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Optimal control of routing to two parallel finite capacity queues or two parallel Erlang loss systems with dedicated and flexible arrivals

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Abstract

We consider two closely related systems with two parallel service stations to which arriving jobs must be routed. Both stations are subject to blocking. The first system concerns two parallel exponential servers, where each has its own finite capacity queue. The second system concerns two parallel stations, where each has its own set of exponential servers. In this system there is no waiting room at any of the stations or servers. Considering the objective to minimize the total number of blocked jobs, we show that the optimal routing control policy for both systems has a threshold structure. We also show that Least Loaded Routing is the optimal routing policy if the system is symmetrical.

Key words: load balancing, routing control, optimal threshold policies, Markov decision processes.

1 Introduction

There is an extensive literature on the routing of jobs to parallel queues. An extensive overview is given in Hariharan, Kulkarni and Stidham [4]. Stidham and Weber [13] also review the topic of routing control in parallel queues in their survey of Markov decision models for optimal control of queueing systems.

The perhaps most basic routing control problem in parallel queues is studied by Winston [15], who considers a queueing system consisting of a finite number of identical exponential servers in parallel, each having its own queue. Jobs arrive at the system according to a Poisson process. Upon arrival of a job, it must be assigned to one of the queues. Under the assumption that jockeying between queues is not permitted, it is shown that the shortest line discipline is optimal in terms of maximizing throughput.

Hordijk and Koole [5] prove that the shortest line discipline maximizes stochastically the number of jobs served at any time t when the queues have finite buffers. The servers are assumed identical but the buffers may have different capacities. Towsley, Sparaggis and
Cassandras [14] also consider identical servers and finite buffers with unequal capacities. They also allow for buffering to be available at the controller.

Koole, Sparaggis and Towsley [7] show that the shortest line discipline is optimal with respect to various cost functions. Their results cover systems with two parallel queues with infinite or finite capacity, arrivals that are independent of the state of the system but otherwise arbitrary, and a broad class of service time distributions.

Koole [8] studies the static assignment of jobs to parallel, exponential, heterogeneous servers. There is no waiting room at any of these servers, and blocked jobs are lost. The objective is to minimize the average number of blocked jobs. In the case of dynamic assignment it is optimal to route to the fastest available server; see Koole [9]. The static version of the problem is formulated as a stochastic control problem with partial observation. Numerical experiments are conducted and the structure of the optimal policy is studied.

All these models assume a single server at any of the queues. In Johri [6] state-dependent service rates are considered. Under certain regularity conditions on these rates, the shortest line discipline minimizes stochastically the number of jobs at any time \( t \).

Hajek [3] considers two interacting parallel stations with two servers at each station and a fifth server that is shared by the two stations. Both stations have an infinite capacity queue. There are three Poisson arrival streams: two dedicated streams and one flexible stream. Jobs in the first dedicated stream always join station 1 and jobs in the second dedicated stream always join station 2. Jobs in the flexible stream may join either queue. So for each arriving flexible job it must be decided to which queue it is routed.

In Menich and Serfozo [12] the service and arrival rates are functions of all queue lengths. This includes the case of a number of parallel service stations, each having \( s \) identical exponential servers. They do not allow for finite buffers.

By combining the last model with the model considered in [3] and by disallowing buffering, we obtain systems with a number of parallel \( M|M|s \) stations and dedicated as well as flexible arrivals. These are highly suitable for modelling wireless networks; see Alanyali and Hajek [1]. Such a network consists of a number of base stations and of users. The users require communication channels, which are available at the base stations. A station may only serve users that are within geographical range of the station. Users may be in range of several stations and the resource allocation problem concerns the question of station selection. If each location has finite capacity, i.e., a finite number of channels, then a consumer is lost if upon its arrival all channels of all stations in its neighbourhood are already in use. The goal of the allocation policy is to minimize the fraction of lost consumers. The authors provide a lower bound for the consumer loss probability under any allocation policy. Structural properties of the optimal policy are not addressed.

For the specific case of two parallel stations with \( c_1 \) and \( c_2 \) channels, respectively, and \( \exp(\mu) \)-distributed service times at any of the \( c_1 + c_2 \) channels, Van Leeuwaarden, Aalto and Virtamo [10] consider various optimal static routing policies. They also consider dynamic routing, for which they discuss a one-step policy improvement algorithm and its performance. They conclude with a brief discussion of three open problems. The first, although intuitively clear, is to show that the optimal routing policy is of a switch-over type, i.e., is a threshold policy.
Proving that the optimal routing policy in this model, termed Model II in the remainder, has indeed a threshold structure is the main goal of this paper. However, we will first establish the same result for a closely related model, termed Model I, and then extend this result to Model II. Model I will be described in Section 2. In Section 3 we state and prove our main results for this model. In Section 4 we shift our attention to Model II. Its description is taken from [10]. In Section 5 the threshold result obtained for Model I is extended to Model II.

The second open problem posed in [10] is to show that Least Loaded Routing is the optimal routing policy in the symmetrical case, i.e., in the case of equal dedicated arrival rates and equal capacities. We prove this assertion in Section 5.2.

2 Model description Model I

We consider the system depicted in Figure 1. The system consists of two identical parallel servers. Service times are $\exp(\mu)$-distributed. The servers have separate queues, with finite capacity. Station 1 is formed by server 1 and its queue. Station 2 is formed by server 2 and its queue. The maximum number of jobs at station 1 is $c_1 \geq 1$ and the maximum number of jobs at station 2 is $c_2 \geq 1$. So the buffer sizes of stations 1 and 2 are $c_1 - 1$ and $c_2 - 1$, respectively.

![Figure 1: Queueing system corresponding to Model I](image)

There are three Poisson arrival streams: two dedicated streams and one flexible stream. Jobs in dedicated stream $k$ ($k = 1, 2$) arrive according to a Poisson process with rate $\lambda_k$ and automatically join station $k$. Jobs in the flexible stream arrive according to a Poisson process with rate $\nu$ and may join either station. Upon arrival of a flexible job it must be decided to which of the two stations it is routed. We assume that the decision maker has complete information, i.e., he knows the number of jobs at each of the two stations. The structure of the system is that of a (semi-)Markovian decision process. It can be described as follows.

**States:** The state of the system is described by the tuple $(i, j)$, where $i$ ($0 \leq i \leq c_1$) is the number of jobs at the first station and $j$ ($0 \leq j \leq c_2$) is the number of jobs at the second station.

**Events:** We distinguish two possible events: (i) the arrival of a new job and (ii) a service completion.
**Decisions:** If the event is an arrival and it concerns a flexible job, then it has to be decided to which of the two stations the job is routed (decision '1' if station 1, decision '2' if station 2). If the event is an arrival and it concerns a dedicated job from stream 1 (or stream 2), then decision '1' (or decision '2') is taken automatically. If the decision is such that the arriving job is routed to a station that is loaded to capacity, then the job leaves the system immediately. If the event is a service completion, then no decision has to be taken.

**Costs and rewards:** If an arriving job is routed to a station that is loaded to capacity, then blocking costs of 1 are incurred. Alternatively, one can say that blocking yields a reward of -1. These are the only costs; there are no holding costs for jobs residing in the system.

**Criterion:** The objective is to minimize the expected (blocking) costs (i.e., number of blocked jobs) over an n-period time horizon. Alternatively stated, the objective is to maximize the expected reward over an n-period time horizon.

**Uniformization:** Applying uniformization, we can consider that transitions occur at the jump times of a Poisson process with rate $\lambda_1 + \lambda_2 + \nu + 2\mu$. By scaling time, we take $\lambda_1 + \lambda_2 + \nu + 2\mu = 1$ without loss of generality. Then, with probability $\lambda_k$ ($k = 1, 2$) a transition concerns the arrival of a dedicated job from stream $k$, with probability $\nu$ it concerns the arrival of a flexible job, with probability $\mu$ a service completion at station 1 and with the same probability a service completion at station 2. A service completion is either a real service completion or an artificial service completion when the server idles because there are no jobs at the station.

As a result, the times between consecutive events are identically distributed. Such times are called periods and if we reverse the direction of time, we can consider the number $n$ of periods left until the process hits time zero. Uniformization enables us to use induction on the remaining number of periods to prove our results for any finite time horizon. Using a standard argument, these results can then be extended to the infinite time horizon case (average cost criterion); see, e.g., Denardo [2]. Note that since $c_1$ and $c_2$ are finite, the system is a finite state system.

Besides dynamic programming we will occasionally make use of sample path arguments. For an exposition of the sample path approach we refer to Liu, Nain and Towsley [11].

### 2.1 Dynamic programming formulation

In this section we complete the model in terms of a mathematical formulation. After that, we successively state and prove our main results.

Recapitulating, $i$ and $j$ denote the number of jobs at station 1 and 2, respectively, and $(i, j)$ is the state of the system, $0 \leq i \leq c_1$ and $0 \leq j \leq c_2$. We will use the following notation:

- $W_n(i, j)$ denotes the maximum expected $n$-period reward when the current state is $(i, j)$. State $(i, j)$ may be the result of an arrival—where the system is observed immediately after the new job has been routed to one of the two stations—or a real or artificial service completion.

- $W_n(i, j; \pi)$ denotes the maximum expected $n$-period reward when the current state is $(i, j)$, given that there is an arrival event at this point in time and given that decision $\pi$
is chosen with respect to the new job; \( \pi = 1 \) if this job belongs to dedicated stream 1, \( \pi = 2 \) if the job belongs to dedicated stream 2 and \( \pi \in \{1, 2\} \) if the job is a flexible job. Let \( \pi^* \) denote the optimal decision. Note that in the notation \( \pi^* \) the dependence on \( i \), \( j \) and \( n \) is suppressed.

Then our model is defined by the following Dynamic Programming Equations (DPEs), where 

\[
-1[\Phi] := \begin{cases} 
-1 & \text{if } \Phi, \\
0 & \text{else}.
\end{cases}
\]

For \( n \geq 0 \) and all \( 0 \leq i \leq c_1 \) and \( 0 \leq j \leq c_2 \):

\[
W_0(i, j) = 0
\]

\[
W_{n+1}(i, j) = \lambda_1 W_n(i, j; 1) + \lambda_2 W_n(i, j; 2) + \nu \max\{W_n(i, j; 1), W_n(i, j; 2)\} + \\
\mu W_n(\max\{i - 1, 0\}, j) + \mu W_n(i, \max\{j - 1, 0\})
\]

\[
W_n(i, j; 1) = -1[i = c_1] + W_n(\min\{i + 1, c_1\}, j)
\]

\[
W_n(i, j; 2) = -1[j = c_2] + W_n(i, \min\{j + 1, c_2\})
\]

### 3 Main results for Model I

We will prove the following proposition.

**Proposition 1** (Characterization of the optimal routing policy)

Let the remaining number of periods be \( n \). Then the optimal routing policy can be characterized as follows. If it is optimal to route an arriving flexible job to station 1 in state \((i, j)\), then it is optimal as well to route it to station 1 in all states \((i + k, j)\) with \(0 < k \leq c_2 - j\) and in all states \((i, j + k)\) with \(0 < k \leq i\).

**Remark 1** Alternatively stated, Proposition 1 reads that if it is optimal to route an arriving flexible job to station 2 in state \((i, j)\), then it is optimal as well to route it to station 2 in all states \((i + k, j)\) with \(0 < k \leq c_1 - i\) and in all states \((i, j - k)\) with \(0 < k \leq j\).

In order to establish Proposition 1 we will prove the following monotonicity results.

**Proposition 2** (Key Proposition)

For \( n \geq 0 \),

\[
W_n(i, j + 1) - W_n(i, j + 2) \geq W_n(i + 1, j) - W_n(i + 1, j + 1), 
\]

(1)

\[
W_n(i + 1, j) - W_n(i + 1, j + 1) \geq W_n(i, j) - W_n(i, j + 1),
\]

(2)

\[
W_n(i + 1, j + 1) - W_n(i, j + 1) \geq W_n(i + 2, j) - W_n(i + 1, j),
\]

(3)

for all \( i, j \) for which the four states appearing in the respective inequality exist (i.e., (1) holds for all \( 0 \leq i \leq c_1 \) and \( 0 \leq j \leq c_2 - 1 \), (2) for all \( 0 \leq i \leq c_1 \) and \( 0 \leq j < c_2 \), and (3) for all \( 0 \leq i < c_1 - 1 \) and \( 0 \leq j < c_2 \).
Inequalities (1) and (3) correspond to properties (c2) and (c1) of [3], respectively, where $V_n(\cdot)$ is the minimum expected cost, so in our model the inequality signs are read the other way around. Inequality (2) corresponds to property (b) of [3], although the latter is more general.

**Remark 2** Combining (1) with (2) for $i = c_1 - 1$, and combining (3) with (2) for $j = c_2 - 1$, we obtain, for $n \geq 0$,

\[
W_n(c_1, j + 1) - W_n(c_1, j + 2) \geq W_n(c_1, j) - W_n(c_1, j + 1),
\]

\[
W_{n}(i + 1, c_2) - W_{n}(i, c_2) \geq W_{n}(i + 2, c_2) - W_{n}(i + 1, c_2),
\]

where (4) holds for all $0 \leq j < c_2 - 1$ and (5) for all $0 \leq i < c_1 - 1$.

The proof of the Key Proposition uses induction on the remaining number of periods and runs as follows. **Step 0:** Observe that (1), (2) and (3) hold for $n = 0$. **Step 1:** Assuming (1), (2) and (3) to hold for some $n \geq 0$, prove that (1), (2) and (3) hold for $n + 1$ as well. Note carefully that it suffices to prove (1) and (2) for $n + 1$. Namely, once these two inequalities have been established, (3) follows by interchanging the names of the two stations (i.e., station 1 is now termed station 2 and vice versa) and then rearranging the terms of (1) such that it reads (3).

In Step 1 of the proof we will need the following lemma and proposition. The latter contains inequalities of the form $W_n(\cdot) \leq 1 + W_n(\cdot)$. The idea to use such inequalities is also found in [5].

**Lemma 1** Let $s_m = (i_m, j_m)$, $m = 1, \ldots, 4$, given that there is an arrival of a flexible job, and let $\phi, \psi \in \{1, 2\}$ and recall that $\pi^* \in \{1, 2\}$ denotes the optimal decision in state $s_m$. Then,

\[
W_n(s_1; \phi) - W_n(s_2; \pi^*) \geq W_n(s_3; \pi^*) - W_n(s_4; \psi)
\]

implies

\[
W_n(s_1; \pi^*) - W_n(s_2; \pi^*) \geq W_n(s_3; \pi^*) - W_n(s_4; \pi^*).
\]

**Proof.** Immediate from $W_n(s_1; \pi^*) \geq W_n(s_1; \phi)$ and $W_n(s_4; \pi^*) \geq W_n(s_4; \psi)$ for all $\phi, \psi$. \hfill \Box

We will use Lemma 1 in the following way. When distinguishing between all possible combinations of optimal decisions in certain states $s_2$ and $s_3$, we choose $\phi$ and $\psi$ such that (6) holds. Then (7) holds as well.

**Proposition 3** For all $n \geq 0$,

\[
0 \leq W_n(i, j) - W_n(i, j + 1) \leq 1, \quad 0 \leq i \leq c_1, 0 \leq j < c_2,
\]

\[
-1 \leq W_n(i + 1, j) - W_n(i, j) \leq 0, \quad 0 \leq i < c_1, 0 \leq j \leq c_2.
\]

**Proof.** By coupling and a sample path argument. We first consider the right-hand inequality of (8). Consider two $n$-period instances of our model, instance $\mathcal{I}_0$ starting in $(i, j)$ and instance $\mathcal{I}_1$ starting in $(i, j + 1)$. We couple all events and all decisions. Instance $\mathcal{I}_0$ follows the optimal
policy and instance $I_1$ copies all decisions taken in $I_0$. In particular, if $I_0$ idles at station 2 because it has run out of jobs at that station, then $I_1$ takes its additional job into service.

Then the costs are the same for both instances as long as $I_0$ does not route a new job to station 2 in some state $(k, c_2 - 1)$ for $0 \leq k \leq c_1$. If this does not occur before time hits zero, then the difference in reward is 0. Now suppose it does occur before time zero, at time $T$ say. If, even earlier in time, at time $T'$ say, $I_1$ completed service at station 2 while $I_0$ was idling at that station, then $I_0$ and $I_1$ became identical at $T'$, and hence their difference in reward is 0. Alternatively, assume $I_0$ has not witnessed an artificial service completion at station 2 before $T$. Then, at $T$, the job routed to station 2 in $I_0$ is blocked in $I_1$, incurring a reward of $-1$ for $I_1$ and causing $I_0$ and $I_1$ to become identical immediately afterwards. So the difference in reward between $I_0$ and $I_1$ is $0 - (-1) = 1$.

The reasoning is almost the same for the left-hand inequality of (8). Again, let instance $I_0$ start in $(i, j)$ and instance $I_1$ in $(i, j + 1)$. But now let $I_1$ follow the optimal policy and let $I_0$ copy all decisions taken in $I_1$. In particular, if $I_1$ starts serving its additional job at station 2, then $I_0$ idles at station 2.

Finally, (9) follows from (8) by interchanging the names of the two stations.

We note that the left-hand inequality of (8) and the right-hand inequality of (9) correspond to property (a) of [3].

We further note that one may easily verify that from Proposition 3 the following intermediate result can be obtained (cf. Remark 3 in [10]).

**Corollary 1** {**The optimal routing policy is greedy**}

For any $n$, the optimal routing policy will route an arriving flexible job to station 1 if $i < c_1$ and $j = c_2$ and to station 2 if $i = c_1$ and $j < c_2$.

We now return to the Key Proposition.

**Proof of the Key Proposition.**

**Step 0.** Inequalities (1), (2) and (3) hold by definition for $n = 0$.

**Induction hypothesis.** Assume that for some $n \geq 0$, (1) holds for all $0 \leq i < c_1$ and $0 \leq j < c_2 - 1$, (2) for all $0 \leq i < c_1$ and $0 \leq j < c_2$ and (3) for all $0 \leq i < c_1 - 1$ and $0 \leq j < c_2$. This will be our induction hypothesis.

**Step 1.** Under the induction hypothesis, we show that (1) and (2) hold for $n + 1$ as well. Then (3) also holds for $n + 1$; cf. the paragraph below Remark 2.

**Proof of (1).** Let $0 \leq i < c_1$ and $0 \leq j < c_2 - 1$. Then,

$$W_{n+1}(i, j + 1) - W_{n+1}(i, j + 2) = \mu[W_n(\max{i - 1, 0}, j + 1) - W_n(\max{i - 1, 0}, j + 2)] \oplus + \mu[W_n(i, j) - W_n(i, j + 1)] \oplus + \lambda_1[W_n(i, j + 1; 1) - W_n(i, j + 2; 1)] \oplus + \lambda_2[W_n(i, j + 1; 2) - W_n(i, j + 2; 2)] \oplus + \nu[\max\{W_n(i, j + 1; 1), W_n(i, j + 1; 2)\} - \max\{W_n(i, j + 2; 1), W_n(i, j + 2; 2)\}] \oplus$$
\[ J_1 \{ W_n(i, j) - W_n(i, j + 1) \} + J_1 \{ W_n(i + 1, \max\{j - 1, 0\}) - W_n(i + 1, j) \} + \mu[W_n(i, j) - W_n(i, j + 1)] \Theta + \mu[W_n(i + 1, \max\{j - 1, 0\}) - W_n(i + 1, j)] \Theta + \lambda_1[W_n(i + 1, j; 1) - W_n(i + 1, j + 1; 1)] \Theta + \lambda_2[W_n(i + 1, j; 2) - W_n(i + 1, j + 1; 2)] \Theta + \nu[\max\{W_n(i + 1, j; 1), W_n(i + 1, j; 2)\} - \max\{W_n(i + 1, j + 1; 1), W_n(i + 1, j + 1; 2)\}] \Theta \]

\[ = W_{n+1}(i + 1, j) - W_{n+1}(i + 1, j + 1). \]

\( \Theta \geq \Theta \) By (1) if \( i > 0 \), and by (1) and subsequently (2) if \( i = 0 \).

\( \Theta \geq \Theta \) By (1) if \( j > 0 \), and by (8) if \( j = 0 \).

\( \Theta \geq \Theta \) By executing decision 1 and subsequently by (1) if \( i < c_1 - 1 \), and by (4) if \( i = c_1 - 1 \).

\( \Theta \geq \Theta \) By executing decision 2 and subsequently by (8) if \( j = c_2 - 2 \), and by (1) if \( j < c_2 - 2 \).

\( \Theta \geq \Theta \) The next decision, \( d_1 \) say, prescribed by the (optimal) policy corresponding to \( W_n(i + 2) \), is either 1 or 2. The same can be said about the next decision, \( d_2 \) say, prescribed by the (optimal) policy corresponding to \( W_n(i + 1, j) \). There are at most four joint cases \( (d_1, d_2) \), namely: \( (1, 1) \), \( (1, 2) \), \( (2, 1) \) and \( (2, 2) \).

Under \( (1, 1) \), we choose 1 in the other two states appearing in (1) as well. Then the result follows by \( \Theta \geq \Theta \) and Lemma 1. Analogously, under \( (2, 2) \), we choose 2 in the other two states as well, after which the result follows by \( \Theta \geq \Theta \) and Lemma 1.

Thirdly, under \( (1, 2) \),

\[ W_n(i, j + 1; 1) - W_n(i, j + 2; 1) \]

\[ W_n(i, j + 1; 2) - W_n(i, j + 2; 2) \]

\[ = W_n(i + 1, j + 1) - W_n(i + 1, j + 2) \]

\[ = W_n(i + 1, j; 2) - W_n(i + 1, j + 1; 2), \]

to which we subsequently apply Lemma 1.

Fourthly, under \( (2, 1) \),

\[ W_n(i, j + 1; 2) - W_n(i, j + 2; 2) \]

\[ = W_n(i, j + 2) - W_n(i, \min\{j + 3, c_2\}) + 1[j = c_2 - 2] \]

\[ = \{ \text{induction hypothesis; } (8) \text{ if } j = c_2 - 2; (1), (4) \text{ if } j < c_2 - 2 \text{ and } i = c_1 - 1; \}

\( (1) \text{ twice if } j < c_2 - 2 \text{ and } i < c_1 - 1 \}

\[ W_n(\min\{i + 2, c_1\}, j) - W_n(\min\{i + 2, c_1\}, j + 1) \]

\[ = W_n(i + 1, j; 1) - W_n(i + 1, j + 1; 1), \]

to which we subsequently apply Lemma 1.

This concludes our proof of (1) for \( n + 1 \).

\textbf{Proof of (2).} Let \( 0 \leq i < c_1 \) and \( 0 \leq j < c_2 \). Then,

\[ W_{n+1}(i + 1, j) - W_{n+1}(i + 1, j + 1) \]

\[ = \mu[W_n(i, j) - W_n(i, j + 1)] \Theta + \mu[W_n(i + 1, \max\{j - 1, 0\}) - W_n(i + 1, j)] \Theta + \]
\[
\begin{align*}
&\lambda_1[W_n(i+1, j; 1) - W_n(i+1, j+1; 1)] + \\
&\lambda_2[W_n(i+1, j; 2) - W_n(i+1, j+1; 2)] + \\
&\nu[\max\{W_n(i+1, j; 1), W_n(i+1, j; 2)\} - \\
&\max\{W_n(i+1, j+1; 1), W_n(i+1, j+1; 2)\}] \\
\geq &\{\text{induction hypothesis; } \circ \geq \bullet; \circ \geq \bullet; \circ \geq \bullet; \circ \geq \bullet; \text{ see below}\} \\
&\mu[W_n(\max\{i-1, 0\}, j) - W_n(\max\{i-1, 0\}, j+1)] + \\
&\lambda_1[W_n(i, j; 1) - W_n(i, j+1; 1)] + \\
&\nu[\max\{W_n(i, j; 1), W_n(i, j; 2)\} - \max\{W_n(i, j+1; 1), W_n(i, j+1; 2)\}] \\
&= W_{n+1}(i, j) - W_{n+1}(i, j+1). \\
\end{align*}
\]

3. By (2) if \(i > 0\), and with equality if \(i = 0\).

4. By (2) if \(j > 0\), and by \(\circ = \bullet = 0\) if \(j = 0\).

5. By executing decision 1 and subsequently with equality if \(i = c_1 - 1\), and by (2) if \(i < c_1 - 1\).

6. By executing decision 2 and subsequently by (2) if \(j < c_2 - 1\), and by \(\circ = \bullet = 1\) if \(j = c_2 - 1\).

7. Analogous to the proof of \(\circ \geq \bullet\) for (1) for \(n + 1\), we distinguish the cases (1, 1), (2, 2), (1, 2) and (2, 1). Again, the first case can be dealt with by choosing 1 in the other two states as well, and the second by choosing 2 in the other two states as well.

Thirdly, under (1, 2),
\[
\begin{align*}
W_n(i+1, j; 2) - W_n(i+1, j+1; 1) &= W_n(i+1, j+1) - W_n(\min\{i+2, c_1\}, j+1) + 1[i = c_1 - 1] \\
&\geq \{\text{induction hypothesis; } (9) \text{ if } i = c_1 - 1; (2), (3) \text{ if } i < c_1 - 1\} \\
&\quad W_n(i, j+1) - W_n(i+1, j+1) \\
&= W_n(i, j+2) - W_n(i, j+1; 1),
\end{align*}
\]
to which we subsequently apply Lemma 1.

Fourthly, under (2, 1),
\[
\begin{align*}
W_n(i+1, j; 2) - W_n(i+1, j+1; 2) &= W_n(i+1, j+1) - W_n(i+1, \min\{j+2, c_2\}) + 1[j = c_2 - 1] \\
&\geq \{\text{induction hypothesis; } (8) \text{ if } j = c_2 - 1; \} \\
&\quad (4) \text{ if } j < c_2 - 1 \text{ and } i = c_1 - 1; (1), (2) \text{ if } j < c_2 - 1 \text{ and } i < c_1 - 1\} \\
&\quad W_n(i+1, j) - W_n(i+1, j+1) \\
&= W_n(i, j+1) - W_n(i, j+1; 1),
\end{align*}
\]
to which we subsequently apply Lemma 1.

This concludes our proof of (2) for \(n + 1\) and hence our proof of the Key Proposition.
We now derive Proposition 1 from the Key Proposition by means of two corollaries. Note that Corollary 3 is exactly Proposition 1.

**Corollary 2** For all \( n \geq 0, \)

\[
W_n(i,j;2) - W_n(i,j+1;2) \geq W_n(i,j;1) - W_n(i,j+1;1), \quad 0 \leq i \leq c_1, 0 \leq j < c_2, \tag{10}
\]

\[
W_n(i+1,j;2) - W_n(i,j;2) \geq W_n(i+1,j;1) - W_n(i,j;1), \quad 0 \leq i < c_1, 0 \leq j \leq c_2. \tag{11}
\]

**Proof.** One may easily verify that (10) follows from (8) for \( 0 \leq i \leq c_1 \) and \( j = c_2 - 1, \) from (1) for \( 0 \leq i < c_1 \) and \( 0 \leq j < c_2 - 1, \) and from (4) for \( i = c_1 \) and \( 0 \leq j < c_2 - 1. \) Analogously, (11) follows from (9) for \( i = c_1 - 1 \) and \( 0 \leq j \leq c_2, \) from (3) for \( 0 \leq i < c_1 - 1 \) and \( 0 \leq j < c_2, \) and from (5) for \( 0 \leq i < c_1 - 1 \) and \( j = c_2. \)

**Corollary 3** Let \( n \geq 0, 0 \leq i \leq c_1 \) and \( 0 \leq j \leq c_2. \) If it is optimal to route an arriving flexible job to station 1 in state \((i,j),\) then it is optimal to route it to station 1 in state \((i,j+1),\) provided \( j < c_2, \) and in state \((i-1,j),\) provided \( i > 0. \)

**Proof.** Let \( n \geq 0. \) It suffices to show that

\[
W_n(i,j;1) \geq W_n(i,j;2) \implies W_n(i,j+1;1) \geq W_n(i,j+1;2), \quad 0 \leq i \leq c_1, 0 \leq j < c_2, \tag{12}
\]

\[
W_n(i,j;1) \geq W_n(i,j;2) \implies W_n(i-1,j;1) \geq W_n(i-1,j;2), \quad 0 < i \leq c_1, 0 \leq j \leq c_2. \tag{13}
\]

One can easily verify that implications (12) and (13) are immediate from inequalities (10) and (11), respectively.

### 3.1 Extension to heterogeneous service rates

In our model we considered homogeneous service rates, i.e., the service rates at stations 1 and 2 were both equal to \( \mu. \) Now assume the service rates are heterogeneous and equal to \( \mu_1 \) and \( \mu_2, \) respectively, where \( \mu_1 \neq \mu_2. \) Then, by replacing each occurrence of \( \mu \) in the DPEs and all proofs by \( \mu_1 \) or \( \mu_2 \) (depending on which of the two is applicable there), it is readily verified that all results and proofs remain intact.

### 4 Model description Model II

Consider the system depicted in Figure 2. The system consists of two parallel Erlang loss stations. The first has \( c_1 \geq 1 \) parallel servers and the second has \( c_2 \geq 1 \) parallel servers. All \( c_1 + c_2 \) servers are identical, and service times are \( \exp(\mu) \)-distributed. Since the stations are loss stations, there is no queueing. The maximum number of jobs at station 1 is \( c_1 \) and the maximum number of jobs at station 2 is \( c_2, \) as in Model I.
Figure 2: Queueing system corresponding to Model II

The arrival process is the same as in Model I, i.e., there are three Poisson arrival streams: two dedicated streams and one flexible stream. Jobs in dedicated stream $k$ ($k = 1, 2$) arrive according to a Poisson process with rate $\lambda_k$ and automatically join station $k$. Jobs in the flexible stream arrive according to a Poisson process with rate $\nu$ and may join either station. Upon arrival of a flexible job it must be decided to which of the two stations it is routed. The decision maker has complete information, i.e., he knows the number of jobs at each of the two stations.

The structure of the system is that of a (semi-)Markovian decision process, and its description is almost identical to that of Model I. We distinguish the same states, events, decisions, costs (or rewards) and assume the same criterion, i.e., minimizing the expected number of blocked jobs over an $n$-period time horizon. The only essential difference lies in the total service rate of a station, which is $\mu \cdot 1[i > 0]$ in Model I when there are $i$ jobs present at that station, and which now becomes $i\mu$. Consequently, also the uniformization is slightly different. We take $\lambda_1 + \lambda_2 + \nu + (c_1 + c_2)\mu = 1$ without loss of generality. Then, with probability $(i+j)\mu$ there is a real service completion, whereas with probability $(c_1 + c_2 - (i+j))\mu$ there is an artificial service completion.

### 4.1 Dynamic programming formulation

Since we have the same states, events, decisions, costs and criterion, the dynamic programming formulation is almost the same as for Model I. We use the same notation, i.e., the same value functions $W_n(i,j)$ and $W_n(i,j;\pi)$, and we only need to modify the DPE for $W_{n+1}(i,j)$. Namely, for $n \geq 0$ and all $0 \leq i \leq c_1$ and $0 \leq j \leq c_2$ (under the convention that $i\mu W_{n}(i-1,j)$ is zero for $i = 0$ and any $j$, and $j\mu W_{n}(i,j-1)$ is zero for $j = 0$ and any $i$):

$$W_{n+1}(i,j) = \lambda_1 W_n(i,j;1) + \lambda_2 W_n(i,j;2) + \nu \max\{W_n(i,j;1), W_n(i,j;2)\} + i\mu W_n(i-1,j) + j\mu W_n(i,j-1) + (c_1 + c_2 - (i+j))\mu W_n(i,j)$$
5 Main results for Model II

We claim that Propositions 1, 2 and 3 (and thus Corollaries 1, 2 and 3) all remain intact. Consequently, the optimal routing policy for Model II has a threshold structure. Note that it suffices to show that Propositions 2 and 3 remain valid.

It can easily be verified that Proposition 3 holds for Model II as well. The proof is analogous to the proof given for Model I. For example, consider the right-hand inequality of (8). Consider two n-period instances of Model II, instance $I_0$ starting in $(i, j)$ and instance $I_1$ starting in $(i, j + 1)$. We couple all events and all decisions, and also all servers. Instance $I_0$ follows the optimal policy and instance $I_1$ copies all decisions taken in $I_0$. Let $S$ denote the server that is idle in $I_0$ at time $n$ and serving a job in $I_1$ at time $n$.

Then the costs are the same for both instances as long as $I_0$ does not route a new job to server $S$. If this does not occur before time hits zero, then the difference in reward is 0. Now suppose it does occur before time zero, at time $T$ say. If, even earlier in time, at time $T'$ say, $I_0$ witnessed an artificial service completion at server $S$, then $I_0$ and $I_1$ became identical at $T'$, and hence their difference in reward is 0. Alternatively, assume $I_0$ has not witnessed an artificial service completion at server $S$ before $T$. Then, at $T$, the job routed to server $S$ in $I_0$ is blocked in $I_1$, incurring a reward of $-1$ for $I_1$ and causing $I_0$ and $I_1$ to become identical immediately afterwards. So the difference in reward between $I_0$ and $I_1$ is 0 - ($-1$) = 1.

We will now focus on Proposition 2, the Key Proposition. For Model I, for each of the inequalities (1) and (2), Step 1 of the proof consisted of 5 parts (namely, establishing $\diamondsuit \geq \bullet$ through $\otimes \geq \varnothing$). One may easily verify that the last three parts ($\otimes \geq \varnothing$, $\otimes \geq \bullet$ and $\otimes \geq \varnothing$), i.e., the parts concerning the three arrival processes, remain intact for Model II, since we have the same induction hypothesis and the same DPEs at arrival times as for Model I.

It remains to show that $\otimes + \varnothing \geq \bullet + \varnothing$ for the terms $\otimes$, $\varnothing$, $\bullet$ and $\varnothing$ corresponding to (1) and (2) for $n + 1$ for Model II. (Again, (3) will follow from (1) by interchanging the names of the two stations.)

Proof. We first consider (1). Let $0 \leq i < c_1$ and $0 \leq j < c_2 - 1$. Then,

$$\otimes + \varnothing = i\mu[W_n(i - 1, j + 1) - W_n(i - 1, j + 2)] \otimes_1 + \mu[W_n(i, j + 1) - W_n(i, j + 2)] \otimes_2 + (c_1 - (i + 1))\mu[W_n(i, j + 1) - W_n(i, j + 2)] \otimes_3 + j\mu[W_n(i, j) - W_n(i, j + 1)] \otimes_4 + \mu[W_n(i, j) - W_n(i, j + 1)] \otimes_5 + \mu[W_n(i, j + 1) - W_n(i, j + 2)] \otimes_6 + (c_2 - (j + 2))\mu[W_n(i, j + 1) - W_n(i, j + 2)] \otimes_7$$

$$\geq \{\text{induction hypothesis; } \otimes_2 = \varnothing; \otimes_3 \geq \varnothing_1, \otimes_3 \geq \varnothing_2, \otimes_2 \geq \varnothing_3 \text{ by (1); } \otimes_2 = \varnothing; \otimes_2 \geq \varnothing_3 \text{ by (1)}\}$$

$$i\mu[W_n(i, j) - W_n(i, j + 1)] \varnothing_1 + \mu[W_n(i, j) - W_n(i, j + 1)] \varnothing_2 + (c_1 - (i + 1))\mu[W_n(i + 1, j) - W_n(i + 1, j + 1)] \varnothing_3 + j\mu[W_n(i + 1, j - 1) - W_n(i + 1, j)] \varnothing_4 + \mu[W_n(i + 1, j) - W_n(i + 1, j + 1)] \varnothing_5 + \mu[W_n(i + 1, j) - W_n(i + 1, j + 1)] \varnothing_6 + (c_2 - (j + 2))\mu[W_n(i + 1, j) - W_n(i + 1, j + 1)] \varnothing_7$$

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Remark 3 In the derivation above we have used explicitly that the service rates of the servers at station 1 are equal to the service rates of the servers at station 2.

Next, consider (2). Let \( 0 < i < c_1 \) and \( 0 < j < c_2 \). Then,

\[
\begin{align*}
\odot &= i \mu[W_n(i,j) - W_n(i,j+1)] \odot_1 + \mu[W_n(i,j) - W_n(i,j+1)] \odot_2 + (c_1 - i + 1) \mu[W_n(i + 1, j) - W_n(i + 1, j + 1)] \odot_3 \\
&\geq \{ \text{induction hypothesis; } \odot_2 = \odot_i; \odot_1 \geq \odot_1, \odot_3 \geq \odot_3 \text{ by (2)} \}
\end{align*}
\]

\[
= \odot_i,
\]

and

\[
\begin{align*}
\oslash &= j \mu[W_n(i+1, j-1) - W_n(i+1, j)] \oslash_1 + \mu[W_n(i+1, j) - W_n(i+1, j+1)] \oslash_2 + (c_2 - j + 1) \mu[W_n(i + 1, j) - W_n(i + 1, j + 1)] \oslash_3 \\
&\geq \{ \text{induction hypothesis; } \oslash_2 = \oslash_i; \oslash_1 \geq \oslash_1, \oslash_3 \geq \oslash_3 \text{ by (2)} \}
\end{align*}
\]

\[
= \oslash_i,
\]

so \( \odot \geq \odot_i \) and \( \oslash \geq \oslash_i \), and thus \( \odot + \oslash \geq \odot_i + \oslash_i \).

This concludes our proof of the Key Proposition for Model II.

\[\square\]

5.1 Extension to heterogeneous service rates

Although not particularly interesting from a practical point of view (the service times in a wireless network are determined by the users, not by the communication channels of the base stations), a natural question would be whether the threshold structure also holds in case of heterogeneous service rates, i.e., in case each of the \( c_1 \) servers at station 1 has service rate \( \mu_1 \) and each of the \( c_2 \) servers at station 2 has service rate \( \mu_2 \), where \( \mu_1 \neq \mu_2 \). This remains an open problem. Our approach for (1) will not work directly; cf. Remark 3. In fact, (1) need not even hold in case \( \mu_2 > \mu_1 \); see the following (counter)example.

Example 1 Consider the following instance of Model II with heterogeneous service rates:

\[
\begin{align*}
\lambda_1 = \lambda_2 &= 0, \quad \nu = \frac{5}{17}, \quad \mu_1 = \frac{1}{17}, \quad \mu_2 = \frac{3}{17} \quad \text{and} \quad c_1 = c_2 = 3.
\end{align*}
\]

For this instance we have calculated the optimal routing policy for the average reward criterion. The desired accuracy was reached after 116 iterations (the accuracy in the calculations is \( 10^{-5} \)). We found

\[
W_{116}(2,1) - W_{116}(2,2) = 0.094 < 0.097 = W_{116}(3,0) - W_{116}(3,1),
\]

which violates inequality (1).
Despite the fact that (1) does not hold in general if \( \mu_2 > \mu_1 \), the threshold structure of the optimal policy may still very well apply, because Proposition 2 is not a necessary condition for Proposition 1. For example, the optimal policy for the instance considered in Example 1 (clearly) is to route to station 1 only if station 2 is loaded to capacity, so it is still of a threshold type.

Perhaps counterexamples can be constructed that show that the optimal policy need not be of a threshold type in general, although we have not yet found any. Alternatively, if one wishes to prove that the optimal policy always possesses the threshold structure, then an alternative approach is required than the one used in this paper, since the Key Proposition is lost when \( \mu_1 \neq \mu_2 \).

5.2 Optimality of Least Loaded Routing in the symmetrical case

In this section we prove the following proposition, which states that in case of equal dedicated arrival rates and equal capacities, Least Loaded Routing, i.e., routing to the station with the least number of occupied channels, is the optimal routing policy.

**Proposition 4 {Optimality of Least Loaded Routing}**
Assume \( \lambda_1 = \lambda_2 \) and \( c_1 = c_2 \) (and \( \mu_1 = \mu_2 \)). Then, for any \( n \), Least Loaded Routing (LLR) is the optimal routing policy.

The assertion follows from Corollary 1 if one of the two stations is loaded to capacity. Define \( \lambda := \lambda_1 = \lambda_2 \) and \( c := c_1 = c_2 \). It is readily verified that for \( i < c \) and \( j < c \) the assertion is a corollary of the following proposition.

**Proposition 5** For all \( n \geq 0 \) and \( 0 \leq i < j < c \),

\[
W_n(i + 1, j) \geq W_n(i, j + 1). \tag{14}
\]

**Proof.** By induction. Inequality (14) holds by definition for \( n = 0 \). Assume that for some \( n \geq 0 \), (14) holds for all \( 0 \leq i < j < c \); this is our induction hypothesis. Then, for \( 0 \leq i < j < c \),

\[
W_{n+1}(i + 1, j) = \lambda[W_n(i + 1, j; 1) + W_n(i + 1, j; 2)] + \nu \max\{W_n(i + 1, j; 1), W_n(i + 1, j; 2)\} + \\
(i + 1)\mu W_n(i, j) + j\mu W_n(i + 1, j - 1) + (2c - (i + 1) - j))\mu W_n(i + 1, j) \\
\geq \{\text{induction hypothesis; } W_n(i + 1, j; 2) \geq W_n(i, j + 1; 2) \text{ by (9) if } j = c - 1\} \\
W_n(i + 2, j) = W_n(i + 1, j + 1) \text{ by symmetry if } i = j - 1 \\
\lambda[W_n(i, j + 1; 1) + W_n(i, j + 1; 2)] + \nu \max\{W_n(i, j + 1; 1), W_n(i, j + 1; 2)\} + \\
i\mu W_n(i - 1, j + 1) + (j + 1)\mu W_n(i, j) + (2c - (i + 1) - j))\mu W_n(i, j + 1) \\
= W_{n+1}(i, j + 1).
\]

Following the same steps, one may easily verify that LLR (which can then be referred to as the shortest line discipline) is also the optimal routing policy in the symmetrical case of Model 1.
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