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Published: 01/01/1981

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

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Link to publication

Citation for published version (APA):
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by

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Eindhoven, October 1981

The Netherlands
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by

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Abstract

Harsanyi introduced regular equilibrium points in [3] and proved that for almost all noncooperative n-person games in normal form all equilibrium points are regular. In this paper it is shown that regular equilibrium points are essential, quasi-strong, proper and perfect. Hence, for almost all games in normal form all equilibrium points are quasi-strong, essential, proper and perfect.

1. Introduction

It is well-known that for noncooperative n-person games in normal form not all Nash equilibrium points are equally suited to be chosen as the solution. The reason for this fact is, that (in general) some equilibrium points are more stable than others. Therefore, in the literature various refinements of the equilibrium concept have been introduced. Among these refinements there are:

- the essential equilibrium point, introduced by Wu Wen-tsün and Jiang Jia-he [8],
- the quasi-strong equilibrium point, introduced by Harsanyi [2],
- the regular equilibrium point, introduced by Harsanyi [3],
- the perfect equilibrium point, introduced by Selten [7], and
- the proper equilibrium point, introduced by Myerson [5].
Loosely speaking we may say that for a general n-person game \( \Gamma \) in normal form:

- an equilibrium point \( p \) is essential, if any game near to \( \Gamma \) has an equilibrium point near to \( p \),
- an equilibrium point is quasi-strong, if each player uses each pure best reply (against the strategy combination played by the others) with a positive probability,
- an equilibrium point \( p \) is regular, if the Jacobian of a certain mapping, associated with the game, evaluated at \( p \) is nonsingular,
- an equilibrium point is perfect, if each player plays a strategy, which is a best reply against small perturbations of the strategy combination used by the others, and
- an equilibrium point is proper, if each player plays a strategy, which is a best reply against special small perturbations of the strategy combination used by the others.

In the literature the following results can be found with respect to these refinements:

(1.1) the set of games for which all equilibrium points are essential is dense in the set of all games; if a game has finitely many equilibrium points, then it has at least one essential equilibrium point [8],

(1.2) for almost all games all equilibrium points are quasi-strong and regular [3],

(1.3) any game possesses at least one perfect equilibrium point [7], and

(1.4) any game possesses at least one proper equilibrium point; if an equilibrium point is proper, then it is perfect [5].
In this paper we concentrate on regular equilibrium points. However, in this paper a slightly different definition of regularity is given. The reason for this fact is, that (1.2) is incorrect when regularity is defined as in [3] (example 3.6), whereas (1.2) is correct, when regularity is defined as in this paper. The main results of the paper are that regular equilibrium points are quasi-strong, that regular equilibrium points are essential and that regular equilibrium points are proper (and hence also perfect). Hence, we can conclude from (1.2) that for almost all games all equilibrium points are essential, proper and perfect.

The organization of the paper is as follows: In section 2 the notation and some basic concepts are introduced. In section 3 regular equilibrium points are defined and some properties of these equilibria are derived. The main theorems are proved in section 4.

2. Preliminaries

A finite noncooperative n-person game in normal form is a 2n-tuple

\[ \Gamma = (A_1, \ldots, A_n, U_1, \ldots, U_n) \]

where (for any \( i \in \{1, \ldots, n\} \)) \( A_i \) is a finite \( n \)-set and \( U_i \) is a function, \( U_i : \prod A_i \to \mathbb{R} \). \( A_i \) is the set of all pure strategies of player \( i \) and \( U_i \) is the payoff function of this player. It will be convenient to have the sets \( A_i \) disjoint, so we will assume (without loss of generality) that \( \bigcap_{i=1}^n A_i = \emptyset \). In the following, we will introduce some notation with respect to such a game \( \Gamma \). Mostly we will follow Harsanyi [3].

Let \( N = \{1, \ldots, n\} \). For a finite set \( S \) we denote by \( |S| \) the number of elements of this set. For \( i \in N \), let \( K_i = |A_i| \). Let \( A = \prod_{i=1}^n A_i \) and \( K = |A| = \prod_{i=1}^n K_i \).
We assume that the elements of A_i are numbered consecutively as 
\[ a_1, \ldots, a_i, \ldots, a_n \]. If no confusion can result we will write k instead 
of \( a_i \). A **mixed strategy** of player i is a probability distribution on A_i.
Hence, any mixed strategy of player i can be identified with a probability 
vector \( p_i = (p_i^1, \ldots, p_i^k, \ldots, p_i^K_i) \). We write:

\[
(2.1) \quad p_i = \left\{ p_i \in \mathbb{R}^{K_i} ; \sum_{k=1}^{K_i} p_i^k = 1, \ p_i^k \geq 0 \text{ for all } k \in \{1, \ldots, K_i\} \right\}
\]

We will write \( P = \prod_{i=1}^{n} P_i \) and \( \bar{P} = \prod_{j \neq i}^{n} P_j \). P is a subset of \( \mathbb{R}^{K^*} \), where 
\( K^* = \sum_{i=1}^{n} K_i \). If \( x \in \mathbb{R}^{K^*} \), then we will write \( x = (x_i, \bar{x_i}) \), where \( x_i \in \mathbb{R}^{K_i} \)
and \( \bar{x_i} \in \mathbb{R}^{K - K_i} \). Hence, if \( p \in P \), then \( p = (p_i, \bar{p}_i) \) with \( p_i \in P_i \) and 
\( \bar{p}_i \in \bar{P}_i \). The set \( C(p_i) \) of all pure strategies to which \( p_i \) assigns a 
positive probability is called the **carrier** of \( p_i \). \( p_i \) is completely mixed 
if \( C(p_i) = A_i \). If \( p = (p_1, \ldots, p_i, \ldots, p_n) \in P \), then \( C(p) = \bigcup_{i=1}^{n} C(p_i) \) and 
p is completely mixed if \( C(p) = \bigcup_{i=1}^{n} A_i \).

Let \( a = (a_1, \ldots, a_i, \ldots, a_n) \in A \) and \( x_i = (x_i^1, \ldots, x_i^k, \ldots, x_i^{K_i}) \in \mathbb{R}^{K_i} \). We 
define \( q_i^a(x_i) := x_i^k \), where \( k \) is such that \( a_i = a_i^k \). Hence, if \( p_i \in P_i \), 
then \( q_i^a(p_i) \) is the probability that \( p_i \) assigns to \( a_i \). So, in particular, 
if \( a_i^k \in A_i \), we have

\[
q_i^a(a_i^k) = 1, \quad \text{if } a_i = a_i^k,
\]
\[
q_i^a(a_i^k) = 0, \quad \text{if } a_i \neq a_i^k.
\]

If the players play the mixed strategy combination \( p = (p_1, \ldots, p_i, \ldots, p_n) \in P \),
then the payoff to player $i$ will be:

\[(2.2) \quad U_i(p) = \sum_{a \in A} \left[ \prod_{j=1}^{K} q_j^a(p_j) \right] U_i(a). \]

Formula (2.2) can be interpreted in terms of payoffs only if $p \in P$. However, since $q_j^a(p_j)$ is well defined for any $p_j = \mathbb{R}^K$ we actually have that by (2.2) a mapping $U_i : \mathbb{R}^K \to \mathbb{R}$ is defined. It will be clear that $U_i$ is a polynomial and hence that $U_i$ is infinitely often differentiable.

A strategy $p_i \in P_i$ is a best reply against a strategy combination $\bar{p}_i \in \bar{P}_i$ if

\[(2.3) \quad U_i(p_i, \bar{p}_i) = \max_{p'_i \in P_i} U_i(p'_i, \bar{p}_i). \]

A pure best reply against $\bar{p}_i$ is a best reply, which is an element of $A_i$. The set of all pure best replies against $\bar{p}_i$ will be denoted by $B(\bar{p}_i)$. A strategy combination $p = (p_1, \ldots, p_i, \ldots, p_n)$ is an equilibrium point ([6]) of $\Gamma$ if $p_i$ is a best reply against $\bar{p}_i$ for all $i \in N$. It is easily seen, that $p$ is an equilibrium point if and only if

\[(2.4) \quad C(p_i) \subset B(\bar{p}_i) \quad \text{for all} \quad i \in N. \]

We will denote the set of equilibrium points of $\Gamma$ by $E(\Gamma)$. We have $E(\Gamma) \neq \emptyset$ ([6]).
Let $G = G(n,K_1,\ldots,K_n)$ be the set of all games in which player $i$ has exactly $K_i$ pure strategies (for any $i \in N$). If we impose a fixed ordering on $N \times A$ then each specific game $\Gamma \in G$ is completely characterized by its $nK$-dimensional payoff vector $u = \langle u_i(a) \rangle_{(i,a) \in N \times A}$, and we will sometimes write $\Gamma_u$ for the game which has $u$ as its payoff vector. Hence, the set of all games of a particular size can be viewed as an Euclidian space.

For any $m \in N$, we will use $\rho$ to denote the Euclidean distance on $\mathbb{R}^m$ and if $x \in \mathbb{R}^m$ and $\varepsilon > 0$, then $\rho_\varepsilon(x) = \{y \in \mathbb{R}^m; \rho(x,y) < \varepsilon\}$. We say that a certain mathematical statement $S$ is true for almost all games $\Gamma \in G$, if the set of all games for which $S$ is false is (in $\mathbb{R}^{nK}$) a closed set with Lebesgue-measure zero.

We close this section by giving the formal definitions of the refinements of the equilibrium concept, which were mentioned in the introduction. Let $\Gamma \in G$ be fixed.

$p \in P$ is an essential equilibrium point of $\Gamma$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $E(\Gamma') \cap \rho_\varepsilon(p) \neq \emptyset$ for any $\Gamma' \in \rho_\delta(\Gamma)$. There exist games without essential equilibrium points, as we see by considering the two person game $\Gamma$ with $|A_1| = |A_2| = 2$ and $U_i(a_1^k,a_2^l) = 1$, for all $i,k,l \in \{1,2\}$.

$p \in P$ is a quasi-strong equilibrium point of $\Gamma$, if $C(p_i) = B(p_i)$ for all $i \in N$. It is still an open question whether any game possesses at least one quasi-strong equilibrium point.

$p \in P$ is a perfect equilibrium point of $\Gamma$, if there exists a sequence $(p(k))_{k \in N}$ of completely mixed strategy combinations, which converges to $p$, such that $p_i$ is a best reply against $\overline{p}_i(k)$ for any $i \in N$ and $k \in N$.

Any game possesses at least one perfect equilibrium point ([7]).
p \in P is an \( \varepsilon \)-proper equilibrium point \((\varepsilon > 0)\) if

i) \( p \) is completely mixed, and

ii) if \( U_i(a_i^{k}, p_i) < U_i(a_i^{\ell}, p_i) \), then \( p_i^{k} \leq \varepsilon p_i^{\ell} \); for all \( i \in \mathbb{N} \) and \( k, \ell \in \{1, \ldots, K_i\} \).

p \in P is a proper equilibrium point of \( \Gamma \), if there exists a sequence \( \{\varepsilon_k, p(k)\}_{k \in \mathbb{N}} \) such that \( \varepsilon_k \) converges to 0, \( p(k) \) converges to \( p \) and \( p(k) \) is an \( \varepsilon_k \)-proper equilibrium point of \( \Gamma \). Any game possesses at least one proper equilibrium point and any proper equilibrium point is perfect ([5]).

3. Regular equilibrium points

In this section the concept of regular equilibrium points is introduced. The definition of regularity given in this paper is slightly different from the definition given in [3]. The reason for such a different definition is illustrated in example 3.6. In this example it is shown that (1.2) is not correct if regularity is defined as in [3]. However, from Harsanyi's analysis ([3]) it easily follows that (1.2) is indeed correct if regularity is defined as in this paper. Furthermore, we derive in this section some elementary properties of regular equilibrium points and investigate what regularity means for bimatrix games.

We consider the set \( G = G(n, K_1, \ldots, K_n) \) of all games of a particular size. Let \( K = \prod_{i=1}^{n} K_i \) and \( K^* = \bigcup_{i=1}^{n} K_i \). Let \( u \in \mathbb{R}^{nK} \) be the payoff vector of some game \( \Gamma \in G \). Let \( a = (a_1, \ldots, a_i, \ldots, a_n) \in A \). We define a mapping
\[ a_F(x;u) : \mathbb{R}^K \to \mathbb{R}^K \] by:

\[ a_{F_i}^k(x;u) = x_i \sum_{k=1}^{k_i} \left[ U_i(a_i, x_i) - U_i(a_{i_k}, x_i) \right], \quad i \in N, \text{ and for each } i: \]

\[ k \in \{1, \ldots, k_i\}, \; k \neq k_i. \]

\[ a_{F_i}^k(x;u) = \sum_{k=1}^{k_i} x_i^{k_i} - 1, \quad i \in N. \]

We will sometimes write \( a_{F_i}^k(x;u) \) for \( U_i(a_{i_k}, x_i) - U_i(a_i, x_i) \). From (2.4) we deduce that for any \( a \in A \) and \( u \in \mathbb{R}^{nK} \):

\[ E(\Gamma_u) \subset \{ x \in \mathbb{R}^K \ ; \ a_F(x;u) = 0 \}. \]

The inclusion in (3.3) may be strict, since \( a_F(a';u) = 0 \) for all \( a, a' \in A \) and \( u \in \mathbb{R}^{nK} \). By \( J^a(p;u) \) we will denote the Jacobian of \( a_F(\cdot;u) \) evaluated at \( p \). Hence,

\[ J^a(p;u) = \left. \frac{\partial a_F(x;u)}{\partial x} \right|_{x=p}. \]

Furthermore, if \( M \) is a matrix, then \( |M| \) is its determinant.

**Definition 3.5.** Let \( \Gamma \in G \), with payoff vector \( u \in \mathbb{R}^{nK} \). Let \( p \) be an equilibrium point of \( \Gamma \). \( p \) is a regular equilibrium point if \( |J^a(p;u)| \neq 0 \) for some \( a \in C(p) \).
Harsanyi [3] defined regular equilibrium points as equilibrium points $p$ for which $|J^a(p;u)| \neq 0$, where $a = (a_1^1, \ldots, a_i^1, \ldots, a_n^1)$. The difference between these two regularity concepts is illustrated in example 3.6.

**Example 3.6.** Consider the set $G(1,3)$ of 1 person games in which player 1 has exactly 3 pure strategies $a_1^1, a_1^2, a_1^3$. We will write $u_k$ for $U_1(a_1^k)$ ($k \in \{1,2,3\}$). Let $U \subseteq G(1,3)$ be defined by:

$$U = \{(u_1,u_2,u_3) ; u_1, u_2 \in [0,1] ; u_3 \in [2,3]\}.$$  

Each element of $U$ has exactly one equilibrium point, namely $p = (0,0,1)$. If $u \in U$, $a = a_1^1$ and $a = a_1^3$ we have:

$$J^a(p;u) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ u_3 - u_1 & 0 & 0 \end{pmatrix} ; \quad J^a(p;u) = \begin{pmatrix} u_1 - u_3 & 0 & 0 \\ 0 & u_2 - u_3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Hence, for any $u \in U$ we have $|J^a(p;u)| = 0$ and $|J^a(p;u)| \neq 0$.

In example 3.6 we see that (1.2) is not correct if regularity is defined as in [3] (the set $U$ has positive Lebesgue-measure in $G(1,3)$). However, in his analysis Harsanyi everywhere implicitly assumes that any equilibrium point $p$ is such that $p_i^1 > 0$ for all $i \in N$, and so he actually works with our definition of regularity. Since Harsanyi's arguments can be easily adjusted to our definition of regularity, we can conclude:

**Theorem 3.7.** For almost all games all equilibrium points are regular.
To check whether an equilibrium point is regular one has to find some \( a \in C(p) \) with \( |J^a(p;u)| \neq 0 \). In theorem 3.8 we show that \( |J^a(p;u)| \neq 0 \) for all \( a \in C(p) \), if \( p \) is regular. Let \( a = (a_1, \ldots, a_i, \ldots, a_n) \in A \) and \( p = (p_1, \ldots, p_i, \ldots, p_n) \in P \). In Theorem 3.8 we write \( J^a(p;u) \) for the matrix that results from \( J^a(p;u) \) when we cross out the rows and columns corresponding to the pure strategies not belonging to \( C(p) \).

Theorem 3.8. Let \( \Gamma \in G \) with payoff vector \( u \in \mathbb{R}^{nK} \). Let \( a = (a_1, \ldots, a_n) \in A \) and \( a' = (a_1', \ldots, a_n') \in A \). Let \( p \) be an equilibrium point of \( \Gamma \). We have:

i) If \( a \in C(p) \), then

\[
|J^a(p;u)| = \left\{ \prod_{i=1}^{n} \prod_{k \in C(p_i)} \delta_{k_i}^i (p_i;u) \right\} |J^a(p;u)|
\]

ii) If \( a, a' \in C(p) \), then \( |J^a(p;u)| = 0 \) if and only if \( |J^{a'}(p;u)| = 0 \).

iii) If \( a \in C(p) \), and \( p \) is not quasi-strong, then \( |J^a(p;u)| = 0 \); hence any regular equilibrium point is quasi-strong.

Proof.

i) If \( a \in C(p) \) and \( a_i \not\in C(p_i) \), then

\[
\frac{\partial^k F_i(x;u)}{\partial x_i^k} \bigg|_{x=p} = 0 \quad \text{if} \quad x_i^k \neq x_i^k \quad \text{and} \quad x_i^k \not\in C(p_i;u).
\]

\[
\frac{\partial^k F_i(x;u)}{\partial x_i^k} \bigg|_{x=p} = p_i \delta_{k_i}^i (p_i;u).
\]
From this the statement follows immediately.

ii) Let $a, a' \in C(p)$. It is clear that

$$\prod_{i=1}^{n} \prod_{k \in C(p_i)} \delta_{k_i}(p_i; u) = 0$$

if and only if

$$\prod_{i=1}^{n} \prod_{k \in C(p_i)} \delta_{k_i}(p_i; u) = 0 .$$

Hence, we have to show that $|J^a(p; u)| = 0$ if and only if $|J^{a'}(p; u)| = 0$.

This follows from the following observations:

$$\frac{\partial^{a_i} F_i(x; u)}{\partial x_i} |_{x=p} = 1 = \frac{\partial^{a_i'} F_i(x; u)}{\partial x_i} |_{x=p}$$

for all $i \in \mathbb{N}$ and $k \in \{1, \ldots, K_i\}$

$$\frac{\partial^{a_i} F_i(x; u)}{\partial x_i} |_{x=p} = 0 = \frac{\partial^{a_i'} F_i(x; u)}{\partial x_i} |_{x=p}$$

for all $i \in \mathbb{N}$; $k, m, m' \in C(p_i)$; $m \neq k_i; m' \neq k_i$.

$$\frac{\partial^{a_i} F_i(x; u)}{\partial x_i} |_{x=p} = - \frac{\partial^{a_i} F_i(x; u)}{\partial x_i} |_{x=p}$$

for all $i, j \in \mathbb{N}$; $j \neq i; \in C(p_j)$. 
\[
\frac{\partial^{a,k}(x;u)}{\partial x_j} \bigg|_{x=p} \frac{\partial^{a,k}(x;u)}{\partial x_j} \bigg|_{x=p} \quad \text{for all } i,j \in \mathbb{N}; i \neq j; k \in C(p_i); \ell \in C(p_j); k \neq k_i, \ell_i.
\]

iii) This statement follows immediately from i).

Next, we will investigate what regularity means for bimatrix games. Let \( \Gamma \in G(2,K_1,K_2) \), with payoff vector \( u \). Assume \( p \) is a regular equilibrium point of \( \Gamma \). Then \( p \) is a quasi-strong equilibrium point. Without loss of generality we may assume that \( C(p_1) = \{ a_1^1, \ldots, a_1^m \} \) and \( C(p_2) = \{ a_2^1, \ldots, a_2^n \} \). Let \( a = (a_1^1, a_2^2) \). Then

\[
\tilde{J}^a(p;u) = \begin{pmatrix}
e_m & 0 \\
\emptyset & \Delta^1 \\
0_m & e_n \\
\emptyset^2 & \emptyset
\end{pmatrix},
\]

where

\( e_k(0_k) \) is the row vector in \( \mathbb{R}^k \) with all coefficients equal to 1 (0),

\( \emptyset \) is a matrix with all entries 0,

\( \Delta^1 \) is an \((m-1)\)-by-\( n \) matrix with

\[
\Delta^1_{ij} = p_1^i p_1^j [U_1(a_1^i, a_2^j) - U_1(a_1^1, a_2^j)] \quad (i \in \{2, \ldots, m\}, j \in \{1, \ldots, n\})
\]

and

\( \Delta^2 \) is an \((n-1)\)-by-\( m \) matrix with

\[
\Delta^2_{ij} = p_2^i p_2^j [U_2(a_1^i, a_2^j) - U_2(a_1^j, a_2^j)] \quad (i \in \{2, \ldots, n\}, j \in \{1, \ldots, m\})
\]
\( J^a(p;u) \) is nonsingular if and only if the matrices

\[
\begin{pmatrix}
\Delta^2 \\
e_m
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\Delta^1 \\
e_n
\end{pmatrix}
\]

are nonsingular.

From this it immediately follows that \(|C(p_1)| = m = n = |C(p_2)|\). Moreover, by applying some elementary linear algebra and using the fact that \( p \in E(\Gamma) \) it is seen that 3.9 is equivalent with:

the matrices:

\[
\tilde{U}_1 := [U_1(a_i^1, a_j^1)]_{i \in C(p_1), j \in C(p_2)} \quad \text{and} \quad \tilde{U}_2 := [U_2(a_i^2, a_j^2)]_{i \in C(p_1), j \in C(p_2)}
\]

are nonsingular.

We can conclude:

**Theorem 3.11.** If \( \Gamma = (A_1,A_2,U_1,U_2) \) is a bimatrix game and \( p \) is an equilibrium point of \( \Gamma \), then \( p \) is a regular equilibrium point if and only if:

i) \( p \) is a quasi-strong equilibrium point, and

ii) \(|C(p_1)| = |C(p_2)|\) and the matrices \( \tilde{U}_1 \) and \( \tilde{U}_2 \) (as in (3.10)) are nonsingular.

In [1] equilibrium points which satisfy the conditions i) and ii) of theorem 3.11 are called nondegenerate equilibrium points. In that paper it is proved that (for all bimatrix games):

(3.12) any nondegenerate equilibrium point is essential, and

(3.13) any nondegenerate equilibrium point is proper.

The proofs in [1] are based on linear algebra and hence these proofs
heavily use the 2-person character of the game. In section 4 of this paper we will prove the analogous statements of (3.12) and (3.13) for general n-person games in normal form, by using arguments based on $J^a(p;u)$. The proofs given in this paper are a considerable improvement of those given in [1]. In section 4 we first prove that regular equilibrium points are essential. After that, we prove that regular equilibrium points are proper, using the fact that regular equilibrium points are essential.

4. The main theorems

Theorem 4.1. Let $\hat{\eta} \in G(n,K_1,\ldots,K_n)$ with defining vector $\hat{u} \in \mathbb{R}^{nK}$. Assume $\hat{\eta}$ is a regular equilibrium point of $\hat{\eta}$. Then there exists a neighborhood (neighborhood) $V_0$ of $\hat{\eta}$, such that for any neighborhood $V$ of $\hat{\eta}$, with $V \subseteq V_0$, there exists a neighborhood $U$ of $\hat{u}$ such that for any $u \in U$ we have $|E(\eta_u) \cap V| = 1$.

Proof. Let $a \in C(\eta)$ be fixed. For any $u \in \mathbb{R}^{nK}$ we will write $F(\cdot;u)$ and $J(\cdot;u)$ for $aF(\cdot;u)$ and $J^a(\cdot;u)$, respectively. Consider the mappings:

$$\varphi : \mathbb{R}^{nK} \times \mathbb{R}^K \to \mathbb{R}^{nK} \times \mathbb{R}^K$$

defined by $\varphi(u,x) = (u,F(x,u))$, and

$$\pi : \mathbb{R}^{nK} \times \mathbb{R}^K \to \mathbb{R}^K$$

defined by $\pi(u,x) = x$.

We have

$$\left| \frac{\partial \varphi(u,x)}{\partial (u,x)}(u_0,x_0) \right| = |J_{u_0}(x_0)|.$$
By the implicit function theorem and the fact that $\hat{p}$ is a regular equilibrium point of $\hat{r}$, there exists a nbhd. $U_1$ of $\hat{u}$, a nbhd. $V_1$ of $\hat{p}$ and a nbhd. $W_1$ of 0, such that $\varphi$ is a diffeomorphism (= a differentiable mapping which is bijective) from $U_1 \times V_1$ to $U_1 \times W_1$. Let $U_0 \subset U_1$, $V_0 \subset V_1$ and $W_0 \subset W_1$ be nbhd's of $\hat{u}, \hat{p}$ and 0, respectively, such that

(4.2) $\varphi$ is a diffeomorphism from $U_0 \times V_0$ to $U_0 \times W_0$,

(4.3) for all $i \in N$ and $k \in \{1, \ldots, K_i\}$: $p_i^k > 0$ implies $x_i^k > 0$, for all $x \in V_i$ and

(4.4) for all $i \in N$ and all $k, \ell \in \{1, \ldots, K_i\}$: $U_i(a_i; \bar{p}) < U_i(a_i; \bar{p})$ implies $U_i(a_i; x_i) < U_i(a_i; x_i)$, for all $u \in U_0$ and $x \in V_0$.

Let $u \in U_0$. Define $x(u) := \pi^{-1}_u(u, 0)$. Then $x(u) \in V_0$ and $F(x(u); u) = 0$. Since $\varphi$ is a diffeomorphism we have that $x(u)$ is the only element $x$ of $V_0$ for which $F(x; u) = 0$. From (4.3) and (4.4) it follows that actually $x(u) \in E(\Gamma_u)$. From (3.3) it follows that $\{x(u)\} = E(\Gamma_u) \cap V_0$. The assertion from the theorem now follows from the fact that the mapping $U_0 \ni u \mapsto x(u) \in V_0$ is continuous.

Corollary 4.5. Any regular equilibrium point is essential.

Theorem 4.6. Any regular equilibrium point is proper.

Proof. Let $\hat{r}, \hat{u}, U_0, V_0$ and $x(u)$ ($u \in U_0$) be as in (the proof of) theorem 4.1. For $\varepsilon \in [0, 1]$ and $\pi = (\pi_1, \ldots, \pi_n) \in P$, define $u^{(\varepsilon, \pi)} \in \mathbb{R}^{nK}$ by:

(4.7) $u^{(\varepsilon, \pi)}(a) = \sum_{\mathcal{S} \ni i} (1 - \varepsilon)^{|\mathcal{S}| - 1} \varepsilon^{-n - |\mathcal{S}|} U_i(a_S, \pi_S c), (a, \pi, (a_1, \ldots, a_n) \in A)$.
where \((a, \pi)^c\) is that element of \(P\) where any player \(j\) in \(S\) plays \(a_j\) and any player \(j\) in \(S^c(=N\setminus S)\) plays \(\pi_j\). The reader can verify that for any \(a \in A:\)

\[
U_i^{(\varepsilon, \pi)}(a) = U_i(a_i; (1 - \varepsilon)a_i^\varepsilon + \varepsilon\pi_i),
\]

where \((1 - \varepsilon)a_i^\varepsilon + \varepsilon\pi_i\) is that element of \(\bar{P}_i\) where any player \(j\) in \(N\setminus\{i\}\) plays \((1 - \varepsilon)a_j + \varepsilon\pi_j\). Moreover, since for any \(p = (p_1, \ldots, p_n) \in P, S \subset N\) and \(i \in N:\)

\[
\hat{U}_i(p_S, \pi^c_S) = \sum_{s \in A} \left[ \prod_{j=1}^{n} q_j(p_j) \right] \hat{U}_i(a_S, \pi^c_S),
\]

we have for any \(p = (p_1, \ldots, p_n) \in P\) and \(i \in N:\)

\[
U_i^{(\varepsilon, \pi)}(p) = \hat{U}_i(p_i; (1 - \varepsilon)p_i^\varepsilon + \varepsilon\pi_i).
\]

It is clear that for \(\varepsilon\) sufficiently small (say \(\varepsilon \in [0, \varepsilon_0]\)) \(u^{(\varepsilon, \pi)}\) will be in \(U_0\), for all \(\pi \in P\). Let \(\varepsilon \in (0, \varepsilon_0)\). For \(\pi \in P\) let \(p(\varepsilon, \pi) = x(u^{(\varepsilon, \pi)})\) be such that \(\{p(\varepsilon, \pi)\} = E(T_u^{(\varepsilon, \pi)}) \cap V_0\). Let \(\delta = \varepsilon^{K/K}\) and define \(P_i(\delta)\) by:

\[
P_i(\delta) = \{x_i \in P_i : x_i^k \geq \delta \text{ for all } k \in \{1, \ldots, K_i\}\}.
\]

Let \(P(\delta) = \bigcap_{i=1}^{n} P_i(\delta)\). We define a multivalued mapping \(G^{\varepsilon} : P(\delta) \rightrightarrows P(\delta)\).
Then $G^e_i(\pi) \neq \emptyset$ for any $\pi \in P(\delta)$. For, if $k \in \{1, \ldots, K_i\}$, let

$$
(4.13) \quad \mu(k) = \left\{ \ell \in \{1, \ldots, K_i\} : U_i^{(e, \pi)}(a_i, p_i(\ell, \pi)) > U_i^{(e, \pi)}(a_i, p_i(k, \pi)) \right\},
$$

and define $x_i \in P_i(\delta)$ by

$$
(4.14) \quad x_i^k = \frac{\varepsilon^{\mu(k)}}{\sum_{\ell=1}^{K_i} \varepsilon^{\mu(\ell)}}.
$$

Then $x_i \in G^e_i(\pi)$.

Furthermore, $G^e_i(\pi)$ is a closed and convex set, for any $\pi \in P(\delta)$. From the proof of theorem 4.1 it follows that the mapping

$$
U_0 \times P \ni (u, \pi) \mapsto (u^{(e, \pi)}, p(\varepsilon, \pi)) \in U_0 \times P
$$

is continuous, and this implies that $G^e$ is upper semi continuous. Hence $G^e$ satisfies the conditions of the Kakutani fixed point theorem [4]; so there exists a fixed point of $G^e$. Let $\pi$ be such a fixed point.
For any \(i \in \mathbb{N}, k, \ell \in \{1, \ldots, K_i\}\), we have:

\[
\begin{align*}
(4.15) \quad & U_{i}(\varepsilon, \pi) (a_{i}^{k}, p_{i}(\varepsilon, \pi)) < U_{i}(\varepsilon, \pi) (a_{i}^{\ell}, p_{i}(\varepsilon, \pi)) \\
& \text{implies } \pi_{i}^{k} \leq \varepsilon \pi_{i}^{\ell}.
\end{align*}
\]

From (4.10) it follows that for any \(i \in \mathbb{N}\) and \(k, \ell \in \{1, \ldots, K_i\}\):

\[
(4.16) \quad \hat{U}_{i}(a_{i}^{k}, (1 - \varepsilon)p_{i}(\varepsilon, \pi) + \varepsilon \pi_{i}) < \hat{U}_{i}(a_{i}^{\ell}, (1 - \varepsilon)p_{i}(\varepsilon, \pi) + \varepsilon \pi_{i})
\]

implies \(\pi_{i}^{k} \leq \varepsilon \pi_{i}^{\ell}\).

Define \(\hat{p}(\varepsilon, \pi) \in \mathbb{P}\) by \(\hat{p}(\varepsilon, \pi) = (1 - \varepsilon)p(\varepsilon, \pi) + \varepsilon \pi\). From (4.16) it follows that, for all \(i \in \mathbb{N}\) and \(a_{i}^{k}, a_{i}^{\ell} \in A_i \setminus C(p_{i}(\varepsilon, \pi))\):

\[
(4.17) \quad \hat{U}_{i}(a_{i}^{k}, \hat{p}_{i}(\varepsilon, \pi)) < \hat{U}_{i}(a_{i}^{\ell}, \hat{p}_{i}(\varepsilon, \pi))
\]

implies

\[
\hat{p}_{i}(\varepsilon, \pi) \leq \varepsilon \hat{p}_{i}(\varepsilon, \pi).
\]

Since \(p(\varepsilon, \pi)\) is an equilibrium point of \(u(\varepsilon, \pi)\), we have for any \(i \in \mathbb{N}\) and \(a_{i}^{k} \in C(p_{i}(\varepsilon, \pi))\):

\[
(4.18) \quad \hat{U}_{i}(a_{i}^{k}, \hat{p}_{i}(\varepsilon, \pi)) = \max_{\ell \in \{1, \ldots, K_i\}} \hat{U}_{i}(a_{i}^{\ell}, \hat{p}_{i}(\varepsilon, \pi)).
\]

Hence, if

\[
(4.19) \quad \eta(\varepsilon) := \varepsilon(1 - \varepsilon)^{-(1)} \left\{ \min_{i \in \mathbb{N}} \min_{\ell \in C(p_{i}(\varepsilon, \pi))} p_{i}^{\ell}(\varepsilon, \pi) \right\}^{-(1)},
\]
then, we have for all $i \in N$ and $k, \ell \in \{1, \ldots, K_i\}$:

$$
(4.20) \quad U_i(a_i^k, \bar{p}_i^k(\epsilon, \pi)) < U_i(a_i^{\ell}, \bar{p}_i^{\ell}(\epsilon, \pi))
$$

implies

$$
\bar{p}_i^{k}(\epsilon, \pi) \leq \eta(\epsilon)\bar{p}_i^{\ell}(\epsilon, \pi).
$$

Hence, $\bar{p}(\epsilon, \pi)$ is an $\eta(\epsilon)$-proper equilibrium point of $\Gamma$. Since,

$$
(4.21) \quad \bar{p} \text{ is a quasi-strong equilibrium point of } \Gamma = \Gamma_u,
$$

$$
(4.22) \quad p(\epsilon, \pi) \text{ is an equilibrium point of } \Gamma_{u(\epsilon, \pi)}, \text{ and}
$$

$$
(4.23) \quad \lim_{\epsilon \to 0} u(\epsilon, \pi) = 0, \lim_{\epsilon \to 0} p(\epsilon, \pi) = \hat{p},
$$

we have that $C(p(\epsilon, \pi) = C(\hat{p})$, for $\epsilon$ sufficiently small.

Hence,

$$
(4.24) \quad \lim_{\epsilon \to 0} \min_{i \in N} \min_{\epsilon \in C(p_i(\epsilon, \pi))} \bar{p}_i^\epsilon(\epsilon, \pi) = \min_{i \in N} \min_{\epsilon \in C(\hat{p}_i)} \bar{p}_i^\epsilon > 0
$$

and so $\lim_{\epsilon \to 0} \eta(\epsilon) = 0$. Since

$$
(4.25) \quad \bar{p}(\epsilon, \pi) \text{ is completely mixed, for } \epsilon > 0.
$$

$$
(4.26) \quad \bar{p}(\epsilon, \pi) \text{ is an } \eta(\epsilon)-\text{proper equilibrium point of } \Gamma, \text{ and}
$$

$$
(4.27) \quad \lim_{\epsilon \to 0} \eta(\epsilon) = 0; \lim_{\epsilon \to 0} \bar{p}(\epsilon, \pi) = \hat{p},
$$

we have that $\hat{p}$ is a proper equilibrium point of $\Gamma$. \[ \square \]

From the theorems 3.7, 3.8, 4.1 and 4.6 and the fact that any proper equilibrium point is perfect, we can conclude:
Corollary 4.28. For almost all finite noncooperative n-person games in normal form, we have

i) all equilibrium points are quasi-strong,

ii) all equilibrium points are essential,

iii) all equilibrium points are proper, and

iv) all equilibrium points are perfect.

References


