Solution to Problem 80-1: A determinant and an identity

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Editorial note. Both conjectures have been proved valid by the proposers and J. BUSTOZ (Arizona State University) and S. LINNAINMAA (University of Helsinki) in their joint paper, *Improved trailing digits estimates applied to optimal computer arithmetic*, J. Assoc. Comput. Mach., 26 (1979), pp. 716–730 (in particular, see pp. 721–723).

A Matrix Eigenvalue Problem

Problem 79-2, by G. EFROYMSON, A. STEGER and S. STEINBERG (University of New Mexico).

Let M_n denote the $n \times n$ matrix whose (j, k) entry $M_n(j, k)$ is given by

$$\omega^{(j-1)(k-1)}/\sqrt{n}, \quad 1 \leq j, k \leq n,$$

where $\omega = e^{2\pi i/n}$. Determine all the eigenvalues of M_n . The matrix M_n arises in some work on finite Fourier transforms.

(Previous solution given in this Journal, 22 (1980), pp. 99–100).

Comment by B. PARLETT (University of California, Berkeley, California). Another solution appears in L. AUSLANDER and R. TOLIMIERI, *Is computing with the finite Fourier transform pure or applied mathematics?*, Bull. Amer. Math. Soc., 1 (1979), pp. 855–857.

A Determinant and an Identity

Problem 80-1, by A. V. BOYD (University of the Witwatersrand, Johannesburg, South Africa).

(a) Prove that

$$\det |A_{rs}| = (-1)^{n+1} (2^{2n} - 2) B_{2n} / (2n)!$$

where $r, s = 1, 2, \dots, n$,

$$A_{rs} = \begin{cases} 1/(2r-2s+3)!, & s \leq r+1, \\ 0, & s > r+1, \end{cases}$$

and B_n is the Bernoulli number defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}.$$

(b) Prove that if n is odd,

$$t^n = \sum_{m=0}^{(n-1)/2} \frac{1-2^{2m-1}}{n-2m+1} \binom{n}{2m} h^{2m-1} B_{2m} \{(t+h)^{n+1-2m} - (t-h)^{n+1-2m}\}.$$

Solution by O. G. RUEHR (Michigan Technological University).

From the given generating function for the Bernoulli numbers, the well-known expansion, $x \operatorname{csch} x = \sum_{n=0}^{\infty} q_n x^{2n}$, $q_n = (2-2^{2n})B_{2n}/(2n)!$, is easily obtained. Part (a) is now an immediate consequence of Wronski's determinant for the reciprocal of a power series, i.e.,

$$\left[\sum_{n=0}^{\infty} q_n t^n \right]^{-1} = \sum_{n=0}^{\infty} c_n t^n \Rightarrow c_n = \frac{(-1)^n}{q_0^{n+1}} \begin{vmatrix} q_1 & q_0 & 0 & \cdots & 0 \\ q_2 & q_1 & q_0 & \cdots & 0 \\ \vdots & & \ddots & & q_0 \\ q_n & q_{n-1} & q_{n-2} & \cdots & q_1 \end{vmatrix}.$$

Part (b) results from the elementary identity

$$[(xh \operatorname{csch} xh)/2h][e^{x(t+h)} - e^{x(t-h)}] = xe^{xt}$$

upon multiplying the series corresponding to the bracketed quantities and employing the formula

$$\sum_{m=0}^{\infty} A_m \sum_{n=1}^{\infty} B_n = \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} A_m B_{n+1-2m}.$$

The stated result (b) is correct for nonnegative integers n , provided that $(n-1)/2$ is replaced by $[n/2]$ as the upper limit of summation.

Remark. This solver recently rediscovered the following generalization of Wronski's determinant which had been quoted by Muir [1] without proof. Let

$$\left(\sum_{n=0}^{\infty} q_n t^n \right)^{-m} = \sum_{n=0}^{\infty} c_n(m) t^n.$$

Then, $c_n(m) = ((-1)^n / q_0^{n+m}) \det |A_{rs}(m)|$, where

$$A_{rs} = \begin{cases} [(r-s+1)m + (s-1)]q_{r-s+1}/r, & s \leq r+1, \\ 0, & s > r+1. \end{cases}$$

The proof [2], depends upon elementary properties of lower Hessenberg matrices and uses the J. C. P. Miller formula [3].

REFERENCES

1. T. MUIR, *A Treatise on the Theory of Determinants*, Dover, New York, 1960, p. 722.
2. O. G. RUEHR, *A Generalization of Wronski's Determinant*, unpublished memorandum.
3. P. HENRICI, *Applied and Computational Complex Analysis*, Wiley-Interscience, New York, 1974, p. 42.

Solution by O. P. LOSSERS (Technische Hogeschool Eindhoven, the Netherlands).

a) Let

$$D_n = \det |A_{rs}|, \quad D_0 = 1.$$

Then by expanding the determinant by the last column and iterating this procedure, we find

$$D_n = \sum_{i=0}^n \frac{(-1)^{i-1}}{(2i+1)!} D_{n-i}$$

i.e.,

$$(1) \quad \sum_{i=0}^n \frac{(-1)^{i-1}}{(2i+1)!} D_{n-i} = 0.$$

Since one easily checks the assertion for $n=1, 2$ it is sufficient to see whether the asserted value of D_n also satisfies (1), i.e., whether for $n > 0$ we have

$$(2) \quad \sum_{i=0}^n \binom{2n+1}{2i+1} (2^{2(n-i)} - 2) B_{2(n-i)} = 0.$$

For the well-known Bernoulli polynomials, we have

$$\begin{aligned} (3) \quad B_{2n+1}(t) &= \sum_{k=0}^{2n+1} \binom{2n+1}{k} B_{2n+1-k} t^k \\ &= -\frac{1}{2} \binom{2n+1}{2n} t^{2n} + \sum_{i=0}^n \binom{2n+1}{2i+1} B_{2(n-i)} t^{2i+1}. \end{aligned}$$

Now substitute $t = \frac{1}{2}$ and $t = 1$ in (3) and use $B_{2n+1}(\frac{1}{2}) = B_{2n+1}(1) = 0$ ($n > 0$).

Then (2) follows by subtracting the two equations.

b) The right-hand side of (b) has the form

$$\sum_{j=0}^n a_j h^j t^{n-j}.$$

Clearly $a_j = 0$ if j is odd. For j even, we find

$$a_j = \frac{1}{n} \binom{n+1}{2j+1} \sum_{m=0}^j (1-2^{2m-1}) \binom{2j+1}{2m} B_{2m},$$

which is 0 if $j > 0$ according to (2). Substituting $h = 0$, we find for the right-hand side

$$\lim_{h \rightarrow 0} \frac{1-2^{-1}}{n+1} \frac{(t+h)^{n+1} - (t-h)^{n+1}}{h} = t^n.$$

This proves (b).

Also solved by C. GIVENS (Michigan Technological University), A. A. JAGERS (Technische Hogeschool Twente, Enschede, the Netherlands), S. L. LEE (University of Alberta) *and the proposer*.

Additionally, LEE provides the generalization

$$\det |A_{rs}| = (-1)^{k(2n+k+1)/2} \det |B_{ij}|,$$

where

$$A_{rs} = 1/[2(r-s+k+1)!], \quad r, s = 1, 2, \dots, n-k+1, \quad k = 0, 1, \dots, n$$

($1/p! = 0$ for $p < 0$), and

$$B_{ij} = \beta_{2(n+2-i-j)}, \quad i, j = 1, 2, \dots, k,$$

where

$$\beta_{2m} = (2^m - 2)B_{2m}/(2m)!.$$

His proof uses Sylvester's identity and induction.

A Matrix Stability Problem

Problem 80-3*, by K. SOURISSEAU (University of Minnesota) and M. F. DOHERTY (University of Massachusetts).

Let

$$J = \begin{bmatrix} A_1 & B_1 & & & & \\ C_2 & A_2 & B_2 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & C_n & A_n & B_n \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & B_{N-1} \\ & & & & & & C_N & A_N \end{bmatrix}$$