Solution to Problem 80-1: A determinant and an identity

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Editorial note. Both conjectures have been proved valid by the proposers and J. Bustoz (Arizona State University) and S. Linna (University of Helsinki) in their joint paper, Improved trailing digits estimates applied to optimal computer arithmetic, J. Assoc. Comput. Mach., 26 (1979), pp. 716–730 (in particular, see pp. 721–723).

A Matrix Eigenvalue Problem

Problem 79-2, by G. Efroymson, A. Steger and S. Steinberg (University of New Mexico).

Let $M_n$ denote the $n \times n$ matrix whose $(j, k)$ entry $M_n(j, k)$ is given by

$$\omega^{((j-1)(k-1))/\sqrt{n}}, \quad 1 \leq j, k \leq n,$$

where $\omega = e^{2\pi i/n}$. Determine all the eigenvalues of $M_n$. The matrix $M_n$ arises in some work on finite Fourier transforms.


Comment by B. Parlett (University of California, Berkeley, California).


A Determinant and an Identity

Problem 80-1, by A. V. Boyd (University of the Witwatersrand, Johannesburg, South Africa).

(a) Prove that

$$\det |A_{rs}| = (-1)^{n+1}(2^{2n} - 2)B_{2n}/(2n)!$$

where $r, s = 1, 2, \ldots, n$,

$$A_{rs} = \begin{cases} 1/(2r - 2s + 3)!, & s \leq r + 1, \\ 0, & s > r + 1, \end{cases}$$

and $B_n$ is the Bernoulli number defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}.$$

(b) Prove that if $n$ is odd,

$$t^n = \sum_{m=0}^{(n-1)/2} \frac{1 - 2^{2m-1}}{n - 2m + 1} \binom{n}{2m} h^{2m-1} B_{2m}((t + h)^n + 1 - 2^{n+1} - (t - h)^n + 1 - 2^{n+1}).$$

Solution by O. G. Ruehr (Michigan Technological University).

From the given generating function for the Bernoulli numbers, the well-known expansion, $x \csch x = \sum_{n=0}^{\infty} q_n x^{2n}$, $q_n = (2 - 2^{2n})B_{2n}/(2n)!$, is easily obtained. Part (a) is now an immediate consequence of Wronski's determinant for the reciprocal of a power series, i.e.,

$$\left[ \sum_{n=0}^{\infty} q_n t^n \right]^{-1} = \sum_{n=0}^{\infty} c_n t^n \Rightarrow c_n = \frac{(-1)^n q_1}{q_{n+1}} q_0 \cdots q_0.$$

Part (b) results from the elementary identity

$$[(xh \csch xh)/2h][e^{x(t+h)} - e^{x(t-h)}] = xe^{xt}.$$
upon multiplying the series corresponding to the bracketed quantities and employing
the formula
\[
\sum_{m=0}^{\infty} A_m \sum_{n=1}^{\infty} B_n = \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} A_m B_{n+1-2m}.
\]
The stated result (b) is correct for nonnegative integers \( n \), provided that \( (n-1)/2 \) is
replaced by \([n/2]\) as the upper limit of summation.

Remark. This solver recently rediscovered the following generalization of Wronski's determinant which had been quoted by Muir [1] without proof. Let
\[
\left( \sum_{n=0}^{\infty} q_n t^n \right)^{-m} = \sum_{n=0}^{\infty} c_n(m) t^n.
\]
Then, \( c_n(m) = (-1)^n / q_0^{n+m} \) \det |A_{rs}(m)|, where
\[
A_{rs} = \begin{cases} 
[(r-s+1)m+(s-1)]q_{r-s+1}/r, & s \leq r+1, \\
0, & s > r+1.
\end{cases}
\]
The proof [2], depends upon elementary properties of lower Hessenberg matrices and
uses the J. C. P. Miller formula [3].

REFERENCES


Solution by O. P. Lossers (Technische Hogeschool Eindhoven, the Netherlands).

a) Let
\[
D_n = |A_{rs}|, \quad D_0 = 1.
\]
Then by expanding the determinant by the last column and iterating this procedure, we
find
\[
D_n = \sum_{i=0}^{n} \frac{(-1)^{i-1}}{(2i+1)!} D_{n-i}
\]
i.e.,
\[
\sum_{i=0}^{n} \frac{(-1)^{i-1}}{(2i+1)!} D_{n-i} = 0.
\]
Since one easily checks the assertion for \( n = 1, 2 \) it is sufficient to see whether the
asserted value of \( D_n \) also satisfies (1), i.e., whether for \( n > 0 \) we have
\[
\sum_{i=0}^{n} \frac{(-1)^{i-1}}{2i+1} (2^{2(n-i)} - 2) B_{2(n-i)} = 0.
\]
For the well-known Bernoulli polynomials, we have
\[
B_{2n+1}(t) = \sum_{k=0}^{2n+1} \binom{2n+1}{k} B_{2n+1-k} t^k
\]
\[
= -\frac{1}{2} \left( \frac{2n+1}{2n} \right) t^{2n} + \sum_{i=0}^{n} \binom{2n+1}{2i+1} B_{2(n-i)} t^{2i+1}.
\]
Now substitute $t = \frac{1}{2}$ and $t = 1$ in (3) and use $B_{2n+1}(\frac{1}{2}) = B_{2n+1}(1) = 0$ ($n > 0$).

Then (2) follows by subtracting the two equations.

b) The right-hand side of (b) has the form

$$\sum_{j=0}^{n} a_j h^{1-t^{n-j}}.$$ 

Clearly $a_j = 0$ if $j$ is odd. For $j$ even, we find

$$a_j = \frac{1}{n} \binom{n+1}{2j+1} \sum_{m=0}^{j} (1-2^{2m-1})(\frac{2j+1}{2m})B_{2m},$$

which is 0 if $j > 0$ according to (2). Substituting $h = 0$, we find for the right-hand side

$$\lim_{h \to 0} \frac{1-2^{-1}((t+h)^{n+1}-(t-h)^{n+1})}{n+1} = t^n.$$ 

This proves (b).

*Also solved by C. GIVENS (Michigan Technological University), A. A. JAGERS (Technische Hogeschool Twente, Enschede, the Netherlands), S. L. LEE (University of Alberta) and the proposer.*

Additionally, LEE provides the generalization

$$\det |A_{rs}| = (-1)^{k(2n+k+1)/2} \det |B_{ij}|,$$

where

$$A_{rs} = 1/[2(r-s+k)+1]!, \quad r, s = 1, 2, \ldots, n-k+1, \quad k = 0, 1, \ldots, n$$

(1/p! = 0 for p < 0), and

$$B_{ij} = \beta_{2(n+2-i-j)}, \quad i, j = 1, 2, \ldots, k,$$

where

$$\beta_{2m} = (2^m - 2)B_{2m}/(2m)!.$$ 

His proof uses Sylvester's identity and induction.

**A Matrix Stability Problem**

*Problem 80-3*, by K. SOURISSEAU (University of Minnesota) and M. F. DOHERTY (University of Massachusetts).

Let

$$J = \begin{bmatrix}
A_1 & B_1 \\
C_2 & A_2 & B_2 \\
& & \ddots \\
& & & C_n & A_n & B_n \\
& & & & & \ddots \\
& & & & & & C_N & A_N
\end{bmatrix}$$