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The $H_{\infty}$ control problem with zeros on the boundary of the stability domain

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The $H_\infty$ control problem with zeros on the boundary of the stability domain

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Abstract

In this paper we study the discrete and continuous time $H_\infty$ control problems without any assumptions on the system parameters. Our approach yields necessary and sufficient conditions for the existence of a suitable controller.

Keywords: $H_\infty$ optimal control, Riccati equation, discrete time systems, continuous time systems.

1 Introduction

The $H_\infty$ control problem has been studied extensively over the last decade. Necessary and sufficient conditions for the existence of suitable controllers have been derived and all kinds of properties of $H_\infty$ controllers are by now well-known. See for instance the books [1, 5, 8, 16]. A number of standard assumptions are made in the literature. For example, requiring the system to be left-invertible or excluding zeros on the imaginary axis. Several people have contributed in the relaxation of these assumptions. See, for instance [6, 10, 13, 14, 16, 17]. The conditions in [14] basically solve the general continuous time $H_\infty$ control problem. Using the bilinear transform this paper could also give necessary and sufficient conditions for the discrete time but, as far as I know, there is no paper studying the discrete-time $H_\infty$ control problem without any assumptions on the system matrices ([17] comes closest). These papers work with Riccati equations and other conditions which can be reduced to Riccati equations in a suitable basis. However, an alternative approach where existence conditions are expressed in terms of linear matrix inequalities, has been derived in [3, 7] and in these papers no assumptions need to be imposed on the system parameters. There is quite a substantial difference in the type of existence conditions. Using linear matrix inequalities requires convex optimization techniques while Riccati equations can be solved using Schur decompositions. Both methods have their merit and each one gives a different kind of insight into the $H_\infty$ control problem.

The objective of this paper is to present necessary and sufficient conditions for the existence of suitable controllers given either a discrete or continuous time $H_\infty$ control problem. Our approach is based on algebraic Riccati equations. The characterization we present has the nice feature that it treats the difficulties of zeros at infinity and zeros on the imaginary axis very transparently.

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Suppose we have a continuous time problem. In the regular case we have two algebraic Riccati equations. If we have a continuous time problem with zeros at infinity then we replace the Riccati equation by a quadratic matrix inequality. However, note that this quadratic matrix inequality reduced to a Riccati equation (of smaller dimension) in a suitable basis. If we have zeros on the boundary of the stability domain then there do not exist stabilizing solutions of the algebraic Riccati equation. However, we know from the $H_2$ optimal control problem that we express the minimal achievable $H_2$ norm in terms of semi-stabilizing solutions of the Riccati equations (we allow for eigenvalues on the imaginary axis). This paper shows that we can also characterize the minimal achievable $H_\infty$ norm in terms of semi-stabilizing solutions of the Riccati equation. Except that this time we have to add an additional condition for each invariant zero on the imaginary axis. A similar statement can be made for discrete time problems.

This paper presents conditions, which in our view, are quite elegant but we do not have a nice approach to actually construct controllers. This remains an interesting research topic.

2 The continuous time $H_\infty$ control problem

We consider the linear, time-invariant, finite-dimensional system:

\[
\Sigma: \begin{cases}
\dot{x} = Ax + Bu + Ew, \\
y = C_1x + D_1w, \\
z = C_2x + D_2u,
\end{cases}
\]

where for all $t$ we have that $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^l$ is the unknown disturbance, $y(t) \in \mathbb{R}^p$ is the measured output and $z(t) \in \mathbb{R}^q$ is the unknown output to be controlled. $A$, $B$, $E$, $C_1$, $C_2$, $D_1$, and $D_2$ are matrices of appropriate dimensions. Note that we have two direct feedthrough matrices identical to 0 (from $w$ to $z$ and from $u$ to $y$). These can be handled using some standard techniques as e.g. described in [16].

We would like to minimize the effect of the disturbance $w$ on the output $z$ by finding an appropriate control input $u$. We seek a stabilizing controller of the form:

\[
\Sigma_F: \begin{cases}
\dot{p} = Kp + Ly, \\
u = Mp + Ny,
\end{cases}
\]

such that after applying the feedback $\Sigma_F$ to the system (2.1), the resulting closed-loop system is internally stable and has $H_\infty$ norm strictly less than some a priori given bound 1. We shall derive necessary and sufficient conditions under which such a compensator exists.

A central role in our study of the above problem will be played by the quadratic matrix inequality. For $P \in \mathbb{R}^{n \times n}$ we consider the following matrix:

\[
F(P) := \begin{pmatrix}
A^TP + PA + C_1^TC_2 + PE^TP & PB + C_1^TD_2 \\
B^TP + D_1^TC_2 & D_1^TD_2 
\end{pmatrix}.
\]

If $F(P) \geq 0$, we say that $P$ is a solution of the quadratic matrix inequality. We also define a dual version of this quadratic matrix inequality. For any matrix $Q \in \mathbb{R}^{n \times n}$ we define the following matrix:

\[
G(Q) := \begin{pmatrix}
AQ + QA^T + EE^T + QC_2C_1 & QC_1 + ED_1^T \\
C_1Q + D_1E^T & D_1D_1^T 
\end{pmatrix}.
\]
If \( G(Q) \geq 0 \), we say that \( Q \) is a solution of the dual quadratic matrix inequality. In addition to these two matrices, we define two matrices pencils, which play dual roles:

\[
L(P, s) := \begin{pmatrix} sI - A - EE^TP & -B \\ -C_1 \end{pmatrix},
\]

\[
M(Q, s) := \begin{pmatrix} sI - A - QC_1C_2 \\ -C_1 \end{pmatrix}.
\]

Finally, we define the following two transfer matrices:

\[
G_{ci}(s) := C_2 (sI - A)^{-1} B + D_2,
\]

\[
G_{di}(s) := C_1 (sI - A)^{-1} E + D_1.
\]

Let \( \rho(\cdot) \) denote the spectral radius. We define the invariant zeros of a system \((A, B, C, D)\) as those points \( \lambda \in \mathbb{C} \) for which

\[
\text{rank} \begin{pmatrix} \lambda I - A & -B \\ C & D \end{pmatrix} < \text{normalrank} \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix}.
\]

We are now in a position to formulate the main result for continuous time systems.

**Theorem 2.1** : Consider the system (2.1). Then the following two statements are equivalent:

(i) For the system (2.1) a controller of the form (2.2) exists such that the resulting closed-loop system, with transfer matrix \( G_F \), is internally stable and has \( H_\infty \) norm less than \( \gamma \), i.e. \( \| G_F \|_\infty < 1 \).

(ii) There are positive semi-definite solutions \( P, Q \) of the quadratic matrix inequalities \( F(P) \geq 0 \) and \( G(Q) \geq 0 \) satisfying \( \rho(PQ) < 1 \), such that the following rank conditions are satisfied

(a) \( \text{rank} F(P) = \text{normalrank} G_{ci} \),

(b) \( \text{rank} G(Q) = \text{normalrank} G_{di} \),

(c) All the zeros of the matrix pencil

\[
\begin{pmatrix} L(P, s) \\ F(P) \end{pmatrix}
\]

are in the closed left half plane. Moreover, the number of invariant zeros on the imaginary axis, counting multiplicity, is equal to the number of invariant zeros of \((A, B, C_2, D_2)\) on the imaginary axis.

(d) All the zeros of the matrix pencil

\[
\begin{pmatrix} M(Q, s) & G(Q) \end{pmatrix}
\]

are in the closed left half plane. Moreover, the number of invariant zeros, counting multiplicity, on the imaginary axis is equal to the number of invariant zeros of \((A, E, C_1, D_1)\) on the imaginary axis.
Finally, for any invariant zero $\lambda$ on the imaginary axis of the system $(A, B, C_2, D_2)$ or of the system $(A, E, C_1, D_1)$ there exists a matrix $K$ such that $\lambda I - A - BKC_1$ is invertible and
\[
\|(C_2 + D_2KC_1)(\lambda I - A - BKC_1)^{-1}(E + BKD_1) + D_2KD_1\| < 1
\]
(2.5)

Remarks:

(i) If $D_2$ is injective the quadratic matrix inequality $F(P) \succeq 0$ together with the first rank-condition is equivalent to stating that $P$ satisfies the following standard algebraic Riccati equation:
\[
A^TP + PA - (PB + C_2^T D_2)(D_2^T D_2)^{-1}(B^TP + D_2^T C_2) + PEE^TP + C_2^T C_2 = 0.
\]
Moreover, in that case the second rank condition is equal the requirement that the matrix
\[
A + EE^TP - B(D_2^T D_2)^{-1}(B^TP + D_2^T C)
\]
has all eigenvalues in the closed left half plane. Moreover, the number of eigenvalues on the imaginary axis should be equal to the number of invariant zeros of the system $(A, B, C_2, D_2)$. Similar comments can be made for the dual quadratic matrix inequality in case $D_1$ is surjective.

(ii) The quadratic matrix inequality $F(P) \succeq 0$ in combination with rank condition (a) reduces to a standard algebraic Riccati equation in a suitable basis. Condition (c) guarantees that we have a semi-stabilizing solution in the same way as in remark (i). For details we refer to [16]. Of course a similar remark can be made for the dual quadratic matrix inequality $G(Q) \succeq 0$.

(iii) In [10] the special case is considered where $D_2$ need not be injective but $\text{normrank } G_{ci} = \text{rank } D_2$. In that case the quadratic matrix inequality $F(P) \succeq 0$ together with the first rank condition reduces to a standard algebraic Riccati equation albeit with a generalized inverse:
\[
A^TP + PA - (PB + C_2^T D_2)(D_2^T D_2)^\dagger(B^TP + D_2^T C_2) + PEE^TP + C_2^T C_2 = 0.
\]
Moreover, in that case the second rank condition is equal the requirement that there exists a matrix $F$ such that
\[
A + EE^TP - B(D_1^T D_1)^\dagger(B^TP + D_1^T C) + B(I - D_1^T D_1)F
\]
has all eigenvalues in the closed left half plane. Moreover the number of eigenvalues on the imaginary axis is equal to the number of invariant zeros on the imaginary axis of the system $(A, B, C_2, D_2)$.

(iv) If the systems $(A, B, C_2, D_2)$ and $(A, E, C_1, D_1)$ do not have zeros on the imaginary axis then clearly the last condition of part (ii) regarding invariant zeros on the imaginary axis becomes void. Moreover, condition (c) reduces to the requirement that the matrix pencil (2.3) has all zeros in the open left half plane. Similarly condition (d) reduces to the requirement that the matrix pencil (2.4) has all eigenvalues in the open left half plane.

(v) In the state feedback case, we have $C_1 = I$ and $D_1 = 0$. Then it is easy to see that $Q = 0$ satisfies $G(Q) \succeq 0$. Moreover, $Q = 0$ also satisfies conditions (b) and (c) and clearly the coupling condition $\rho(PQ) < 1$ is automatically satisfied. Hence part (ii) reduces to:

- There exists a matrix $P \succeq 0$ satisfying $F(P) \succeq 0$ and conditions (a) and (c).
- For any invariant zero $\lambda$ on the imaginary axis of the system $(A, B, C_2, D_2)$, there exists a matrix $K$ such that $\lambda I - A - BKC_1$ is invertible and (2.5) is satisfied.
3 The discrete time $H_\infty$ control problem

We consider the linear, time-invariant, finite-dimensional system:

$$\Sigma: \begin{cases} \sigma x = Ax + Bu + Ew, \\
y = C_1x + D_1w, \\
z = C_2x + D_2u + D_3w, \end{cases}$$

where $\sigma$ denotes the shift operator: $(\sigma x)(k) = x(k+1)$. Moreover, for all $t$ we have that $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^l$ is the unknown disturbance, $y(t) \in \mathbb{R}^p$ is the measured output and $z(t) \in \mathbb{R}^q$ is the unknown output to be controlled. $A$, $B$, $E$, $C_1$, $C_2$, $D_1$, $D_2$, and $D_3$ are matrices of appropriate dimensions. We do not have a direct feedthrough matrix from $u$ to $y$. This can be handled via techniques as given in e.g. [16].

We would like to minimize the effect of the disturbance $w$ on the output $z$ by finding an appropriate control input $u$. We seek a stabilizing controller of the form:

$$\Sigma_F: \begin{cases} \sigma p = Kp + Ly, \\
u = Mp + Ny. \end{cases}$$

such that after applying the feedback $\Sigma_F$ to the system (3.1), the resulting closed-loop system is internally stable and has $H_\infty$ norm strictly less than 1. We shall derive necessary and sufficient conditions under which such a compensator exists. By $H^+$ we denote the Moore-Penrose generalized inverse of a matrix $H$.

Theorem 3.1: Consider the system (3.1). Assume the systems $(A, B, C_2, D_2)$ and $(A, E, C_1, D_1)$ have no invariant zeros on the unit circle. The following statements are equivalent:

(i) There exists a dynamic compensator $\Sigma_F$ of the form (3.2) such that the resulting closed loop system is internally stable and the closed loop transfer matrix $G_F$ satisfies $\|G_F\|_\infty < 1$.

(ii) There exist symmetric matrices $P \succeq 0$ and $Q \succeq 0$ such that

(a) We have $R > 0$ where

$$V := B^TPB + D_2^TD_2,$$

$$R := I - D_3D_3^T - E^TP + (E^TPB + D_2^TD_2)V^+(B^TPE + D_3^TD_3).$$

(b) $P$ satisfies the discrete algebraic Riccati equation:

$$P = A^TPA + C_2^T C_2 - \left( B^TPA + D_2^T C_2 \right) \left( E^TPA + D_3^T C_2 \right)^T G(P)^+ \left( B^TPA + D_2^T C_2 \right);$$

where

$$G(P) := \begin{pmatrix} D_2^TD_2 & D_2^TD_3 \\ D_3^TD_2 & D_3^TD_3 - I \end{pmatrix} + \left( B^T \right)^T P \begin{pmatrix} B & E \end{pmatrix}. \tag{3.4}$$
(c) All the zeros of the matrix pencil

\[
\begin{pmatrix}
zI - A & -B & -E \\
B^T PA + D_2^T C_2 & B^T PB + D_2^T D_2 & B^T PE + D_2^T D_3 \\
E^T PA + D_2^T C_2 & E^T PB + D_2^T D_2 & E^T PE + D_2^T D_3 - I
\end{pmatrix}
\] (3.5)

are inside or on the unit circle. Moreover, the number of invariant zeros on the unit circle, counting multiplicity, is equal to the number of invariant zeros of \((A, B, C_2, D_2)\) on the unit circle.

(d) We have \(S > 0\) where

\[
W := D_1 D_1^T + C_1 QC_1, \\
S := I - D_3 D_3^T - C_2 QC_2^T + (C_2 QC_2^T + D_3 D_3^T)W^+(C_1 QC_1^T + D_1 D_1^T).
\]

(e) \(Q\) satisfies the following discrete algebraic Riccati equation:

\[
Q = AQA^T + EE^T - \begin{pmatrix}
C_1 QA^T + D_1 E^T \\
C_2 QA^T + D_3 E^T
\end{pmatrix} H(Q)^+ \begin{pmatrix}
C_1 QA^T + D_1 E^T \\
C_2 QA^T + D_3 E^T
\end{pmatrix},
\] (3.6)

where

\[
H(Q) := \begin{pmatrix}
D_1 D_1^T & D_1 D_1^T \\
D_3 D_3^T & D_3 D_3^T - I
\end{pmatrix} + \begin{pmatrix}
C_1 \\
C_2
\end{pmatrix} Q \begin{pmatrix}
C_1^T \\
C_2^T
\end{pmatrix}.
\] (3.7)

(f) All the zeros of the matrix pencil

\[
\begin{pmatrix}
zI - A & AQC_1^T + ED_1^T & AQC_2^T + ED_3^T \\
-C_1 & C_1 QC_1^T + D_1 D_1^T & C_1 QC_2^T + D_1 D_3^T \\
-C_2 & C_2 QC_1^T + D_3 D_1^T & C_2 QC_2^T + D_3 D_3^T - I
\end{pmatrix}
\] (3.8)

are inside or on the unit circle. Moreover, the number of invariant zeros on the unit circle, counting multiplicity, is equal to the number of invariant zeros of \((A, E, C_1, D_1)\) on the unit circle.

(g) \(\rho(PPQ) < 1\).

Finally, for any invariant zero \(\lambda\) on the unit circle of the system \((A, B, C_2, D_2)\) or of the system \((A, E, C_1, D_1)\) there exists a matrix \(K\) such that \(\lambda I - A - BK C_1\) is invertible and

\[
\|(C_2 + D_2 K C_1)(\lambda I - A - BK C_1)^{-1}(E + K D_1) + D_2 KD_1\| < 1 \] (3.9)

Remarks:

(i) If the system \((A, B, C_2, D_2)\) is left-invertible then \(G(P)\) is invertible and we can replace the generalized inverse by a standard inverse. Moreover, in that case the rank condition is equal the requirement that the matrix

\[
A - \begin{pmatrix}
B & E
\end{pmatrix} G(P)^{-1} \begin{pmatrix}
B^T PA + D_2^T C_2 \\
E^T PA
\end{pmatrix}
\]
has all eigenvalues in the closed unit circle. Moreover, the number of invariant zeros on the unit circle, counting multiplicity, should be equal to the number of invariant zeros of \((A, B, C_2, D_2)\) on the unit circle.

Similar comments can be made for the dual algebraic Riccati equation in case \((A, E, C_1, D_1)\) is right-invertible.

(ii) The Riccati equation contains in general a generalized inverse. As shown in [17] we can find a suitable basis such that in this basis the Riccati equation reduces to a Riccati equation of lower dimension which has a standard instead of a generalized inverse.

(iii) If the systems \((A, B, C_2, D_2)\) and \((A, E, C_1, D_1)\) do not have zeros on the unit circle then clearly the last condition of part (ii) regarding invariant zeros on the unit circle becomes void. Moreover, condition (c) reduces to the requirement that the matrix pencil (3.5) has all zeros inside the unit circle. Similarly condition (d) reduces to the requirement that the matrix pencil (3.8) has all eigenvalues inside the unit circle.

(iv) In the state feedback case, we have \(C_1 = I\) and \(D_1 = 0\). Then it is easy to see that \(Q = EE^T\) satisfies conditions (d), (e), and (f). Hence part (ii) reduces to:

- The existence of a matrix \(P \geq 0\) satisfying \(E^TPE < I\) and conditions (b) and (c).
- For any invariant zero \(\lambda\) on the imaginary axis of the system \((A, B, C_2, D_2)\), there exists a matrix \(K\) such that \(\lambda I - A - BKc_1\) is invertible and (3.9) is satisfied.

(v) For the full-information case where \(C_1 = (I \quad 0)^T\) and \(D_1 = (0 \quad I)^T\) we have that \(Q = 0\) satisfies conditions (d), (e), and (f). Hence part (ii) reduces to:

- The existence of a matrix \(P \geq 0\) satisfying conditions (a), (b), and (c).
- For any invariant zero \(\lambda\) on the imaginary axis of the system \((A, B, C_2, D_2)\), there exists a matrix \(K\) such that \(\lambda I - A - BKc_1\) is invertible and (3.9) is satisfied.

4 Necessity of the continuous-time conditions

4.1 The state feedback control problem

First of all we note that if the measurement feedback problem is solvable then certainly the state feedback control problem must be solvable. In this section we will show that this implies the existence of a solution to the first quadratic matrix inequality.

It is well-known that we can find a preliminary state feedback \(u = Fx + v\) such that in a suitable basis the system matrices have a nice form:

\[
A + BF = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \quad C_2 + D_2F = \begin{pmatrix} C_2 & 0 \end{pmatrix},
\]

such that the eigenvalues of \(A_{22}\) are precisely the invariant zeros of the system \((A, B, C_2, D_2)\) on the imaginary axis. Note that the states associated to \(A_{22}\) do not affect the output and hence if we can make the \(H_\infty\) norm less than 1 for the original system then we must also be able to achieve an \(H_\infty\) norm less than 1 for the following reduced system:

\[
\Sigma_r : \begin{cases} \dot{x}_r = A_{11}x_r + B_1v + E_1w \\ z = C_{21}x_r + D_2v \end{cases}
\]
Note that the $H_{\infty}$ norm we can achieve for this system might actually be smaller because we dropped the requirement to stabilize the dynamics associated to $A_{22}$. Since the system $(A_{11}, B_1, C_{21}, D_2)$ does not have any invariant zeros on the imaginary axis we can apply the results from [16] and we obtain that there exists a matrix $P_1 \geq 0$ satisfying the following conditions:

- $F_r(P_1) := \begin{pmatrix} A_{11}^T P_1 + P_1 A_{11} + C_{21}^T C_{21} + P_1 E_1 E_1^T P_1 & P_1 B_1 + C_{21}^T D_2 \\ B_1^T P_1 + D_2^T C_{21} & D_2^T D_2 \end{pmatrix} \geq 0.$
- $\text{rank } F_r(P_1) = \text{normalrank} \left( sI - A_{11} \right)^{-1} B_1 + D_2,$
- All the zeros of the matrix pencil
  
  $\begin{pmatrix} sI - A_{11} - E_1 E_1^T P_1 & -B_1 \\ A_{11}^T P_1 + P_1 A_{11} + C_{21}^T C_{21} + P_1 E_1 E_1^T P_1 & P_1 B_1 + C_{21}^T D_2 \\ B_1^T P_1 + D_2^T C_{21} & D_2^T D_2 \end{pmatrix}$
  
  are in the open left half plane.

It is then easy to check that $P$ defined by

$$P := \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}$$

satisfies the conditions in part (ii) of theorem 2.1.

We would like to note that the above can not be proved straightforwardly using cheap-control arguments since it is hard to keep a check on the number of zeros on the imaginary axis. Finally, we can easily show using a dual argument that there exists a solution $Q \geq 0$ satisfying the dual quadratic matrix inequality $G(Q) \geq 0$ and the associated two rank conditions.

### 4.2 A system transformation

We again consider the system in the basis corresponding to (4.1). Consider the system $\Sigma_r$ but this time with a measurement equation:

$$\dot{x}_r = A_{11} x_r + B_1 v + E_1 w$$

$$y = C_{11} x_r + D_1 w$$

$$z = C_{21} x_r + D_2 v$$

where $C_1 = (C_{11} \ C_{12})$. The system $(A_{11}, B_1, C_{21}, D_2)$ has no invariant zeros on the imaginary axis. Hence we can study the first system transformation of [16, section 5.3].

We define the following system:

$$\Sigma_{P,r} : \begin{cases} \dot{x}_r = (A_{11} + E_1 E_1^T P_1) x_r + B_1 v + E_1 w \\ y = C_{11} x_r + D_1 w \\ z = C_{P,r} x_r + D_{P,r} v \end{cases}$$

8
where $C_{P,r}$ and $D_{P,r}$ are arbitrary matrices satisfying:

$$
\begin{pmatrix}
C_{P,r}^T & D_{P,r}^T
\end{pmatrix}
\begin{pmatrix}
C_{P,r} & D_{P,r}
\end{pmatrix} = F_r(P_1) \geq 0
$$

According to [16] there exists a system $\Sigma_{U,r}$ with inputs $w_u, v_u$ and outputs $z_u, y_u$ where:

$$
\begin{pmatrix}
z_u \\
y_u
\end{pmatrix} = U_r
\begin{pmatrix}
w_{P,r} \\
v
\end{pmatrix} =
\begin{pmatrix}
U_{11,r} & U_{12,r} \\
U_{21,r} & U_{22,r}
\end{pmatrix}
\begin{pmatrix}
w_u \\
v_u
\end{pmatrix}
$$

such that

- $\Sigma_{U,r}$ is stable and inner
- $U_{21,r}$ is invertible and has a stable inverse

Most importantly if we consider the following interconnections:

$$
\begin{align*}
& \Sigma_r \\
& \Sigma_{F}
\end{align*}
\quad
\begin{align*}
& \Sigma_{U,r} \\
& \Sigma_{P,r}
\end{align*}
\quad
\begin{align*}
& \Sigma_{F}
\end{align*}
$$

then the system on the left and the system on the right have the same transfer matrix while the only difference in internal dynamics is some additional, stable but uncontrollable, dynamics in the system on the right.

Next, we define the following system:

$$
\Sigma_p : \begin{cases}
\dot{x} = A_p x + Bu + Ew, \\
y = C_{1,p} x + D_1 w, \\
z = C_{2,p} x + D_p u,
\end{cases}
$$

where $A_p = A + EE^T P$, $C_{1,p} = C_1 + D_1 E^T P$, while $C_{2,p}$ and $D_p$ are arbitrary matrices satisfying

$$
\begin{pmatrix}
C_{2,p}^T \\
D_p^T
\end{pmatrix}
\begin{pmatrix}
C_{2,p} & D_p
\end{pmatrix} = F(P) \geq 0.
$$

Using the specific form of the system matrices as pointed out in (4.1) and the special form of $P$ as
given in (4.2) we can compare the following interconnections:

We find that the system on the left and the system on the right have the same transfer matrix while the only difference in internal dynamics is some additional, stable but uncontrollable, dynamics in the system on the right.

If we now apply Redheffer's lemma (see [12]) to the system on the right we obtain the following lemma:

Lemma 4.1: Given the systems (2.1) and (4.6). For any arbitrary controller $\Sigma_F$ of the form (2.2), the following statements are equivalent:

(i) $\Sigma_F$ stabilizes $\Sigma$ and yields a closed loop system with $H_\infty$ norm less than 1.

(ii) $\Sigma_F$ stabilizes $\Sigma_p$ and yields a closed loop system with $H_\infty$ norm less than 1.

However, we shall later need a particular corollary of the above lemma

Corollary 4.2: Let a matrix $P$ satisfy the conditions of part (ii) in theorem 2.1 such that, in the basis associated with (4.1), $P$ has the special form (4.2).

Let the systems (2.1) and (4.6) be given. For any arbitrary controller $\Sigma_F$ of the form (2.2), the following statements are equivalent:

(i) All poles of the interconnection of $\Sigma_F$ and $\Sigma$ are in the closed left half plane and the closed loop transfer matrix is stable and has $H_\infty$ norm less than 1.

(ii) All poles of the interconnection of $\Sigma_F$ and $\Sigma_p$ are in the closed left half plane and the closed loop transfer matrix is stable and has $H_\infty$ norm less than 1.

Note that in the above corollary we require a closed loop system which is input-output stable but we allow for pole-zero cancellations on the imaginary axis. This is precisely what happens if we ignore the zero-dynamics of $A_{22}$ and derive our controller on the basis of the reduced system (4.3).

Secondly note that we ask for a particular matrix $P$ which satisfies an additional condition besides the conditions of part (ii) of theorem 2.1. This is related to the problem that in contrast with the stabilizing solutions of the quadratic matrix inequality studied in [16], the semi-stabilizing, rank-minimizing solution of the quadratic matrix inequality need not be unique. However the existence of a semi-stabilizing, rank-minimizing solution of the quadratic matrix inequality guarantees (as we will see later) the existence of a semi-stabilizing, rank-minimizing solution of the quadratic matrix inequality satisfying the additional constraint (4.2).
4.3 The coupling condition

We have derived a semi-stabilizing, rank-minimizing solution $P$ of the state feedback quadratic matrix inequality. Using a dual argument, we have derived the existence of a semi-stabilizing, rank-minimizing solution $Q$ of the filter quadratic matrix inequality. Remains to prove the necessity of the coupling condition $\rho(PQ) < 1$.

We know that if a suitable controller exists for the original system $\Sigma$ then this controller also stabilizes $\Sigma_p$ and yields a closed loop $H_\infty$ norm strictly less than 1. Therefore, using the necessary conditions derived in subsection 4.1, we find that there exists solutions to the quadratic matrix inequalities associated with $\Sigma_p$.

For arbitrary matrices $X$ and $Y$ in $\mathbb{R}^{n \times n}$ we define the following matrices:

\[
\bar{F}(X) := \begin{pmatrix} 
A_p^T X + X A_p + C_{2,p}^T C_{2,p} + X E E^T X & X B + C_{2,p}^T D_p \\
B^T X + D_{1}^T C_{2,p} & D_{1}^T D_p 
\end{pmatrix},
\]

\[
\bar{G}(Y) := \begin{pmatrix} 
A_p Y + Y A_p^T + E E^T + Y C_{2,p}^T C_{2,p} Y & Y C_{1,p}^T + E D_{1}^T \\
C_{1,p} Y + D_{1} E^T & D_{1} D_{1}^T 
\end{pmatrix},
\]

\[
\bar{L}(X, s) := \begin{pmatrix} 
s I - A_p - E E^T X & -B 
\end{pmatrix},
\]

\[
\bar{M}(Y, s) := \begin{pmatrix} 
s I - A_p - Y C_{2,p}^T C_{2,p} & -C_{1,p} 
\end{pmatrix}.
\]

Moreover we define two new transfer matrices:

\[ \bar{G}_{ci}(s) := C_{2,p}(s I - A_p)^{-1} B + D_p, \]
\[ \bar{G}_{di}(s) := C_{1,p}(s I - A_p)^{-1} E + D_{1}. \]

Lemma 4.3: Let $P$ and $Q$ satisfy the conditions of theorem 2.1 except for possibly the coupling condition $\rho(PQ) < 1$. Then we have the following two results:

(i) $X := 0$ is a solution of the quadratic matrix inequality $\bar{F}(X) \succeq 0$ and the following two rank conditions are satisfied:

(a) rank $\bar{F}(X) = \text{normalrank} \bar{G}_{ci}$,
(b) All the zeros of the matrix pencil

\[
\begin{pmatrix} 
\bar{L}(X, s) \\
\bar{F}(X) 
\end{pmatrix}
\]

are in the closed left half plane. Moreover, the number of invariant zeros on the imaginary axis, counting multiplicity, is equal to the number of invariant zeros of $(A_p, B, C_{2,p}, D_p)$ on the imaginary axis.

(ii) We are looking for a matrix $Y$ which satisfies the quadratic matrix inequality $\bar{G}(Y) \succeq 0$ together with the following two rank conditions:
(a) \( \text{rank } \tilde{G}(Y) = \text{normal rank } \tilde{G}_{di} \),
(b) All the zeros of the matrix pencil

\[
\begin{pmatrix}
\tilde{M}(Y,s) & \tilde{G}(Y)
\end{pmatrix}
\]

are in the closed left half plane. Moreover, the number of invariant zeros, counting multiplicity, on the imaginary axis is equal to the number of invariant zeros of \((A_F, E, C_{1,F}, D_1)\) on the imaginary axis.

Such a matrix \( Y \) exists if, and only if, \( I - QP \) is invertible. Moreover, in this case, a solution is given by \( Y := (I - QP)^{-1} Q \). There exists a positive semi-definite solution \( Y \) if, and only if,

\[
\rho (PQ) < 1.
\]

**Proof**: Part (i) of this lemma is straightforward. For part (ii) we rely on the reduction technique described in subsection 4.1. However, since we basically have a dual problem, we either dualize the system \( \Sigma_F \) and apply the reduction technique directly or we dualize the reduction technique and apply it directly to the system \( \Sigma_F \). In any case we obtain a reduced without zeros on the imaginary axis and we can directly apply the results from [16].

### 4.4 Invariant zeros on the imaginary axis

Assume there exists a stabilizing controller with transfer matrix \( F \) which achieves a closed loop transfer matrix \( G_F \) with \( H_\infty \) norm strictly less than 1. Without loss of generality we assume that \( F \) does not have poles on the imaginary axis. Otherwise, we apply a small perturbation which, when sufficiently small, will preserve closed loop stability and the \( H_\infty \) norm bound. Since the \( H_\infty \) norm is less than 1 we find that for each \( \lambda \) on the imaginary axis, we have \( \|G_F(\lambda)\| < 1 \). Hence it is easy to see that, if \( \lambda \) is an invariant zero on the imaginary axis of \((A, B, C_2, D_2)\) or \((A, E, C_1, D_1)\) then \( K = F(\lambda) \) is such that \( \lambda - A - BKC_1 \) is invertible and such that (2.5) is satisfied. This completes the proof of the implication \( (i) \implies (ii) \) of theorem 2.1.

### 5 Necessity of the discrete-time conditions

#### 5.1 The state feedback control problem

We note that the discrete time \( H_\infty \) control problem with measurement feedback is solvable then the full-information \( H_\infty \) control problem is solvable. Here we allow a feedback of both the state and the disturbance.

We apply a technique which is completely analogous to the continuous-time. First we apply a preliminary state feedback so that in a suitable basis the system matrices have the form (4.1). But this time the eigenvalues of \( A_{22} \) correspond to the invariant zeros on the unit circle. We then note that the full-information \( H_\infty \) control problem for the original system is solvable then the full-information \( H_\infty \) control problem for the following reduced system is solvable:

\[
\Sigma_r : \begin{cases}
\sigma x_r = A_{11} x_r + B_1 v + E_1 w \\
z = C_{21} x_r + D_2 v + D_3 w
\end{cases}
\]
By applying the results from [17], we find the existence of a solution $P_1$ of the algebraic Riccati equation associated to this reduced system. We can then show that $P$, defined by (4.2) satisfies the conditions of part (ii) of theorem 3.1.

5.2 A system transformation

Our technique is completely similar to the continuous time results but this time we rely on results from [17]. Assume we have a solution $P \geq 0$ satisfying the conditions (a)-(c) of theorem 3.1. Moreover, assume that, with respect to the special basis associated to the decomposition (4.1), $P$ has the special form (4.2). We define the following transformed system:

$$
\Sigma_P : \begin{cases}
\sigma x_p &= A_p x_p + B u_p + E_p w_p, \\
y_p &= C_{1,p} x_p + D_{1,p} u_p, \\
z_p &= C_{2,p} x_p + D_{2,p} u_p + D_{3,p} w_p,
\end{cases}
$$

(5.1)

where,

$$
A_x := A - BV^\top \left[ B^T PA + D_2^2 C_2 \right], \\
C_x := C_2 - D_2 V^\top \left[ B^T PA + D_2^2 C_2 \right], \\
A_p := A + ER^{-1} \left[ E^T PA_x + D_3^2 C_x \right], \\
E_p := ER^{-1/2}, \\
C_{1,p} := C_1 + D_1 R^{-1} \left[ E^T PA_x + D_3^2 C_x \right], \\
C_{2,p} := (V^{1/2})^\top \left( B^T PA + D_2^2 C_2 + [B^T PE + D_2^2 D_3] R^{-1} [E^T PA_x + D_3^2 C_x] \right), \\
D_{1,p} := D_1 R^{-1/2}, \\
D_{2,p} := V^{1/2}, \\
D_{3,p} := (V^{1/2})^\top \left[ B^T PE + D_2^2 D_3 \right] R^{-1/2},
$$

with $R$ and $V$ as defined in theorem 3.1.

We can derive the following discrete-time analogues of lemma 4.1 and corollary 4.2:

**Lemma 5.1:** Given the systems (3.1) and (5.1). For any arbitrary controller $\Sigma_F$ of the form (3.2), the following statements are equivalent:

(i) $\Sigma_F$ stabilizes $\Sigma$ and yields a closed loop system with $H_\infty$ norm less than 1.

(ii) $\Sigma_F$ stabilizes $\Sigma$ and yields a closed loop system with $H_\infty$ norm less than 1.  \(\square\)

Like in the continuous time we will need a particular corollary of the above lemma:

**Corollary 5.2:** Let a matrix $P$ satisfy the conditions of part (ii) in theorem 3.1 such that, in the basis associated with (4.1), $P$ has the special form (4.2). Given the systems (3.1) and (5.1). For any arbitrary controller $\Sigma_F$ of the form (3.2), the following statements are equivalent:

13
(i) All poles of the interconnection of $\Sigma_F$ and $\Sigma$ are in the closed unit disc and the closed loop transfer matrix is stable and has $H_\infty$ norm less than 1.

(ii) All poles of the interconnection of $\Sigma_F$ and $\Sigma_p$ are in the closed unit disc and the closed loop transfer matrix is stable and has $H_\infty$ norm less than 1.

5.3 The coupling condition

We have derived a semi-stabilizing, solution $P$ of the state feedback Riccati equation. Using a dual argument, we have derived the existence of a semi-stabilizing solution $Q$ of the filter Riccati equation. Like in the continuous time the final step is to prove the necessity of the coupling condition $\rho(PQ) < 1$. We use the same technique as in the continuous time by investigating the solutions of the Riccati equations associated to the system $\Sigma_p$.

Since there exists a stabilizing controller for the system $\Sigma$ which yields an $H_\infty$ norm strictly less than 1, this controller has the same properties for $\Sigma_p$. Hence if we apply the results from subsection 5.1, we obtain the existence of semi-stabilizing solutions of the two algebraic Riccati equations associated with $\Sigma_p$. It is easy to check that 0 is a stabilizing solution of the state feedback algebraic Riccati equation. But the above also guarantees the existence of a matrix $Y \geq 0$ satisfying the following conditions

(i) $Y$ is such that $S_p > 0$ where

$$
W_p := D_{1,p}D_{1,p}^T + C_{1,p}Y C_{1,p}^T
$$

$$
S_p := I - D_{2,p}D_{3,p}^T - C_{2,p}Y C_{2,p}^T + (C_{2,p}Y C_{1,p}^T + D_{3,p}D_{1,p}^T) W_p^T (C_{1,p}Y C_{2,p}^T + D_{1,p}D_{3,p}^T).
$$

(ii) $Y$ satisfies the following discrete algebraic Riccati equation:

$$
Y = A_p Y A_p^T + E_p E_p^T - \left( \begin{array}{l} 
C_{1,p}YA_p^T + D_{1,p}E_p^T \\
C_{2,p}YA_p^T + D_{3,p}D_{1,p}^T
\end{array} \right)^T H_p(Y) \left( \begin{array}{l} 
C_{1,p}YA_p^T + D_{1,p}E_p^T \\
C_{2,p}YA_p^T + D_{3,p}D_{1,p}^T
\end{array} \right).
$$

where

$$
H_p(Y) := \left( \begin{array}{cc} 
D_{1,p}D_{1,p}^T & D_{1,p}D_{3,p}^T \\
D_{3,p}D_{1,p}^T & D_{3,p}D_{3,p}^T
\end{array} \right) + \left( \begin{array}{c}
C_{1,p} \\
C_{2,p}
\end{array} \right) Y \left( \begin{array}{c}
C_{1,p} \\
C_{2,p}
\end{array} \right)^T.
$$

(iii) All the zeros of the matrix pencil

$$
\begin{pmatrix}
2I - A & C_{1,p}YA_p^T + D_{1,p}E_p^T & C_{2,p}YA_p^T + D_{3,p}E_p^T \\
-C_{1,p} & C_{1,p}YA_p^T + D_{1,p}D_{1,p}^T & C_{1,p}YA_p^T + D_{1,p}D_{3,p}^T \\
-C_{2,p} & C_{2,p}YA_p^T + D_{3,p}D_{1,p}^T & C_{2,p}YA_p^T + D_{3,p}D_{3,p}^T - I
\end{pmatrix}
$$

are inside or on the unit circle. Moreover, the number of invariant zeros on the unit circle, counting multiplicity, is equal to the number of invariant zeros of $(A_p, E, C_{1,p}, D_1)$ on the unit circle.
The following lemma relates the existence and the solution of the above conditions to the conditions in theorem 3.1:

**Lemma 5.3:** Assume there exist matrices $P \geq 0$ and $Q \geq 0$ satisfying the conditions in part (ii) of theorem 3.1 except for possibly the coupling condition $\rho(PQ) < 1$. There exist a matrix $Y \geq 0$ satisfying the above conditions if and only if $\rho(PQ) < 1$. Moreover, in that case we have:

$$Y = (I - QP)^{-1} Q$$

**Proof:** The proof of this lemma goes along the same line as the proof of lemma 4.3 except that this time we rely on [17,19] instead of [16].

### 5.4 Invariant zeros on the unit circle

Assume there exists a stabilizing controller with transfer matrix $F$ which achieves a closed loop transfer matrix $G_F$ with $H_\infty$ norm strictly less than 1. Without loss of generality we assume that $F$ does not have poles on the unit circle. Otherwise, we apply a small perturbation which when sufficiently small will preserve closed loop stability and the $H_\infty$ norm bound. Since the $H_\infty$ norm is less than 1 we find that for each $\lambda$ on the unit circle, we have $\|G_F(\lambda)\| < 1$. Hence it is easy to see that, if $\lambda$ is an invariant zero on the unit circle of $(A, B, C_1, D_1)$ or $(A, E, C_1, D_1)$ then $K = F(\lambda)$ is such that $\lambda - A - BK C_1$ is invertible and such that (3.9) is satisfied. This completes the proof of the implication $(i) \implies (ii)$ of theorem 3.1.

### 6 Sufficiency of the continuous-time conditions

#### 6.1 Solutions of the quadratic matrix inequality with additional structure

Suppose we are given matrices $P$ and $Q$ satisfying the conditions of part (ii) of theorem 2.1. We would like to use a technique based on the system transformation used in subsection 4.2. However, this requires that in the bases associated to the factorization (4.1), we have that $P$ takes the special form (4.2). On the other hand, via simple examples we can see that a matrix $P$ satisfying the conditions of part (ii) of theorem 2.1 does not necessarily satisfy this additional property.

We are given a matrix satisfying the properties of part (ii) of theorem 2.1 which we will denote by $\bar{P}$. We will show that there exists another matrix $P$ satisfying the properties of part (ii) of theorem 2.1 while having the special form (4.2) in the bases associated with (4.1).

We define the matrix $L$ as the solution of the linear matrix inequality

$$S(L) := \begin{pmatrix} A^T L + LA + C_1^T C_2 & LB + C_1^T D_2 \\ B^T L + D_1^T C_2 & D_2^T & D_1^T & D_2 \\ \end{pmatrix} \geq 0$$

such that rank $S(L) = \text{normalrank } G_{ci}$ and if we define the matrix pencil:

$$\begin{pmatrix} sI - A & -B \\ A^T L + LA + C_1^T C_2 & LB + C_1^T D_2 \\ B^T L + D_1^T C_2 & D_2^T & D_1^T & D_2 \\ \end{pmatrix}$$
then it has all its zeros in the closed left half plane. This is called a semi-stabilizing solution of the linear matrix inequality and $L$ is uniquely defined. We know $L$ is the optimal cost of the following linear quadratic control problem (see e.g. [4, 21]):

$$
\xi^T L \xi = \inf_u \left\{ \int_0^\infty z^T(t)z(t) \, dt \mid x(t) \to 0 \, (t \to \infty) \right\}
$$

where

$$
\begin{align*}
\dot{x} &= Ax + Bu, \quad x(0) = \xi \\
z &= C_2 x + D_2 u
\end{align*}
$$

(6.1)

In a similar way we can show that $\bar{P}$ is the optimal cost of the following linear quadratic control problem:

$$
\xi^T \bar{P} \xi = \inf_u \left\{ \int_0^\infty z^T(t)z(t) + x(t)\bar{P}E E^T \bar{P} \, dt \mid x(t) \to 0 \, (t \to \infty) \right\}
$$

subject to the same dynamical system (6.1). This immediately implies that $\bar{P} \geq L$.

Next we define two differential games. We have (see e.g. [15]):

$$
\xi^T \bar{P} \xi = \sup_w \inf_u \left\{ \int_0^\tau z^T(t)z(t) - w^T(t)w(t) \, dt + x^T(\tau)\bar{P} x(\tau) \right\}
$$

for all $\tau > 0$ where

$$
\begin{align*}
\dot{x} &= Ax + Bu + Ew, \quad x(0) = \xi \\
z &= C_2 x + D_2 u
\end{align*}
$$

(6.2)

On the other hand we can consider the following optimization problem:

$$
\mathcal{J}(\xi, \tau) = \sup_w \inf_u \left\{ \int_0^\tau z^T(t)z(t) - w^T(t)w(t) \, dt + x^T(\tau)L x(\tau) \right\}
$$

subject to (6.2). Given that $\bar{P} \geq L$ we immediately find that $\mathcal{J}(\xi, \tau) \leq \bar{P}$. Moreover using the interpretation of $L$ as the cost function of a linear quadratic control problem we find:

$$
\mathcal{J}(\xi, \tau) = \sup_w \inf_u \left\{ \int_0^\infty z^T(t)z(t) - w^T(t)w(t) \, dt + x^T(\tau)L x(\tau) \right\}
$$

Moreover, using some results from [11]) we get that there exist a matrix-valued function $P_L$ such that:

$$
\mathcal{J}(\xi, \tau) = \xi^T P_L(\tau) \xi
$$

It is then immediate to note that $P_L(\tau)$ is increasing and bounded by $\bar{P}$. Hence there is a matrix $P$ such that: $P_L(\tau) \to P$ as $\tau \to \infty$. Obviously, $P \leq \bar{P}$.

Using the continuous-time equivalent of some techniques in [18] we can then show that $P$ satisfies the conditions of theorem 2.1. But we also obtain that $P$ has the special form (4.2) in the basis associated with (4.1). The latter can be seen by checking that $P_L(\tau)$ has this special form for all $\tau > 0$. Note that it is known from e.g. [4] that $L$ has this special form.
6.2 Existence of semi-stabilizing controllers

In subsection 4.3 we showed existence of solutions to the quadratic matrix inequalities for the system $I$. We can apply a transformation to $I$ which is dual to the transformation from $I$ to $I: p$. By lemma 4.3 we know $Y = (I - QP)^{-1} Q \geq 0$ satisfies $\tilde{G}(Y) \geq 0$. We factorize $\tilde{G}(Y)$:

$$\tilde{G}(Y) = \begin{pmatrix} E_{p,Q} & E_{p,Q}^T \\ D_{p,Q} & D_{p,Q}^T \end{pmatrix}. \quad (6.3)$$

where $E_{p,Q}$ and $D_{p,Q}$ are matrices of suitable dimensions. We define the following system:

$$\Sigma_{p,Q} : \begin{cases} \dot{x}_{p,Q} = A_{p,Q}x_{p,Q} + B_{p,Q}u_{p,Q} + E_{p,Q}w, \\ y_{p,Q} = C_{1,p}x_{p,Q} + D_{p,Q}w, \\ z_{p,Q} = C_{2,p}x_{p,Q} + D_{p}u_{p,Q}, \end{cases} \quad (6.4)$$

where

$$A_{p,Q} := A_p + YC_{2,p}^T C_{2,p},$$

$$B_{p,Q} := B + YC_{2,p}^T D_p.$$  

Since the transformation from $I$ to $\Sigma_{p,Q}$ is dual to the transformation from $I$ to $I$, we immediately obtain the following result (basically as a consequence of corollary 4.2 and a dual version of this corollary).

**Theorem 6.1:** Let the systems (2.1) and (6.4) be given. For any arbitrary controller $\Sigma_F$ of the form (2.2), the following statements are equivalent:

(i) All poles of the interconnection of $\Sigma_F$ and $\Sigma$ are in the closed left half plane and the closed loop transfer matrix is stable and has $H_\infty$ norm less than 1.

(ii) All poles of the interconnection of $\Sigma_F$ and $\Sigma_{p,Q}$ are in the closed left half plane and the closed loop transfer matrix is stable and has $H_\infty$ norm less than 1.

Next, we need to study the specific properties of this system $\Sigma_{p,Q}$ we have thus constructed. We can easily check that:

$$\text{rank} \begin{pmatrix} sI - A_{p,Q} & -B_{p,Q} \\ C_{2,p} & D_p \end{pmatrix} = n + \text{rank} \begin{pmatrix} C_{2,p} & D_p \end{pmatrix}, \quad \forall s \in \mathbb{C}^+ \quad (6.5)$$

and

$$\text{rank} \begin{pmatrix} sI - A_{p,Q} & -E_{p,Q} \\ C_{1,p} & D_{p,Q} \end{pmatrix} = n + \text{rank} \begin{pmatrix} E_{p,Q} & D_{p,Q} \end{pmatrix}, \quad \forall s \in \mathbb{C}^+. \quad (6.6)$$

We obtain the following theorem which is a special case of a result from [20]

**Theorem 6.2:** Suppose the system (6.4) satisfies (6.5) and (6.6). Then for each $\varepsilon > 0$ there exists a controller $\Sigma_F$, of the form (2.2), such that all poles of the interconnection of $\Sigma_F$ and $\Sigma$ are in the closed left half plane and the closed loop transfer matrix is stable and has $H_\infty$ norm less than $\varepsilon$.

This implies that we can certainly find a controller which is semi-stabilizing and which yields an $H_\infty$ norm less than 1 when applied to $\Sigma_{p,Q}$. Theorem 6.1 then implies that this controller has the same properties when applied to the original system $\Sigma$. 

17
6.3 Approximation with stabilizing controllers

In this section we will prove that the existence of a semi-stabilizing controller which yields an $H_\infty$ norm less than 1 in combination with the pointwise conditions (2.5), guarantees the existence of a stabilizing controller which yields an $H_\infty$ norm strictly less than 1.

We will use the Youla parameterization. Suppose the system (2.1) is given. Let $F$ and $H$ be such that $A + BF$ and $A + HC_1$ are asymptotically stable. The closed loop transfer matrix can be written as:

$$T_1 - T_2 QT_3$$  \hspace{1cm} (6.7)

where

$$T_1(z) = \begin{pmatrix} C_2 + D_2 F & -D_2 F \\ 0 & zI - A - BF \end{pmatrix} \begin{pmatrix} zI - A - BF & BF \\ zI - A - H C_1 \end{pmatrix}^{-1} \begin{pmatrix} E \\ E + HD_1 \end{pmatrix}$$

$$T_2(z) = (C_2 + D_2 F)(zI - A - BF)^{-1}B + D_2$$

$$T_3(z) = C_1(zI - A - HC_1)^{-1}(E + HD_1) + D_1$$

Note that all the $T_i$ are asymptotically stable. Moreover the invariant zeros of $[A, B, C_2, D_2]$ are precisely the invariant zeros of $T_2$ while the invariant zeros of $[A, E, C_1, D_1]$ are precisely the invariant zeros of $T_3$.

$Q$ is closely related to our controller. If we have a controller $\Sigma_F$ given by (2.2) then $Q$ is the transfer matrix of the interconnection $\Sigma_L \times \Sigma_F$ where

$$\Sigma_L : \begin{cases} \dot{x} = Ax + Bu - Hw \\ y = C_1x + w \\ z = Fx - u \end{cases}$$

It is easy to check that $\Sigma_F$ stabilizes $\Sigma$ if and only if $Q$ is stable. Moreover the interconnection has all poles in the closed left half plane if and only if $Q$ has all poles in the closed left half plane. Finally, we should note that for any $Q$ we can find a controller $\Sigma_F$ such that the interconnection $\Sigma_L \times \Sigma_F$ has transfer matrix $Q$. For details we refer to [2].

The above basically implies that we can concentrate on finding a stable matrix $Q$ such that the rational matrix (6.7) has $H_\infty$ norm strictly less than 1. Moreover, the results from the previous subsection guarantee that we can find a $Q$ with poles in the closed left half plane such that rational matrix (6.7) is stable and has $H_\infty$ norm strictly less than 1. Finally, the conditions (2.5) guarantee that for any zero $\lambda$ of $T_2$ or $T_3$ on the imaginary axis there exists a matrix $M$ such that

$$\|T_1(\lambda) - T_2(\lambda)MT_3(\lambda)\| < 1.$$

We have the following lemma:

Lemma 6.3 : Assume there exists a proper rational matrix $Q$ such that

- $Q$ has all its poles in the closed left half plane,
- The rational matrix (6.7) is stable and has $H_\infty$ norm strictly less than 1,
- The condition (2.5) is satisfied for any zero $\lambda$ of $T_2$ or $T_3$ on the imaginary axis.
In that case there exists a stable rational matrix $Q_s$ such that

$$\|T_1 - T_2 Q_s T_3\|_\infty < 1$$ \quad (6.8)

Note that the above lemma completes the proof of the implication (ii) $\Rightarrow$ (i) of theorem 2.1.

**Proof:** We assume the existence of a matrix $Q$ satisfying the conditions of the lemma. We factorize $T_2$ and $T_3$ as:

$$T_2 = T_{2,i} T_{2,r} T_{2,o}, \quad T_3 = T_{3,o} T_{3,r} T_{3,i},$$

where $T_{2,i}$ is stable and inner, $T_{2,r}$ is stable, square and has full rank and $T_{2,o}$ is stable and has a stable right-inverse. Similarly $T_{3,i}$ is stable and co-inner, $T_{3,r}$ is stable, square, and has full rank and $T_{3,o}$ is stable and has a stable left-inverse. We define $S = T_{2,o} Q T_{3,o}$ and note that $S$ has all its eigenvalues in the closed left half plane and:

$$\|T_1 - T_{2,i} T_{2,r} S T_{3,r} T_{3,i}\|_\infty < 1 \quad (6.9)$$

Note that it is sufficient to find a stable matrix $\tilde{S}$ satisfying (6.9) since a $\tilde{Q}$ satisfying the conditions of the lemma is then given by $\tilde{Q} = T_{2,o}^R \tilde{S} T_{3,o}^L$ where $T_{2,o}^R$ and $T_{3,o}^L$ denote a stable right and left-inverse of $T_{3,o}$ and $T_{2,o}$ respectively.

Note that we can find stable rational matrices $T_{2,c}$ and $T_{3,c}$ such that

$$\begin{pmatrix} T_{2,i} \\ T_{2,c} \end{pmatrix}, \quad \begin{pmatrix} T_{3,i} \\ T_{3,c} \end{pmatrix}$$

are both square and inner. In this way we can rewrite (6.9) in the following form:

$$\left\| \begin{pmatrix} T_{2,i} T_1 T_{3,i} - T_{2,r} S T_{3,r} & T_{2,i}^\sim T_1 T_{3,c}^\sim \\ T_{2,c} T_1 T_{3,i} & T_{2,c} T_1 T_{3,c}^\sim \end{pmatrix} \right\|_\infty < 1 \quad (6.10)$$

where $H^\sim(s) = H^T(-s)$. Note that $T_{2,c} T_1 T_{3,c}^\sim$ need not be stable but does not have poles on the imaginary axis. Moreover the above norm is the $L_\infty$ norm instead of the $H_\infty$ norm. There exists a spectral factor $R$ satisfying:

$$R^{-1} R = I - \begin{pmatrix} T_{3,i} T_{3,c}^\sim & T_{2,i} T_{3,c}^\sim \\ T_{3,c} T_{1} T_{2,c} & T_{2,c} T_1 T_{3,c}^\sim \end{pmatrix} \begin{pmatrix} T_{2,i} T_1 T_{3,i} \end{pmatrix}.$$  

Hence $R$ is stable and has a stable inverse. We can rewrite (6.10) in the following form:

$$\left\| \begin{pmatrix} T_{2,i} T_1 T_{3,i} R_1 - T_{2,r} S T_{3,r} R_1 & T_{2,i} T_1 T_{3,c} R_2 \\ T_{3,c} T_1 T_{2,c} R_1 & T_{2,c} T_1 T_{3,c} R_2 \end{pmatrix} \right\|_\infty < 1 \quad (6.11)$$

where $(R_1 \quad R_2)$ is the inverse of $R$. Next, we note there exists a spectral factor $V$ satisfying:

$$V V^\sim = I - T_{2,i} T_1 T_{3,i} R_2 R_2 T_{3,c} T_1 T_{2,i}$$

Again $V$ and $V^{-1}$ are stable. We rewrite (6.11) in the following form:

$$\left\| V^{-1} T_{2,i} T_1 T_{3,i} R_1 - V^{-1} T_{2,r} S T_{3,r} R_1 \right\|_\infty < 1 \quad (6.12)$$
Note that $V^{-1}T_{2,i}^\sim T_1 T_{3,i}^\sim R_1$ has no poles on the imaginary axis and hence we can find a square, stable, inner matrix $W$ such that

$$W V^{-1} T_{2,i}^\sim T_1 T_{3,i}^\sim R_1$$

is asymptotically stable. Finally we can transform 6.12 in the following form:

$$\| \tilde{T}_1 - \tilde{T}_2 \tilde{Q} \tilde{T}_3 \|_\infty < 1$$

(6.13)

where

$$\tilde{T}_1 := W V^{-1} T_{2,i}^\sim T_1 T_{3,i}^\sim R_1$$

$$\tilde{T}_2 := W V^{-1} T_{2,r}$$

$$\tilde{T}_3 := T_{3,r} R_1$$

Note that $\tilde{T}_1$, $\tilde{T}_2$ and $\tilde{T}_3$ are stable. Moreover $\tilde{T}_2$ and $\tilde{T}_3$ are square and have full rank. Finally the zeros on the imaginary axis of $T_2$ and $T_3$ are equal to the zeros of $\tilde{T}_2$ and $\tilde{T}_3$.

We will show that the existence of $\tilde{Q}$ satisfying (6.13) and having unstable poles on the imaginary axis implies the existence of another matrix $Q$, satisfying (6.13) which has strictly less unstable poles than $\tilde{Q}$. Using a recursive argument, this implies the existence of a stable matrix $Q_s$ satisfying (6.8).

Assume $j\omega$ is an unstable pole of $\tilde{Q}$. Since $\tilde{T}_2$ and $\tilde{T}_3$ are square and have full rank as a rational matrix there exists a rational matrix $K$ having only poles on $j\omega$ such that $T_1 - T_2 K T_3$ is stable and:

$$\left( \tilde{T}_1 - \tilde{T}_2 K \tilde{T}_3 \right) (j\omega) = 0.$$ 

Define for $\varepsilon > 0$

$$Q_{\text{new}}(s) = \frac{s - j\omega}{s - j\omega + \varepsilon} \tilde{Q}(s) + \frac{\varepsilon}{s - j\omega + \varepsilon} K(s)$$

(Note that in this way we obtain a complex controller. However, it is easy albeit technical to adapt this algorithm and obtain a real compensator.) Choose $\mu$ such that:

$$\mu < 1 - \| \tilde{T}_1 - \tilde{T}_2 \tilde{Q} \tilde{T}_3 \|_\infty$$

Then there exists $\delta$ such that for all we have:

$$\| (\tilde{T}_1 - \tilde{T}_2 K \tilde{T}_3)(s) \| < \delta$$

for all $s \in B_\delta$ where:

$$B_\delta := \{ s \in \mathbb{C}^0 \mid |s - j\omega| < \delta \}$$

Then for all $s \in B_\delta$ we find:

$$\| (\tilde{T}_1 - \tilde{T}_2 Q_{\text{new}} \tilde{T}_3)(s) \| < 1$$

On the other hand on $\mathbb{C}^0 \setminus B_\delta$ we have:

$$(\tilde{T}_1 - \tilde{T}_2 Q_{\text{new}} \tilde{T}_3)(s) \rightarrow (\tilde{T}_1 - \tilde{T}_2 \tilde{Q} \tilde{T}_3)(s)$$

uniformly in $s$ as $\varepsilon \to 0$. Therefore for $\varepsilon$ small enough we find:

$$\| \tilde{T}_1 - \tilde{T}_2 Q_{\text{new}} \tilde{T}_3 \|_\infty < 1$$
and
\[(\check{\mathbf{T}}_1 - \check{\mathbf{T}}_2 Q_{\text{new}} \check{\mathbf{T}}_3)(j\omega) = 0\] (6.14)

Next we will construct a rational matrix \(\check{\mathbf{Q}}_s\) satisfying (6.13) which does not have poles in \(j\omega\). We know there exists a matrix \(\check{\mathbf{K}}\) and \(\mu > 0\) such that:
\[\| (\check{\mathbf{T}}_1 - \check{\mathbf{T}}_2 \check{\mathbf{K}} \check{\mathbf{T}}_3)(j\omega) \| = 1 - \mu < 1\] (6.15)

Factorize \(Q_{\text{new}}\):
\[Q_{\text{new}} = Q_1 + \frac{1}{(s-j\omega+\varepsilon)^p} Q_2\]

such that \(Q_1\) has no poles in \(j\omega\) and \(Q_2\) is stable. Note that
\[\frac{1}{(s-j\omega+\varepsilon)^p} \check{\mathbf{T}}_2 Q_2 \check{\mathbf{T}}_3 = \check{\mathbf{T}}_2 Q_{\text{new}} \check{\mathbf{T}}_3 - \check{\mathbf{T}}_2 Q_1 \check{\mathbf{T}}_3\]
is stable. This is easily seen by noting that the left-hand side can only have unstable poles in \(j\omega\) while the right-hand side cannot have poles in \(j\omega\).

We define, for \(\varepsilon > 0\), \(Q_s\) by:
\[Q_s(s) = \frac{s-j\omega}{s-j\omega+\varepsilon} \left[ Q_1 + \frac{1}{(s-j\omega+\varepsilon)^p} Q_2(s) \right] + \frac{\varepsilon}{s-j\omega+\varepsilon} K\]

Obviously \(Q_s\) has no poles in \(j\omega\) and we did not introduce new poles on the imaginary axis. Hence the number of unstable poles has indeed been reduced. Remains to show that for \(\varepsilon\) small enough the \(H_{\infty}\) norm is strictly less than 1.

We have:
\[(\check{\mathbf{T}}_1 - \check{\mathbf{T}}_2 Q_s \check{\mathbf{T}}_3)(s) = \frac{s-j\omega}{s-j\omega+\varepsilon} (\check{\mathbf{T}}_1 - \check{\mathbf{T}}_2 Q_{\text{new}} \check{\mathbf{T}}_3)(s) + \frac{\varepsilon}{s-j\omega+\varepsilon} (\check{\mathbf{T}}_1 - \check{\mathbf{T}}_2 \check{\mathbf{K}} \check{\mathbf{T}}_3)(s)\]
\[+ \frac{s-j\omega}{s-j\omega+\varepsilon} \check{\mathbf{T}}_2(s) \left[ \frac{1}{(s-j\omega+\varepsilon)^p} - \frac{1}{(s-j\omega)^p} \right] Q_2(s) \check{\mathbf{T}}_3(s)\]

Because of (6.14) there exists \(\delta_1\) such that
\[\| (\check{\mathbf{T}}_1 - \check{\mathbf{T}}_2 Q_{\text{new}}(s) \check{\mathbf{T}}_3)(s) \| < \mu/3\]
on \(B_{\delta_1}\). Similarly, because of (6.15), there exists \(\delta_2\) such that:
\[\| (\check{\mathbf{T}}_1 - \check{\mathbf{T}}_2 \check{\mathbf{K}} \check{\mathbf{T}}_3)(s) \| < 1 - 2\mu/3\]
on \(B_{\delta_2}\). Finally, we have on \(B_1\):
\[\| \check{\mathbf{T}}_2(s) \left[ \frac{1}{(s-j\omega+\varepsilon)^p} - \frac{1}{(s-j\omega)^p} \right] Q_2(s) \check{\mathbf{T}}_3(s) \| = \left| \frac{(s-j\omega+\varepsilon)^p - (s-j\omega)^p}{(s-j\omega+\varepsilon)^p} \right| \| \check{\mathbf{T}}_2(Q_{\text{new}} - Q_1) \check{\mathbf{T}}_3 \| \leq M \left| \frac{\varepsilon^2}{s-j\omega+\varepsilon^2} \right|\]

Let \(\delta := \min \{ \delta_1, \delta_2, 1 \} \). We have:
\[\| (\check{\mathbf{T}}_1 - \check{\mathbf{T}}_2 Q_s \check{\mathbf{T}}_3)(s) \| = 1 - \mu/3 + M \left| \frac{\varepsilon^2(s-j\omega)}{(s-j\omega+\varepsilon^2)(s-j\omega+\varepsilon)} \right|\]

Therefore there exists \(\varepsilon_1\) such that for all \(\varepsilon < \varepsilon_1\) we have for all \(s \in B_\delta\):
\[\| (\check{\mathbf{T}}_1 - \check{\mathbf{T}}_2 Q_s \check{\mathbf{T}}_3)(s) \| < 1\]
On $\mathbb{C} \setminus B_3$ we have that
\[
(\tilde{T}_1 - \tilde{T}_2 Q_1 \tilde{T}_3)(s) \rightarrow (\tilde{T}_1 - \tilde{T}_2 Q_{\text{new}} \tilde{T}_3)(s)
\]
uniformly in $s$ and we know $\|\tilde{T}_1 - \tilde{T}_2 Q_{\text{new}} \tilde{T}_3\| < 1$. Therefore for $\epsilon < \epsilon_1$ small enough we have
\[
\|\tilde{T}_1 - \tilde{T}_2 Q_1 \tilde{T}_3\|_{\infty} < 1
\]
$Q_1$ has no poles in $j\omega$ and we did not introduce unstable poles. By repeating this argument we can remove all unstable poles and in the end we obtain a stable $Q_1$ satisfying (6.8).

\section{Sufficiency of the discrete-time conditions}

\subsection{Solutions of the quadratic matrix inequality with additional structure}

Suppose we are given matrices $P$ and $Q$ satisfying the conditions of part (ii) of theorem 3.1. We would like to use a technique based on the system transformation used in subsection 5.2. However, as in the continuous time, this requires that in the bases associated to the factorization (4.1), we have that $P$ takes the special form (4.2). On the other hand, also in the discrete time a matrix $P$ satisfying the conditions of part (ii) of theorem 2.1 does not necessarily satisfy this additional property.

We are given a matrix $P$ satisfying the properties of part (ii) of theorem 3.1 which we will denote by $\hat{P}$. Using the same arguments as in the continuous time, we can show that there exists another matrix $P$ which also satisfies the properties of part (ii) of theorem 3.1 while having the special form (4.2) in the bases associated with 4.1.

\subsection{Existence of semi-stabilizing controllers}

We use a similar technique as in the continuous time. In subsection 5.3 we showed existence of solutions to the Riccati equations for the system $\Sigma_p$. We then apply a transformation to $\Sigma_p$ and obtain the following system:

\[
\begin{align*}
\Sigma_{p,Y} : & \begin{cases} 
\sigma x_{p,Y} = A_{p,Y} x_{p,Y} + B_{p,Y} u_{p,Y} + E_{p,Y} w_{p,Y}, \\
y_{p,Y} = C_{1,p,Y} x_{p,Y} + D_{1,p,Y} w_{p,Y}, \\
z_{p,Y} = C_{2,p,Y} x_{p,Y} + D_{2,p,Y} u_{p,Y} + D_{3,p,Y} w_{p,Y},
\end{cases}
\end{align*}
\tag{7.1}
\]

where

\[
\begin{align*}
\tilde{Z} & : = A_p Y C_{2,p}^T + E_p D_{1,p}^T - (A_p Y C_{1,p}^T + E_p D_{1,p}^T) W_p^{-1} (C_{1,p} Y C_{2,p}^T + D_{1,p} D_{3,p}^T), \\
A_{p,Y} & : = A_p + \tilde{Z} S_p^{-1} C_{2,p}, \\
B_{p,Y} & : = B + \tilde{Z} S_p^{-1} D_{2,p}, \\
E_{p,Y} & : = (A_p Y C_{1,p}^T + E_p D_{1,p}^T) W_p^{-1/2} + \tilde{Z} S_p^{-1} (C_{2,p} Y C_{1,p}^T + D_{3,p} D_{1,p}^T) W_p^{-1/2}, \\
C_{2,p,Y} & : = S_p^{-1/2} C_{2,p}, \\
D_{1,p,Y} & : = W_p^{1/2}, \\
D_{2,p,Y} & : = S_p^{-1/2} D_{2,p}, \\
D_{3,p,Y} & : = S_p^{-1/2} (C_{2,p} Y C_{1,p}^T + D_{3,p} D_{1,p}^T) W_p^{-1/2},
\end{align*}
\]
and $W_p > 0$ and $S_p > 0$ as defined in subsection 5.3.
Since the transformation from $\Sigma_p$ to $\Sigma_{p,Y}$ is dual to the transformation from $\Sigma$ to $\Sigma_p$, we immediately obtain the following result (basically as a consequence of corollary 5.2 and a dual version of this corollary).

**Theorem 7.1**: Let the systems (3.1) and (7.1) be given. For any arbitrary controller $\Sigma_F$ of the form (3.2), the following statements are equivalent:

(i) All poles of the interconnection of $\Sigma_F$ and $\Sigma$ are in the closed unit circle and the closed loop transfer matrix is stable and has $H_\infty$ norm less than 1.

(ii) All poles of the interconnection of $\Sigma_F$ and $\Sigma_{p,Y}$ are in the closed unit circle and the closed loop transfer matrix is stable and has $H_\infty$ norm less than 1.

Next, we need to study the specific properties of this system $\Sigma_{p,Y}$ we have thus constructed. We can easily check that for all $Z$ outside the unit circle:

$$\text{rank} \begin{pmatrix} zI - A_{p,Y} & -B_{p,Y} \\ C_{2,p,Y} & D_{2,p,Y} \end{pmatrix} = n + \text{rank} \begin{pmatrix} C_{2,p,Y} & D_{2,p,Y} \end{pmatrix},$$

and

$$\text{rank} \begin{pmatrix} zI - A_{p,Y} & -E_{p,Y} \\ C_{1,p} & D_{1,p,Y} \end{pmatrix} = n + \text{rank} \begin{pmatrix} E_{p,Y} \\ D_{1,p,Y} \end{pmatrix}. \quad (7.2)$$

Moreover,

$$\text{rank} \begin{pmatrix} C_{2,p,Y} & D_{2,p,Y} \end{pmatrix} = \text{rank} D_{2,p,Y}, \quad \text{rank} \begin{pmatrix} E_{p,Y} \\ D_{1,p,Y} \end{pmatrix} = \text{rank} D_{1,p,Y}.$$

and, finally, there exists a matrix $T$ such that

$$D_{3,p,Y} + D_{2,p,Y}TD_{1,p,Y} = 0.$$

We then obtain the following theorem which is an adaption of results in [20]

**Theorem 7.2**: Let the system (7.1) as constructed above be given. Then for each $\varepsilon > 0$ there exists a controller $\Sigma_F$, of the form (3.2), such that all poles of the interconnection of $\Sigma_F$ and $\Sigma$ are in the closed unit circle and the closed loop transfer matrix is stable and has $H_\infty$ norm less than $\varepsilon$. 

23
7.3 Approximation with stabilizing controllers

In this section we will prove that the existence of a semi-stabilizing controller which yields an $H_{\infty}$ norm less than 1 in combination with the pointwise conditions (3.9), guarantees the existence of a stabilizing controller which yields an $H_{\infty}$ norm strictly less than 1.

We will use the Youla parameterization. Suppose the system (3.1) is given. Like in the continuous time, the closed loop transfer matrix can be written as:

$$T_1 - T_2 QT_3$$

(7.4)

where

$$T_1(z) = \begin{pmatrix} C_2 + D_2 F & -D_2 F \end{pmatrix} \begin{pmatrix} zI - A - BF & BF \\ 0 & zI - A - HC_1 \end{pmatrix}^{-1} \begin{pmatrix} E \\ E + HD_1 \end{pmatrix} + D_3$$
$$T_2(z) = (C_2 + D_2 F)(zI - A - BF)^{-1} B + D_2$$
$$T_3(z) = C_1(zI - A - HC_1)^{-1}(E + HD_1) + D_1$$

However, this time $F$ and $H$ be such that $A + BF$ and $A + HC_1$ are asymptotically stable in the discrete time sense, i.e. all its eigenvalues are inside the unit circle. Note that then all the $T_i$ are asymptotically stable. Finally, the invariant zeros of $[A, B, C_2, D_2]$ are precisely the invariant zeros of $T_2$ while the invariant zeros of $[A, E, C_1, D_1]$ are precisely the invariant zeros of $T_3$. $Q$ is closely related to our controller. If we have a controller $\Sigma_F$ given by (3.2) then $Q$ is the transfer matrix of the interconnection $\Sigma_L \times \Sigma_F$ where

$$\Sigma_L : \begin{cases} \sigma x = Ax + Bu - Hw \\ y = C_1 x + w \\ z = Fx - u \end{cases}$$

It is easy to check that $\Sigma_F$ stabilizes $\Sigma$ if and only if $Q$ is stable. Moreover the interconnection has all poles in the closed unit circle if and only if $Q$ has all poles in the closed unit circle. Finally, we should note that for any $Q$ we can find a controller $\Sigma_F$ such that the interconnection $\Sigma_L \times \Sigma_F$ has transfer matrix $Q$.

The above basically implies that we can concentrate on finding a stable matrix $Q$ such that the rational matrix (6.7) has $H_{\infty}$ norm strictly less than 1. Moreover, the results from the previous subsection guarantee that we can find a $Q$ with poles in the closed left unit circle such that rational matrix (7.4) is stable and has $H_{\infty}$ norm strictly less than 1. Finally, the conditions (3.9) guarantee that for any zero $\lambda$ of $T_2$ or $T_3$ on the unit circle there exists a matrix $M$ such that

$$\|T_1(\lambda) - T_2(\lambda) QT_3(\lambda)\| < 1.$$ 

We have the following lemma, the proof of which is completely identical to the proof of lemma 7.3.

**Lemma 7.3**: Assume there exists a proper rational matrix $Q$ such that

- $Q$ has all its poles in the closed unit circle,
- The rational matrix (7.4) is stable and has $H_{\infty}$ norm strictly less than 1,
- The condition (3.9) is satisfied for any zero $\lambda$ of $T_2$ or $T_3$ on the unit circle.
In that case there exists a stable rational matrix \( Q_s \) such that

\[ \| T_1 - T_2 Q_s T_3 \|_\infty < 1 \]

Note that the above lemma completes the proof of the implication (ii) \( \Rightarrow \) (i) of theorem 3.1.

8 Conclusion

In this paper we have derived necessary and sufficient conditions for the existence of suitable \( H_\infty \) controllers. Note that a cheap control argument can be used to determine such a suitable compensator. This immediately shows for instance that we can suffice with controllers of the same complexity as the system. However, the Riccati equations are in general very stiff with respect to the cheap control parameter \( \varepsilon \) which might cause problems.

The main contribution of this paper is in our view the nice analytical expressions. We believe them to very powerful analysis tools. Already in [9] we were able to make effective use of these conditions (it was actually the motivation to derive the results in this paper).

References


