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Published: 01/01/1978

Document Version
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

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Citation for published version (APA):

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Download date: 18. Dec. 2018
Memorandum COSOR M78-09

Repetitive schemes with performance bounds
for the economic-lot scheduling
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by

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Eindhoven, April 1978
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Summary: In this paper the deterministic economic-lot scheduling problem is reconsidered for a single machine producing two homogeneous groups of products. It has been proved that so-called repetitive schemes are very efficient for such problems. In this paper we will consider the extra difficulty of set-up times which interfere critically with the optimal repetitive production scheme. It will be shown how a modified repetitive scheme can be found which is still very efficient.

1. Introduction: In this paper we will consider the problem of determining lot sizes and scheduling production periods for several products on a single machine. For constant and deterministic demands it has been proved in [2], that under certain conditions, so-called repetitive production schemes are nearly optimal. The most important aspects of the approach in [2] are: a) an optimal repetitive scheme can be constructed easily, b) estimates are obtained for the deviation from the optimum of the average costs for the constructed scheme.

In this paper we will try to weaken the conditions of [2] in order to incorporate the situation in which set-up times make the optimal repetitive scheme infeasible. In fact it will be shown how a modified repetitive scheme can be constructed (section 2) and how this modified scheme performs (section 3). It will appear that the optimal modified repetitive scheme is (under certain conditions to be specified below) never more than 3% more expensive than necessary and usually much less.

In the rest of this introductory section we will describe the model and present a short review of repetitive schemes.

We will restrict attention to the situation with two homogeneous groups of products, since this situation is the most essential one for performance studies and moreover it is the most clear situation. Let us first specify the assumptions:

a) at any time instant only one of m+n products can be produced on the available machine (there are m products in the first group and n in the second).

b) the demand rates d, e for the products in the first and second group respectively are deterministic and constant in time. The same holds for respective production rates p and r.

c) any production cycle for a product in the first group requires a set-up cost $F_0$ and a set-up time $s_0$. For the products in the second group the corresponding data are $G_0$ and $t_0$. 
d) inventory costs per unit of time are \( u,v \) for a unit of a product from the first and second group respectively.

e) the production level should guarantee a service level \( \beta, \gamma \ (0 < \beta, \gamma \leq 1) \) for a product from the first and second group respectively. So, \( \beta \) is the minimally required fraction of time with a positive inventory for any of the products from the first group individually.

The aim is to construct a production schedule (i.e. lot sizes and production instants) with minimal costs per time unit.

In [2] the situation has been studied with \( s_0 = t_0 = 0 \), at least with set-up times which don't essentially interfere. For that situation the so-called repetitive schemes - which generalize pure rotational schemes - have been introduced and analyzed in [2].

In this type of problem the difficult point in finding optimal production schemes is that optimal scheduling and optimal determination of lot-sizes is required simultaneously. Because of this difficulty it is an interesting approach to refrain from optimality and work heuristically. Usually (see e.g. Doll and Whybark [1]), heuristic procedures start by determining sensible lot sizes and proceed by constructing schedules with these given cycle times. In such procedures the scheduling part remains rather cumbersome and it is very difficult to obtain useful assertions on the quality of the schedules constructed in such a way. Especially for comparison with less smooth situations it is essential to have such assertions. This can be obtained in a way by proceeding the other way around: consider only simply structured production schedules and determine the optimal one within this class. If the structure is chosen sufficiently simple, then it may be possible to optimize simultaneously with respect to lot sizes and scheduling. Moreover, the more structure is required for the schedule, the more likely it becomes that indications can be obtained about the deviation from the optimum. In [2] it has been shown (essentially for the case \( s_0 = t_0 = 0 \)) that a restriction to repetitive schemes works well in either respect.

Now suppose for the rest of this section: \( s_0 = t_0 = 0 \). For this situation we will give a short review of repetitive schemes in order to use the results in the other sections.

A \( k \)-repetitive scheme is a production schedule prescribing cyclic production for all products in such a way, that the products of the second group are produced \( k \) times as often as the products of the first group. All products in the same group have the same cycle time. If this cycle time is \( T \) for the products in the first group, then it is \( Tk^{-1} \) for the products in the second group and after \( T \) time units the whole schedule starts again (see the figure for an example).
fig.: a 2-repetitive scheme for $m = 4$, $n = 3$

The optimal $k$-repetitive scheme is obtained by minimizing the costs per unit of time as a function of $T$. The function to be minimized is

$$(1.1) \quad [F + kG + \frac{1}{2} \xi T^2 + \frac{1}{4} n T^2 k^{-2} ] T^{-1},$$

where $F := m F_0$, $G := n G_0$, $\xi := m s^2 (1 - \frac{2}{p}) du$, $n := n \gamma^2 (1 - \frac{2}{e}) ev$.

This minimization gives a cycle length $T(k)$ with average costs $K(k)$ for the optimal $k$-repetitive scheme:

$$(1.2) \quad T(k) = \sqrt{2} (F + kG)^{\frac{1}{2}} (\xi + k^{-1} n)^{-\frac{1}{2}},$$

$$(1.3) \quad K(k) = \sqrt{2} (F + kG)^{\frac{1}{2}} (\xi + k^{-1} n)^{\frac{1}{2}}.$$

Now it may be determined which choice of $k$ is the best. It may be expected that this depends on the ratio of the optimal cycle times of the individual products of both groups (if only one product was to be made). These optimal cycle times for the individual products can easily be obtained via the Wilson or Camp lot size formula:

$$(1.4) \quad T_a = \left( \frac{2F}{\xi} \right)^{\frac{1}{2}}, \quad T_b = \left( \frac{2G}{n} \right)^{\frac{1}{2}},$$

where $T_a$ holds for the products of the first group and $T_b$ for the products of the second group (note that $F$ and $\xi$ both contain a factor $m$). Introducing $\tau$ by

$$\tau := \frac{T_a T_b^{-1} = (F n)^{\frac{1}{2}} (G e)^{-\frac{1}{2}}},$$

we obtain that the optimal $k$-repetitive scheme is also the optimal repetitive scheme if $\sqrt{k(k-1)} < \tau \leq \sqrt{k(k+1)}$.

If each product would be produced according to its individually optimal cycle (which will usually be infeasible) the costs $K$ per unit of time would be:

$$(1.5) \quad K = (2F \xi)^{\frac{1}{2}} + (2G n)^{\frac{1}{2}}.$$

$K$ provides an under estimate for the costs of an overall optimal scheme. Hence, $K(k)$ provides a measure for the nonoptimality of the $k$-repetitive scheme. Using this measure it has been shown in [2], that the costs of the optimal repetitive scheme never exceed $K$ by more than $1.5\%$, but usually the excess is far less.

For the precise measure see [2].
2. Positive set-up times: In this section it is supposed that at least one of the set-up times $s_0$, $t_0$ is positive. This might make the optimal repetitive scheme infeasible. Namely, suppose $\sqrt{k(k-1)} < \tau \leq \sqrt{k(k+1)}$ and $k$ divides $m$, then in each subcycle with length $T(k)$ a similar product set has to be produced requiring a total set-up time of $mk^{-1}s_0 + nt_0$, which is not necessarily available. Difficulties arise if the effective utilization rate $\rho := mdp^{-1} + ner^{-1}$ of the machine is so high that no time is left for the set-up times i.e.

\[
1 - \rho < (s + kt)T(k)^{-1}, \text{ with } s := ms_0, t = nt_0.
\]

However, if this occurs it can be repaired by a simple modification, viz. do not choose the optimal $k$-repetitive cycle time $T = T(k)$, but choose $T = aT(k)$ with the smallest $a$ such that the scheme can be fitted. This gives the following choice $a_k$ for $a$:

\[
a_k = (s + kt)T(k)^{-1}(1 - \rho)^{-1}.
\]

The $k$-repetitive scheme with cycle time $a_kT(k)$ incurs the average costs $\bar{K}(k)$:

\[
\bar{K}(k) = \frac{1}{2} (a_k + a_k^{-1})K(k).
\]

Under the supposition that $k$ divides $m$, this scheme is feasible. In [2] it has already been demonstrated what can be done (without much loss) if $m$ is not a multiple of $k$. So for clearness of the argument we will further suppose in this paper that $m$ is a multiple of any relevant $k$.

Now we have determined what might be called an optimal modified $k$-repetitive scheme if the optimal $k$-repetitive scheme is not feasible. However, if such a modification is necessary, a $(k+1)$-scheme or a $(k-1)$-scheme might be better. Therefore, it becomes relevant to minimize $\bar{K}(k)$ with respect to $k$. This may be done by answering the question: when does $\bar{K}(k) = \bar{K}(k+1)$ hold? For $a_k, a_{k+1} > 1$ we obtain as condition:

\[
k^{-1}(k+1)^{-1} = (sn)^{-1}e + 2(1 - \rho)sn(s + kt)^{-1}(s + (k + 1)t)^{-1}(Ft - Gs).
\]

The second term in the right hand side of (2.4) vanishes if $Ft = Gs$, i.e. if the set-up costs are proportional to the set-up times: $s_0 = cF_0$, $t_0 = cG_0$. For this special case (2.4) becomes:

\[
k(k+1) = \frac{Ft}{Gs} = \tau^2 \text{ or } \sqrt{k(k+1)} = \tau.
\]

So, in this situation of time-proportional set-up costs there is no necessity to shift to another value of $k$. In other situations such a shift may very well be advisable as follows from (2.4).
In the special situation considered above, the costs for the optimal modified k-repetitive scheme (2.3) can easily be worked out and appear to be:

\[ \tilde{K}(k) = \frac{1}{l}c(1-\rho)^{-1}(\xi+k^{-1}\eta)(F+kG) + (1-\rho)c^{-1}. \]

These costs are the best obtainable with a repetitive scheme if \( \sqrt{k(k-1)} < \tau \leq \sqrt{k(k+1)}. \)

3. Performance of the modified repetitive schemes: The performance of the optimal modified repetitive scheme might be analyzed by comparing \( \min K(k) \) where \( \alpha_k \) is replaced by 1 if it is smaller than 1 with \( K \), the costs if all products would be manufactured according to their individually best cycle times. Although \( \frac{1}{2}(\alpha_k + \alpha_k^{-1}) \) does not increase very fast with \( \alpha_k \) e.g. it is 1.0167 for \( \alpha_k = 1.2 \) it may eventually become too large in case a very large \( \alpha_k \) is needed. An idea for finding a better under estimate than \( K \) is the following: if the optimal repetitive scheme is not feasible, then it is likely that also the Camp-cycle times give a utilization (including set-up times) greater than one. So, the Camp of Wilson cycles for an under estimate have also to be blown up. The cheapest combination of cycle times \( vT_a, \mu T_b \) (for the products of the first and second group respectively), having a utilization of at most one, gives certainly an under estimate for the costs of the overall optimal scheme. So, find \( v, \mu \)

minimizing \( \frac{1}{2}(v+v)^{-1}(2Ft)^{\frac{1}{2}} + \frac{1}{2}(\mu+\mu)^{-1}(2Gt)^{\frac{1}{2}} \)

with the condition: \( 1-\rho \geq \frac{s}{vT_a} + \frac{t}{\mu T_b} \) or \( (1-\rho)T_a \geq \frac{s}{v} + \frac{t}{\mu} \).

Supposing that the Wilson cycles imply a utilization greater than one, the minimum will be attained on the boundary. Hence the minimization problem can easily be solved by using a Lagrange multiplier \( \lambda \), which gives the set of equations (3.1) en (3.2):

\[ \begin{align*}
\frac{1}{2}(1-v^{-2})(2Ft)^{\frac{1}{2}} - sv^{-2}z &= 0 \\
\frac{1}{2}(1-\mu^{-2})(2Gt)^{\frac{1}{2}} - \tau \mu^{-2}z &= 0 \\
(1-\rho)T_a &= sv^{-1} + \tau \mu^{-1}.
\end{align*} \]

Elimination of \( z \) from (3.1) and replacement of \( \tau \) by \( (Ft)^{\frac{1}{2}}(Gt)^{-\frac{1}{2}} \) gives:

\[ (v^2-1)Fs^{-1} = (\mu^2-1)Gt^{-1}. \]

So, according to (3.3), we have to increase \( v \) and \( \mu \) starting in 1 in such a way that \( v^2-1 \) and \( \mu^2-1 \) remain in fixed proportion. This increase should go on until (3.2) is satisfied. Again everything simplifies in the case of time-proportional set-up costs as in section 2, viz. then (3.3) and (3.2) are solved by
(3.4) \[ \nu = \mu = \frac{s + \tau t}{(1-p)T_a} = c \frac{F + tG}{(1-p)T_a}. \]

As in the foregoing section, we give for this special - but typical - case the costs per time unit \( \bar{K} \), which are an under estimate for the optimal costs:

(3.5) \[ \bar{K} = \frac{1}{2} (\nu + \nu^{-1}) K, \text{ with } \nu \text{ given by (3.4)}. \]

If \( \sqrt{k(k-1)} < \tau \leq \sqrt{k(k+1)} \), the quotient \( K(k) \bar{K}^{-1} \) gives an upper estimate for the performance of the optimal modified repetitive scheme (for the case of time-proportional set-up costs). (2.3) and (3.5) imply:

(3.6) \[ \frac{K(k)}{K} = \frac{\alpha_k + \alpha_k^{-1}}{\nu + \nu^{-1}} K(k) = \frac{\nu}{\alpha_k} \frac{\alpha_k^2 + 1}{K(k)}. \]

We know already how \( K(k) \bar{K}^{-1} \) behaves (see section 1 and for details see [2]). So, now the most interesting thing is: how large can \( \nu \alpha_k^{-1} \) be? (2.2) and (3.4) give the answer (together with (1.2), (1.4) and the definition of \( \tau \) for the second equality):

(3.7) \[ \frac{\nu}{\alpha_k} = \frac{F + tG}{T_a} \frac{T(k)}{F + kG} = \frac{K}{K(k)}. \]

Hence (3.6) and (3.7) give:

(3.8) \[ \frac{K(k)}{K} = \frac{\alpha_k^2 + 1}{\nu^2 + 1} = \frac{\nu^2 + 1}{\nu^2 + 1}, \text{ where } \sigma := \frac{\alpha_k}{\nu} = \frac{K(k)}{K}. \]

(3.8) gives a sharp upper estimate for the quality of the optimal modified repetitive scheme for the case of time-proportional set-up costs. It implies for example (see section 2):

\[ \frac{K(k)}{K} \leq \sigma^2 \leq 1.03, \quad \frac{K(k)}{K} \leq \frac{1.03 \nu^2 + 1}{\nu^2 + 1}, \quad \frac{K(k)}{K} = 1 \text{ if } \tau = k. \]

For this derivation it has been supposed that \( \nu \geq 1 \) (and \( \alpha_k \geq 1 \)). What happens if \( \alpha_k \geq 1 \) and \( \nu < 1 \)? Well, then (3.7) implies \( \alpha_k \leq K(k) \bar{K}^{-1} = \sigma \). Hence

(3.9) \[ \frac{K(k)}{K} = \frac{1}{2} (\alpha_k + \alpha_k^{-1}) K(k) \leq 1.0001. \]

References:
