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A Problem of Optimal Control

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1. Introduction

Let $S$ be a compact subset of the open first quadrant of the $xy$-plane, i.e.
\[ x > 0, \quad y > 0 \quad \text{for all} \quad (x, y) \in S. \quad (1.1) \]
Since $S$ is compact, this implies that there exist positive numbers $a, b, A, B$ such that
\[ a \leq x \leq A, \quad b \leq y \leq B \quad \text{for all} \quad (x, y) \in S. \quad (1.2) \]
We shall consider the following system of linear differential equations
\[ \frac{du}{dt} = -\frac{1}{\eta(t)} v, \quad \frac{dv}{dt} = \xi(t) u. \quad (1.3) \]
with initial condition $u(0) = 1, v(0) = 0$.

Here $\xi(t)$ and $\eta(t)$ (referred to as "control functions") are subject to the constraint
\[ (\xi(t), \eta(t)) \in S \quad \text{for all} \quad t \geq 0. \quad (1.4) \]
The question to be considered in this paper is to find $\xi$ and $\eta$ that optimalize for large values of $T$. The answer will be that there is a number $\alpha > 0$ such that the optimal value of $J_T$ (under the constraint (1.4)) has the form $e^{\alpha T + o(T)}$. Moreover we shall show how to calculate $\alpha$. It will turn out that $\alpha > 0$, apart from a trivial special case (see example (c) at the end of this introduction).

The number $\alpha$ is equal to the maximum in the following problem: Find $\xi, \eta$ such that (1.4) holds, and such that $P/Q$ is maximal. Here $Q$ denotes the smallest positive number for which $v(Q) = 0$ ($u, v$ represent the solution of (1.3) with $u(0) = 1, v(0) = 0$), and $P = \log |u(Q)|$. In formula
\[ \alpha = \sup_{(\xi, \eta) \in S} Q^{-1} \log |u(Q)|. \quad (1.5) \]
In a previous paper ([1], Theorem 2) we showed in a similar situation how (1.5) leads to the optimum $e^{\int_{0}^{T} f_0(t) dt}$ of $f_T$. We shall not repeat these (comparatively simple) arguments and we shall restrict ourselves here to the determination of $x$ from (1.5).

We shall make a few comments.

1. Equations (1.3) are somewhat unsymmetrical. Needless to say, if we replace $\eta^{-1}(t)$ by $\lambda(t)$, $\xi(t) \eta^{-1}(t)$ by $\mu(t)$, then $(\lambda, \mu)$ is again restricted to a compact subset of the first quadrant. We did not adopt this more symmetric presentation since the unsymmetric form (1.3) is easier for explaining our method and results.

2. We mention some special cases:

   (a) The case where $S$ is the horizontal line segment defined by

   \[ 1 - b \leq \xi \leq 1 + b, \quad \eta = 1 \]  

   (where $b$ is a constant, $0 < b < 1$) is the one considered in [1]. There we studied the second order equation $u'' + (1 + f(t)) u = 0$, with $u(0) = 1$, $u'(0) = 0$, $-b \leq f(t) \leq b$. Putting $u' = -v$, we obtain the form (1.3).

   In the introduction of [1] it was stated that the optimization problem connected with $u'' + (1 + f(t)) u = 0$, $-b \leq f(t) \leq b$ finds its physical interpretation in the problem of raising the amplitude of the oscillations when standing on a swing. In this interpretation the author was mistaken, for altering the length of the pendulum in the course of its movement introduces a Coriolis acceleration that may not be neglected. A correct physical interpretation would be the case of the torsional pendulum, if we are assumed to be able to control the moment of inertia between given limits. We get this equation $u'' + (1 + f(t)) u = 0$ if we put the angular momentum equal to $u$.

   (b) A correct formulation of the problem of the swing leads to equations of the form (1.3). Let $l(t)$ be the variable length of the pendulum, and allow $l(t)$ to vary between the positive constants $l_1, l_2$ ($0 < l_1 \leq l_2$). Denoting the angle between the pendulum and the vertical by $\theta$, and the angular momentum by $u$, we have

   \[ u' = -mg l \sin v, \quad v' = m^{-1} l^{-2} u, \]  

   where $m$ is the mass, and $g$ the gravitational constant. If we linearize, replacing $\sin v$ by $v$, we obtain (1.3), where $S$ is a piece of a third degree curve:

   \[ \xi = (mg)^{-1} \lambda^3, \quad \eta = (mg)^{-1} \lambda \]  

   \[ (l_2^{-1} \leq \lambda \leq l_1^{-1}). \]  

   (c) Next we consider the case that $S$ is a piece of a vertical line segment, i.e.

   \[ S: \xi = \xi_0, \quad \eta_1 \leq \eta \leq \eta_2, \]  

   \[ (1.9) \]
where $\xi_0$, $\eta_1$, $\eta_2$ are positive constants. It is easy to see that now $\xi_0u^2 + v^2$ is constant, whence $\xi_0u^2 + v^2 = \xi_0$ for all $t$. So if $Q$ is the first positive zero of $v$, we have $u(Q) = 1$, for every function $\eta$ satisfying (1.9). Hence $\alpha = 0$.

If, however, $S$ is not a part of a vertical line, i.e., if (3.1) holds, we shall show that the control can be used in order to get a positive effect, that is, we shall show that $\alpha > 0$.

2. **Optimization of $P - \lambda Q$**

As in the Introduction, let $S$ be a compact subset of the first quadrant, let $u$ and $v$ satisfy (1.3), with $u(0) = 1$, $v(0) = 0$, let $Q$ be the least positive number with $v(Q) = 0$, and let $P = \log u(Q)$. $P$ and $Q$ depend on the control functions $\xi, \eta$.

If $\lambda$ is any positive constant we wish to determine the maximum of $P - \lambda Q$ under the constraint (1.4).

Introducing polar coordinates $u = r \cos \varphi$, $v = r \sin \varphi$, we obtain from (1.3) (the dash denotes differentiation with respect to $t$):

$$\frac{r'}{r} = \frac{\xi - 1}{\eta} \sin \varphi \cos \varphi, \quad \varphi' = \frac{1}{\eta} \sin^2 \varphi + \frac{\xi}{\eta} \cos^2 \varphi.$$ 

We have $\varphi = 0$ at $t = 0$, $\varphi = \pi$ at $t = Q$. Hence

$$P = \int_0^Q \frac{r'}{r} \, dt = \int_0^\pi \frac{r'}{r} \, d\varphi, \quad Q = \int_0^\pi \frac{d\varphi}{\varphi'}.$$ 

Introducing $q = \tan \varphi$ we finally obtain

$$P = \int_{-\infty}^{\infty} \frac{(\xi - 1)q}{\xi + q^2} \, dq, \quad Q = \int_{-\infty}^{\infty} \frac{\eta dq}{\xi + q^2}. \quad (2.1)$$

Since $q' > 0$, the relation between $t$ and $q$ is one-to-one. Therefore $\xi$ and $\eta$ can be as well described as arbitrary functions of $q$, subject to $(\xi, \eta) \in S$ for all $q$.

We easily obtain

$$P - \lambda Q = \int_{-\infty}^{\infty} \left( \frac{q}{1 + q^2} - \frac{\lambda \eta + q}{\xi + q^2} \right) \, dq.$$ 

Obviously $P - \lambda Q$ is maximal if $(\lambda \eta + q)/(\xi + q^2)$ is minimal for every separate value of $q$. The point $(- q^2, \lambda^{-1}q)$ lies in the closed left half-plane, and $S$ is a compact subset of the open right half-plane. So for every $q$ the minimum

$$\min_{(\xi, \eta) \in S} \frac{\lambda \eta + q}{\xi + q^2} \quad (2.2)$$
is attained at a certain point $\xi_\lambda(q), \eta_\lambda(q)$ of $S$. Geometrically (see Fig. 1), this is the point of contact of the lower tangent drawn from the point $(-q^2, -\lambda^{-1}q)$ to $S$. If $q$ varies the point $(-q^2, -\lambda^{-1}q)$ describes a parabola.

If the boundary of $S$ contains a straight line segment, it may happen that there is no unique point of contact. In that case we take $\xi_\lambda(q), \eta_\lambda(q)$ as one of the points of contact, chosen arbitrarily. Since there are at most countably many line segments in the boundary, this arbitrary choice has no influence on our integrals.

The functions $\xi_\lambda(q), \eta_\lambda(q)$ are easily seen to have bounded total variation on $-\infty < q < \infty$.

We shall write

$$P_\lambda = \int_{-\infty}^{\infty} \frac{(\xi_\lambda(q) - 1)}{\xi_\lambda(q) + q^2} \frac{dq}{q^2 + 1}, \quad Q_\lambda = \int_{-\infty}^{\infty} \frac{\eta_\lambda(q)}{\xi_\lambda(q) + q^2} dq.$$
3. Optimization of $P/Q$

We shall assume that $S$ has positive width, i.e.

$$m_2 - m_1 > 0$$ (3.1)

where

$$m_1 = \min_{(\xi, \eta) \in S} \xi, \quad m_2 = \max_{(\xi, \eta) \in S} \xi.$$

If $q$ is a fixed positive number then we have (see (2.2)) $\xi_{\lambda} \to m_2$ if $\lambda \to 0$; if $q$ is negative, however, then $\xi_{\lambda} \to m_1$ if $\lambda \to 0$. It follows (e.g., by Lebesgue's theorem on dominated convergence) that

$$\lim_{\lambda \to 0} P_\lambda = \int_{-\infty}^{0} \frac{(m_1 - 1) q dq}{m_1 + q^2} + \int_{0}^{\infty} \frac{(m_2 - 1) q dq}{m_2 + q^2} = (m_2 - m_1) \int_{0}^{\infty} \frac{dq}{(m_1 + q^2)(m_2 + q^2)} > 0.$$

As $Q_{\lambda}$ is bounded for $0 < \lambda < \infty$, we infer that $P_\lambda - \lambda Q_{\lambda}$ has a positive limit if $\lambda \to 0$. On the other hand, if $\lambda \to \infty$ then $P_\lambda - \lambda Q_{\lambda}$ tends to $-\infty$ since $P_\lambda$ is bounded and

$$Q_{\lambda} \geq \int_{-\infty}^{\infty} b(A + q^2)^{-1} dq > 0$$

(see (1.2)) for all $\lambda > 0$.

Remarking that $P_\lambda - \lambda Q_{\lambda}$ is a continuous function of $\lambda$, we now infer that there exists a number $\alpha$ with

$$\alpha > 0, \quad P_\lambda - \alpha Q_{\lambda} = 0.$$ (3.2)

It follows from Section 2 that for every admissible pair of control functions $\xi, \eta$ we have

$$P - \alpha Q \leq P_\lambda - \alpha Q_{\lambda},$$ (3.3)

whence by (3.2)

$$P - \alpha Q \leq 0.$$

This means

$$\frac{P}{Q} \leq \alpha = \frac{P_\lambda}{Q_{\lambda}},$$

and since the control functions $\xi_\alpha, \eta_\alpha$ are also admissible, we infer that

$$\alpha = \max_{(\xi, \eta) \in S} \left( \frac{P}{Q} \right);$$ (3.4)
that is, our \( \alpha \) is the one promised in the introduction (see (1.5)). At the same time we have obtained the optimal control functions \( \xi_\alpha(q) \), \( \eta_\alpha(q) \). If we want to have them as functions of the original variable \( t \), we have to integrate

\[
dt = \frac{\eta_\alpha(q) \, dq}{\xi_\alpha(q) + q^2},
\]

and finally we get \( u^2 + v^2 \) as a function of \( q = u/v \) by

\[
\frac{1}{2} \, d \log (u^2 + v^2) = \left( \frac{\xi_\alpha(q) - 1}{\xi_\alpha(q) + q^2} \right) \frac{dq}{q^2 + 1}.
\]

It is not immediately trivial from (3.2) that \( \alpha \) is uniquely determined, but this fact is obvious from (3.4).

4. Final Remarks

It is obvious that the solution of our control problem depends only on the boundary of the convex hull of \( S \). It even depends only on the lower part of that boundary (see the heavy line in Fig. 1).

In the case of the swing \( S \) is a piece of a curve which is convex upwards (see (1.8)). It follows that the point \( \xi_\alpha \), \( \eta_\alpha \) either coincides with the left endpoint or with the right end-point of this curve. That is, the control is of the so-called "bang-bang" type.

We have the same bang-bang situation in the case of (1.6). There the problem is computationally simpler: the fact that the line connecting the endpoints is horizontal implies that one of its points of intersection with the parabole lies at infinity, i.e., corresponds with \( q = \infty \), \( u = 0 \). This means that the control function \( f \) has to switch to \(+ b\) a moment before \( v = 0 \), whereas it switches back to \(- b\) exactly at the point \( u = 0 \). In the case of the swing (\( S \) described by (1.8)), a similar "advanced ignition" has to be applied a moment before \( v = 0 \) as well as a moment before \( u = 0 \).

Reference