha, \beta} \). So there exists \( G_{\alpha, \beta} \in \sigma(X_{\alpha, \beta}, (A_{\alpha, \beta})^{\frac{1}{2}}) \), \( G_{\alpha, \beta} = \sum_{n=0}^{\infty} \xi_n^{(\alpha, \beta)} R_n^{(\alpha, \beta)}(a, \beta) \), such that for all \( w \in E^*_t, t_{\alpha, \beta} > r \),

\[
(\ell \circ \exp(-t_{\alpha, \beta}(A_{\alpha, \beta})^{\frac{1}{2}}))(\exp(t_{\alpha, \beta}(A_{\alpha, \beta})^{\frac{1}{2}})r_w) = \frac{\exp(t_{\alpha, \beta}(A_{\alpha, \beta})^{\frac{1}{2}})r_w \cdot G_{\alpha, \beta}(t_{\alpha, \beta})}{\exp(t_{\alpha, \beta}(A_{\alpha, \beta})^{\frac{1}{2}})r_w} = \sum_{n=0}^{\infty} \xi_n^{(\alpha, \beta)} R_n^{(\alpha, \beta)}(\alpha, \beta)(w).
\]

The series \( \sum_{n=0}^{\infty} \xi_n^{(\alpha, \beta)} R_n^{(\alpha, \beta)}(\alpha, \beta) \) are convergent in the sense that
\[ \| \exp(-t_{\alpha,\beta}(A_{\alpha,\beta}^{1/2})) \left( \sum_{n=0}^{\infty} \xi_n^{(a,\beta)} R_n^{(a,\beta)} \right) - G_{\alpha,\beta}(t_{\alpha,\beta}) \|_{X,\alpha,\beta} \to 0 \]

as \( N \to \infty \). Following the classification theorem (1.1) there exists \( r_{\alpha,\beta} > 0 \) such that the operator \( \exp(r_{\alpha,\beta} A^{1/2}) \circ \exp(-t_{\alpha,\beta}(A_{\alpha,\beta}^{1/2})) \) is bounded. Hence the sequence

\[ \left( \exp(-r_{\alpha,\beta} A^{1/2}) \left( \sum_{n=0}^{\infty} \xi_n^{(a,\beta)} R_n^{(a,\beta)} \right) \right)_{N \in \mathbb{N}} \]

is convergent in \( X \). But then it follows from Lemma (5.18) that

\[ \sum_{n=0}^{\infty} \xi_n^{(a,\beta)} R_n^{(a,\beta)} \]

is uniformly convergent on \( \mathbb{R}_{r_{\alpha,\beta}} \).

Let \( \alpha = \beta = -\frac{1}{2} \). If \( \theta \) is analytic on \( \mathbb{R}^{*} \), then for each \( t > r \) the functional \( \ell \circ \exp(-tA^{1/2}) \) can be extended to a continuous linear functional on \( X \). So there exists \( G \in \sigma(X,A^{1/2}) \), \( G = \sum_{n=0}^{\infty} \xi_n R_{n}^{(-\frac{1}{2},-\frac{1}{2})} \),

\[ (\ell \circ e^{-tA^{1/2}})(e^{tA^{1/2}} \tau_{\omega}) = (e^{tA^{1/2}} \tau_{\omega}, G(t))_{X} = \sum_{n=0}^{\infty} \xi_n R_{n}^{(-\frac{1}{2},-\frac{1}{2})}(\omega) \].

Since

\[ \| e^{-tA^{1/2}} \left( \sum_{n=0}^{N} \xi_n R_{n}^{(-\frac{1}{2},-\frac{1}{2})} \right) - G(t) \|_{X} \to 0 \]

it follows from Lemma (4.18) that the series \( \sum_{n=0}^{\infty} \xi_n R_{n}^{(-\frac{1}{2},-\frac{1}{2})} \) are uniformly convergent on \( \mathbb{R}^{*} \).

For the definition of \( \widetilde{R}_{n}^{(a,\beta)} \) we refer to Chapter 4, p. 97. The functions \( \widetilde{R}_{n}^{(a,\beta)} \) are closely related to the Jacobi functions of the second kind. The mentioned classical result has also been proved in an extended form by Szego [9], p. 250.
6 REPRESENTATION OF SOME LARGE CONTINUOUS GROUPS AS GROUPS OF CONTINUOUS LINEAR OPERATORS ON SPACES OF (ULTRA-) HYPERFUNCTIONS

In this chapter we consider two continuous groups of analytic functions. For the product operation in these groups we take the composition of functions. We shall not introduce the (rather obvious) topologies on these groups. We employ the same notation as used in Chapter 4 and Chapter 5. Hence $X_{0,0}$ denotes the Hilbert space $L_2([-1,1])$ and $A_{0,0}$ the positive self-adjoint operator in $X_{0,0}$ given by $A_{0,0} = -\frac{d}{dx} (1-x^2) \frac{d}{dx}$. Moreover, $\lambda_F$ denotes the hyperfunction in $H([-1,1])$ corresponding to $F \in T_{X_{0,0}}(A_{0,0})$. Further $U([-1,1])$ denotes the class of ultra-hyperfunctions corresponding to $\sigma(X_{0,0},(A_{0,0}))$.

(6.1) Definition

$G_I$ denotes the set of all functions $\rho$ which are analytic on an open neighbourhood of the interval $[-1,1]$ which map $[-1,1]$ bijectively onto $[-1,1]$ and also satisfy $\frac{d\rho}{dw} \neq 0$ on $[-1,1]$.

From this definition it is clear that for all $\rho_1, \rho_2 \in G_I$ the inverse $\rho_1^+$ of $\rho_1$ and the composition $\rho_1 \circ \rho_2$ defined by $(\rho_1 \circ \rho_2)(w) = \rho_1(\rho_2(w))$ belong to $G_I$ also. Hence endowed with the composition mapping as the product operation, $G_I$ becomes a group.

In the next definition we introduce the action of the group $G_I$ on the analyticity space $S_{X_{0,0},(A_{0,0})}$. We recall that this space consists of all function which are analytic on an open neighbourhood of $[-1,1]$. 
(6.2) Definition

Let $\rho \in G_1$. Then we define the linear operator $R_{\rho}$ on $\mathcal{S}_{X_0,0,(A_0,0)^1}$ by

$$(R_{\rho} f)(w) = f(\rho(w)), \quad f \in \mathcal{S}_{X_0,0,(A_0,0)^1}.$$  

In the next theorem we show that the mapping $\rho \mapsto R_{\rho}$, $\rho \in G_1$, is a group representation of $G_1$. Moreover, this group representation can be lifted to the class of hyperfunctions $H([-1,1])$.

(6.3) Theorem

I. For each $\rho \in G_1$ the operator $R_{\rho}$ maps $\mathcal{S}_{X_0,0,(A_0,0)^1}$ continuously into itself.

II. For all $\rho_1, \rho_2 \in G$ we have $R_{\rho_1} \circ R_{\rho_2} = R_{\rho_2 \circ \rho_1}$. Moreover, the representation $R$ is faithful.

III. The representation $R$ can be lifted to a faithful representation $R^e$ such that $R^e_{\rho}$ is a continuous linear mapping on $H([-1,1])$. The operator $R^e_{\rho}$ is explicitly given by

$$(R^e_{\rho} \theta)(w) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\theta(\zeta)}{w - \rho^+(\zeta)} (D^+_{\rho})(\zeta)d\zeta.$$  

Here $D$ denotes the differential operator $D = \frac{d}{dz}$, and $\mathcal{C}$ denotes a clockwise oriented contour which is taken sufficiently narrow around $[-1,1]$.

Proof.

I. From the characterization of $\mathcal{S}_{X_0,0,(A_0,0)^1}$ in Chapter 3, it easily follows that $R$ maps $\mathcal{S}_{X_0,0,(A_0,0)^1}$ into itself. Further, let $f,g \in \mathcal{S}_{X_0,0,(A_0,0)^1}$. Then
\[ (R^*_\rho f, g)_{X_0} = \left\{ \begin{array}{l} \int_{-1}^{1} f(\rho(x)) g(x) \, dx = \int_{-1}^{1} f(x) g(\rho^*(x)) \, (D\rho^*)(x) \, dx = \\
= (f, R^*_\rho g)_{X_0} \end{array} \right. \]

where \( R^*_\rho \) is defined by

\[ (R^*_\rho g)(w) = \overline{(D\rho^*)(w)} g(\rho^*(w)) , \quad g \in \mathcal{S}_{X_{0,0}, (A_{0,0})^\frac{1}{2}} . \]

Observe that the adjoint \( R^*_\rho \) is the composition of the operator \( (R^*_\rho)^{-1} = R^*_{\rho^*} \) and the multiplier \( D\rho^* \). It is clear that also \( R^*_\rho \) maps \( \mathcal{S}_{X_{0,0}, (A_{0,0})^\frac{1}{2}} \) into itself. Then from [6], Corollary 4.9, it follows that \( R^*_\rho \) and \( R^*_\rho \) are continuous on \( \mathcal{S}_{X_{0,0}, (A_{0,0})^\frac{1}{2}} \) and also continuously extendible to \( T_{X_{0,0}, (A_{0,0})^\frac{1}{2}} \).

II. It is clear that \( R: \rho \mapsto R^*_\rho, \rho \in G_1, \) is a homomorphism. The faithfulness of the representation \( R \) follows by considering \( R^*_\rho f \) with \( f(\omega) = \omega \). We then get \( R^*_\rho = \rho, \rho \in G_1 \). So \( \rho_1 \neq \rho_2 \) iff \( \rho_1 \neq \rho_2 \).

III. The extendibility of the mapping \( R^*_\rho \) to the trajectory space \( T_{X_{0,0}, (A_{0,0})^\frac{1}{2}} \) has been shown in part I of the proof. By Corollary (4.16) we get

\[ \Lambda_{R^*_\rho} F(\omega) = \left< \overline{\tau_{R^*_\rho} F} , \tau_{R^*_\rho} F \right> = \left< R^*_\rho \tau_{R^*_\rho} F, F \right> = \]

\[ = \frac{1}{2\pi i} \oint_{C} \frac{1}{w - \rho^*(\zeta)} (D\rho^*)(\zeta) \Lambda_F(\zeta) \, d\zeta \]

with \( \Lambda_F(u) = \left< \overline{\tau_u} F \right> \). The contour \( C \) is sufficiently narrow around \([-1,1]\) and clockwise oriented. Since each \( \theta \in \mathcal{U}([-1,1]) \) corresponds to a unique \( F \in T_{X_{0,0}, (A_{0,0})^\frac{1}{2}}, \theta(w) = \left< \overline{\tau_w} F \right> \), the operator \( R^e_{\rho} \) defined by
(R^e_\rho \theta)(w) = \frac{1}{2\pi i} \oint \frac{1}{w - \rho^+(\zeta)} \left( D\rho^+(\zeta) \theta(\zeta) \right) d\zeta

maps U([-1,1]) continuously into itself.

Moreover, R^e is a faithful representation of G_1.

(6.4) Definition

G_2 denotes the set of all analytic functions \sigma defined on a neighbourhood of 0 with the property \sigma(0) = 0 and \sigma'(0) \neq 0. The domain of definition of each element in G_2 depends on this element.

We observe that G_2 may be properly called a set of germs of analytic functions. The operations of composition and of inversion of functions turn G_2 into a group.

In the next definition we introduce the action of G_2 on the space of ultrahyperfunctions U([-1,1]). Remind that U([-1,1]) consists of all analytic functions \Omega which are analytic at infinity and satisfy \Omega(\pm) = 0.

(6.5) Definition

Let \sigma \in G_2. Then the operator P_\sigma on U([-1,1]) is defined by

(P_\sigma \Omega)(w) = \Omega\left(\frac{1}{(\frac{\sigma}{w})}\right), \quad \Omega \in U([-1,1]).

(6.6) Theorem

I. For each \sigma \in G_2 the operator P_\sigma maps U([-1,1]) continuously into itself.

II. For all \sigma_1, \sigma_2 \in G_2 we have

P_{\sigma_1} \circ P_{\sigma_2} = P_{\sigma_2 \circ \sigma_1}.

The representation of P is faithful.
Proof

I. It is clear that $P_{\sigma} \Omega \in U([-1,1])$ whenever $\Omega \in U([-1,1])$. Following Lemma (5.18) the spaces $U([-1,1])$ and $\sigma(X_{0,0},(A_{0,0})^{1})$ are homeomorphic. Hence continuity of $P_{\sigma}$ can be checked by only considering sequences.

(See [2], Theorem 4.13.) Take a converging sequence $(\Omega_{n})_{n \in \mathbb{N}}$ in $U([-1,1])$. It means that there exists sufficiently large $R > 0$ such that $\Omega_{n} \to \Omega$ uniformly on $|w| > R$, for some $\Omega \in U([-1,1])$. Next take $R_{1} > R$ so large that $\sigma(\frac{1}{w})$ is defined for $|w| > R_{1}$ and $|\sigma(\frac{1}{w})| > R$ for all $w$, $|w| > R_{1}$. It follows that $P_{\sigma}(\Omega_{n})$ converges to $P_{\sigma}(\Omega)$ uniformly for all $w$ with $|w| > R_{1}$.

II. By writing out the definitions it easily follows that the mapping $P$:

$$\sigma \mapsto P_{\sigma}$$

is a homomorphism. The faithfulness follows by considering $P_{\sigma}(\Omega)$ with $\Omega(w) = \frac{1}{w}$. We get $(P_{\sigma}(\Omega))(w) = \sigma(\frac{1}{w})$. So $P_{\sigma_{1}} \neq P_{\sigma_{2}}$ iff $\sigma_{1} \neq \sigma_{2}$. □

Remark. It is not likely that the representation $P$ of the preceding theorem can be restricted to the corresponding entireness space $\sigma(X_{0,0},(A_{0,0})^{1})$ which consists of all entire functions. However, by duality it follows that there is a continuous faithful representation of $G_{2}$ into the continuous linear mappings of $\Lambda(\mathbb{C}) = \tau(X_{0,0},(A_{0,0})^{1})$ into itself:

$$\sigma \mapsto P'_{\sigma}, \quad P'_{\sigma_{2}} \circ \sigma_{1} = P'_{\sigma_{2}} \circ P'_{\sigma_{1}}.$$  

We do not know the explicit expression for this representation.
APPENDIX

In this appendix we compute the matrix of the differential operators \( D = \frac{d}{dx} \) and \( xD \) with respect to the orthonormal basis established by the Jacobi polynomials \( R_n^{(\alpha, \beta)} \) where \( \alpha, \beta > -1 \) arbitrary and fixed. In [8], p. 213, the following relations can be found

\[
(a.1) \quad Dp_n^{(\alpha, \beta)} = \frac{1}{2}(n+\alpha+\beta+1)p_{n-1}^{(\alpha+1, \beta+1)} , \quad n = 1, 2, \ldots .
\]

We express the polynomials \( p_n^{(\alpha+1, \beta+1)} \) as finite combinations of the polynomials \( p_k^{(\alpha, \beta)} \), \( k = 0, 1, 2, \ldots, n-1 \). So we write

\[
p_n^{(\alpha+1, \beta+1)} = \sum_{k=0}^{n-1} \gamma_{n-1,k}^{(\alpha, \beta)} p_k^{(\alpha, \beta)} .
\]

In order to compute the coefficients \( \gamma_{n,k}^{(\alpha, \beta)} \), \( n = 0, 1, 2, \ldots, k = 0, 1, 2, \ldots, n \), we use the following relations which can be derived from [8], p. 213

\[
(a.2.i) \quad p_{n-1}^{(\alpha+1, \beta+1)} = c_{n-1}^{(\alpha+1, \beta+1)} + d_{n-1}^{(\alpha+1, \beta+1)}
\]

\[
(a.2.ii) \quad p_n^{(\alpha, \beta+1)} = a_n^{(\alpha, \beta+1)} + b_n^{(\alpha, \beta+1)}
\]

where

\[
a_n^{(\alpha, \beta+1)} = \frac{\ell + \alpha + \beta + 1}{\ell + \alpha + \beta + 1} , \quad c_{n-1}^{(\alpha+1, \beta+1)} = \frac{\ell + \alpha + \beta + 2}{\ell + \alpha + \beta + 2} ,
\]

\[
b_n^{(\alpha, \beta+1)} = -\frac{\ell + \alpha}{\ell + \alpha + \beta + 1} , \quad d_{n-1}^{(\alpha+1, \beta+1)} = \frac{\ell + \beta + 1}{\ell + \alpha + \beta + 2} .
\]

So starting from \( p_{\ell}^{(\alpha+1, \beta+1)} \) we get \( c_{\ell}^{(\alpha+1, \beta+1)} p_{\ell}^{(\alpha, \beta+1)} + d_{\ell}^{(\alpha+1, \beta+1)} p_{\ell-1}^{(\alpha+1, \beta+1)} \) by (a.2.i)

and then by (a.2.ii) \( p_{\ell}^{(\alpha, \beta+1)} = a_{\ell}^{(\alpha, \beta+1)} p_{\ell}^{(\alpha, \beta+1)} + b_{\ell}^{(\alpha, \beta+1)} p_{\ell-1}^{(\alpha, \beta+1)} \), and also by (a.2.i) \( p_{\ell-1}^{(\alpha+1, \beta+1)} = c_{\ell-1}^{(\alpha+1, \beta+1)} p_{\ell-1}^{(\alpha, \beta+1)} + d_{\ell-1}^{(\alpha+1, \beta+1)} p_{\ell-2}^{(\alpha+1, \beta+1)} \), etc.
The sketched process terminates, because \( p_{n-1}(p,q) \equiv 1 \). It can be described by the following directed graph.

The graph (a.3) shows the following:

- \( c_z \) is multiplied by \( a_z \) or \( b_z \)
- \( d_z \) is multiplied by \( d_{z-1} \) or \( c_{z-1} \)
- \( b_z \) is multiplied by \( b_{z-1} \) or \( a_{z-1} \)
- every factor ends with some \( a_q \)

The above examinations yield the following result:

\[
p_n^{(\alpha+1, \beta+1)} = \sum_{p=0}^{n} \left( \sum_{k=0}^{p} d_n \ldots d_{n-k+1} c_{n-k} b_{n-k} \ldots b_{n-p+1} a_{n-p} \right) p_n^{(\alpha, \beta)}
\]

with the convention \( d_n d_{n+1} = 1 \) and \( b_{n-p} b_{n-p+1} = 1 \); equivalently

\[
(a.4) \quad p_n^{(\alpha+1, \beta+1)} = \sum_{z=0}^{\gamma} \left( \sum_{k=0}^{n-\gamma} d_n \ldots d_{n-k+1} c_{n-k} b_{n-k} \ldots b_{\gamma+1} a_{\gamma} \right) p_z^{(\alpha, \beta)}.
\]

Thus we find that

\[
\gamma_n^{(\alpha, \beta)} = \sum_{k=0}^{n-\gamma} (d_n \ldots d_{n-k+1} c_{n-k} b_{n-k} \ldots b_{\gamma+1} a_{\gamma}).
\]
A simple calculation yields

\[(a.5) \quad \gamma_{n,k}^{(\alpha, \beta)} = (-1)^k \frac{\Gamma(k+\alpha+\beta+1)\Gamma(n+\beta+2)}{\Gamma(n+\alpha+\beta+3)\Gamma(k+\alpha+1)} \left(2k+\alpha+\beta+1\right)^n \sum_{k=\ell}^{n} (-1)^k (2k+\alpha+\beta+2) \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\beta+2)} .\]

Let \(O_k^{(\alpha, \beta)}\) denote the \(X_{\alpha, \beta}\)-normalization factor for the Jacobi polynomials,

\[(a.6) \quad O_k^{(\alpha, \beta)} = \left(\frac{2k + \alpha + \beta + 1}{2^{\alpha+1}} \frac{\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}\right)^{1/2} .\]

Then we obtain for the matrix of \(D\) with respect to the orthonormal basis \(\{(R_n^{(\alpha, \beta)})_n^{\infty}\}_{n=0}\) of \(X_{\alpha, \beta}\)

\[(a.7) \quad (D_{R_n^{(\alpha, \beta)}}^{(\alpha, \beta)})_{X_{\alpha, \beta}}^{\gamma_{n,k}^{(\alpha, \beta)}} = \begin{cases} 0 & \text{if } \ell \geq n, \ell, n \in \mathbb{N} \cup \{0\} \\ \frac{O_n^{(\alpha, \beta)}}{O_k^{(\alpha, \beta)}} \gamma_{n-1, \ell}^{(\alpha, \beta)} (n+\alpha+\beta+1) & \text{if } \ell = 0, 1, \ldots, n-1, n \in \mathbb{N} . \end{cases}\]

With similar methods we next compute the matrix of the differential operator \(xD\) with respect to \(\{(R_n^{(\alpha, \beta)})_n^{\infty}\}_{n=0}\). From [8], p.213, we obtain the following identities:

\[(1-x)p_n^{(\alpha, \beta)}(x) = \frac{2(n+\alpha)}{2n + \alpha + \beta + 1} p_n^{(\alpha-1, \beta)}(x) - \frac{2(n+1)}{2n + \alpha + \beta + 1} p_{n+1}^{(\alpha-1, \beta)}(x) \]

and

\[(1+x)p_n^{(\alpha, \beta)}(x) = \frac{2(n+\beta)}{2n + \alpha + \beta + 1} p_n^{(\alpha, \beta-1)}(x) + \frac{2(n+1)}{2n + \alpha + \beta + 1} p_{n+1}^{(\alpha, \beta-1)}(x) .\]

Adding these relations, we obtain the following formula
Thus it follows that
\[
(xD)_n p(a, \beta) = \frac{1}{2} (n + a + \beta + 2) x p(a + 1, \beta + 1) =
\]
\[
= \frac{1}{2} \left( \frac{n + a + \beta + 2}{2n + a + \beta + 3} \right) \left[ - (n + a + 1) p(a + 1, \beta + 1) + (n + 1) p(a + 1, \beta + 1) + (n + 1) p(a, \beta + 1) \right].
\]

With the relations (a.2.i) and (a.2.ii) we get
\[
P_k^{(a+1, \beta)} = \sum_{\ell=0}^{k} \tilde{a}_{\ell} \cdots \tilde{a}_{\ell+1} \tilde{c}_\ell \ p_k^{(a, \beta)}
\]
and
\[
P_k^{(a, \beta+1)} = \sum_{\ell=0}^{k} \tilde{b}_{\ell} \cdots \tilde{b}_{\ell+1} \tilde{a}_\ell \ p_k^{(a, \beta)}
\]
where
\[
\tilde{a}_{\ell} = \frac{2\ell + a + \beta}{\ell + a + \beta + 1}, \quad \tilde{b}_{\ell} = -\frac{\ell + a}{\ell + a + \beta + 1},
\]
\[
\tilde{c}_{\ell} = \frac{2\ell + a + \beta + 1}{\ell + a + \beta + 1}, \quad \tilde{d}_{\ell} = \frac{\ell + \beta}{\ell + a + \beta + 1}.
\]

Finally, substituting the above values in (a.8) we get for the \( \ell \)-th coefficient, \( 0 \leq \ell \leq n \), in the expression of \((xD)P_n^{(a, \beta)}\)
\[ \frac{1}{2n} \left\{ (-1)^{n-\lambda-1} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)} \frac{\Gamma(\lambda+\alpha+\beta+1)}{\Gamma(\lambda+\alpha+\beta+2)} \right\} + \frac{1}{2n} \left\{ (-1)^{n-\lambda+1} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)} \frac{\Gamma(\lambda+\alpha+\beta+1)}{\Gamma(\lambda+\alpha+\beta+2)} \right\} + \frac{1}{2n} \left\{ (-1)^{n-\lambda+1} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)} \frac{\Gamma(\lambda+\alpha+\beta+1)}{\Gamma(\lambda+\alpha+\beta+2)} \right\} = \]

\[ \frac{1}{2n} \left\{ (-1)^{n-\lambda+1} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)} \frac{\Gamma(\lambda+\alpha+\beta+1)}{\Gamma(\lambda+\alpha+\beta+2)} \right\} = \]

The \((n-1)\)-th coefficient in (a.8) is given by

\[ \frac{1}{2n} \left\{ (-1)^{n-\lambda+1} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)} \frac{\Gamma(\lambda+\alpha+\beta+1)}{\Gamma(\lambda+\alpha+\beta+2)} \right\} = n + 1. \]

Remark. If \(\alpha = \beta = \lambda = \frac{1}{2}\), then the polynomials \(P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}\) lead to the so-called Gegenbauer polynomials

\[ C_n^{(\lambda)}(x) = \frac{\lambda^n}{\Gamma(n+1)} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x). \]

From the above computation we obtain

\[ (xD)C_{2n}^{(\lambda)} = \sum_{k=0}^{n-1} (4k+2\lambda)C_{2k}^{(\lambda)} + 2n C^{(\lambda)}_{2n} \]

\[ (xD)C_{2n+1}^{(\lambda)} = \sum_{k=0}^{n-1} (4k+2\lambda-2)C_{2k+1}^{(\lambda)} + (2n+1)C^{(\lambda)}_{2n+1}. \]

This result corresponds with the well-known formula.
Now for \( 0 \leq \varepsilon < n+1 \) we put

\[
\phi_{n+1, \varepsilon}^{(\alpha, \beta)} = \frac{1}{2}(2\varepsilon+\alpha+\beta+1) \Gamma(\varepsilon+\alpha+\beta+2) \left( (-1)^{n-\varepsilon+1} \frac{\Gamma(n+\alpha+2) + \Gamma(n+\beta+2)}{\Gamma(\varepsilon+\alpha+1) + \Gamma(\varepsilon+\beta+1)} \right) .
\]

Then the matrix of the operator \((xD)\) with respect to \( (R_n^{(\alpha, \beta)})_n^{\infty} \) is given by

\[
\begin{cases}
0 & \text{if } \varepsilon > n, \ n \in \mathbb{N} \cup \{0\} \\
n & \text{if } \varepsilon = n, \ n \in \mathbb{N} \cup \{0\} \\
\phi_{n, \varepsilon}^{(\alpha, \beta)} \frac{O_n^{(\alpha, \beta)}}{O_{n, \varepsilon}^{(\alpha, \beta)}} & \text{if } 0 \leq \varepsilon < n \text{ and } n \in \mathbb{N}
\end{cases}
\]

Above we have computed the explicit values of the matrix elements of the operators \( D \) and \((xD)\) with respect to each orthonormal basis \( (R_n^{(\alpha, \beta)})_n^{\infty} \).

The next step is the derivation of sufficiently sharp upper bounds for these values. Therefore we need the following result.

\[(a.11)\ \text{Lemma}\]

Let \( c, d > 0 \). Then there exists a positive constant \( K_{c, d} > 0 \) such that for all \( m \in \mathbb{N} \)

\[
\frac{\Gamma(m+c)}{\Gamma(m+d)} \leq K_{c, d} m^{c-d} .
\]

\textbf{Proof}

From [12] we take the following inequality:

\[
\forall m \in \mathbb{N} \quad \forall s, 0 \leq s \leq 1 : \quad \frac{\Gamma(m+1)}{\Gamma(m+s)} \leq (m+1)^{1-s} .
\]
We proceed as follows. Let \( m \in \mathbb{N} \). Then

\[
\frac{\Gamma(m+c)}{\Gamma(m+d)} = \frac{\Gamma(m+c)}{\Gamma(m+1)} \cdot \frac{\Gamma(m+1)}{\Gamma(m+d)}.
\]

Moreover we have

\[
\frac{\Gamma(m+c)}{\Gamma(m+1)} = (m+c-1) \cdots (m+1) \frac{\Gamma(m+c-[c])}{\Gamma(m+1)} \leq (m+c-1)[c] \frac{m-c-1}{m}.
\]

and, also

\[
\frac{\Gamma(m+1)}{\Gamma(m+d)} = \frac{1}{(m+d-1) \cdots (m+d-[d])} \frac{\Gamma(m+1)}{\Gamma(m+d-[d])} \leq \left( \frac{1}{m+d-[d]} \right)_{(m+1)}^{1-d+[d]}.
\]

Since

\[
\frac{(m+c-1)[c]}{(m+d-[d])} = [c] \frac{1 + \frac{c-1}{m}}{1 + \frac{d-[d]}{m}} \leq (c)[c] \frac{m}{m-d} \frac{c-1}{m} 
\]

we finally get

\[
\frac{\Gamma(m+c)}{\Gamma(m+d)} \leq (c)[c] \frac{m}{m-d} \frac{c-1}{m} (m+1)^{1-d+[d]} \leq (c)[c] 2^{1-d+[d]} m^{c-d}.
\]

The previous lemma gives rise to the following estimates

\[(a.13.i) \quad |O_k^{(\alpha,\beta)}| = \left( \frac{2k + \alpha + \beta + 1}{2^{\alpha+\beta+1}} \frac{\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)} \right)^{1/2}.
\]
\[ \left( \frac{\Gamma(k+\alpha+\beta+1)(k+\alpha+1)(k+\beta+1)}{\Gamma(k+\alpha+\beta+3)} \right)^{1/2} \leq \sup_{k \in \mathbb{N} \cup \{0\}} \left\{ \frac{\Gamma(k+\alpha+2)\Gamma(k+\beta+2)}{\Gamma(k+2)^{3/2}} \right\}^{1/2} \]

\[ \left( \frac{\Gamma(k+\alpha+\beta+1)(k+\alpha+1)(k+\beta+1)}{\Gamma(k+\alpha+\beta+3)} \right)^{1/2} \leq \left( \sup_{k \in \mathbb{N} \cup \{0\}} \left\{ \frac{\Gamma(k+\alpha+2)\Gamma(k+\beta+2)}{\Gamma(k+2)^{3/2}} \right\}^{1/2} \right)^{1/2} \]

\[ =: C_{\alpha, \beta}(k+1)^{1/2} \]

(a.13.ii) \[ |(a, \beta)|^{-1} \leq \left( \frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+\beta+3)} \right)^{1/2} \left( \frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(\ell+\alpha+1)} \right)^{1/2} \left( 2k+\alpha+\beta+1 \right)^{1/2} \]

\[ \leq (a+1)^{1/2} \left( \frac{(2k+\alpha+\beta+2)(\ell+\alpha+1)}{(k+\alpha+2)(\ell+\alpha+1)} \right)^{1/2} \left( \frac{(2k+\alpha+\beta+1)}{(k+\alpha+2)(\ell+\alpha+1)} \right)^{1/2} \]

\[ \leq \sum_{k=\ell}^{n} \left( \frac{2k+\alpha+\beta+1}{k+\alpha+2} \right)^{1/2} \left( \frac{k+\alpha+1}{\ell+\alpha+1} \right)^{1/2} \]

\[ \leq \left( \sup_{k \in \mathbb{N} \cup \{0\}} \left\{ \frac{\Gamma(k+\alpha+2)(\ell+\alpha+1)}{(\ell+\alpha+1)(\ell+\alpha+2)} \right\}^{1/2} \right)^{1/2} \left( \sup_{k \in \mathbb{N} \cup \{0\}} \left\{ \frac{\Gamma(k+\alpha+2)(\ell+\alpha+1)}{(k+\alpha+2)(\ell+\alpha+1)} \right\}^{1/2} \right)^{1/2} \]

\[ \left( \gamma_{n, \ell}^\alpha \right)^{1/2} \leq \left[ \frac{(\ell+\alpha+1)(n+1)}{(n+\alpha+3)} \right]^{1/2} \left[ \frac{(n+\alpha+2)}{(n+\alpha+3)} \right]^{1/2} \left( \frac{(2k+\alpha+\beta+1)}{(k+\alpha+2)(\ell+\alpha+1)} \right)^{1/2} \left( \frac{(2k+\alpha+\beta+2)}{(k+\alpha+2)} \right)^{1/2} \]

\[ \leq \left\{ \sup_{k \in \mathbb{N} \cup \{0\}} \left\{ \frac{(\ell+\alpha+1)}{(n+\alpha+3)} \right\} \right\}^{1/2} \left\{ \sup_{k \in \mathbb{N} \cup \{0\}} \left\{ \frac{(n+\alpha+2)}{(n+\alpha+3)} \right\} \right\}^{1/2} \left( \frac{(2k+\alpha+\beta+2)}{(k+\alpha+1)} \right)^{1/2} \left( \frac{(2k+\alpha+\beta+1)}{(k+\alpha+2)} \right)^{1/2} \]

\[ \leq (n+1)^{-1/2} \left( \gamma_{n, \ell}^\alpha \right)^{1/2} \leq C_{\alpha, \beta}(n+1)^{1/2} \]

\[ \leq \sum_{k=\ell}^{n} \left( \frac{k+1}{n+1} \right)^{\alpha+1} \left( \frac{\ell+1}{k+1} \right)^{\beta+1} \leq \sum_{k=\ell}^{n} \left( \frac{k+1}{n+1} \right)^{\alpha+1} \left( \frac{\ell+1}{k+1} \right)^{\beta+1} \]

\[ \leq (n+1)^{-1/2} \left( \gamma_{n, \ell}^\alpha \right)^{1/2} \leq C_{\alpha, \beta}(n+1)^{1/2} \]

\[ =: E_{\alpha, \beta} \sum_{k=\ell}^{n} \left( \frac{k+1}{n+1} \right)^{\alpha+1} \left( \frac{\ell+1}{k+1} \right)^{\beta+1} \leq E_{\alpha, \beta}(n+1)^{1/2} \]
\[(a.13.\text{iv}) \quad |g^{(n+\alpha+\beta+2)}_{n+1, \ell}| \leq \frac{1}{2} (n+\alpha+\beta+2) \left[ \frac{2 \ell+\alpha+\beta+1}{\ell+\alpha+1} \right] \frac{\Gamma(n+\alpha+2) \Gamma(\ell+\alpha+\beta+3)}{\Gamma(\ell+\alpha+1) \Gamma(n+\alpha+3) \Gamma(\ell+\alpha+2)} + \\
+ \frac{1}{2} \frac{2 \ell+\alpha+\beta+1}{\ell+\alpha+1} \frac{\Gamma(n+\beta+2) \Gamma(\ell+\alpha+\beta+3)}{\Gamma(\ell+\alpha+1) \Gamma(n+\alpha+3) \Gamma(\ell+\alpha+2)} \leq \\
\leq \frac{1}{2} (n+\alpha+\beta+2) \left[ \sup_{\ell \in \mathbb{N} \cup \{0\}} \left( \frac{2 \ell+\alpha+\beta+1}{\ell+\alpha+1} \right) \right] \cdot \\
\cdot K_{\alpha+1, \alpha+\beta+2} K_{\alpha+\beta+2, \alpha+1} \left( \frac{\beta+1}{n+1} \right)^{\beta+1} + \\
+ \left( \sup_{\ell \in \mathbb{N} \cup \{0\}} \left( \frac{2 \ell+\alpha+\beta+1}{\ell+\alpha+1} \right) \right) \cdot \\
\cdot K_{\beta+1, \alpha+\beta+2} K_{\alpha+\beta+2, \alpha+1} \left( \frac{\beta+1}{n+1} \right)^{\beta+1} \leq \\
\leq F_{\alpha, \beta}(n+1)\]

for some well-chosen positive constant \(F_{\alpha, \beta}\).

With the estimates (a.13.i-iv) we find

\[(a.14) \quad \left| (D_{n, \alpha, \beta} \cdot R_{n, \alpha, \beta} \cdot)_{\alpha, \beta} \right| \leq \begin{cases} 0 & \text{if } k \geq n \\ G_{\alpha, \beta}^{3/2} \frac{(n-k)}{(k+1)^{1/2}} & \text{if } 0 \leq k < n \end{cases}\]

Here \(G_{\alpha, \beta} > 0\) is a constant dependent on \(C_{\alpha, \beta}, D_{\alpha, \beta}\) and \(E_{\alpha, \beta}\). Also

\[(a.15) \quad \left| (xD_{n, \alpha, \beta} \cdot R_{n, \alpha, \beta} \cdot)_{\alpha, \beta} \right| \leq \begin{cases} 0 & \text{if } k > n \\ n & \text{if } k = n \\ H_{\alpha, \beta}^{3/2} (n-k)^{-1/2} & \text{if } 0 \leq k < n \end{cases}\]
REFERENCES


   Part b: Analyticity spaces, trajectory spaces and their pairing.
      To appear in KNAW.
   Part c: Linear mappings, tensor products and kernel theorems.
      To appear in KNAW.


