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A SIMPLE PROOF OF HEYMANN'S LEMMA

of

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Abstract. Heymann's lemma is proved by a simple induction argument.

The problem of pole assignment by state feedback in the system

\[ x_{k+1} = Ax_k + Bu_k ; \quad (k = 0, 1, \ldots) \]

where \( A \) is an \( n \times n \)-matrix and \( B \) an \( n \times m \)-matrix, has been considered by many authors. The case \( m = 1 \) has been dealt with by Rissanen [3] in 1960. In 1964 Popov [2] showed the pole assignability for complex systems (more generally systems over an algebraically closed field). In 1967 Wonham gave a proof valid for real systems (or more generally for systems over an infinite field). Finally, in 1968, Heymann [1] gave a proof which is valid for systems over an arbitrary field. Heymann's proof depends on the following result.

**Lemma 1.** If \((A, b)\) is controllable and \( b = Bv \neq 0 \), then there exists \( F \) such that \((A + BF, b)\) is controllable.

By means of this result the multivariable problem can be reduced to the single variable problem.

It is the aim of this correspondence to give a simple proof of this lemma. The result follows immediately from

**Lemma 2.** If \((A, B)\) is controllable and \( b = Bv \neq 0 \), then there exists \( u_1, \ldots, u_{n-1} \) such that the sequence defined by

\[ \begin{align*}
    x_1 &= b, \\
    x_{k+1} &= Ax_k + Bu_k ,
\end{align*} \]

for \( k = 1, \ldots, n - 1 \) is independent.

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Indeed, if Lemma 2 is shown we can choose $u_n$ arbitrary and define $F$ by $Fx_k = u_k$. Then it is easily seen that $(A + BF)^b = x_k$, so that $(A + BF, b)$ is controllable.

PROOF OF LEMMA 2. We proceed stepwise. $x_1 \neq 0$ and hence independent. Suppose that $x_1, \ldots, x_k$ have been constructed according to (1) and are independent. Denote by $\mathcal{L}$ the linear space generated by $x_1, \ldots, x_k$. We have to choose $u_k$ such that $x_{k+1} = Ax_k + Bu_k \notin \mathcal{L}$. If this is not possible, then

$$ (2) \quad Ax_k + Bu \in \mathcal{L} $$

for all $u$. Choosing in particular $u = 0$ we find

$$ (3) \quad Ax_k \in \mathcal{L} $$

and consequently, by the linearity of $\mathcal{L}$, $Bu \in \mathcal{L}$ for all $u$. That is, $\text{im} B \subseteq \mathcal{L}$. Also, for $i < k$ we have

$$ Ax_i = x_{i+1} - Bu_i \in \mathcal{L} $$

Hence $Ax_i \in \mathcal{L}$ for $i = 1, \ldots, k$, and, consequently, $\mathcal{L}$ is $A$-invariant. From the controllability of $(A, B)$ it follows that $\mathcal{L}$ must be the whole state space, which implies that $k = n$. \hfill $\square$

REMARK. In [1] and in [5, Lemma 2.2] proofs of Lemma 1 were given by constructing a particular sequence $u_k$ satisfying the condition of Lemma 2. These constructions may suggest that such a special $u_k$ is essential for the calculation of $F$, which is not the case as follows from the proof of Lemma 2. It also follows that the $u_k$'s can be constructed recursively in the following sense: Once $u_1, \ldots, u_{k-1}$ have been chosen so as to render $x_1, \ldots, x_k$ independent, one can always continue the construction of the remaining $u_k$'s.

The may be useful when it comes to an actual numerical computation of $F$. 
REFERENCES


