Note on the factorization of a square matrix into two hermitian or symmetric matrices

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Note on the factorization of a square matrix into two Hermitian or symmetric matrices

by

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Eindhoven, the Netherlands

November 1984
NOTE ON THE FACTORIZATION OF A SQUARE MATRIX INTO TWO HERMITIAN OR SYMMETRIC MATRICES

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1. Introduction

Although the results already have been published (partially) by Frobenius in 1910 (see [5]), these are still not very known to mathematicians. I even could not find them in modern textbooks on matrix theory or linear algebra. These results and their proofs (see [1], [2], [3]) are not very accessible for non-mathematicians. But they need the results. Applications can be found in system theory and in problems in mechanics concerning systems of differential equations. The aim of this paper is to give elementary proofs as well as a clear summary of the conditions. The basis of all proofs is the Jordan normal form. As we will see: every square matrix (real or complex) is a product of two symmetric (real resp. complex) matrices. However, not every complex square matrix is a product of two hermitian matrices.
2. Notations

A is a complex or real matrix of order $n \times n$;

$\Lambda$ is a diagonal matrix of eigenvalues; $A^T$ is the transpose of $A$;

$A^* = A^T$ the conjugate transpose of $A$;

$H$ denotes an hermitian matrix: $H^* = H$; $U$ a unitary matrix: $UU^* = I$;

$S$ a real symmetric matrix: $S^T = S$; $C$ a complex symmetric matrix $C^T = C$;

$A \sim D$ means: $A$ is similar to the matrix $D$ or $A = BDB^{-1}$;

$H > 0$ means $H$ is positive definite: for all vectors $x \neq 0 : x^* H x > 0$.

3. Preliminaries (for Theorems 1 and 3 see [4])

Theorem 1: Let $H_1$ be an hermitian matrix. Then there exists a unitary matrix $U$ such that $H_1 = U \Lambda U^*$ with $\Lambda$ real.

Moreover, is $H_1 > 0$ then all eigenvalues $\lambda_i$ are positive.

Theorem 2: Let $H_1 > 0$. Then there exists an $H > 0$ such that $H_1 = H^2$.

Proof: $H_1 = U \Lambda U^* = (U \Lambda^{1/2} U^*)(U \Lambda^{1/2} U^*) = H^2$ and $H > 0$.

Theorem 3: (Jordan normal form). Let $A$ be an arbitrary $n \times n$-matrix.

Then $A = B J B^{-1}$ where

$$J := \begin{bmatrix}
J_{k_1}(\lambda_1) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & J_{k_r}(\lambda_r)
\end{bmatrix}; \quad J_k(\lambda) \text{ is a } k \times k \text{-matrix}
$$

$J_1(\lambda) = \lambda$; The $\lambda_i$ are not necessarily different.
Definition: A Jordan matrix $J$ is called balanced when $J_k(\lambda)$ is a Jordan-block in $J$, $J_k(\bar{\lambda})$ is also in $J$. This means that each complex $\lambda$ and $\bar{\lambda}$ have the same "Jordan structure", and $J \sim J$, or equivalently $A \sim \bar{A}$.

4. Lemmas on factorization

**Lemma 1:** Every complex $n \times n$-matrix $A$ is a product of two complex symmetric matrices: $A = C_1 C_2$, where $C_1$ or $C_2$ is nonsingular.

**Proof:**

$$J_k(\lambda) = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & \lambda \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \lambda \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix} =: S_k C_k;$$

$$J = \begin{pmatrix}
S_k C_k \\
& 0 \\
& & S_k C_k \\
& & & 0 \\
& & & & S_k C_k
\end{pmatrix} =: \tilde{S} \tilde{C}.$$

$$A = B \tilde{S} \tilde{C} B^{-1} = (B \tilde{S} B^T) (B^{-T} \tilde{C} B^{-1}) =: C_1 C_2 \text{ with } C_1 \text{ nonsingular.}$$

**Corollary 1:** $A \sim A^T$.

**Proof:** $A = C_1 C_2$, suppose $C_1$ nonsingular

$$C_1^{-1} A C_1 = C_2 C_1 = A^T,$$

hence $A \sim A^T$.

*) Thanks to Dr. Laffey, Dublin, for this corollary and as a consequence, the improvement in the proof of Lemma 2,1.
Lemma 2: A complex matrix $A$ of order $n \times n$ is a product of two Hermitian matrices: $A = H_1 H_2$, where $H_1$ or $H_2$ is nonsingular, iff $A \simeq \overline{A}$.

Proof:

i: $A = H_1 H_2$; $A^* = H_2 H_1^{-1} (H_1 H_2) H_2 H_1 = H_1^{-1} A H_1$.

Hence $A^* \simeq A$. With Corollary 1: $A^* \simeq (A^*)^T = \overline{A}$, so $A \simeq \overline{A}$.

ii: $A \simeq \overline{A}$ or $A = BJ B^{-1}$ with $J$ balanced: for each $J_k(\lambda)$ in $J$ there is a $J_k(\bar{\lambda})$ in $J$. By permutation of the columns of $B$, it is always possible that $J_k(\bar{\lambda})$ comes directly after $J_k(\lambda)$ for each complex $\lambda$.
Hence $J = \tilde{S}\tilde{H}$ and $A = B\tilde{S}\tilde{H}^{-1} = (BSB^*) (B^{-1}\tilde{H}B^{-1}) = H_1H_2$

where $H_1$ is nonsingular.

**Corollary 2:** The characteristic polynomial of $H_1H_2$, $\det(H_1H_2 - \lambda I)$, has only real coefficients, and specially $\text{tr}(H_1H_2)$ and $\det(H_1H_2)$ are real.

**Example:**

$A = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \neq H_1H_2$ because $\text{tr} A = 2i$ is not real.

$A = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \neq H_1H_2$ because $\det A = 1 - i$ is not real.

$A = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} = H_1H_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix}$.

**Lemma 3:** Every complex matrix $A$ with real eigenvalues, is a product of two hermitian matrices: $A = H_1H_2$, where $H_1$ or $H_2$ is nonsingular.

**Proof:** This follows directly from Lemma 2. $A$ is real, so $J$ is real and $J = \bar{J}$. The condition for Lemma 2 is fulfilled.

**Lemma 4:** Every real $n \times n$-matrix $A$ is a product of two real matrices:

$A = S_1S_2$ where $S_1$ or $S_2$ is nonsingular.

**Proof 1:** $A = BJB^{-1}$. $A$ is real, $A = \bar{A}$. The condition of Lemma 2 holds.

By permutation of the columns of $B$, it is always possible that

$$
J = \begin{pmatrix}
J_{k_1}(\lambda) & \cdots & 0 \\
0 & \ddots & \cdots \\
0 & \cdots & J_{k_2}(\lambda)
\end{pmatrix}
$$

, with all real $J_{k}(\lambda)$ in the middle of $J$. 
From $AB = BJ$ we see that

$$Ab_i = \lambda_i b_i + \delta_i b_{i-1} \quad (\delta_i = 0 \text{ or } 1)$$

$$\bar{Ab}_i = \bar{\lambda}_i \bar{b}_i + \bar{\delta}_i \bar{b}_{i-1}.$$

This means that, if $b_i$ is a column of $B$, $\bar{b}_i$ also.

Then $B = (b_1 \ldots b_p \overline{b_1} \ldots \overline{b_1} \ldots b_q \overline{b_1} \ldots \overline{b_1})$; The $b_i, i = p + 1, \ldots, q$ are real columns corresponding with the real eigenvalues of $B$ (this set can be empty as well as the set $\{b_1, \ldots, b_p\}$).

As in the proof of Lemma 1, $J = \tilde{S} \tilde{C}$.

$$A = BS \tilde{C}B^{-1} = (B\tilde{S})^T (B^{-T}\tilde{C}B^{-1}) =: S_1 S_2 \text{ where } S_1 \text{ is nonsingular.}$$

Indeed:

$$S_1 := BSB^T = (b_1 \ldots b_p \overline{b_1} \ldots \overline{b_1} \ldots b_q \overline{b_1} \ldots \overline{b_1})(\overline{b_1} b \ldots \overline{b_1} b \ldots \overline{b_1} b) =$$

$$= \sum_{i=1}^{p} b_i b_{p+1-i} + \sum_{i=p+1}^{p+q} b_i b_{p+q+1-i} + \sum_{i=p+1}^{p+q} \bar{b}_i \bar{b}_{p+1-i} \text{ is real.}$$

$$S_2 = S_1^{-1} A \text{ is, a product of two real matrices, also real.}$$

Proof 2: (suggested by Dr. Laffey, Dublin).

With Lemma 1: $A = C_1 C_2$; suppose $C_1 = S_0 + i S_3$ nonsingular

($S_0, S_3$ real symmetric) $AC_1 = C_1 C_2 C_1 = C_1 A^T$; $A(S_0+iS_3) = (S_0+iS_3)A^T$;

$AS_0 = S_0 A^T$ and $AS_3 = S_3 A^T$.

So, for all real numbers $r$: $A(S_0 + r S_3) = (S_0 + r S_3)A^T$.

If $S_3 = 0$, then $C_1 = S_0$ real and $A = S_1 S_2$. So suppose $S_3 \neq 0$.

Define \( f(z) := \det(S_0 + z S_3) \); \( \det(S_0 + i S_3) = \det C_1 \neq 0 \).

Hence \( f(z) \) is not the zero-polynomial, or \( \exists r \in \mathbb{R} \text{ with } f(r) \neq 0 \)

or \( \det(S_0 + r S_3) \neq 0 \), \( S_0 + r S_3 =: S_1 \) nonsingular.

$$AS_1 = S_1 A^T; A = S_1 A^T S_1^{-1}; S_1 = S_1 S_2 \text{ with } S_2 := A^T S_1^{-1}; S_2^{-1} A = A^T S_1^{-1} = S_2.$$
**Lemma 5:** A complex $n \times n$-matrix $A$ is a product of two hermitian matrices:

$$A = H_1H_2,$$

where $H_1$ or $H_2$ is positive definite, iff $A$ is similar to a real or $A \simeq \Lambda$ real.

**Proof:**

i: Only if: suppose $H_1 > 0$. $H_1 = H^2$ (Theorem 2); $H_1H_2 = H(HH_2H)^{-1}$. $HH_2H$ is hermitian, so $= U \Lambda U^*$ with $\Lambda$ real (Theorem 1).

$$A = H(U \Lambda U^*)^{-1} = (HU) \Lambda (HU)^{-1} = B \Lambda B^{-1} \text{ or } A \simeq \Lambda \text{ real.}$$

ii: If: $A \simeq \Lambda$ real; $A = B \Lambda B^{-1}$; $A = (BB^*)(B^* \Lambda B^{-1}) =: H_1H_2$

with $H_1 > 0$.

**Remark:** If $H_1$ is semi-positive definite, then the "only if" part does not hold:

Example: $H_1H_2 = \begin{pmatrix} 0 & 0 \\ 0 & \phi \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}$ is defective, hence not similar to a diagonal matrix.

**Lemma 6:** Every real $n \times n$-matrix $A$ is a product of two real symmetric matrices: $A = S_1S_2$, where $S_1$ or $S_2$ is positive definite, iff $A$ is similar to a real.

**Proof:** This follows directly from the proof in Lemma 5:

i: Replace each $H$ by $S$ and $U$ by an orthogonal matrix $G$.

ii: $A$ and $\Lambda$ real, hence $B$ is real. So $H_i = S_1$ and $A = S_1S_2$.

**Remark:** If we weaken the iff-condition and cancel the word ("real", then of course $A \neq H_1H_2$ (see Lemma 5), but $A = H_1N_2$ with $H_1 > 0$ and $N_2$ such that $N_2H^*_1N_2 = N_2^*H_1N_2$. 
Summary

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References


