Introduction to a behavioral approach for continuous-time systems

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**INTRODUCTION TO
A BEHAVIORAL APPROACH
FOR
CONTINUOUS-TIME SYSTEMS**

HAN SOETHOUDT

Introduction to a behavioral approach for continuous-time systems

Proefschrift

ter verkrijging van de graad van doctor aan de
Technische Universiteit Eindhoven, op gezag van
de Rector Magnificus, prof. dr. J.H. van Lint,
voor een commissie aangewezen door het college
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en de copromotor:

dr. ir. S.J.L. van Eijndhoven

And the earth was without form, and void;
and darkness was upon the face of the deep.

GENESIS 1:2

*aan mijn ouders
voor Lillian en Job*

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Chapter 1

Introduction

Mathematical system theory finds its applications in various fields such as electrical engineering, economics and physics. To many people working in these areas, system theory in its widest sense is related to the analysis of mathematical models which are formulated in terms of sets of equations. The mathematical model describes the time evolution of an e.g. electrical circuit, an economy, or a physical system. As an illustration, we take the classical *state-space model*, which is used often in control theory. This model describes the evolution of the input u and the output y of an *input/output*-system, say Σ :

$$\Sigma: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, t \in \mathbb{R} \quad (1.1)$$

where x is the state (the memory of the system) and A, B, C, D are constant matrices of appropriate dimensions. The input u and the output y are the external variables, through which the system communicates with its environment. In spite of being a natural starting point in control theory, the classical approach to modelling dynamical systems in terms of an input/output representation is not always appropriate. For example, an externally defined autonomous system does not fit in such an input/output-framework. Furthermore, when modelling dynamical systems, there are situations where this causal dependence cannot be explicated in advance. The disadvantages of this classical approach to system theory were recognized by J.C.Willems, and about fifteen years ago he introduced the so-called *behavioral approach* (see [19]). In this new approach to dynamical system theory, one starts off from a suitable axiomatic foundation. The basis is the behavior of a system (i.e.

the set of trajectories) and not one of its representations. Hence one talks about properties of a system on the level of its behavior and translates these into some characterization of that system. The input/output-representation can be a consequence of the modelling by identifying which variables constitute causes and which effects. And hence, being a modelling question itself, it should be dealt with on a conceptual level.

It turns out that in the behavioral setting various notions from the field of system theory like 'controllability' and 'autonomy' can be defined rigorously in such a way that they are consistent with classical ones. A complete foundation of the behavioral approach can be found in a number of papers of Willems (see [20],[21]), where a framework has been created that provides general and intuitively clear definitions of many system-theoretical concepts. At the moment, not every one in the field of system theory is familiar with this 'new' approach and hence we can not leave out the introduction of its basic definitions in this thesis.

We start by defining a dynamical system in the behavioral set up.

Definition 1.0.1 A *dynamical system* Σ is a triple $\Sigma = (T, W, \mathcal{B})$, where $T \subseteq \mathbb{R}$ is the *time axis*, W is an abstract set, the *signal alphabet*, and $\mathcal{B}$, a subset of W^T , the *behavior*.

This definition is based on the following observation, and here we quote Willems: '*one should view a dynamical system as an object which is embedded in its environment, is abstracted from it, but which can interact with it through a set of system variables*'. In fact, the evolution in time of these relevant variables is just what we want to describe.

To do so we select a time-axis T and a signal alphabet W in which the external variables take their values. Then we can describe the time-evolution of the system variables by functions from T to W . Such a function is called a *trajectory*. A behavior $\mathcal{B}$ is simply a subset of a family of trajectories. Elements of $\mathcal{B}$ are precisely the trajectories compatible with the laws governing the system.

Although this theory has won its spurs throughout the years, a number of fundamental problems are still open, and it is our goal to tackle one of them. We require some definitions in order to be able to formulate the problem.

Definition 1.0.2 We will call a dynamical system $\Sigma = (T, W, \mathcal{B})$ *linear* if W is a vector space and $\mathcal{B}$ is a linear subspace of W^T (viewed as a vector space in a natural way with pointwise addition and scalar multiplication).

Definition 1.0.3 Let $(T, +)$ be an additive semigroup in $\mathbb{R}$, then the *shift operator* σ^t , $t \in T$, is defined by $\sigma^t f(\cdot) := f(\cdot + t)$ for all functions f from T to W . We will write σ instead of σ^1 .

Definition 1.0.4 Let $(T, +)$ be an additive semigroup in $\mathbb{R}$. We will call the dynamical system $\Sigma = (T, W, \mathcal{B})$ *time invariant* if $\sigma^t \mathcal{B} \subseteq \mathcal{B}$ for all $t \in T$.

In many applications, the signal alphabet equals the finite-dimensional vector space $\mathbb{R}^q$ and the behavior is described by a finite number of linear equations, often differential or difference-equations. *AR-relations* are well-known examples of this.

Definition 1.0.5 The class of linear differential operators of order l with constant coefficients in the field $\mathbf{K}$ will be denoted by $LDO_l(\mathbf{K})$. If we do not want to specify the order we talk about $LDO(\mathbf{K})$.

If we do not want to specify the order we talk about $LDO(\mathbf{K})$.

Definition 1.0.6 An $m \times n$ matrix with entries in the ring $LDO(\mathbf{K})$ will be denoted by $LDO(m, n, \mathbf{K})$.

Corresponding to these differential operators $R(\frac{d}{dt}) = A_0 + A_1 \frac{d}{dt} + \dots + A_n (\frac{d}{dt})^n$ we can define polynomial matrices $R(s) = A_0 + A_1 s + \dots + A_n s^n$, $s \in \mathbf{C}$. As usual, this class of matrices in the indeterminate s is denoted by $\mathbb{R}[s]$.

Definition 1.0.7 If $T = \mathbb{R}$ or $T = \mathbb{Z}$ an *AR-relation* of a dynamical system $\Sigma = (T, \mathbb{R}^q, \mathcal{B})$ is a description of the form:

$$w \in \mathcal{B} \Leftrightarrow R(s)w = 0,$$

where $R(s) = R_l s^l + R_{l-1} s^{l-1} + \dots + R_1 s + R_0$ is a polynomial matrix with $R_j \in \mathbb{R}^{q \times q}$ ($j = 0, 1, \dots, l$). Here $s = \frac{d}{dt}$ if $T = \mathbb{R}$ and $s = \sigma$ if $T = \mathbb{Z}$, corresponding to the continuous-time and discrete-time case, respectively.

From an AR-relation we can always obtain input/output representations and also state-space descriptions as in (1.1). This is a well-known result in system theory (see theorem 5.1, in [22]).

It is of course interesting to investigate when a behavior can be described by such an AR-relation. In the discrete-time case, this problem has been completely solved. To give the result we first need some definitions.

The *restriction* $f|_{A_0}$ of a mapping $f : A \rightarrow B$ to a subset $A_0 \subseteq A$ is the mapping from A_0 to B defined by $f|_{A_0}(t) := f(t)$, for all $t \in A_0$.

The *restriction of a behavior* B to the interval $[a, b]$, denoted by $B|_{[a, b]}$, is defined as follows

$$B|_{[a, b]} = \{g|_{[a, b]} \mid \exists w \in B [g|_{[a, b]} = w|_{[a, b]}\}$$

Definition 1.0.8 A dynamical system $\Sigma = (T, W, B)$ will be called *complete* if

$$\{w|_{[t_0, t_1]} \in B|_{[t_0, t_1]} \text{ for all } t_0, t_1 \text{ satisfying } -\infty < t_0 \leq t_1 < \infty\} \Leftrightarrow \{w \in B\}$$

So a complete behavior B is determined by the set of all trajectories in B truncated to bounded intervals.

Definition 1.0.9 Let $L > 0$. A dynamical system $\Sigma = (T, W, B)$ will be called *L-complete* if

$$\{w|_{[t_0, t_0+L]} \in B|_{[t_0, t_0+L]} \text{ for all } t_0 \in \mathbb{R}\} \Leftrightarrow \{w \in B\}$$

Definition 1.0.10 A dynamical system $\Sigma = (T, W, B)$ will be called *locally specified* if it is L-complete for all $L > 0$.

Clearly if $L > 0$,

B is locally specified $\Rightarrow B$ is L-complete $\Rightarrow B$ is complete.

The following theorem shows the one-to-one correspondence between the possibility of describing a discrete-time system by an AR-relation and intrinsic properties of such a system, i.e. properties corresponding to the set of trajectories representing the behavior. For a proof see [19] (p.86,89).

Theorem 1.0.11 *A discrete-time dynamical system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B})$ is linear, time invariant and complete if and only if $\mathcal{B}$ can be described by an AR-relation.*

If $T = \mathbb{R}$ we have a continuous-time system and we would like to derive conditions for a similar one-to-one characterization to be possible. A reasonable conjecture for the analog would be the condition that $\mathcal{B}$ be linear, time invariant, closed and locally specified. Unfortunately, this conjecture has been disproven by Hörmander (see chapter 6). Hence a similar result as in the discrete-time case does not exist.

Going through the proof of theorem 1.0.11 above one can see clearly that it cannot be used for the continuous-time case. The arguments are based on the dual of the space of time series from $\mathbb{Z}$ to $\mathbb{R}^q$ equipped with the topology of pointwise convergence, whereas the dual of the mappings from $\mathbb{R}$ to $\mathbb{R}^q$ with the same topology equals the space of functions from $\mathbb{R}$ to $\mathbb{R}^q$ that are nonzero in only finitely many points. Obviously, this dual is not appropriate to represent continuous-time dynamical systems. Moreover the relation " $\mathcal{B}$ is complete $\Leftrightarrow \mathcal{B}$ is closed", is a very crucial and easy statement in the discrete-time case. Unfortunately these arguments do not work in the continuous-time case. A completely new set up is needed.

In a very general setting, a linear continuous-time dynamical system Σ is a triple $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{B})$ where the behavior $\mathcal{B}$ is a subset of the vector space of all functions from $\mathbb{R}$ to the finite-dimensional vector space $\mathbb{R}^q$. This set up is too extensive to deal with and mostly the external signals of a continuous-time system are taken in, for instance, $L_{1,loc}(\mathbb{R})$, $C^\infty(\mathbb{R})$ or $C(\mathbb{R})$. These spaces have one more thing in common: they are all Fréchet spaces.

Definition 1.0.12 *A Fréchet space is a complete locally convex topological vector space in which the topology is generated by a countable family of seminorms.*

Example: The vector space of continuous functions becomes a Fréchet space if we impose the topology generated by the family of seminorms $(p_n)_{n \in \mathbb{N}}$, where for all $f \in C(\mathbb{R})$ and $n \in \mathbb{N}$ we define $p_n(f) := \max_{t \in [-n, n]} |f(t)|$.

Assuming signals to be in this kind of spaces is in fact imposing some topological structure on the behavior in order to obtain more grip on it. In a

Fréchet space, the *closed-graph-theorem* holds and continuity of a linear mapping can be expressed in terms of sequences. Especially the first property turns out to be a powerful mathematical tool in many situations.

This suggests us to redefine continuous-time dynamical systems as quadruples $\Sigma = (T, W, \mathcal{F}, \mathcal{B})$ where T and W are as before, $\mathcal{F}$ is a Fréchet space, and $\mathcal{B} \subseteq \mathcal{F}$. Moreover we require that $C^\infty(T, W)$ is dense in $\mathcal{F}$ with respect to the $\mathcal{F}$ -topology. Clearly this includes all standard continuous-time systems and hence the condition $\mathcal{B} \subseteq \mathcal{F}$ is hardly restrictive. Note that we drop the idea of $\mathcal{B} \subseteq W^T$ and hence $\mathcal{B}$ does not necessarily consist of functions. Nevertheless the specification of T and W is needed for the condition that $C^\infty(T, W)$ be dense in $\mathcal{F}$. This enables us to derive a crucial property for linear, shift-invariant and closed behaviors $\mathcal{B}$, namely that the restriction of the set of signals in $\mathcal{B}$ to $C^\infty(T, W)$ is dense in $\mathcal{B}$.

Often, a complication for continuous-time dynamical systems described by an AR-relation (see definition 1.0.7), is the interpretation of the set of differential equations describing it. Contrary to the discrete-time case, the equations for continuous-time systems impose smoothness-conditions on the external signals if differentiation is not understood in some generalized sense. The usual way out is embedding the space $\mathcal{F}$ in $\mathcal{D}'(\mathbb{R}, \mathbb{R}^q)$, whose elements are called $\mathbb{R}^q$ -valued distributions. (This is the distribution space of Schwartz). $\mathcal{D}'(\mathbb{R}, \mathbb{R}^q)$ is the dual space of $\mathcal{D}(\mathbb{R}, \mathbb{R}^q)$, the space of $C^\infty(\mathbb{R}, \mathbb{R}^q)$ -functions from $\mathbb{R}$ to $\mathbb{R}^q$ with compact support, also denoted by $C_c^\infty(\mathbb{R}, \mathbb{R}^q)$. As an example consider the Fréchet space $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$. This being a subspace of $\mathcal{D}'(\mathbb{R}, \mathbb{R}^q)$, a representation of an element in $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$ can be given in the following way: associate with each element $f \in L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$ the continuous linear functional (or distribution) on $C_c^\infty(\mathbb{R}, \mathbb{R}^q)$

$$L_f(\varphi) := \int_{\mathbb{R}} \langle f(t), \varphi(t) \rangle dt = \sum_{j=1}^q \int_{\mathbb{R}} f_j(t) \varphi_j(t) dt$$

where $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{R}^q)$, the dual space of $\mathcal{D}'(\mathbb{R}, \mathbb{R}^q)$.

Note that in the set up, as exposed here, $\mathcal{B}$ consists of functions from $\mathbb{R}$ to $\mathbb{R}^q$ and hence, strictly speaking, $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$ -signals are excluded in this set up, because, in fact the elements of $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$ are no functions but equivalence classes. However, as we showed a few lines earlier, using the embedding in $\mathcal{D}'(\mathbb{R}, \mathbb{R}^q)$ allows us to consider this kind of signals too, which

are very common in mathematical system theory. Unfortunately, this effort to avoid interpretation problems has a drawback. The space $\mathcal{D}'(\mathbb{R}, \mathbb{R}^q)$ is only a locally convex vector space and there is not much topological structure to go on, whereas the signal space $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$ is a Fréchet space. This is the reason for us to look at continuous-time dynamical systems as the quadruples we defined earlier.

If $\mathcal{F}$ is a Fréchet space then the *product space* $\mathcal{F} \times \dots \times \mathcal{F}$ (k -times) with the product topology will be denoted by $[\mathcal{F}]^k$.

In this thesis, we will assume that the signals are in the Fréchet space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^q$. We have made this choice to avoid technicalities. The results we shall derive for behaviors with signals in $C(\mathbb{R}, \mathbb{R}^q)$ also hold for behaviors in $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$ and only small (but technical) modifications in the proof are needed. At the end of the thesis we shall be more specific about this. Now we can give the concrete problem formulation.

Problem formulation

Find necessary and sufficient conditions for a continuous-time dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^q, C(\mathbb{R}, \mathbb{R}^q), \mathcal{B})$, to satisfy an AR-relation (to be understood in some generalized sense).

We give two ways to tackle the problem:

Approach 1: In this approach, the use of $\mathcal{D}'(\mathbb{R})$ to solve the interpretation problem is avoided. We start from a behavior in the Fréchet space $C(\mathbb{R}, \mathbb{R}^q)$ and look for a generalized interpretation of the AR-relation (i.e. in fact of the differential operator) without losing the topological structure of a Fréchet space. To this extent, we introduce new families of seminorms on $C^\infty(\mathbb{R})$ and define the completion of $C^\infty(\mathbb{R})$ with respect to the generated topology, thus obtaining the Fréchet spaces $C^{-k}(\mathbb{R})$ ($k \in \mathbb{N}$). We will call $C^{-k}(\mathbb{R})$ the space of k -irregular distributions. The above construction yields a chain of Fréchet spaces $C^\infty(\mathbb{R}) \subset \dots \subset C^m(\mathbb{R}) \subset \dots \subset C(\mathbb{R}) \subset \dots \subset C^{-k}(\mathbb{R}) \subset \dots$ where the relation between $C^k(\mathbb{R})$ and $C(\mathbb{R})$ is similar to the one between $C(\mathbb{R})$ and $C^{-k}(\mathbb{R})$. It is possible to define a generalized notion of differentiability that fits in with the classical one, and no smoothness-conditions are necessary. Every element of $C^m(\mathbb{R})$ ($m \in \mathbb{Z}$) can be differentiated ar-

bitrarily many times. However, the complete set up of such a framework involves quite some effort. Especially the choice of the families of seminorms is nontrivial.

Approach 2: Here we do use $\mathcal{D}'(\mathbb{R}, \mathbb{R}^q)$ to interpret the AR-relation. If we suppose that B is described by an AR-relation (now to be understood in $\mathcal{D}'(\mathbb{R}, \mathbb{R}^q)$ -sense) we can show that the behavior B must be linear, time-invariant and closed in $C(\mathbb{R}, \mathbb{R}^q)$ -sense. Exactly these three properties enable us to overcome the interpretation difficulties we met earlier. Specifically, these conditions are necessary and sufficient for $B^\infty := B \cap C^\infty(\mathbb{R}, \mathbb{R}^q)$ to be dense in B (with respect to the topology of $C(\mathbb{R}, \mathbb{R}^q)$). Since $C^\infty(\mathbb{R})$ is a Fréchet space, there are no problems interpreting an AR-relation for $C^\infty(\mathbb{R})$ -signals. Moreover B^∞ is also linear, time invariant and closed and hence it is easier to deal with than B . It turns out that, because B^∞ is dense in B , the AR-relation corresponding to B^∞ is of the same form as the one describing the original behavior. So all the intrinsic properties specifying such a behavior are already present in the restriction of the behavior to $C^\infty(\mathbb{R})$ -signals.

If we compare the approaches, the choice is easily made. Of course we prefer the second set up. Instead of gathering mathematical tools to tackle the interpretation difficulties with respect to smoothness we just simplify the problem by considering B^∞ . It is the feeling of getting to the core of a system by abandoning the external signals that are not necessary for the identification of its properties.

Hence although the first approach seems quite a logical step to make, based on a machinery that can be developed for this particular type of problems, we prefer the second one. Nevertheless, besides giving insight (which has lead to the second method!), the first set up provides a foundation to more general problems. For, if the behavior is not linear, time invariant or closed the second method may fail, because B^∞ is not necessarily dense anymore. The introduction of the new spaces $C^{-k}(\mathbb{R})$, ($k \in \mathbb{N}$) can be of great interest whenever similar smoothness-conditions occur in continuous-time systems. Therefore in chapter 5 we expose the whole framework in great detail and solve a special case of our problem to show how to deal with it.

Let us explain globally, using the second approach, how the problem can be

solved. From now on, we assume:

Assumption 1: The behavior $\mathcal{B}$ we consider is linear, time-invariant and closed with respect to the topology of $C(\mathbb{R}, \mathbb{R}^q)$.

Going through this thesis we will see that convolution operators take a very important part. It is possible to show that on $C(\mathbb{R})$, a convolution operator is the pointwise limit of a sequence in $\text{span}\{\sigma^t | t \in \mathbb{R}\}$. Using the linearity, time invariance and closedness of $\mathcal{B}$ this almost proves that $\mathcal{B}^\infty$ is dense in $\mathcal{B}$. We already noted that $\mathcal{B}^\infty$ is in fact the core of the system and therefore, from now on we restrict ourselves to $\mathcal{B}^\infty$. In this preliminary overview assumptions will be made on $\mathcal{B}^\infty$. The transition into assumptions on the original behavior will be performed later on.

Remark 1.0.13 Sometimes our assumptions seem some kind of ad hoc choices in order to be able to continue. However, later on, they are shown to be necessary and a system theoretical motivation is presented. $\square$

We want to view a behavior as the *graph* of an operator. This can be done in the following way: let $(w_1, \dots, w_q)$ represent the external signal. Find the maximal number $p \in \mathbb{N}$ satisfying $0 \leq p \leq q$ of external variables such that if the rest of the variables is taken zero we obtain a finite-dimensional subspace. Without loss of generality we take the first p , otherwise we apply a permutation. I.e., if we define $z := (w_1, \dots, w_p)$, $x := (w_{p+1}, \dots, w_q)$, $w = (z \mid x)$ then

$$E := \{z \in C^\infty(\mathbb{R}, \mathbb{R}^p) \mid (z \mid 0) \in \mathcal{B}^\infty\} \text{ is finite dimensional} \quad (1.2)$$

Note that such a p always exists.

Definition 1.0.14 A *Bohl function* is a linear combination of functions of the form $P(t)e^{\lambda t}$, where $\lambda \in \mathbb{C}$ and P a polynomial.

It will be proved that any finite-dimensional, linear, shift-invariant subspace in $C(\mathbb{R}, \mathbb{R}^q)$ consists only of Bohl functions and can be characterized as the kernel of a square differential matrix S with entries in $LDO(\mathbb{R})$. Moreover S can be chosen upper triangular. Hence, $E = \ker S$ for some $S \in LDO(q, q, \mathbb{R})$.

Assumption 2: $\forall_{x \in C^\infty(\mathbb{R}, \mathbb{R}^{q-p})} \exists_{z \in C^\infty(\mathbb{R}, \mathbb{R}^0)} [(z | x) \in \mathcal{B}^\infty]$

We demand that $q - p$ variables are *free* in $\mathcal{B}^\infty$. Hence, to every x corresponds precisely one Sz , with $(z|x) \in \mathcal{B}^\infty$.

Definition 1.0.15 A dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^q, [\mathcal{F}]^q, \mathcal{B})$ is called *n-free* ($n \in \mathbb{N}, n \leq q$) if the maximum number of variables taken zero, such that the remaining ones span a finite dimensional space, equals $q - n$.

Example 1: The behavior $\mathcal{B} \subseteq C^\infty(\mathbb{R}, \mathbb{R}^2)$ described by the equation

$$(w_1, w_2) \in \mathcal{B} \Leftrightarrow Sw_1 = w_2, \quad S \neq 0$$

where $S \in LDO(\mathbb{R})$, is a 1-free behavior, since $\{w_1 \in C^\infty(\mathbb{R}) | (w_1, 0) \in \mathcal{B}\}$ is a finite-dimensional space, but $\{(w_1, w_2) \in C^\infty(\mathbb{R}, \mathbb{R}^2) | (w_1, w_2) \in \mathcal{B}\}$ is not. Note that we do not specify which variable is free and which isn't.

Example 2: The behavior $\mathcal{B} \subseteq C^\infty(\mathbb{R}, \mathbb{R}^2)$ described by the equations

$$(w_1, w_2) \in \mathcal{B} \Leftrightarrow \begin{cases} S_{11}w_1 + S_{12}w_2 &= 0 \\ S_{21}w_1 &= 0 \end{cases}$$

(where S_{11}, S_{12}, S_{21} are nonzero elements of $LDO(\mathbb{R})$) is a 0-free behavior.

Using the relationship between x and Sz , we can define the operator K from $C^\infty(\mathbb{R}, \mathbb{R}^{q-p})$ to $C^\infty(\mathbb{R}, \mathbb{R}^p)$ as follows: $Kx = Sz$ where x and z are such that $(z | x) \in \mathcal{B}^\infty$. Hence

$$(z | x) \in \mathcal{B}^\infty \Leftrightarrow Kx = Sz \quad (1.3)$$

So here, for the first time, we have a kind of representation for the behavior $\mathcal{B}^\infty$. The operator K remains to be determined. Since $\mathcal{B}^\infty$ is linear and time invariant, it follows from equation (1.3) that K is linear and commutes with all shift-operators σ^t ($t \in \mathbb{R}$). The closedness of $\mathcal{B}^\infty$ and the *closed graph theorem* imply that K is continuous. Now we have sufficient information to determine K . K turns out to be an $p \times q$ -matrix operator with entries of the form TC_μ where $T \in LDO(\mathbb{R})$ and C_μ a convolution operator of the following kind: $C_\mu f := \mu * f$, where μ is in the dual space of $C(\mathbb{R})$. Hence, in order to arrive at an AR-relation we need to get rid of the convolution

operators in the matrix representation of K .

To this extent, we split up the behavior into subsystems with each one free variable. For all $j \in \{p+1, \dots, q\}$ consider the subsystems

$$B_j^\infty := \{w \in B^\infty \mid w_i = 0, p+1 \leq i \leq q, i \neq j\}$$

where e_i is the i -th element of the standard basis and w^t is the transposed of w . Furthermore define for vectors $w \in B_j^\infty$ the projection on the i -th component by

$$\mathcal{E}_i B_j^\infty := \{w_i \in C^\infty(\mathbb{R}) \mid w \in B_j^\infty\} \quad \text{for } i = 1, \dots, m.$$

The subsystems can be used to recover the original system because of the superposition principle. Because of assumption 2, we know that for $j = p+1, \dots, q$ and for every $w_j \in C^\infty(\mathbb{R})$ there exists a $z \in C^\infty(\mathbb{R}, \mathbb{R}^p)$ such that $(z_j \mid (0, \dots, 0, w_j, 0, \dots, 0))$ is in B^∞ . This yields

$$\begin{pmatrix} T_{1j} C_{\mu_{1j}} \\ \vdots \\ T_{mj} C_{\mu_{mj}} \end{pmatrix} w_j = S z_j \quad (j = p+1, \dots, q) \quad (1.4)$$

Definition 1.0.16 If $n \in \mathbb{N}$ and $v \in \mathbb{R}^n$ then for all $i, j \in \{1, \dots, n\}$ the permutation operator $(i, j)v$ swaps the i -th and j -th component of the vector v .

Assumption 3: For all $j \in \{p+1, \dots, q\}$, $i \in \{1, \dots, p\}$ $\mathcal{E}_i B_j^\infty$ or $\{w \in B_j^\infty \mid w_i = 0\}$ is finite dimensional.

If $\{w \in B_j^\infty \mid w_i = 0\}$ is finite dimensional, then it is possible to show that $\mathcal{E}_i B_j^\infty = C^\infty(\mathbb{R})$ and the roles of w_i and w_j can be 'exchanged'. By 'exchanged' we mean that it is possible to consider w_j as the free variable instead of w_i . Define $(i, j)w = \tilde{w}$ and split up the vector $\tilde{w}$ as before in $(\tilde{z} \mid \tilde{x})$. Then

$$E_{i,j} := \{\tilde{z}_j \mid (\underbrace{\tilde{z}_j}_m \mid \underbrace{0}_{q-m}) \in B_j^\infty\}$$

is finite-dimensional and

$$\forall \tilde{x} \in C^\infty(\mathbb{R}) \exists \tilde{z}_j \in C^\infty(\mathbb{R}, \mathbb{R}^p) [(\tilde{z}_j \mid \tilde{x}) \in B_j^\infty]$$

In words this means that either w_i is a free variable in this subsystem $\mathcal{B}_j^\infty$ or it is independent of w_j . As an example take the behavior

$$\mathcal{B}^\infty := \{(w_1, w_2) \in C^\infty(\mathbb{R}, \mathbb{R}^2) \mid w_1 = w_2\}$$

One could regard $\mathcal{B}^\infty$ as the graph of the identity-operator acting on w_1 , but also vice versa because $\mathcal{E}_2 \mathcal{B}_1^\infty = C^\infty(\mathbb{R})$. Hence the roles of w_1 and w_2 can be exchanged.

Assumption 4: $\mathcal{B}^\infty$ is locally-specified.

The conditions 3 and 4 are sufficient to assure that the convolution-operators in equation (1.4) are identities. This equation now reads

$$\begin{pmatrix} T_{1j} \\ \vdots \\ T_{mj} \end{pmatrix} w_j = S z_j \quad (j = p+1, \dots, q) \quad (1.5)$$

Consequently all the convolution-operators disappear and we end up with a finite set of differential equations. Possibly there is an overlap between the assumptions. We comment on that later in this thesis.

Thus, under the conditions 1,2,3 and 4, $\mathcal{B}^\infty$ satisfies an AR-relation:

$$w \in \mathcal{B}^\infty \Leftrightarrow R\left(\frac{d}{dt}\right)w = 0$$

for some polynomial matrix $R\left(\frac{d}{dt}\right)$.

If we interpret this equation in a weak sense we obtain

$$\forall_{\varphi \in D(\mathbb{R}, \mathbb{R}^q)} \left[\int_{\mathbb{R}} \left\langle R\left(\frac{d}{dt}\right)w, \varphi \right\rangle = \int_{\mathbb{R}} \left\langle w, R\left(\frac{d}{dt}\right)^* \varphi \right\rangle = 0 \right] \quad (1.6)$$

Define

$$\mathcal{B}_{D'} := \{w \in C(\mathbb{R}, \mathbb{R}^q) \mid (1.6) \text{ is satisfied}\}$$

Then $\mathcal{B}_{D'}$ is closed in $C(\mathbb{R}, \mathbb{R}^q)$ and, very importantly, the restriction of $\mathcal{B}_{D'}$ to $C^\infty(\mathbb{R}, \mathbb{R}^q)$ equals $\mathcal{B}^\infty$. Because $\mathcal{B}^\infty$ is dense in $\mathcal{B}$, this yields that $\mathcal{B}_{D'} = \mathcal{B}$. It is possible to turn the conditions we impose on $\mathcal{B}^\infty$ into similar conditions on the original behaviors. Summarizing we have the following result:

Theorem 1.0.17 *B is a subspace of $C(\mathbb{R}, \mathbb{R}^q)$ satisfying*

- *The behavior B we consider is linear, time-invariant and closed with respect to the topology of $C(\mathbb{R}, \mathbb{R}^q)$.*
- *$\forall_{x \in C^\infty(\mathbb{R}, \mathbb{R}^{q-p})} \exists_{z \in C^\infty(\mathbb{R}, \mathbb{R}^p)} [(z \mid x) \in B]$*
- *For all $j \in \{p+1, \dots, q\}$, $i \in \{1, \dots, p\}$ $\mathcal{E}_i B_j$ or $\{w \in B_j \mid w_i = 0\}$ is finite dimensional.*
- *B is locally-specified.*

if and only if B satisfies

$$w \in B \Leftrightarrow R\left(\frac{d}{dt}\right)w^t = 0$$

where $R\left(\frac{d}{dt}\right)$ is in $LDO(p, q, \mathbb{R})$. (I.e. B satisfies an AR-relation).

This completes the overview of the solution method we will use. Of course, some mathematical background is necessary to prove the results. This leads to the following set up of the thesis.

Chapter 2: Convolution and differential operators

The contents of this chapter are purely mathematical. There is no relation with system theory, but it is the foundation for important results that will follow in the next chapters. We start with a small introduction to Fréchet spaces (like $C(\mathbb{R})$, $C^\infty(\mathbb{R})$) and their duals. The Fourier transform will be defined on several spaces and some properties will be derived. As stated before, an important role is played by a special kind of convolution operators and many results on this class of operators, as well as on differential operators, will be proven. Especially topological properties and the relation with the Fourier-transformation will have our attention. This chapter concludes with the characterization of translation-invariant operators, which are closely related to our problem.

Chapter 3: AR-descriptions for systems consisting of $C^\infty(\mathbb{R})$ -signals.

We have stated that all the intrinsic properties of a behavior B are present in its restriction B^∞ to $C^\infty(\mathbb{R}, \mathbb{R}^q)$. The derivation of the AR-relation will be done for behaviors B^∞ , with a finite-dimensional signal alphabet. Initially, to catch the idea, we apply the chosen method to a behavior with

two external variables. Besides the main theorem and some proofs a system-theoretical interpretation will be given of the conditions we use.

Chapter 4: AR-descriptions for systems consisting of $C(\mathbb{R})$ -signals.

In this chapter the transition is made from B^∞ to B . We will show that if we interpret the AR-relation we found for B^∞ properly, it is indeed the AR-relation describing the original behavior B . For this purpose we introduce weak forms. Furthermore the necessary and sufficient conditions we have found for B^∞ are transformed into conditions on the original behavior. A nice result is, when dealing with linear time-invariant and closed behaviors, that for several notions as *locally specifiedness* and *L-completeness*, there is a strong relation between B and its restriction to $C^\infty(\mathbb{R}, \mathbb{R}^q)$, B^∞ . For instance, B is locally specified if and only if B^∞ is.

Chapter 5: The spaces $C^{-k}(\mathbb{R})$, ($k \in \mathbb{N}$) of *k-irregular* distributions.

We present a new framework in which the problem can be solved. It involves the introduction of a new chain of Fréchet spaces that can be used to interpret differential equations in case the signals are not differentiable. The advantage of this approach over an approach in a pure distributional setting is that you can talk about concrete signals instead of linear functionals. Moreover, the topological structure of a Fréchet space is available, in contrast to the distribution-space $\mathcal{D}'(\mathbb{R})$, where the topological structure is very complex. The complete calculus of differentiation and integration can be given in a natural and consistent way. However, some difficulties have to be taken care of, and very soon the notation will cloud the view. That is why we omit the solution in this context. We present the whole framework in detail, not only for completeness but also because it can be applied in case one is not dealing with behaviors that are linear, time-invariant and closed.

The spaces $C^{-k}(\mathbb{R})$, ($k \in \mathbb{N}$) can be linked with $\mathcal{D}'(\mathbb{R})$ and more insight is given by relating them to $L_{1,loc}(\mathbb{R})$, $C(\mathbb{R})$. At the end, we show by a simple example of a behavior with two external signals how the technique works in this context. This chapter stands on itself and can be read separately from the rest.

Chapter 6: Conclusions, suggestions and additions.

We summarize the main results and indicate how they could be generalized to other Fréchet spaces. Since, in system theory, $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$ is very often used as a signal space we are more specific about this generalization. Furthermore we give the Hörmander-counterexample to the conjecture (the continuous-time analog of theorem 1.0.11) on page 6 and directions for further research are proposed.

Chapter 2

Convolution and differential operators

In system theory, very often the function spaces $L_{p,loc}(\mathbb{R})$, $C(\mathbb{R})$, $C^k(\mathbb{R})$ and $C^\infty(\mathbb{R})$ occur. These spaces are used for their algebraic structure but their topological structure does not seem to play an essential role in most cases. In this manuscript, topological structure is a crucial point. We hope that this chapter will throw some light upon this part.

Another important issue is shift invariance. Looking at a behavior as the graph of an operator clarifies the interest for operators that commute with all shift operators $\sigma^t, t \in \mathbb{R}$. Besides, of course the differential operators, there will be another kind of operator satisfying this property. Namely, the convolution operators as we define them in section 4. There is a strong relation between the family of convolution operators and the group $(\sigma^t)_{t \in \mathbb{R}}$. In fact, we could have taken the commutation-property as a definition for the convolution operators instead of the explicit form we use in the definition now.

A full characterization is presented of all continuous operators from $C^k(\mathbb{R})$ to $C^l(\mathbb{R})$ ($k, l \in \mathbb{N} \cup \{\infty\}$), that commute with all the shifts. Closely related to convolution operators is the Fourier transformation. It will be used to prove all kind of properties of the convolution. In the last section, a very general result on closed, shift-invariant operators is stated. It will be needed only in case we use the approach where the spaces $C^{-k}(\mathbb{R})$ ($k \in \mathbb{N}$) are involved. However, we think it is an interesting result by itself.

The reader will notice that several results in this chapter are quite standard. However we believe, that it is worthwhile to present these results here. They show relations between the calculus, we built up here, and the existing theory. Wherever possible this link will be made.

2.1 Fréchet spaces

First of all note that the natural topologies for the vector spaces $L_{p,loc}(\mathbb{R})$, $C(\mathbb{R})$, $C^k(\mathbb{R})$ and $C^\infty(\mathbb{R})$ turns them into Fréchet spaces.

In this section we give an enumeration of results on Fréchet spaces. These results will be used later on but are independent of the special structure of our problem. Hence we state these results separately thus giving them some extra attention. For a short and elementary introduction to Fréchet spaces we refer to Friedlander ([6], appendix pp.144-151) or Yosida ([23], pp.52-55).

Let $\mathcal{F}$ be a Fréchet space with topology generated by a system of seminorms $(p_j)_{j \in \mathbb{N}}$. In the sequel we assume, without loss of generality, that the system is ordered, i.e. $p_j \leq p_{j+1}$ for all $j \in \mathbb{N}$ (if necessary, instead of p_j take $\tilde{p}_j = p_1 + \dots + p_j$). We see that Banach spaces are a special kind of Fréchet spaces.

Lemma 2.1.1 *A linear mapping A from a Fréchet space $\mathcal{F}_1$ into another Fréchet space $\mathcal{F}_2$ is continuous if and only if*

$$\forall_{j \in \mathbb{N}} \exists_{i \in \mathbb{N}} \exists_{C \in \mathbb{R}} \forall_{f \in \mathcal{F}_1} [p_j(Af) \leq Cq_i(f)]$$

where $(p_j)_{j \in \mathbb{N}}$ and $(q_j)_{j \in \mathbb{N}}$ are complete sets of ordered seminorms on $\mathcal{F}_2$ and $\mathcal{F}_1$ respectively.

Proof: See Schaefer ([15], theorem 1.1, p.74). □

As a consequence we have the following

Corollary 2.1.2 *A linear functional L on a Fréchet space $\mathcal{F}$ is continuous if and only if*

$$\exists_j \in \mathbb{N} \exists_C \in \mathbb{R} \forall_f \in \mathcal{F} [|L(f)| \leq C p_j(f)]$$

Lemma 2.1.3 *Let V be a topological vector space and $\mathcal{A} : V \rightarrow \mathbb{R}^m$ a continuous linear mapping. Then there are continuous linear functionals L_i ($i = 1, \dots, m$) on V such that for all $v \in V$, $\mathcal{A}v = (L_1v, \dots, L_mv)$.*

Proof: Let $\{e_1, \dots, e_m\}$ be the standard basis for $\mathbb{R}^m$ with $\{e^1, \dots, e^m\}$ as dual basis. Then $e^j \circ \mathcal{A}$ is a continuous linear functional on V , say L_j . But also

$$L_j v = (e^j \circ \mathcal{A})(v) = e^j \left(\sum_{i=1}^m (\mathcal{A}v)_i e_i \right) = (\mathcal{A}v)_j$$

Hence $\mathcal{A}v = (L_1v, \dots, L_mv)$. □

In chapter 1 we already introduced $[\mathcal{F}]^k$ as the k -fold product space of a Fréchet space $\mathcal{F}$, with corresponding product topology. This topology is generated by the family of seminorms

$$p_j^{(k)}(f_1, \dots, f_k) = \sum_{i=1}^k p_j(f_i) \quad (j \in \mathbb{N})$$

where $(p_j)_{j \in \mathbb{N}}$ are the seminorms on $\mathcal{F}$. Then $[\mathcal{F}]^k$ is a Fréchet space. The dual of $[\mathcal{F}]^k$ can be characterized in terms of the dual of $\mathcal{F}$. Note that $\mathcal{F}$ is by definition a topological vector space.

Lemma 2.1.4 *Let V be a topological vector space. Then a linear functional $\mathcal{L}$ on V^n is continuous if and only if $\mathcal{L}$ is of the form*

$$\mathcal{L}(v_1, v_2, \dots, v_n) = \mathcal{L}_1(v_1) + \mathcal{L}_2(v_2) + \dots + \mathcal{L}_n(v_n)$$

where $\mathcal{L}_i$ is a continuous linear functional on V , $i = 1, 2, \dots, n$.

Proof: The "if"-part is obvious. For $i = 1, 2, \dots, n$, $v_i \mapsto \mathcal{L}(0, 0, \dots, v_i, \dots, 0)$ defines a continuous linear functional on V . Hence $\mathcal{L}(0, 0, \dots, v_i, \dots, 0) = \mathcal{L}_i(v_i)$ for some $\mathcal{L}_i$ in the dual of V . The linearity of $\mathcal{L}$ completes the proof. □

Lemma 2.1.5 *Let V be a topological vector space and $m, n \in \mathbb{N}$. If $A : V^n \rightarrow \mathbb{R}^m$ is a continuous linear mapping then there are continuous linear functionals L_{ij} ($i = 1, \dots, m; j = 1, \dots, n$) on V such that for all $v \in V^n$*

$$Av = \left(\sum_{j=1}^n L_{1j}v_j, \dots, \sum_{j=1}^n L_{mj}v_j \right)$$

Proof: If $\{e_1, \dots, e_m\}$ is the standard basis for $\mathbb{R}^m$ again with $\{e^1, \dots, e^m\}$ as dual basis, then $e^i \circ A$ is a continuous linear functional on V^n and hence (see lemma 2.1.4) $e^i \circ A = \sum_{j=1}^n L_{ij}$ with L_{ij} are continuous linear functionals on V . Applying lemma 2.1.3 finishes the proof. $\square$

In general a subspace of a Fréchet space $\mathcal{F}$ is not closed and a closed subspace cannot be complemented. In this sense the finite-dimensional subspaces of a Fréchet space are special.

Lemma 2.1.6 *Let $\mathcal{M}$ be a finite-dimensional subspace of a Fréchet space $\mathcal{F}$. Then $\mathcal{M}$ is closed. Furthermore there exists a closed subspace $\mathcal{N}$ of $\mathcal{F}$ such that $\mathcal{M} \oplus \mathcal{N} = \mathcal{F}$ and the corresponding projections onto $\mathcal{M}$ along $\mathcal{N}$ are continuous.*

Proof: Since a Fréchet space is a Hausdorff topological vector space the closedness of $\mathcal{M}$ is immediate from Treves ([18], cor.2, p.80). We can define a continuous projection on $\mathcal{M}$ in the following way: if $\langle x_1, x_2, \dots, x_n \rangle$ is a basis for $\mathcal{M}$, then there exists a dual basis of continuous linear functionals l_j on $\mathcal{M}$ which satisfy $l_j(x_i) = \delta_{ij}$ (see Kreyszig, [11], pp.113-114). From the Hahn-Banach theorem (see Schaefer, [15], theorem 4.2, pp.49) we know that these l_j can be extended to L_j continuously to the whole space $\mathcal{F}$. Define $P(x) = \sum_{j=1}^n L_j(x)x_j$, $x \in \mathcal{F}$. Then P is a continuous projection from $\mathcal{F}$ onto $\mathcal{M}$ and hence $\mathcal{F} = \mathcal{M} \oplus \ker(P)$. $\square$

Definition 2.1.7 Let $\mathcal{F}_1, \mathcal{F}_2$ be Fréchet spaces. An operator A defined on a dense subspace $\mathcal{D}(A)$ of $\mathcal{F}_1$ with $A : \mathcal{D}(A) \rightarrow \mathcal{F}_2$ is said to be *closed* if its graph

$$\{(x, y) | x \in \mathcal{D}(A), y = Ax\}$$

is closed in $\mathcal{F}_1 \times \mathcal{F}_2$.

A well known theorem for Banach spaces holds true for Fréchet spaces also.

Theorem 2.1.8 (THE CLOSED-GRAPH THEOREM).

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Fréchet spaces and A an everywhere defined linear mapping from $\mathcal{F}_1$ to $\mathcal{F}_2$ such that the graph $(A) = \{(\xi, A\xi) | \xi \in \mathcal{F}_1\}$ is a closed subspace in $\mathcal{F}_1 \times \mathcal{F}_2$ (with respect to the product topology). Then A is continuous.

Proof: See Yosida ([23], theorem 1, p.79). □

Lemma 2.1.9 Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ be Fréchet spaces. If A is a closed operator from $\mathcal{F}_2$ to $\mathcal{F}_3$ and B continuous from $\mathcal{F}_1$ to $\mathcal{F}_2$ together with $\text{Range}(B) \subseteq \text{Dom}(A)$, then AB is a closed operator from $\mathcal{F}_1$ to $\mathcal{F}_3$.

Proof: Obvious. □

Lemma 2.1.10 Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Fréchet spaces and T be a closed operator from $\mathcal{F}_1 \rightarrow \mathcal{F}_2$ such that T is surjective and $\ker(T)$ is finite dimensional. Then, if $\ker(T) \neq \{0\}$, T has infinitely many right-inverses that are continuous.

Proof: Take an operator T that satisfies the conditions. From lemma 2.1.6 we know that there exists a closed subspace $\mathcal{N}$ such that $\mathcal{F}_1 = \ker(T) \oplus \mathcal{N}$. Define the operator $\tilde{T} : \mathcal{N} \rightarrow \mathcal{F}_2$ as follows:

$$\text{Dom}(\tilde{T}) = \text{Dom}(T) \cap \mathcal{N}, \quad \tilde{T}f = Tf \quad (f \in \mathcal{N} \cap \text{Dom}(T))$$

Note that $\text{Dom}(\tilde{T}) \neq \{0\}$ since $\text{Dom}(T) = \ker(T) \oplus (\text{Dom}(T) \cap \mathcal{N})$. Since T is surjective it is clear that $\tilde{T}$ is a bijection. Now $\tilde{T}$ is closed since it is the intersection of two closed sets

$$\text{Graph}(\tilde{T}) = \text{Graph}(T) \cap (\mathcal{N} \oplus \mathcal{F}_2)$$

Thus $\tilde{T}^{-1}$ is also a closed operator, defined on the whole $\mathcal{F}_2$. Hence $\tilde{T}^{-1}$ is a continuous operator from $\mathcal{F}_2$ to $\mathcal{N}$ (see theorem 2.1.8). In fact $\tilde{T}^{-1}$ maps $\mathcal{F}_2$ onto $\mathcal{D}(\tilde{T})$ and thus $T\tilde{T}^{-1} = I$ is well defined.

There are infinitely many continuous right-inverses if $\mathcal{N} \neq \{0\}$, because in that case there are infinitely many spaces complementary to $\ker(T)$. □

Since we will deal with behaviors with signals in $C(\mathbb{R})$ we pay special attention to this Fréchet space. Consider the vector space $C(\mathbb{R})$ of complex-valued functions on $\mathbb{R}$. In first instance, we prefer complex-valued functions,

but this is only from a technical point of view. Later on, when we study behaviors, we restrict ourselves to real-valued continuous functions. The vector space $C(\mathbb{R})$ becomes a Fréchet space when endowed with the topology generated by the ordered family of seminorms:

$$p_n(f) = \max_{t \in [-n, n]} |f(t)| \quad (n \in \mathbb{N}, f \in C(\mathbb{R})) \quad (2.1)$$

This is the topology of uniform convergence on compact sets in $\mathbb{R}$. The characterization of certain operators on $C(\mathbb{R})$ utilizes the dual space, i.e. the space of all continuous linear functionals on $C(\mathbb{R})$. Determining this dual is a good opportunity to get acquainted with some techniques for Fréchet spaces. Especially proving convergence (with respect to seminorms) is a little different from the concept of convergence for normed spaces. Moreover several results in this chapter are proven in an elementary way such that the techniques can be applied to other Fréchet spaces, such as $C^{-k}(\mathbb{R})$ ($k \in \mathbb{N}$), later on.

The following definition is from Dunford & Schwartz ([4]).

Definition 2.1.11 The space of all functions from $\mathbb{R} \rightarrow \mathbb{C}$ of bounded variation is denoted by $ba(\mathbb{R})$. A function is of *bounded variation* if

$$\exists K > 0 \forall n \in \mathbb{N} \forall \{t_0, \dots, t_n\} \subset \mathbb{R} [\forall i \in \{1, \dots, n\} (t_{i-1} < t_i) \wedge \sum_{i=1}^n |f(t_i) - f(t_{i-1})| < K]$$

Definition 2.1.12 The subclass in $ba(\mathbb{R})$ of right-continuous functions μ such that there exists $T \in \mathbb{R}$, with

$$\begin{cases} \mu(t) = 0 & t \leq -T \\ \mu(t) = \mu(T) & t \geq T \end{cases}$$

is denoted by $ba_c(\mathbb{R})$.

Lemma 2.1.13 A linear functional $\mathcal{L}$ on $C(\mathbb{R})$, that is continuous, corresponds uniquely to some $\mu \in ba_c(\mathbb{R})$ in the following way

$$\forall f \in C(\mathbb{R}) \left[\mathcal{L}(f) = \int_{\mathbb{R}} f d\mu =: \mu(f) \right]$$

(One should interpret this integral as a Riemann-Stieltjes integral)

Proof: The seminorms on $C(\mathbb{R})$ (see 2.1) are ordered and hence we can apply lemma 2.1.2. The functional $\mathcal{L}$ on $C(\mathbb{R})$ is continuous if and only if

$$\exists n \in \mathbb{N} \exists C \in \mathbb{R} \forall f \in C(\mathbb{R}) [|\mathcal{L}(f)| \leq C \cdot p_n(f)]$$

We define the continuous linear functional $\mathcal{L}_r$ on $C([-n, n])$ in the following way: extend $g \in C([-n, n])$ to a continuous function on $\mathbb{R}$, say g_{ext} , and define $\mathcal{L}_r(g) = \mathcal{L}(g_{ext})$. This is independent of the extension. Namely, if f_1 and f_2 are two distinct extensions of g such that

$$f_1|_{[-n, n]} = f_2|_{[-n, n]} = g$$

then we have

$$|\mathcal{L}(f_1 - f_2)| \leq C \cdot p_n(f_1 - f_2) = 0$$

and thus $\mathcal{L}(f_1) = \mathcal{L}(f_2)$.

The Riesz-representation theorem on $C([-n, n])$ (see Kreyszig, [11], theorem 4.4.1, p.227-231) yields that $\mathcal{L}_r$ corresponds one-to-one to a right-continuous function $t \mapsto \mu(t)$ ($t \in [-n, n]$) of bounded variation on $[-n, n]$ with $\mu(-n) = 0$. The relation between $\mathcal{L}_r$ and μ is as follows:

$$\forall f \in C([-n, n]) [\mathcal{L}_r(f) = \int_{-n}^n f(t) d\mu(t)]$$

Define

$$\mu_{ext}(t) = \begin{cases} 0 & t \leq -n \\ \mu(t) & t \in (-n, n] \\ \mu(n) & t \geq n \end{cases}$$

Then μ_{ext} in $ba_c(\mathbb{R})$ and for all $f \in C(\mathbb{R})$

$$\mathcal{L}(f) = \mathcal{L}_r(f|_{[-n, n]}) = \int_{\mathbb{R}} f d\mu_{ext}$$

□

Definition 2.1.14 The *support* of some $\mu \in ba_c(\mathbb{R})$, say $\text{supp}(\mu)$, is defined to be the complement of the union of all open sets $\mathcal{A}$ for which

$$\forall f \in C(\mathbb{R}) [(\text{supp})(f) \subseteq \mathcal{A} \Rightarrow \mu(f) = \int_{\mathbb{R}} f d\mu = 0]$$

So for $\mu \in ba_c(\mathbb{R})$ $\text{supp}(\mu)$ is compact.

Theorem 2.1.15 *The space $ba_c(\mathbb{R})$ is isomorphic to the dual space of $C(\mathbb{R})$. Namely, two corresponding elements $\mathcal{L}$ in the dual of $C(\mathbb{R})$ and μ in $ba_c(\mathbb{R})$ satisfy the identity*

$$\forall f \in C(\mathbb{R}) \quad [\mathcal{L}(f) = \int_{\mathbb{R}} f d\mu = \mu(f)]$$

Proof: One implication is shown in lemma 2.1.13.

On the other hand, if $\mu \in ba_c(\mathbb{R})$ and $\text{supp}(\mu) \subseteq [-a, a]$ then for all $f \in C(\mathbb{R})$

$$|\mathcal{L}(f)| = \left| \int_{\mathbb{R}} f d\mu \right| \leq \text{var}(\mu) \cdot p_a(f)$$

□

Because of this one-to-one relationship we will identify the function μ with its corresponding continuous linear functional.

To simplify formulas we introduce the *Volterra-integral operators*

$$I_0 f = f, \quad I_{k+1} f := \int_0^t (I_k f)(\tau) d\tau \quad k = 0, 1, 2, \dots$$

This can be made explicit

$$I_0 f = f, \quad I_k f := \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} f(\tau) d\tau \quad k = 1, 2, \dots$$

The following relations are immediate

$$D^j I_k = I_{k-j} \quad (0 \leq j \leq k), \quad D^j I_k = D^{j-k} \quad (0 \leq k \leq j) \quad (2.2)$$

Lemma 2.1.16 *The integral operator I_k is continuous from $C(\mathbb{R})$ to $C(\mathbb{R})$.*

Proof: Let $f \in C(\mathbb{R})$. Then for each $m \in \mathbb{N}$

$$\begin{aligned} p_m(I_k f) &= \max_{t \in [-m, m]} \left| \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} f(\tau) d\tau \right| \\ &\leq \max_{t \in [-m, m]} (|f(t)| \cdot \left| \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} d\tau \right|) \\ &\leq p_m(f) \cdot \max_{t \in [-m, m]} \frac{|t|^k}{k!} \\ &= p_m(f) \cdot \frac{m^k}{k!} \end{aligned}$$

from which continuity of I_k follows. $\square$

Similar to $C(\mathbb{R})$ we can introduce the function space $C^k(\mathbb{R})$ the vector space of complex-valued functions that are k times differentiable. The Fréchet topology for $C^k(\mathbb{R})$ is generated by the ordered family of seminorms:

$$p_m^{(k)}(f) = \sum_{j=0}^k p_m(D^j f) \quad (f \in C^k(\mathbb{R}), m \in \mathbb{N}) \quad (2.3)$$

where D denotes differentiation. The following family of seminorms:

$$\tilde{p}_m^{(k)}(f) = \sum_{j=1}^{k-1} |(D^j f)(0)| + p_m(D^k f), \quad (f \in C^k(\mathbb{R}), m \in \mathbb{N}) \quad (2.4)$$

generate the same topology. Namely, if $f \in C^k(\mathbb{R})$, then $\tilde{p}_m^{(k)}(f) \leq p_m^{(k)}(f)$. On the other hand, for all $f \in C^k(\mathbb{R})$

$$f(t) = \sum_{j=0}^{k-1} (D^j f)(0) \frac{t^j}{j!} + (I_k D^k f)(t)$$

and hence (differentiating this equality l times)

$$(D^l f)(t) = \sum_{j=0}^{k-l-1} (D^j f)(0) \frac{t^j}{j!} + (I_{k-l} D^{k-l} D^l f)(t)$$

This yields (using the estimate from the proof in lemma 2.1.16 for $p_m(I_{k-l} f)$)

$$\begin{aligned} p_m(D^l f) &\leq [p_m(\sum_{j=0}^{k-l-1} |(D^j f)(0)|) + p_m(D^k f(t))] \sum_{s=0}^{k-l} \frac{t^s}{s!} \\ &\leq (p_m(\sum_{j=0}^{k-1} |(D^j f)(0)|) + p_m(D^k f)) \cdot e^m \end{aligned}$$

and herewith

$$p_m^{(k)}(f) = \sum_{j=0}^k p_m(D^j f) \leq k e^m \tilde{p}_m^{(k)}(f)$$

We will use either collection of seminorms, whichever turns out to be more convenient.

Lemma 2.1.17 For all $l \geq 0$, $I_k : C^l(\mathbb{R}) \rightarrow C^{k+l}(\mathbb{R})$ is continuous.

Proof: Let f be in $C^l(\mathbb{R})$. For each $m \in \mathbb{N}$

$$\begin{aligned}
 p_m^{(k+l)}(I_k f) &= \sum_{j=0}^{k+l} p_m(D^j I_k f) \\
 &= \sum_{j=0}^k p_m(I_{k-j} f) + \sum_{j=k+1}^{k+l} p_m(D^{j-k} f) \\
 &\leq \sum_{j=0}^k C_j(k, m) p_m(f) + \sum_{j=1}^l p_m(D^j f) \\
 &\leq C(k, m) p_m(f) + p_m^{(l)}(f) \leq D(k, m) p_m^{(l)}(f)
 \end{aligned}$$

where $C_j(k, m)$, $C(k, m)$ and $D(k, m)$ are constants and we used lemma 2.1.16. $\square$

Lemma 2.1.18 A linear functional $\mathcal{L}$ on $C^k(\mathbb{R})$ is continuous if and only if $\mathcal{L}$ is of the form: $\mathcal{L}(f) = (Sf)(0) + \mu(D^k f)$, for some $\mu \in ba_c(\mathbb{R})$ and $S \in LDO_{k-1}(\mathbb{R})$.

Proof: The "if"-part is obvious if one uses the seminorms (2.4). Now let $\mathcal{L}$ be a continuous linear functional on $C^k(\mathbb{R})$. Note that $\mathcal{L} \circ I_k$ is a continuous linear functional on $C(\mathbb{R})$ and thus $\mathcal{L} \circ I_k = \mu$ for some $\mu \in ba_c(\mathbb{R})$. For all $f \in C^k(\mathbb{R})$

$$f = \sum_{j=0}^{k-1} (D^j f)(0) \frac{t^j}{j!} + I_k D^k f$$

and

$$\begin{aligned}
 \mathcal{L}(f) &= \sum_{j=0}^{k-1} \alpha_j (D^j f)(0) + \mu(D^k f) \\
 &= \left(\sum_{j=0}^{k-1} \alpha_j D^j f \right)(0) + \mu(D^k f) \\
 &= (Sf)(0) + \mu(D^k f)
 \end{aligned}$$

where $S := \sum_{j=0}^{k-1} \alpha_j D^j$ and $\mu \in ba_c(\mathbb{R})$. $\square$

Remark 2.1.19 In section 4 of this chapter (corollary 2.4.26) we show that $\mathcal{L}$ is a continuous linear functional on $C^k(\mathbb{R})$ if and only if $\mathcal{L}(f) = \mu(Sf)$ for some $\mu \in ba_c(\mathbb{R})$ and $S \in LDO_k(\mathbb{R})$. $\square$

Consider the vector space $C^\infty(\mathbb{R})$ of infinitely differentiable functions on $\mathbb{R}$, or in formula, $C^\infty(\mathbb{R}) = \bigcap_{k=0}^\infty C^k(\mathbb{R})$. It is a Fréchet space if we use the following ordered family of seminorms to generate the topology (see Treves, [18], p.86)

$$p_n^{(\infty)}(f) = \sum_{j=0}^n p_j^{(j)}(f) \quad (n \in \mathbb{N}, f \in C^\infty(\mathbb{R}))$$

where $p_j^{(j)}$ is the seminorms on $C^j(\mathbb{R})$ defined in (2.3). This family of seminorms on $C^\infty(\mathbb{R})$ immediately yields that $C^\infty(\mathbb{R})$ can be embedded continuously in $C^k(\mathbb{R})$ ($C^\infty(\mathbb{R}) \hookrightarrow C^k(\mathbb{R})$) for all $k \in \mathbb{N}$. This means that $C^\infty(\mathbb{R}) \subseteq C^k(\mathbb{R})$ and the identity is a continuous operator from $C^\infty(\mathbb{R})$ into $C^k(\mathbb{R})$.

Lemma 2.1.20 *A linear functional $\mathcal{L}$ on $C^\infty(\mathbb{R})$ is continuous if and only if there exists a $k \in \mathbb{N}$ such that $\mathcal{L}$ extends continuously to $C^k(\mathbb{R})$.*

Proof: Let $\mathcal{L}$ be a continuous linear functional on $C^\infty(\mathbb{R})$. Because of the ordering of the seminorms, there exists an $k \in \mathbb{N}$, $C \in \mathbb{R}$ such that for all $f \in C^\infty(\mathbb{R})$

$$|\mathcal{L}(f)| \leq \sum_{j=0}^k p_j^{(j)}(f) \leq C \left[\sum_{j=0}^{k-1} |(D^j f)(0)| + \max_{t \in [-n, n]} |(D^k f)(t)| \right]$$

Moreover $C^\infty(\mathbb{R})$ is dense in $C^k(\mathbb{R})$ and consequently $\mathcal{L}$ has a unique continuous extension to $C^k(\mathbb{R})$, say $\mathcal{L}_{ext}$ (see 2.4). The reverse statement is obvious. $\square$

Lemma 2.1.21 *A linear functional $\mathcal{L}$ on $C^\infty(\mathbb{R})$ is continuous if and only if $\mathcal{L}$ is of the form: $\mathcal{L}(f) = (Sf)(0) + \mu(D^k f)$, for some $k \in \mathbb{N}$, $\mu \in ba_c(\mathbb{R})$ and $S \in LDO_{k-1}(\mathbb{R})$.*

Proof: The "if"-part is obvious. From lemma 2.1.20 we know that $\mathcal{L}$ has a continuous extension $\mathcal{L}_{ext}$ to $C^k(\mathbb{R})$ for some $k \in \mathbb{N}$ and hence (see lemma 2.1.18) $\mathcal{L}_{ext}$ is of the form $\mathcal{L}_{ext}(f) = (Sf)(0) + \mu(D^n f)$, for some $\mu \in ba_c(\mathbb{R})$ and $S \in LDO_{k-1}(\mathbb{R})$. Because $\mathcal{L}_{ext}|_{C^\infty(\mathbb{R})} = \mathcal{L}$ we are done. $\square$

Remark 2.1.22 Note that the continuous linear functionals on $C^\infty(\mathbb{R})$ are of the same form as those on $C^k(\mathbb{R})$. The only difference is that on $C^k(\mathbb{R})$ the k is fixed, whereas on $C^\infty(\mathbb{R})$ k depends on the choice for the functional. $\square$

2.2 Fourier transformation

It is well known that whenever convolution operators are involved, the Fourier transformation can simplify calculus. Moreover, in many cases, the space of Fourier transforms of a function space can be dealt with more easily than with the function space itself.

For notational convenience we define

$$e_\omega : t \mapsto e^{-i\omega t}, \mathbb{R} \rightarrow \mathbb{C} \text{ for } \omega \in \mathbb{C} \quad (2.5)$$

Definition 2.2.1 For each $\mu \in ba_c(\mathbb{R})$ its *Fourier transform* $\mathcal{F}_\mu$ is defined by

$$\mathcal{F}_\mu(\omega) = \mu(e_\omega) = \int_{\mathbb{R}} e^{-i\omega t} d\mu(t) \quad (\omega \in \mathbb{C}).$$

Since μ has compact support, $\mathcal{F}_\mu$ is an analytic function on $\mathbb{C}$.

Lemma 2.2.2 *The Fourier transformation is injective on $ba_c(\mathbb{R})$.*

Proof: Let $\mu \in ba_c(\mathbb{R})$ and suppose $\mathcal{F}_\mu = 0$. Then, because μ has compact support, say $\text{supp}(\mu) = [-T, T]$, we obtain for all $n \in \mathbb{N}$

$$\left(\frac{d}{d\omega}\right)^n \int_{-T}^T e^{-i\omega t} d\mu(t) = (-i)^n \int_{-T}^T t^n e^{-i\omega t} d\mu(t) = 0 \text{ for all } \omega \in \mathbb{C}$$

If $\omega = 0$, then we have

$$\forall n \in \mathbb{N} \left[\int_{-T}^T t^n d\mu(t) = 0 \right]$$

and, because the subspace of polynomials is dense in $C([-T, T])$, this yields $\mu = 0$. $\square$

Lemma 2.2.3 $\text{span}\{e_\omega \mid \omega \in \mathbb{R}\}$ is dense in $C(\mathbb{R})$.

Proof: Suppose $W := \overline{\text{span}\{e_\omega \mid \omega \in \mathbb{R}\}} \neq C(\mathbb{R})$. Then, according to the Hahn-Banach theorem, there exists a $\mu \in ba_c(\mathbb{R})$, $\mu \neq 0$ such that $\mu|_W = 0$. Hence, for all $\omega \in \mathbb{R}$, we have $\mu(e_\omega) = \mathcal{F}_\mu(\omega) = 0$. Since $\mathcal{F}_\mu$ is an analytic function, it follows that $\mathcal{F}_\mu(\omega) \equiv 0$ and therefore, using lemma 2.2.2, $\mu = 0$. This yields a contradiction, hence $W = C(\mathbb{R})$. $\square$

Definition 2.2.4 An entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ is of *exponential type* if there are positive constants $A, B \in \mathbb{R}$ such that

$$\forall \omega \in \mathbb{C} \left[|f(\omega)| \leq A \cdot e^{B|\omega|} \right]$$

Definition 2.2.5 An entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ is said to be of *finite order* if there exists a positive number k such that

$$\max\{|f(z)| \mid |z| = r\} \leq e^{r^k}$$

as soon as r is "sufficiently large", i.e. $r > r(k)$. The greatest lower bound of all positive numbers k for which this is true is called the *order* of the function and is denoted by ρ .

It is immediate that for each $\mu \in ba_c(\mathbb{R})$ its Fourier transform $\mathcal{F}_\mu$ is an entire function of exponential type, which is bounded on the real axis. In fact, we have

$$|\mathcal{F}_\mu(\omega)| \leq C \cdot e^{a|\text{Im}(\omega)|} \quad (2.6)$$

for some $C \in \mathbb{R}$ and $a \in \mathbb{R}$ is such that $\text{supp}(\mu) \subseteq [-a, a]$. Unfortunately, the condition in 2.6 is not sufficient for $\mathcal{F}_\mu$ to be the Fourier transform of a $\mu \in ba_c(\mathbb{R})$. However in Kahane ([8], p.14) the following result can be found:

Theorem 2.2.6 *If an entire function $M(\omega)$ ($\omega = u+iv$) satisfies for some $a < b$ and $c > 0$ the conditions*

$$|M(\omega)| \leq K e^{c|\omega|} \quad \text{and} \quad M(u) = O(|u|^{-2})$$

$$|M(iv)| = O(e^{bv}), \quad v > 0; \quad |M(iv)| = O(e^{av}), \quad v < 0$$

Then $M(\omega)$ is the Fourier transform of a $\mu \in ba_c(\mathbb{R})$ satisfying $\text{supp}(\mu) \subseteq [a, b]$.

Lemma 2.2.7 *For all $\mu \in ba_c(\mathbb{R})$ the Fourier transform $\mathcal{F}_\mu$ has either a (countably) infinite number of zeroes in the complex plane or no zeroes at all. Moreover there is no accumulation point in the set of zeroes.*

Proof: Suppose $\mathcal{F}_\mu$ has only finitely many zeroes, then $\mathcal{F}_\mu(\omega) = p(\omega)e^{g(\omega)}$ where $p(\omega)$ is a polynomial containing the zeroes of $\mathcal{F}_\mu$ and $g(\omega)$ is an analytic function. The quotient of two entire functions of exponential type again is an entire function of exponential type (see Malgrange, [12], proposition 3, p.305). Hence

$$\left| \frac{\mathcal{F}_\mu(\omega)}{p(\omega)} \right| = |e^{g(\omega)}| \leq K e^{a|\omega|}$$

for some $K \in \mathbb{R}$, $a \in \mathbb{C}$. Since g is analytic we can write $g(\omega) = g(0) + \omega g'(0) + \omega^2 h(\omega)$ for some analytic function h . We obtain the following estimate

$$|e^{\omega^2 h(\omega)}| \leq \frac{K e^{a|\omega|}}{|e^{g(0) + \omega g'(0)}|} \leq K' e^{b|\omega|}$$

This yields $\lim_{|\omega| \rightarrow \infty} |h(\omega)| = 0$. Hence h is analytic and bounded implying $h(\omega) = \text{constant} = 0$ (Liouville). The expression for the Fourier transform equals $\mathcal{F}_\mu = p(\omega)e^{c\omega+d}$. Clearly $\mathcal{F}_\mu$ cannot be bounded on the real axis, unless there are no zeroes at all. Since $\mathcal{F}_\mu$ is analytic there is no accumulation of zeroes (see Rudin, [14], theorem 10.18, pp.208-209). $\square$

Remark 2.2.8 If μ is real-valued then $\mathcal{F}_\mu(iy) \in \mathbb{R}$ for all $y \in \mathbb{R}$, and hence $\mathcal{F}_\mu(\alpha) = 0 \Leftrightarrow \mathcal{F}_\mu(-\bar{\alpha}) = 0$. The zeroes are symmetric with respect to the imaginary axis. $\square$

We define differentiation on the class $ba_c(\mathbb{R})$ in a weak sense.

Definition 2.2.9 We call $\mu \in ba_c(\mathbb{R})$ *weakly differentiable* with derivative $\nu \in ba_c(\mathbb{R})$ if and only if

$$\forall f \in C^1(\mathbb{R}) \quad [\mu(Df) = -\nu(f)]$$

Similarly for higher order differentiation operators.

Note that ν is unique and consistent with the usual differentiation. Namely, if for all $f \in C^1(\mathbb{R})$ we have $\mu(Df) = -\nu_1(f) = -\nu_2(f)$ then

$$0 = \nu_1(e_\omega) - \nu_2(e_\omega) = (\nu_1 - \nu_2)(e_\omega) = \mathcal{F}_{\nu_1 - \nu_2}(\omega) \text{ for all } \omega \in \mathbb{C}$$

The injectivity of the Fourier transform implies $\nu_1 = \nu_2$ (lemma 2.2.2). Consistency follows immediately by partial integration.

An immediate consequence is the following: if $\mu \in ba_c(\mathbb{R})$ is differentiable with derivative ν if and only if

$$i\omega \mathcal{F}_\mu(\omega) = -\mu(De_\omega) = \nu(e_\omega) = \mathcal{F}_\nu(\omega)$$

Lemma 2.2.10 Let $\nu \in ba_c(\mathbb{R})$. If $\mathcal{F}_\nu(\beta) = 0$ then $\mathcal{F}_\nu(\omega) = i(\omega - \beta)\mathcal{F}_\mu(\omega)$ for some $\mu \in ba_c(\mathbb{R})$.

Proof: Without loss of generality we assume $\beta = 0$ (otherwise we replace $\nu(t)$ by $e^{i\beta t}\nu(t)$). Define $\mu = \nu \circ I_1$, then

$$\mu(e_\omega) = \nu(I_1 e_\omega) = \nu\left(\frac{e_\omega - e_0}{i\omega}\right) = \frac{\nu(e_\omega)}{i\omega}$$

since $\nu(e_0) = 0$. □

As for $ba_c(\mathbb{R})$ we can define the Fourier transformation on the duals of $C^k(\mathbb{R})$ and $C^\infty(\mathbb{R})$ by evaluation in e_ω , $\omega \in \mathbb{C}$.

Definition 2.2.11 For all $\mathcal{L}$ in $(C^k(\mathbb{R}))'$, the dual space of $C^k(\mathbb{R})$, we define the Fourier transform $\mathcal{F}_\mathcal{L}(\omega)$ by $\mathcal{L}(e_\omega)$.

Using lemma 2.1.18 and Hadamard's theorem on factorization for an entire function of exponential type (see Young, [24], th.9, p.74) we can be more explicit:

$$\begin{aligned} \mathcal{L}(e_\omega) &= (Se_\omega)(0) + \mu_2(D^k e_\omega) \\ &= p(-i\omega) + (-i\omega)^k e^{a\omega + b\omega^l} \prod_{j=1}^{\infty} \left(1 - \frac{\omega}{a_j}\right) e^{\frac{\omega}{a_j}} \end{aligned} \quad (2.7)$$

where p is a polynomial of degree $k - 1$.

Again this characterization does not imply that we are dealing with the Fourier transform of an element in $(C^k(\mathbb{R}))'$.

Remark 2.2.12 In remark 2.1.19 we pointed out that a continuous linear functional on $C^k(\mathbb{R})$ is of the form $\mathcal{L}(f) = \mu(Sf)$ for some $S \in LDO_k(\mathbb{R})$ and $\mu \in ba_c(\mathbb{R})$ (see corollary 2.4.26 which is yet to come). Corresponding to this representation we find a different form for the Fourier transform from the one in equation (2.7):

$$\mathcal{F}_\mu(\omega) = p(-i\omega)e^{a\omega+b}\omega^l \prod_{j=1}^{\infty} (1 - \frac{\omega}{a_j})e^{\frac{\omega}{a_j}} \quad (2.8)$$

where $p(-i\omega)$ is a polynomial of order k . □

Definition 2.2.13 For all $\mathcal{L}$ in $(C^\infty(\mathbb{R}))'$ we define the Fourier transform $\mathcal{F}_\mathcal{L}(\omega)$ by $\mathcal{L}(e_\omega)$.

Lemma 2.1.21 yields the same Fourier image as for $(C^k(\mathbb{R}))'$ except that k may depend on $\mathcal{L}$. For some $k \in \mathbb{N}$

$$\begin{aligned} \mathcal{L}(e_\omega) &= (Se_\omega)(0) + \mu(D^k e_\omega) \\ &= (Se_\omega)(0) + (-i\omega)^k \mu(e_\omega) \\ &= p(-i\omega) + (-i\omega)^k e^{a\omega+b}\omega^l \prod_{j=1}^{\infty} (1 - \frac{\omega}{a_j})e^{\frac{\omega}{a_j}} \end{aligned} \quad (2.9)$$

where p is a polynomial of degree $k - 1$.

Similarly as for the Fourier transform of $\mathcal{L}(f) \in (C^k(\mathbb{R}))'$ this equation can be rewritten into equation (2.8).

Another consequence of the representation $\mathcal{L}(f) = \mu(Sf)$ is the following estimate

$$|\mathcal{L}(e_\omega)| = |\mu(Se_\omega)| = \left| \int_{\mathbb{R}} (Se_\omega) d\mu \right| \leq C \cdot |p(-i\omega)| e^{a|Im(\omega)|} \quad (2.10)$$

In case we are dealing with the Fourier transform of some $\mathcal{L} \in (C^\infty(\mathbb{R}))'$ the expression above is also sufficient. I.e an analytic function satisfying (2.10) is the Fourier transform of an element in $(C^\infty(\mathbb{R}))'$. It is known as the (first part of the) *Paley-Wiener theorem* (see Friedlander, [6], th.10.2.2, p.132).

Theorem 2.2.14 (THE PALEY-WIENER THEOREM)

A function $\phi(\omega)$ which is analytic on $\mathbb{C}$ is the Fourier transform of a continuous linear functional on $C^\infty(\mathbb{R})$ supported in $|t| \leq a$ if and only if there is an estimate

$$|\phi(\omega)| \leq C(1 + |\omega|)^N e^{a|\operatorname{Im}(\omega)|} \quad (\omega \in \mathbb{C})$$

for some constants $C, N \geq 0$.

Furthermore, $\phi(\omega)$ is the Fourier transform of a function $f \in C_c^\infty(\mathbb{R})$ with support in $[-a, a]$ if and only if for each $m = 0, 1, \dots$ there is a constant $C_m > 0$ such that

$$\phi(\omega) \leq C_m(1 + |\omega|)^{-m} e^{a|\operatorname{Im}(\omega)|} \quad (\omega \in \mathbb{C})$$

2.3 Differential operators

In this section we prove a few results on differential operators. These results are well known, but we prove them in such a way that these proofs can be extended to prove similar properties of generalized differential operators later on.

Lemma 2.3.1 A differential operator $S \in LDO_k(\mathbb{R})$ is continuous, when viewed as an operator from $C^k(\mathbb{R})$ to $C(\mathbb{R})$.

Proof: This is immediate from the seminorms in (2.3). □

Lemma 2.3.2 The differential operator D^k as an operator from $C(\mathbb{R})$ to $C(\mathbb{R})$, with domain $\operatorname{Dom}(D^k) = C^k(\mathbb{R})$ is closed.

Proof: Take a sequence $(f_n)_{n \in \mathbb{N}}$ in $C^k(\mathbb{R})$ such that

$$\begin{cases} f_n \rightarrow f & (n \rightarrow \infty) \\ D^k f_n \rightarrow g & (n \rightarrow \infty) \end{cases} \quad \text{both in } C(\mathbb{R})\text{-sense}$$

We have the following equality again

$$f_n(t) = \sum_{j=0}^{k-1} (D^j f_n)(0) \frac{t^j}{j!} + (I_k D^k f_n)(t) \quad (2.11)$$

If $n \rightarrow \infty$ then $I_k D^k f_n \rightarrow I_k g$ and $f_n \rightarrow f$.

We see that $\sum_{j=0}^{k-1} f_n^{(j)}(0) \frac{t^j}{j!}$ must converge too and since this is a sequence in the finite-dimensional and thus closed subspace of polynomials of degree $\leq k$ its limit is also in this subspace. Let $n \rightarrow \infty$ in equation (2.11)

$$f(t) = \sum_{j=0}^{k-1} a_j \frac{t^j}{j!} + (I_k g)(t) \text{ for some } a_j \in \mathbb{R}$$

Then we see that $f \in C^k(\mathbb{R})$ and $D^k f = g$. Hence D^k is closed. $\square$

Lemma 2.3.3 *Let p be a polynomial of degree k and $S := p(D)$ an operator from $C(\mathbb{R})$ to $C(\mathbb{R})$ with domain $C^k(\mathbb{R})$. Then S is closed.*

Proof: Let $f \in C^k(\mathbb{R})$. $Sf = p(D)f = \sum_{j=0}^k \alpha_j D^j f$ for some $\alpha_j \in \mathbb{C}$, with $\alpha_k = 1$ (without loss of generality of course). So

$$Sf = D^k(\alpha_k f + \alpha_{k-1} I_1 f + \dots + \alpha_0 I_k f)$$

where we use that $D^k I_m = D^{k-m}$ for all $0 \leq m \leq k$. The operator J

$$J = \alpha_k + \alpha_{k-1} I_1 + \dots + \alpha_0 I_k$$

clearly is continuous as an operator from the whole $C(\mathbb{R})$ to $C(\mathbb{R})$ and for all $f \in C^k(\mathbb{R})$ $Sf = D^k Jf$. Since D^k is a closed operator from $C(\mathbb{R})$ to $C(\mathbb{R})$ with $\text{Dom}(D^k) = C^k(\mathbb{R})$ we can apply lemma 2.1.9 we have $D^k J$ is a closed operator from $C(\mathbb{R})$ to $C(\mathbb{R})$. It is important to see that we do not know yet whether S is closed or not, since we did not prove that Sf is well defined. So the question is, is $f \in \text{Dom}(S) = C^k(\mathbb{R})$?

Note that $\alpha_k f + \alpha_{k-1} I_1 f + \dots + \alpha_0 I_k f$ is in $C^k(\mathbb{R})$ and $\alpha_{k-1} I_1 f + \dots + \alpha_0 I_k f$ is in $C^1(\mathbb{R})$. Hence their difference is in $C^1(\mathbb{R})$. This is $\alpha_k f$, so $f \in C^1(\mathbb{R})$, because $\alpha_k \neq 0$. Similarly, because $\alpha_{k-2} I_2 f + \dots + \alpha_0 I_k f$ is in $C^2(\mathbb{R})$, $\alpha_k f + \alpha_{k-1} I_1 f$ is in $C^2(\mathbb{R})$. We know already that $f \in C^1(\mathbb{R})$ and thus $I_1 f \in C^2(\mathbb{R})$. Hence $f \in C^2(\mathbb{R})$. Repeating this argument leads to $f \in C^k(\mathbb{R}) = \text{Dom}(S)$. Therefore S is a closed operator. $\square$

Besides closedness an operator $S \in LDO(\mathbb{R})$ has an additional important property, that is well known.

Lemma 2.3.4 *Let $S := p(D)$, for some polynomial p of degree k . Then S is a surjective operator from $C^k(\mathbb{R})$ onto $C(\mathbb{R})$.*

Proof: Use the variation-of-constants-formula. □

We can obtain a solution to the differential equation $(D - \lambda I)f = g$ by finding a right inverse for the operator $(D - \lambda I)$. More can be said about these right-inverses.

Definition 2.3.5 The set of all linear mappings from a vector space V into a vector space W is denoted by $\mathcal{L}(V, W)$. If $V = W$ we write $\mathcal{L}(V)$.

For all $S \in LDO(\mathbb{R})$ we can define the sets

$$\text{Inv}(S, C(\mathbb{R})) := \{J \mid J \in \mathcal{L}(C(\mathbb{R})), J \text{ is continuous, } SJ = I \text{ on } C(\mathbb{R})\}$$

$$\text{Inv}(S, C^\infty(\mathbb{R})) := \{J \mid J \in \mathcal{L}(C^\infty(\mathbb{R})), J \text{ is continuous, } SJ = I \text{ on } C^\infty(\mathbb{R})\}$$

Lemma 2.3.6 *Let $S \in LDO(\mathbb{R})$ and S of nonzero order, then $\text{Inv}(S, C(\mathbb{R}))$ and $\text{Inv}(S, C^\infty(\mathbb{R}))$ consist of infinitely many elements. Furthermore we have $\text{Inv}(S, C(\mathbb{R})) \subseteq \text{Inv}(S, C^\infty(\mathbb{R}))$.*

Proof: It is clear that all differential operators (from $C(\mathbb{R})$ to $C(\mathbb{R})$) in $LDO(\mathbb{R})$ have a finite dimensional kernel. Moreover this kernel is a subspace of $C^\infty(\mathbb{R})$. Hence, we can apply lemma 2.1.10 to obtain the first part of the lemma.

The second part involves a decomposition problem. Consider the Fréchet space $C(\mathbb{R})$ and suppose we have a decomposition for it:

$$C(\mathbb{R}) = \ker(S) \oplus \mathcal{N} \tag{2.12}$$

where $\mathcal{N}$ is closed in $C(\mathbb{R})$ -sense. Then, if we intersect it with $C^\infty(\mathbb{R})$, this subspace $\mathcal{N} \cap C^\infty(\mathbb{R})$ is complementary for $\ker(S)$ in $C^\infty(\mathbb{R})$, too. This is a consequence of intersecting equation (2.12) with $C^\infty(\mathbb{R})$:

$$\begin{aligned} C(\mathbb{R}) \cap C^\infty(\mathbb{R}) &= [\ker(S) \cap C^\infty(\mathbb{R})] \oplus [\mathcal{N} \cap C^\infty(\mathbb{R})] \\ C^\infty(\mathbb{R}) &= \ker(S) \oplus [\mathcal{N} \cap C^\infty(\mathbb{R})] \end{aligned}$$

Of course $\mathcal{N} \cap C^\infty(\mathbb{R})$ is closed in $C^\infty(\mathbb{R})$ -sense and the right-inverse operator, as constructed in lemma 2.1.10 can be used for $C^\infty(\mathbb{R})$, too. From

the proof of that lemma we know that finding a continuous right-inverse is the same thing as finding a closed complementary subspace and hence we are done. $\square$

Remark 2.3.7 We show that in general $\text{Inv}(S, C(\mathbb{R})) \neq \text{Inv}(S, C^\infty(\mathbb{R}))$. Let $f \in C^\infty(\mathbb{R})$ and apply Taylor's formula

$$f(t) = f(0) \cdot 1 + (Df)(0) \cdot t + (I_2 D^2 f)(t)$$

Clearly

$$C^\infty(\mathbb{R}) = \langle \{1\}, \{t\} \rangle \oplus \{f \in C^\infty(\mathbb{R}) \mid f(0) = (Df)(0) = 0\}$$

where $\ker(D^2) = \langle \{1\}, \{t\} \rangle$ and $\mathcal{N} = \{f \in C^\infty(\mathbb{R}) \mid f(0) = (Df)(0) = 0\}$. If we want to use $\mathcal{N}$ as a complementary space for $\ker(D^2)$ in $C(\mathbb{R})$ we should take the closure in $C(\mathbb{R})$ -sense and check whether or not

$$C(\mathbb{R}) = \langle \{1\}, \{t\} \rangle \oplus \overline{\{f \in C^\infty(\mathbb{R}) \mid f(0) = (Df)(0) = 0\}}$$

It turns out to be incorrect, since $\{t\}$ is in $\overline{\mathcal{N}}$ and so $\overline{\mathcal{N}} \cap \langle 1, t \rangle \neq \{0\}$. The sequence

$$f_n(t) := t - \frac{t}{1 + n^2 t^2} \quad (n \in \mathbb{N})$$

is in $\mathcal{N}$ and converges to $\{t\}$ in $C(\mathbb{R})$ -sense. Namely,

$$f_n(0) = 0 \quad \text{and} \quad \dot{f}_n(t) = \frac{3n^2 t^2 + n^4 t^4}{(1 + n^2 t^2)^2}$$

and it is easily verified that

$$\forall m \in \mathbb{N} \quad [p_m(f_n(t) - t) < \frac{1}{2m}]$$

$\square$

This decomposition problem will reappear in chapter 5 on the spaces $C^{-k}(\mathbb{R})$.

2.4 Convolution operators

In the space $C(\mathbb{R})$, $C^k(\mathbb{R})$ and $C^\infty(\mathbb{R})$, we introduce the notion of *convolution operator*. Although we cannot avoid some redundancy, we consider these spaces separately. First we consider the space $C(\mathbb{R})$.

Definition 2.4.1 On $C(\mathbb{R})$ the *shift operators* $\sigma^t, t \in \mathbb{R}$ are defined as follows:

$$(\sigma^t f)(s) = f(t + s) \quad (s \in \mathbb{R}, f \in C(\mathbb{R}))$$

Clearly $\sigma^t \sigma^\tau = \sigma^{t+\tau}$ and $\sigma^0 = I$. Furthermore each $\sigma^t : C(\mathbb{R}) \rightarrow C(\mathbb{R})$ is continuous where $\sigma^t f \rightarrow f$ as $t \rightarrow 0$. Of course, the shift operators are continuous on $C^k(\mathbb{R})$ and $C^\infty(\mathbb{R})$ also.

On each of the spaces $C(\mathbb{R})$, $C^k(\mathbb{R})$ and $C^\infty(\mathbb{R})$, we can define convolution operators. This will be done via the dual space. We start with convolution operators on $C(\mathbb{R})$.

Definition 2.4.2 Let $\mu \in ba_c(\mathbb{R})$. Then the associated *convolution operator* C_μ on $C(\mathbb{R})$ is defined by

$$(C_\mu f)(t) := \mu(\sigma^t f) = \int_{\mathbb{R}} f(t + \tau) d\mu(\tau) =: (\mu * f)(t) \quad (2.13)$$

(Observe that integration actually takes place over a finite interval.)

We note that C_μ is continuous, because if $\text{supp}(\mu) \subseteq [-k, k]$ then

$$p_m(C_\mu f) \leq \text{var}(\mu) p_n(f)$$

where $n := m + k$ (see lemma 2.1.1).

Definition 2.4.3 Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Fréchet spaces where the shift operators are well defined and K be an operator from $\mathcal{F}_1$ into $\mathcal{F}_2$ then we call K a *shift-invariant operator* if $\sigma^t(\text{Dom}(K)) = \text{Dom}(K)$ and for all $f \in \text{Dom}(K)$ $K\sigma^t f = \sigma^t K f$.

Convolution operators are characterized by the following property:

Lemma 2.4.4 A continuous operator K from $C(\mathbb{R})$ to $C(\mathbb{R})$ is a convolution operator if and only if K is a shift-invariant operator.

Proof: ($\Rightarrow$): Follows from definition 2.4.1.

($\Leftarrow$): Since K is continuous $\mathcal{L}(f) := (Kf)(0)$ defines a continuous linear functional on $C(\mathbb{R})$. Lemma 2.1.13 yields: there exists a $\mu \in ba_c(\mathbb{R})$

$$\forall f \in C(\mathbb{R}) [(Kf)(0) = \mu(f)]$$

Then

$$(Kf)(t) = (\sigma^t Kf)(0) = (K\sigma^t f)(0) = \mu(\sigma^t f) = C_\mu f$$

□

Now we derive some properties of these convolution operators.

Lemma 2.4.5 *The mapping $\mu \mapsto C_\mu$ from $ba_c(\mathbb{R})$ to $\mathcal{L}(C(\mathbb{R}))$ is linear and injective.*

Proof: Let $C_\mu = 0$. Since for all $f \in C(\mathbb{R}), t \in \mathbb{R}$

$$(C_\mu f)(t) = \mu(\sigma^t f) = 0$$

we obtain, by substituting $t = 0$, for all $f \in C(\mathbb{R}), \mu(f) = 0$. This immediately implies that $\mu \equiv 0$. Linearity is obvious. □

Remark 2.4.6 It is possible to consider a convolution operator as a differential operator of infinite order, i.e. $C_\mu = \mathcal{F}_\mu(D)$. In fact, if we take a polynomial p and evaluate $C_\mu p$ then

$$\begin{aligned} (C_\mu p)(t) &= \int_{\mathbb{R}} p(t + \tau) d\mu(\tau) = \int_{\mathbb{R}} \left[\sum_{l=0}^{\infty} \frac{\tau^l}{l!} (D^l p)(t) \right] d\mu(\tau) \\ &= \sum_{l=0}^{\infty} (D^l p)(t) \int_{\mathbb{R}} \frac{\tau^l}{l!} d\mu(\tau) \\ &= \left(\sum_{l=0}^{\infty} \left[\frac{(\mathcal{F}_\mu^{(l)})(0)}{l!(-i)^l} \right] D^l p \right)(t) = (\mathcal{F}_\mu(D)p)(t) \end{aligned}$$

where $(\mathcal{F}_\mu^{(l)})(0) = \int_{\mathbb{R}} (-i\tau)^l d\mu(\tau)$.

Because p is a polynomial the sums involved are actually finite. □

Let $\mu_1, \mu_2 \in ba_c(\mathbb{R})$. Then $C_{\mu_1} \circ C_{\mu_2}$ is a continuous linear mapping on $C(\mathbb{R})$, which commutes with all $\sigma^t (t \in \mathbb{R})$, and hence (see lemma 2.4.4 and lemma 2.1.13) there exists a unique $\mu \in ba_c(\mathbb{R})$ such that $C_{\mu_1} \circ C_{\mu_2} = C_\mu$. This leads to the following definition:

Definition 2.4.7 If $\mu_1, \mu_2 \in ba_c(\mathbb{R})$ and μ is such that $C_{\mu_1} \circ C_{\mu_2} = C_\mu$. Then $\mu := \mu_1 * \mu_2$.

The following property will be used a lot. For all $\mu \in ba_c(\mathbb{R})$ e_ω (see 2.5) is an eigenfunction of C_μ :

$$(C_\mu e_\omega)(t) = \mu(\sigma^t e_\omega) = \mu(e^{-i\omega t} e_\omega) = e^{-i\omega t} \mu(e_\omega) = \mu(e_\omega) e_\omega(t) = \mathcal{F}_\mu(\omega) e_\omega(t) \quad (2.14)$$

Lemma 2.4.8 For all $\mu_1, \mu_2 \in ba_c(\mathbb{R})$: $\mathcal{F}_{\mu_1 * \mu_2} = \mathcal{F}_{\mu_1} \cdot \mathcal{F}_{\mu_2}$

Proof: For all $\omega \in \mathbb{C}$: $C_{\mu_1 * \mu_2} e_\omega = (\mu_1 * \mu_2)(e_\omega) e_\omega$. But also (by definition)

$$\begin{aligned} C_{\mu_1 * \mu_2} e_\omega &= C_{\mu_1} C_{\mu_2} e_\omega \\ &= \mu_2(e_\omega) C_{\mu_1} e_\omega \\ &= \mu_1(e_\omega) \mu_2(e_\omega) e_\omega \end{aligned}$$

Hence

$$(\mu_1 * \mu_2)(e_\omega) = \mu_1(e_\omega) \mu_2(e_\omega)$$

Or written otherwise

$$\mathcal{F}_{\mu_1 * \mu_2} = \mathcal{F}_{\mu_1} \cdot \mathcal{F}_{\mu_2}$$

□

Lemma 2.4.9 $(ba_c(\mathbb{R}), +, *)$ is a commutative ring with no zero divisors and an identity.

Proof: We only prove that there are no zero divisors. The rest is obvious, since all the integration intervals are finite.

Let $\mu_1, \mu_2 \in ba_c(\mathbb{R})$ and $\mu_1 * \mu_2 = 0$. Then for all $\omega \in \mathbb{C}$

$$0 = (\mu_1 * \mu_2)(e_\omega) = \mu_1(e_\omega) \mu_2(e_\omega) = \mathcal{F}_{\mu_1}(\omega) \mathcal{F}_{\mu_2}(\omega)$$

Since $\mathcal{F}_{\mu_1}, \mathcal{F}_{\mu_2}$ are analytic functions at least one of them must be zero. The injectivity of the Fourier transform on the class $ba_c(\mathbb{R})$ yields the desired

result. Of course, the identity of the ring $(\mu(f) = f(0))$ corresponds to the Heaviside-function H , the function which satisfies

$$\forall f \in C(\mathbb{R}) [(C_H f)(t) := \int_{\mathbb{R}} f(t + \tau) dH(\tau) = f(t)]$$

□

Theorem 2.4.10 *A continuous and injective shift invariant operator K from $C(\mathbb{R})$ to $C(\mathbb{R})$ is a shift operator, up to a multiplicative constant. Consequently, K is invertible.*

Proof: From lemma 2.4.4 we know that for all $f \in C(\mathbb{R})$

$$Kf = C_\mu f$$

for some μ in the dual of $C(\mathbb{R})$. The injectivity of C_μ implies, using equation (2.14), that $\mathcal{F}_\mu$ can have no zeroes. From the proof of lemma 2.2.7 we have $(\mathcal{F}_\mu)(\omega) = e^{a\omega+b}$ for some $a, b \in \mathbb{C}$. Because $\mathcal{F}_\mu$ is bounded on the real axis $a = i\alpha$ for some $\alpha \in \mathbb{R}$, and we have

$$\mathcal{F}_\mu(\omega) = ke^{i\omega\alpha}$$

which clearly implies that $C_\mu e_\omega = ke^{i\omega\alpha} e_\omega = k\sigma^\alpha e_\omega$ for all $\omega \in \mathbb{C}$. This completes the proof. □

Lemma 2.4.11 *Range(C_μ) is dense in $C(\mathbb{R})$ for all nonzero $\mu \in ba_c(\mathbb{R})$.*

Proof: Let $\mu \in ba_c(\mathbb{R})$ and suppose $W := \overline{\text{Range}(C_\mu)} \neq C(\mathbb{R})$. Then there exists a $\nu \in ba_c(\mathbb{R})$, $\nu \neq 0$ such that $\nu|_W = 0$ (Hahn-Banach). Hence $\nu(C_\mu f) = 0$ for all $f \in C(\mathbb{R})$ and thus for all $f \in C(\mathbb{R})$, $C_{\nu*\mu} f = 0$. This implies $C_{\nu*\mu} = 0$. I.e. $\nu * \mu = 0$. Since $(ba_c(\mathbb{R}), +, *)$ has no zero divisors this is a contradiction, which completes the proof. □

Lemma 2.4.12 *Let S be in $LDO(\mathbb{R})$ with characteristic polynomial $p(-i\omega)$. Then for all $\mu, \nu \in ba_c(\mathbb{R})$*

$$C_\nu = SC_\mu \Leftrightarrow \mathcal{F}_\nu(\omega) = p(-i\omega)\mathcal{F}_\mu(\omega)$$

Proof: This can be seen from the following equalities

$$C_\nu e_\omega = \nu(e_\omega) e_\omega \wedge SC_\mu e_\omega = \mu(e_\omega) S(e_\omega) = p(-i\omega) \mu(e_\omega) e_\omega$$

□

Lemma 2.4.13 C_μ maps $C^n(\mathbb{R})$ continuously into $C^n(\mathbb{R})$.

Proof: If $f \in C^1(\mathbb{R})$, then for all $t \neq 0$

$$0 = \sigma^t C_\mu f - C_\mu \sigma^t f = \frac{\sigma^t C_\mu f - C_\mu f}{t} - C_\mu \left(\frac{\sigma^t f - f}{t} \right)$$

We prove that $\lim_{t \rightarrow 0} \frac{\sigma^t f - f}{t}$ exists in $C(\mathbb{R})$ -sense. Let $m \in \mathbb{N}$ and consider

$$p_m \left(\frac{\sigma^t f - f}{t} - Df \right) = p_m \left(\frac{1}{t} \int_0^t [(Df)(s + \tau) - (Df)(s)] d\tau \right)$$

Because Df is uniformly continuous on any compact interval, we obtain

$$\lim_{t \rightarrow 0} p_m \left(\frac{\sigma^t f - f}{t} - Df \right) = 0$$

Using the fact that C_μ is a continuous operator from $C(\mathbb{R})$ to $C(\mathbb{R})$, we have

$$\lim_{t \rightarrow 0} C_\mu \left(\frac{\sigma^t f - f}{t} \right) = C_\mu Df$$

and hence

$$\lim_{t \rightarrow 0} \frac{\sigma^t C_\mu f - C_\mu f}{t} = DC_\mu f$$

Hence $C_\mu f \in C^1(\mathbb{R})$ and $C_\mu Df = DC_\mu f$.

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $C^1(\mathbb{R})$ such that

$$\left. \begin{array}{l} f_n \rightarrow f \\ C_\mu f_n \rightarrow g \end{array} \right\} \text{ in } C^1(\mathbb{R})\text{-sense}$$

Then both sequences converge in $C(\mathbb{R})$ -sense too. Since C_μ is continuous on $C(\mathbb{R})$ we have $g = C_\mu f$. Hence $C_\mu f$ is closed from $C^1(\mathbb{R})$ to $C^1(\mathbb{R})$. The closed-graph theorem implies that C_μ is continuous on $C^1(\mathbb{R})$. Suppose C_μ maps $C^{k-1}(\mathbb{R})$ into $C^{k-1}(\mathbb{R})$ continuously, and $D^{j-1}C_\mu = C_\mu D^{j-1}$ ($0 < j \leq k$) on $C^{k-1}(\mathbb{R})$. By the same reasoning we obtain $C_\mu Df = DC_\mu f$

for all $f \in C^k(\mathbb{R})$, and $C_\mu f \in C^k(\mathbb{R})$. Furthermore $D^k C_\mu f = D^{k-1} C_\mu Df = C_\mu D^k f$. Continuity from $C^k(\mathbb{R})$ into itself follows immediately by the same argumentation as for $C^1(\mathbb{R})$. $\square$

Remark 2.4.14 In the proof of this lemma we obtained the following result:

$$\forall_{\mu \in ba_c(\mathbb{R})} \forall_{f \in C^k(\mathbb{R})} [D^k C_\mu f = C_\mu D^k f]$$

Now let $\mu \in ba_c(\mathbb{R})$ be weakly differentiable with derivative ν . Then by definition 2.2.9 for all $f \in C^1(\mathbb{R})$ $-\nu(f) = \mu(Df)$ and hence $-C_\nu f = C_\mu Df = DC_\mu f$ for all $f \in C^1(\mathbb{R})$. A continuous extension yields $-C_\nu f = DC_\mu f$ for all $f \in C(\mathbb{R})$. Thus C_μ maps $C(\mathbb{R})$ into $C^1(\mathbb{R})$.

Similarly, if $k \in \mathbb{N}$ and $\mu \in ba_c(\mathbb{R})$ is weakly differentiable with derivative ν , then $(-1)^k C_\nu f = D^k C_\mu f$ for all $f \in C(\mathbb{R})$ and C_μ maps $C(\mathbb{R})$ into $C^k(\mathbb{R})$. $\square$

These considerations lead to the following lemma:

Lemma 2.4.15 *Let $\mu \in ba_c(\mathbb{R})$. If μ is k times weakly differentiable ($k > 0$) then $\mu \in C^{k-1}(\mathbb{R})$.*

Proof: Let $\mu \in ba_c(\mathbb{R})$ and let μ be k times weakly differentiable. From the remark above we obtain that C_μ maps $C(\mathbb{R})$ into $C^k(\mathbb{R})$. Take the following function $f \in C(\mathbb{R})$:

$$f(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$

Then

$$(C_\mu f)(t) = \int_{-t}^{\infty} (t + \tau) d\mu(\tau) = (t + \tau)\mu(\tau)|_{-t}^{\infty} - \int_{-t}^{\infty} \mu(\tau) d\tau$$

and $C_\mu f \in C^k(\mathbb{R})$ together with $DC_\mu f = \mu$ proves the result. $\square$

Lemma 2.4.16 *Let $\mu \in ba_c(\mathbb{R})$ and $\lambda \in \mathbb{C}$. The following statements are equivalent:*

1. C_μ is an operator from $C(\mathbb{R})$ into $C^1(\mathbb{R})$.
2. $\exists_{\nu \in ba_c(\mathbb{R})} [(D + i\lambda I)C_\mu = C_\nu]$

3. $\exists_{\nu \in ba_c(\mathbb{R})} [\mathcal{F}_\nu(\omega) = i(\lambda - \omega)\mathcal{F}_\mu(\omega)]$
4. $\exists_{\nu \in ba_c(\mathbb{R})} \forall_{f \in C^1(\mathbb{R})} [\mu((D + i\lambda I)f) = \nu(f)]$

Proof:

- 1 $\Rightarrow$ 2 : The operator $(D + i\lambda I)C_\mu$ is well defined on $C(\mathbb{R})$ and is continuous from $C(\mathbb{R})$ to $C(\mathbb{R})$ (closed-graph-theorem). Moreover it commutes with all shift operators σ^t and hence $(D + i\lambda I)C_\mu = C_\nu$ for some $\nu \in ba_c(\mathbb{R})$.
- 2 $\Rightarrow$ 1 : Since $(D + i\lambda I)C_\mu$ is well defined on $C(\mathbb{R})$, $\text{Range}(C_\mu) \subseteq C^1(\mathbb{R})$.
- 2 $\Rightarrow$ 3 : If $(D + i\lambda I)C_\mu f = C_\nu f$ for all $f \in C(\mathbb{R})$ then for all $\omega \in \mathbb{C}$ we have

$$C_\nu e_\omega = \nu(e_\omega)e_\omega = \mu(e_\omega)(i\lambda I + D)(e_\omega) = i(\lambda - \omega)\mu(e_\omega)e_\omega$$

- 3 $\Rightarrow$ 4 : 3. can be rewritten as $C_\nu e_\omega = (D + i\lambda I)C_\mu e_\omega$ and using lemma 2.2.3, we obtain $\nu(f) = (i\lambda I + D)\mu(f) = \mu((D + i\lambda I)f)$.
- 4 $\Rightarrow$ 2 : For all $f \in C^1(\mathbb{R})$ we have

$$(D + i\lambda I)C_\mu f = C_\mu(D + i\lambda I)f = \mu(\sigma^t(D + i\lambda I)f) = \nu(\sigma^t f) = C_\nu f$$

where we used $DC_\mu f = C_\mu Df$ for all $f \in C^1(\mathbb{R})$, from the proof of lemma 2.4.13. Since $C^1(\mathbb{R})$ is dense in $C(\mathbb{R})$ and $(D + i\lambda I)C_\mu$ is continuous from $C(\mathbb{R})$ to $C(\mathbb{R})$ we are done. $\square$

Similarly, an extension to higher order differential operators can be proven leading to:

Theorem 2.4.17 *Let $\mu \in ba_c(\mathbb{R})$, S in $LDO_k(\mathbb{R})$. The following statements are equivalent:*

1. C_μ is an operator from $C(\mathbb{R})$ into $C^k(\mathbb{R})$.
2. $\exists_{\nu \in ba_c(\mathbb{R})} [SC_\mu = C_\nu]$, i.e. $C_\mu f \in C^k(\mathbb{R})$ and $SC_\mu f = C_\nu f$ for all $f \in C(\mathbb{R})$.
3. $\exists_{\nu \in ba_c(\mathbb{R})} [\mathcal{F}_\nu(\omega) = p(-i\omega)\mathcal{F}_\mu(\omega)]$ with $p(-i\omega)$ the characteristic polynomial of S .
4. $\exists_{\nu \in ba_c(\mathbb{R})} \forall_{f \in C^k(\mathbb{R})} [\mu(Sf) = \nu(f)]$

Corollary 2.4.18 *The operator C_μ maps $C(\mathbb{R})$ into $C^\infty(\mathbb{R})$ if and only if $\mu \in [C^\infty(\mathbb{R}) \cap ba_c(\mathbb{R})]$.*

Proof: Suppose the operator C_μ maps $C(\mathbb{R})$ into $C^\infty(\mathbb{R})$, then C_μ maps $C(\mathbb{R})$ into $C^k(\mathbb{R})$ for all $k \in \mathbb{N}$. Hence, using lemma 2.4.17 (4), μ is infinitely many times weakly differentiable and lemma 2.4.15 implies $\mu \in [C^\infty(\mathbb{R}) \cap ba_c(\mathbb{R})]$. The "if"-part follows from

$$C_\mu f = \int_{\mathbb{R}} f(\tau) \mu'(t - \tau) d\tau$$

The last term is infinitely differentiable, since $\mu' \in \mathcal{D}(\mathbb{R})$. $\square$

The interpretation of the following lemma is that we can exchange zeroes of Fourier transforms of elements in $ba_c(\mathbb{R})$.

Lemma 2.4.19 *If $\mu \in ba_c(\mathbb{R})$ and $\mathcal{F}_\mu(\beta) = 0$, then we have for all $\alpha \in \mathbb{C}$ that $\frac{\omega - \alpha}{\omega - \beta} \cdot \mathcal{F}_\mu(\omega) = \mathcal{F}_\nu(\omega)$ for some $\nu \in ba_c(\mathbb{R})$.*

Proof: Since $\mu \in ba_c(\mathbb{R})$ and $\mathcal{F}_\mu(\beta) = 0$ we have $\frac{1}{i(\omega - \beta)} \mathcal{F}_\mu(\omega) = \mathcal{F}_\rho(\omega)$ for some $\rho \in ba_c(\mathbb{R})$ (see lemma 2.2.10). Hence $i(\omega - \beta) \mathcal{F}_\rho(\omega) = \mathcal{F}_\mu(\omega)$. Applying lemma 2.4.16 yields that C_ρ maps $C(\mathbb{R})$ into $C^1(\mathbb{R})$. The same lemma says that $i(\omega - \alpha) \mathcal{F}_\rho(\omega) = \mathcal{F}_\nu(\omega)$ for some $\nu \in ba_c(\mathbb{R})$. Thus

$$\frac{\omega - \alpha}{\omega - \beta} \cdot \mathcal{F}_\mu(\omega) = \frac{\omega - \alpha}{\omega - \beta} \cdot i(\omega - \beta) \mathcal{F}_\rho(\omega) = i(\omega - \alpha) \mathcal{F}_\rho(\omega) = \mathcal{F}_\nu(\omega)$$

$\square$

Lemma 2.4.20 *In general it is not true that $\ker C_\mu \oplus F = \ker C_\nu$ for some finite dimensional space F , consisting of Bohl functions, and $\mu, \nu \in ba_c(\mathbb{R})$.*

Proof: We will give a counterexample. Take $\mu = \sigma^\pi H + H$, then $\mathcal{F}_\mu(\omega) = e^{i\omega\pi} + 1$. Note that F is not necessarily shift invariant.

Suppose we want to add the function $f = 1$. Then f is not in $\ker C_\mu$. Question: is there a $\nu \in ba_c(\mathbb{R})$ such that

$$\ker C_\mu \oplus \langle 1 \rangle = \ker C_\nu$$

Every Bohl function in $\ker C_\mu \oplus \langle 1 \rangle$ must be in $\ker C_\nu$ and vice versa. Hence $\frac{\mathcal{F}_\nu(\omega)}{\omega \mathcal{F}_\mu(\omega)}$ must be an entire function without zeroes. Using the theorem of Hadamard again (see lemma 2.2.7) we immediately obtain

$$\frac{\mathcal{F}_\nu(\omega)}{\omega \mathcal{F}_\mu(\omega)} = e^{a\omega + b}$$

for some $a, b \in \mathbb{C}$. Thus $\frac{\mathcal{F}_\nu(\omega)}{\omega} = \mathcal{F}_\mu(\omega)e^{a\omega+b}$. Since $\mathcal{F}_\nu(0) = 0$ it is clear that $\mathcal{F}_\mu(\omega)e^{a\omega+b}$ must be bounded on the real axis, like $\mathcal{F}_\nu$. Hence $a = i\alpha$ for some $\alpha \in \mathbb{R}$. This implies

$$\mathcal{F}_\nu(\omega) = \omega \mathcal{F}_\mu(\omega) e^{i\alpha\omega+b} = \omega(e^{i\omega\pi} + 1)e^{i\alpha\omega+b}$$

Now we see that $\mathcal{F}_\nu$ cannot be bounded on the real axis anymore, which is a contradiction. $\square$

Remark 2.4.21 What we see from this lemma and from lemma 2.2.10 is that we obtain another element in $ba_c(\mathbb{R})$ if we divide its Fourier transform by a zero, but when we want to add a zero it is not always true that we end up with an element in $ba_c(\mathbb{R})$. $\square$

As we have already seen convolution operators can be smoothing, i.e. there are $\mu \in ba_c(\mathbb{R})$ such that $\text{Range}(C_\mu) \subseteq C^k(\mathbb{R})$. Then for any $S \in LDO(\mathbb{R})$, with degree k , the linear mapping SC_μ is well defined and continuous from $C(\mathbb{R})$ to $C(\mathbb{R})$. We show that it is possible to find $\mu \in ba_c(\mathbb{R})$ such that C_μ maps $C(\mathbb{R})$ into $C^\infty(\mathbb{R})$. We can construct smoother functions by taking a $\mu \in C^\infty(\mathbb{R}) \cap ba_c(\mathbb{R})$ and evaluate $C_\mu f$ for some $f \in C(\mathbb{R})$.

$$(C_\mu f)(t) = \int_{\mathbb{R}} f(t+\tau) d\mu(\tau) = \int_{\mathbb{R}} f(t+\tau) \mu'(\tau) d\tau$$

It is easy to show that $C_\mu f \in C^\infty(\mathbb{R})$. Integration by parts yields (if F is such that $DF = f$)

$$(C_\mu f)(t) = F(t+\tau) \mu'(\tau) \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} F(t+\tau) \mu^{(2)}(\tau) d\tau$$

and hence $DC_\mu f = -C_{D\mu} f$. Similarly $D^k C_\mu f = (-1)^k C_{D^k \mu} f$.

Lemma 2.4.22 *There exists a sequence $(\mu_n)_{n \in \mathbb{N}} \in [C^\infty(\mathbb{R}) \cap ba_c(\mathbb{R})]$, such that for all $f \in C(\mathbb{R})$: $C_{\mu_n} f \rightarrow f$ in $C(\mathbb{R})$ -sense if $n \rightarrow \infty$. The sequence is called AN APPROXIMATE IDENTITY.*

Proof: Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence in $C^\infty(\mathbb{R}) \cap ba_c(\mathbb{R})$ such that for all $n \in \mathbb{N}$ $\text{supp}(\mu_n) = [-\frac{1}{n}, \frac{1}{n}]$, $\int_{\mathbb{R}} d\mu_n = 1$ and hence $\mu_n(f) \rightarrow f(0)$, ($n \rightarrow \infty$)

for all $f \in C(\mathbb{R})$. Namely

$$\begin{aligned} f(0) - \int_{\mathbb{R}} f d\mu_n &= \int_{\mathbb{R}} (f(0) - f(t)) d\mu_n(t) \\ &\leq \max_{|t| \in [-\frac{1}{n}, \frac{1}{n}]} |f(0) - f(t)| \int_{\mathbb{R}} d\mu_n(t) \rightarrow 0 \quad (n \rightarrow \infty) \end{aligned}$$

Then for all $k \in \mathbb{N}$

$$\begin{aligned} \max_{t \in [-k, k]} |(C_{\mu_n} f - f)(t)| &= \max_{s \in [-k, k]} \left| \int_{\mathbb{R}} (f(t + \tau) - f(t)) d\mu_n(\tau) \right| \\ &\leq \frac{2}{n} \cdot \max_{t \in [-2k, 2k]} 2|f(t)| \rightarrow 0 \quad (n \rightarrow \infty) \end{aligned}$$

A well known example for such a μ_n is the primitive of the following function in $C_c^\infty(\mathbb{R})$:

$$f_n(t) = \begin{cases} n \exp\left(\frac{-1}{1-(nt)^2}\right) & |t| \leq \frac{1}{n} \\ 0 & |t| \geq \frac{1}{n} \end{cases}$$

□

Similar characterizations as in lemma 2.4.4 are also possible for operators on $C^k(\mathbb{R})$ and $C^\infty(\mathbb{R})$. This is one of the main results in this thesis.

Lemma 2.4.23 *Let K be a linear operator from $C^k(\mathbb{R}) \rightarrow C(\mathbb{R})$. Then K is a continuous and shift invariant operator from $C^k(\mathbb{R})$ to $C(\mathbb{R})$ if and only if $K = SC_\mu$ for some S in $LDO_k(\mathbb{R})$ and $\mu \in ba_c(\mathbb{R})$.*

Proof: Let $f \in C^k(\mathbb{R})$. $(Kf)(t) = (\sigma^t Kf)(0) = (K\sigma^t f)(0) = \mathcal{L}(\sigma^t f)$ where the continuous linear functional $\mathcal{L}$ on $C^k(\mathbb{R})$ is defined as $\mathcal{L}(f) = (Kf)(0)$. From lemma 2.1.18, we obtain that for all $f \in C^k(\mathbb{R})$,

$$\mathcal{L}(f) = (Sf)(0) + \mu(D^k f)$$

and hence, $t \in \mathbb{R}$

$$\begin{aligned} \mathcal{L}(\sigma^t f) &= (Sf)(t) + \mu(D^k \sigma^t f) \\ &= (Sf)(t) + (C_\mu D^k f)(t) \end{aligned}$$

Hence $Kf = Sf + C_\mu D^k f = Sf + D^k C_\mu f$ on $C^k(\mathbb{R})$. Clearly K , with domain $\text{Dom}(K) = C^k(\mathbb{R})$, is closed as an operator from $C(\mathbb{R})$ to $C(\mathbb{R})$

and $\sigma^t K = K \sigma^t$. $\text{Ker}(K)$ is a closed subspace in $C^k(\mathbb{R})$. Furthermore it is linear and shift invariant. Since $\text{ker}(K)$ is a closed subspace of a Fréchet space it is a Fréchet space itself when endowed with the induced topology. Let $M_k \subseteq \text{ker } K$ be linear and k -dimensional. Moreover we can select M_k in such a way that it is translation invariant by choosing M_k spanned by Bohl functions and satisfying: $t^l e^{-i\lambda t} \in M_k \Rightarrow t^j e^{-i\lambda t} \in M_k$ for $j = 0, 1, \dots, l-1$. Then M_k has a closed complement in the Fréchet space $\text{ker } K$ (see lemma 2.1.6), say R . So

$$\text{ker } K = M_k \oplus R$$

Note that R is not necessarily shift-invariant.

We have chosen M_k in such a way that it equals the kernel of a linear differential operator with constant coefficients, say S . There exists a continuous operator $J : C(\mathbb{R}) \rightarrow C^k(\mathbb{R})$ such that $SJ = \text{I}$ on $C(\mathbb{R})$ and $\text{I} - JS =: P$ is a projection from $C^k(\mathbb{R})$ onto M_k . Now we have

$$\begin{aligned} \sigma^t J - J \sigma^t &= \sigma^t JSJ - J \sigma^t SJ \\ &= (\sigma^t JS - JS \sigma^t) J \\ &= (\sigma^t (\text{I} - P) - (\text{I} - P) \sigma^t) J \\ &= (P \sigma^t - \sigma^t P) J \end{aligned}$$

Herewith

$$\begin{aligned} K(\sigma^t J - J \sigma^t) &= \sigma^t KJ - KJ \sigma^t \\ &= K(P \sigma^t - \sigma^t P) J \\ &= 0 \end{aligned}$$

since $KP = 0$.

Since J is continuous from $C(\mathbb{R})$ to $C^k(\mathbb{R})$ we obtain that KJ is a continuous operator from $C(\mathbb{R})$ to $C(\mathbb{R})$ that commutes with σ^t for all $t \in \mathbb{R}$. Hence $KJ = C_\mu$ for some $\mu \in ba_c(\mathbb{R})$. This means $K = C_\mu S$ because $KJS = K(\text{I} - P) = K = C_\mu S = SC_\mu$.

On the other hand, operators of the form $K = SC_\mu$, where S is of order k and $\mu \in ba_c(\mathbb{R})$ map $C^k(\mathbb{R})$ into $C(\mathbb{R})$ continuously. So using lemma 2.3.1 and lemma 2.4.13 we are done. $\square$

Corollary 2.4.24 *Let K be a linear operator from $C^k(\mathbb{R})$ to $C^k(\mathbb{R})$. The operator K is continuous and shift invariant if and only if $K = C_\mu$ for some $\mu \in ba_c(\mathbb{R})$.*

Proof: Clearly, K satisfies the requirements from the previous lemma and thus $K = SC_\mu$ for some $S \in LDO(\mathbb{R})$ of order k and $\mu \in ba_c(\mathbb{R})$. Since S maps $C^{k+l}(\mathbb{R})$ into $C^l(\mathbb{R})$ for all $l \in \mathbb{N}$, C_μ maps $C^l(\mathbb{R})$ into $C^{k+l}(\mathbb{R})$ for all $l \in \mathbb{N}$. Hence μ is k times weakly differentiable. The equivalence ' $1 \Leftrightarrow 2$ ' in theorem 2.4.17 closes the proof. $\square$

Lemma 2.4.25 *A linear operator K is shift invariant and continuous from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$ if and only if $K = SC_\mu$ for some S in $LDO(\mathbb{R})$ and $\mu \in ba_c(\mathbb{R})$.*

Proof: Since K is continuous there exists a continuous linear functional $\mathcal{L}$ such that $(Kf)(t) = (\sigma^t Kf)(0) = (K\sigma^t f)(0) = \mathcal{L}(\sigma^t f)$ for all $f \in C^\infty(\mathbb{R})$. $\mathcal{L}$ can be extended to $C^k(\mathbb{R})$ for some $k \in \mathbb{N}$ (see lemma 2.1.20). Using lemma 2.4.23 we have $\mathcal{L}(\sigma^t f) = (SC_\mu f)(t)$ for some S in $LDO(\mathbb{R})$ and $\mu \in ba_c(\mathbb{R})$. Hence for all $f \in C^\infty(\mathbb{R})$ we have $(Kf)(t) = (SC_\mu f)(t)$. $\square$

As a consequence we can characterize the continuous linear functionals on $C^k(\mathbb{R})$ and $C^\infty(\mathbb{R})$ in the way we indicated before (see remark 2.1.19):

Corollary 2.4.26 *A continuous linear functional $\mathcal{L}$ on $C^k(\mathbb{R})$ is of the form $\mathcal{L}f = \mu(Sf)$ for some $S \in LDO(\mathbb{R})$ of order k and $\mu \in ba_c(\mathbb{R})$.*

Corollary 2.4.27 *A continuous linear functional $\mathcal{L}$ on $C^\infty(\mathbb{R})$ is of the form $\mathcal{L}f = \mu(Sf)$ for some $S \in LDO(\mathbb{R})$ and $\mu \in ba_c(\mathbb{R})$.*

This is immediate from $\mathcal{L}f = (Kf)(0)$, where K is a continuous operator from $C^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R})$ that commutes with all the shifts.

The factorization of $\mathcal{L}$ into $\mu \circ S$ depends on the choice of the k -dimensional shift-invariant subspace of $\ker(\mathcal{L})$.

Corollary 2.4.28 *Let K be a nonzero continuous shift invariant operator from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$. Then $\text{Range}(K)$ is dense in $C^\infty(\mathbb{R})$.*

Proof: Lemma 2.4.25 yields $K = SC_\mu$ for some $S \in LDO(\mathbb{R})$ and $\mu \in ba_c(\mathbb{R})$. Suppose $W := \overline{\text{Range}(SC_\mu)} \neq C^\infty(\mathbb{R})$. Then there exists a $\nu \in (C^\infty(\mathbb{R}))'$, $\nu \neq 0$ such that $\nu|_W = 0$. Hence $\nu(SC_\mu f) = 0$ for all $f \in C^\infty(\mathbb{R})$. From corollary 2.4.27 we know $\nu f = \kappa(Tf)$ for some $\kappa \in ba_c(\mathbb{R})$ and $T \in LDO(\mathbb{R})$. Hence

$$C_\kappa(C_\mu T S f) = \kappa(T \sigma^t S C_\mu f) = \kappa(C_\mu \sigma^t T S f) = \kappa(T S C_\mu \sigma^t f) = 0.$$

Since $(ba_c(\mathbb{R}), +, *)$ has no zero divisors and TS is surjective from $C^\infty(\mathbb{R})$ onto $C^\infty(\mathbb{R})$ this is a contradiction, which completes the proof. $\square$

Remark 2.4.29 A consequence of lemma 2.4.25 is another possible definition of convolution operators on $C^\infty(\mathbb{R})$. An operator $\mathcal{L}(\sigma^t f)$ on $C^\infty(\mathbb{R})$ is called a convolution operator on $C^\infty(\mathbb{R})$, where $\mathcal{L} \in (C^\infty(\mathbb{R}))'$ and hence $\mathcal{L}f = SC_\mu f$ for some $S \in LDO(\mathbb{R})$, $\mu \in ba_c(\mathbb{R})$. $\square$

Definition 2.4.30 If $\mu_1, \mu_2 \in ba_c(\mathbb{R})$, $S_1, S_2 \in LDO(\mathbb{R})$ and μ and $S \in LDO(\mathbb{R})$ are such that $S_1 C_{\mu_1} \circ S_2 C_{\mu_2} = SC_\mu$. Then $(\mu_1 \circ S_1) * (\mu_2 \circ S_2) := \mu \circ S$.

Note that this is well defined since $S_1 C_{\mu_1} \circ S_2 C_{\mu_2}$ is a shift-invariant operator again.

Like in $C(\mathbb{R})$ the shift invariant operators on $C^\infty(\mathbb{R})$ form an algebra:

Lemma 2.4.31 If we define $W := (C^\infty(\mathbb{R}))'$ then $(W, +, *)$ is a commutative ring with no zero divisors.

Proof: Let $\mu_1, \mu_2 \in ba_c(\mathbb{R})$ and $S_1, S_2 \in LDO(\mathbb{R})$ corresponding to $\mu_1(S_1)$ and $\mu_2(S_2)$ in W . Both elements in W correspond to convolution operators on $C^\infty(\mathbb{R})$, namely $S_1 C_{\mu_1}$ and $S_2 C_{\mu_2}$ respectively. The operator $K := S_1 C_{\mu_1} + S_2 C_{\mu_2}$ is also shift-invariant and therefore $K = SC_\mu$ for some $\mu \in ba_c(\mathbb{R})$ and $S \in LDO(\mathbb{R})$. Consequently $\mu_1(S_1) + \mu_2(S_2) = \mu(S)$ and $\mu(S) \in W$. To prove that there are no zero divisors just copy the proof of lemma 2.4.9. The only difference is that two polynomials will occur. However they do not invalidate the argumentation. $\square$

Now we have found a characterization for the continuous linear functionals on $C^\infty(\mathbb{R})$, there is a nice link to the Paley-Wiener theorem (theorem 2.2.14)

and a theorem by Kahane (theorem 2.2.6).

Let $\phi(\omega)$ be an analytic function satisfying

$$|\phi(\omega)| \leq C(1 + |\omega|)^N e^{a|\operatorname{Im}(\omega)|}, \quad (\omega \in \mathbb{C})$$

for some constants $C, N \geq 0$. Paley-Wiener tells us that $\phi(\omega)$ is the Fourier transform of a continuous linear functional on $C^\infty(\mathbb{R})$, say μ . We can prove this with the result from Kahane.

Suppose $\phi(\omega)$ has k zeroes, and define $p_\phi(\omega)$ as a polynomial of order k with exactly the same zeroes. If we define $\tilde{\phi}(\omega) = \frac{\phi(\omega)}{p_\phi(\omega)}$ then

$$|\tilde{\phi}(\omega)| \leq C \cdot \frac{1 + |\omega|^N}{|p_\phi(\omega)|} e^{a|\operatorname{Im}(\omega)|} \quad (2.15)$$

and $\tilde{\phi}(\omega)$ is a function of exponential type (since $\phi(\omega)$ is) and has no zeroes. According to Hadamard (compare with equation 2.7) $\tilde{\phi}(\omega) = e^{a\omega+b}$, for some $a, b \in \mathbb{C}$. Because of equation (2.15) considered on the real axis, we find $a = i\alpha$, $\alpha \in \mathbb{R}$ and hence $\tilde{\phi}(\omega)$ is the Fourier transform of a constant times a shift operator. Consequently if ϕ has finitely many zeroes it corresponds to a continuous linear functional $\mu \circ S$ for some $\mu \in ba_c(\mathbb{R})$ and $S \in LDO(\mathbb{R})$.

Now assume ϕ has infinitely many zeroes. Let $p(-i\omega)$ be a polynomial of degree $N + 2$ such that the zeroes of $p(-i\omega)$ are also zeroes of $\phi(\omega)$. Define

$$\tilde{\phi}(\omega) = \frac{\phi(\omega)}{p(-i\omega)}.$$

Then clearly $\tilde{\phi}(\omega)$ satisfies the requirements of the Kahane-theorem. Therefore $\tilde{\phi}$ corresponds to some $\nu \in ba_c(\mathbb{R})$ and $\nu(Sf) = \mu(f)$ for all $f \in C^\infty(\mathbb{R})$ where S corresponds to the characteristic polynomial $p(-i\omega)$. Hence μ is indeed a continuous linear functional on $C^\infty(\mathbb{R})$.

We close this section with the characterization of translation-invariant continuous operators from $C^k(\mathbb{R})$ to $C^l(\mathbb{R})$. We omit the proof.

Lemma 2.4.32 *Let K be a linear operator from $C^k(\mathbb{R})$ to $C^l(\mathbb{R})$. The operator K is continuous and commutes with all shift operators if and only*

if K is either of the form $K = SC_\mu$ ($k \geq l$) where $S \in LDO(\mathbb{R})$ of order $k - l$ and $\mu \in ba_c(\mathbb{R})$ or $K = C_\mu$ ($k \leq l$) and μ is $l - k$ times weakly differentiable.

2.5 Closed, translation-invariant operators in $C(\mathbb{R})$

In the closing section we characterize closed, shift-invariant operators on $C(\mathbb{R})$. We consider closed operators from $C(\mathbb{R})$ to $C(\mathbb{R})$ with a shift-invariant domain. This kind of operator is interesting for the spaces $C^{-k}(\mathbb{R})$ in chapter 5. As a special case, we treat the closed operators from $C(\mathbb{R})$ to $C(\mathbb{R})$ with domain $C^k(\mathbb{R})$ for some $k \in \mathbb{N}$.

For the characterization of the closed linear operators K for which $\sigma^t K = K \sigma^t$, $t \in \mathbb{R}$, the following theorem is crucial. (Later on we will see that this lemma has another important application).

Theorem 2.5.1 *Every convolution operator C_μ on $C(\mathbb{R})$ is the strong (i.e. pointwise) limit of a sequence in $\text{span} \langle \{\sigma^t | t \in \mathbb{R}\} \rangle$. I.e. for all $f \in C(\mathbb{R})$ there exists a sequence $(G_n)_{n \in \mathbb{N}}$ in $\text{span} \langle \{\sigma^t | t \in \mathbb{R}\} \rangle$ such that $G_n f \rightarrow C_\mu f$ if $n \rightarrow \infty$.*

Proof: Let $\mu \in ba_c(\mathbb{R})$ and suppose $\text{supp}(\mu) \subseteq [-N, N]$.

For $n \in \mathbb{N}$ define

$$t_{j,n} = -N + \frac{j}{n}, \quad j = 0, 1, \dots, 2Nn$$

Then $t_{j+1,n} - t_{j,n} = \frac{1}{n}$.

Put

$$a_{j,n} := \mu(t_{j+1,n}) - \mu(t_{j,n}) = \int_{t_{j,n}}^{t_{j+1,n}} d\mu(\tau), \quad j = 0, 1, \dots, 2Nn - 1$$

and define $G_n \in \text{span} \langle \{\sigma^t | t \in \mathbb{R}\} \rangle$ by

$$G_n = \sum_{j=0}^{2Nn-1} a_{j,n} \sigma^{t_{j,n}}$$

Now let $f \in C(\mathbb{R})$ and $k \in \mathbb{N}$. Then there exists for given $\varepsilon > 0$, $n_\varepsilon \in \mathbb{N}$ such that for $n > n_\varepsilon$

$$|f(s) - f(\sigma)| < \frac{\varepsilon}{\text{var}(\mu)}$$

whenever $s, \sigma \in [-N - k, N + k]$ with $|s - \sigma| < \frac{1}{n}$.

Let $\varepsilon > 0$ and choose n_ε as indicated. Then

$$\begin{aligned} p_k(G_n f - C_\mu f) &= \max_{t \in [-k, k]} \left| \sum_{j=0}^{2Nn-1} \int_{t_{j,n}}^{t_{j+1,n}} [f(t_{j,n} + t) - f(\tau + t)] d\mu(\tau) \right| \\ &< \sum_{j=0}^{2Nn-1} \frac{\varepsilon}{\text{var}(\mu)} \int_{t_{j,n}}^{t_{j+1,n}} |d\mu(\tau)| \leq \varepsilon \end{aligned}$$

□

Lemma 2.5.2 Let $G \in \text{span} \langle \{\sigma^t | t \in \mathbb{R}\} \rangle$ and let K be a closed shift invariant linear mapping in $C(\mathbb{R})$. Then G maps $\text{Dom}(K)$ into itself and for all $f \in \text{Dom}(K)$, $GKf = KGf$.

Proof: Suppose $G = \sum_{j=1}^m a_j \sigma^{t_j}$. Then for all $f \in \text{Dom}(K)$, $Gf = \sum_{j=1}^m a_j \sigma^{t_j} f \in \text{Dom}(K)$ and

$$KGf = \sum_{j=1}^m a_j K \sigma^{t_j} f = \sum_{j=1}^m a_j \sigma^{t_j} Kf = GKf$$

□

Lemma 2.5.3 Let K be a closed linear mapping in $C(\mathbb{R})$ with domain $\text{Dom}(K)$. If K is a shift invariant operator, then for all $\mu \in \text{ba}_c(\mathbb{R})$ and for all $f \in \text{Dom}(K)$, $C_\mu f \in \text{Dom}(K)$ and $KC_\mu f = C_\mu Kf$.

Proof: Let $\mu \in \text{ba}_c(\mathbb{R})$ and $(G_n)_{n \in \mathbb{N}}$ be a sequence in $\text{span} \langle \{\sigma^t | t \in \mathbb{R}\} \rangle$ such that for all $h \in C(\mathbb{R})$, $G_n h \rightarrow C_\mu h$ ($n \rightarrow \infty$) (see theorem 2.5.1).

Then if $f \in \text{Dom}(K)$ we have, for all n , $G_n f \in \text{Dom}(K)$ and $G_n f \rightarrow C_\mu f$. Also $KG_n f = G_n Kf \rightarrow C_\mu Kf$. Since K is a closed operator it follows that $C_\mu f \in \text{Dom}(K)$ and $KC_\mu f = C_\mu Kf$. □

Theorem 2.5.4 Let K be a closed operator from $C(\mathbb{R})$ to $C(\mathbb{R})$, with domain $\text{Dom}(K)$, and $C^\infty(\mathbb{R}) \subseteq \text{Dom}(K)$. Then $K|_{C^\infty(\mathbb{R})}$ is continuous from $C^\infty(\mathbb{R})$ to $C(\mathbb{R})$. If in addition K is shift invariant, then

$$\overline{\{(f, Kf) | f \in C^\infty(\mathbb{R})\}} = \text{Graph}(K)$$

where the bar denotes the closure in $C(\mathbb{R}) \times C(\mathbb{R})$.

Proof: Let $f_n \rightarrow f$ in $C^\infty(\mathbb{R})$ -sense, and let $Kf_n \rightarrow g$ in $C(\mathbb{R})$ -sense. Then $f_n \rightarrow f$ in $C(\mathbb{R})$ -sense. Hence $Kf = g$, so that $K|_{C^\infty(\mathbb{R})}$ is a closed operator from $C^\infty(\mathbb{R})$ to $C(\mathbb{R})$. Since $C^\infty(\mathbb{R})$ is a Fréchet space the result follows from the closed-graph-theorem.

The inclusion " $\subseteq$ " is obvious since K is a closed operator from $C(\mathbb{R})$ to $C(\mathbb{R})$. Now let $(f, Kf) \in \text{graph}(K)$. We construct a sequence in $\{(f, Kf) | f \in C^\infty(\mathbb{R})\}$ converging in $C(\mathbb{R}) \times C(\mathbb{R})$ -sense to (f, Kf) . According to lemma 2.4.22 there exists a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R})$ such that $C_{\mu_n}f \rightarrow f$ in $C(\mathbb{R})$ -sense. But also $C_{\mu_n}Kf \rightarrow Kf$ in $C(\mathbb{R})$ -sense. From lemma 2.5.3, we have $C_{\mu_n}Kf = KC_{\mu_n}f$ where $C_{\mu_n}f$ is a sequence in $C^\infty(\mathbb{R})$. The continuity of $K|_{C^\infty(\mathbb{R})}$ implies $KC_{\mu_n}f \rightarrow Kf$ in $C(\mathbb{R})$ -sense. So we found a sequence $(C_{\mu_n}f, KC_{\mu_n}f)$ in $\{(f, Kf) | f \in C^\infty(\mathbb{R})\}$ converging in $C(\mathbb{R}) \times C(\mathbb{R})$ -sense to (f, Kf) . $\square$

Corollary 2.5.5 *Let $S \in LDO(\mathbb{R})$ of order k and $\mu \in ba_c(\mathbb{R})$. Then the operator SC_μ , with domain $\text{Dom}(SC_\mu) = \{f \in C(\mathbb{R}) | C_\mu f \in C^k(\mathbb{R}) = \text{Dom}(S)\}$, is closed, and*

$$\overline{\{(f, SC_\mu f) | f \in C^\infty(\mathbb{R})\}} = \text{Graph}(SC_\mu)$$

Proof: We know already that S is a closed operator from $C(\mathbb{R})$ to $C(\mathbb{R})$ with $\text{Dom}(S) = C^k(\mathbb{R})$. Hence SC_μ is closed. If $f \in \text{Dom}(SC_\mu)$, then $C_\mu \sigma^t f = \sigma^t C_\mu f \in C^k(\mathbb{R})$ for all $t \in \mathbb{R}$, and $SC_\mu \sigma^t f = \sigma^t SC_\mu f$. $\square$

Now we gathered the tools to prove the full characterization result:

Theorem 2.5.6 *Let K be a closed shift invariant operator from $C(\mathbb{R})$ to $C(\mathbb{R})$ and $C^\infty(\mathbb{R}) \subseteq \text{Dom}(K)$. Then there are $\mu \in ba_c(\mathbb{R})$ and $S \in LDO_k(\mathbb{R})$ for some k , such that $K = SC_\mu$, i.e. $\text{Dom}(K) = \{f \in C(\mathbb{R}) | C_\mu f \in C^k(\mathbb{R})\}$ and for all $f \in \text{Dom}(K)$, $Kf = SC_\mu f$.*

Proof: If $K\sigma^t = \sigma^t K$ and $\sigma^t(\text{Dom}(K)) = \text{Dom}(K)$ then $(K|_{C^\infty(\mathbb{R})})\sigma^t = \sigma^t K|_{C^\infty(\mathbb{R})}$. Using lemma 2.4.25 we know that $K|_{C^\infty(\mathbb{R})} = SC_\mu|_{C^\infty(\mathbb{R})}$ for some $\mu \in ba_c(\mathbb{R})$ and S in $LDO(\mathbb{R})$ of some finite order, say k . It follows that

$$\text{Graph}(K) = \overline{\{(f, SC_\mu f) | f \in C^\infty(\mathbb{R})\}} = \text{Graph}(SC_\mu)$$

$\square$

Corollary 2.5.7 *The following statements are equivalent*

- K a closed shift invariant operator from $C(\mathbb{R})$ to $C(\mathbb{R})$ with $\text{Dom}(K) = C^k(\mathbb{R})$
- $K = SC_\mu$ where S is in $LDO_k(\mathbb{R})$ and μ is not weakly differentiable.

Remark 2.5.8 Note that the representation of the operator K is far from unique. In fact, applying lemma 2.4.19, we know that zeroes can be exchanged between the operators S and C_μ . $\square$

2.6 Mean periodic functions

As stated in the introduction we will use the characterization of (finite) shift-invariant subspaces. Strongly related to this subject is the theory of mean-periodic functions as developed by J.Delsarte (see [2]) and L.Schwartz (see [16]). This theory is not necessary for us to derive the results we are looking for, however we do think that it fits very well in the context.

We will need the main result of this theory to characterize shift-invariant subspaces, but before we can state that result some introduction to the nomenclature of this theory is necessary.

Suppose we are dealing with the function space $C(\mathbb{R})$, the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Definition 2.6.1 Let $f \in C(\mathbb{R})$. Then $\tau(f)$ denotes the subspace of $C(\mathbb{R})$ generated by the translates of f (note that in our context a subspace is closed by definition).

Definition 2.6.2 A function $f \in C(\mathbb{R})$ is called *mean-periodic* if $\tau(f) \neq C(\mathbb{R})$.

As an example look at periodic functions which are clearly mean-periodic. An example of a non-mean-periodic function is $\exp(x^2)$. Another example is a nonzero $\mathcal{L}_2$ -function. Namely, if $f \in \mathcal{L}_2$ and f is mean-periodic, then there exists a $\mu \in ba_c(\mathbb{R})$ such that $\mu * f = 0$. If we take the Fourier-transform of this equation we obtain that, since $\mathcal{F}_\mu$ can have only countably many zeroes, $\mathcal{F}_f = 0$ a.e. (Note that $\mathcal{F} : \mathcal{L}_2 \rightarrow \mathcal{L}_2$ is an isometrie). Hence $f = 0$.

Before we proceed with introducing new terminology we state an important result that fits very well in the context.

Lemma 2.6.3 *A function $f \in C(\mathbb{R})$ is mean-periodic if and only if there exists a $\mu \in ba_c(\mathbb{R})$, $\mu \neq 0$ with f in $\ker C_\mu$. I.e.*

$$\forall t \in \mathbb{R} [(C_\mu f)(t) := \int_{\mathbb{R}} f(t + \tau) d\mu(\tau) = 0]$$

Proof: ($\Rightarrow$): Assume $\tau(f) \neq C(\mathbb{R})$, then there exists a continuous linear functional μ , $\mu \neq 0$ on $C(\mathbb{R})$ such that $\mu|_{\tau(f)} \equiv 0$ (Hahn-Banach). From lemma 2.1.13 we obtain

$$\forall t \in \mathbb{R} [\mu(\sigma^t f) = (C_\mu f)(t) = 0]$$

($\Leftarrow$): Let $f \in \ker C_\mu$, then $\mu(\sigma^t f) = 0$ for all t ; $\mu \neq 0$ and μ is in the dual of $C(\mathbb{R})$. This completes the proof. $\square$

Definition 2.6.4 If f is a mean-periodic function then the *spectrum* of f , say $Sp(f)$, is defined as the set of $\lambda \in \mathbb{C}$ such that $e^{-i\lambda t} \in \tau(f)$. The highest possible degree, say p_λ of a polynomial $P(t)$ such that $P(t)e^{-i\lambda t} \in \tau(f)$ (such a p_λ always exists, otherwise $\tau(f) = C(\mathbb{R})$) is called the *multiplicity* of λ in the spectrum of f . Furthermore if $\mu \in ba_c(\mathbb{R})$ then the *cospectrum* of μ , say $CSp(\mu)$, is defined as the set of $\lambda \in \mathbb{C}$ such that $e^{-i\lambda t} \in \ker (C_\mu)$.

Remark 2.6.5 One usually defines the spectrum in another way using a special type of Fourier transforms (the so-called Fourier-Carleman transforms). Since we did not want to elaborate too much we used the above definition which seems to come out of the blue. For a more involved introduction I refer to Kahane (see [9]). $\square$

Lemma 2.6.6 $P(t)e^{-i\lambda t} \in \tau(f)$, where $P(t)$ a polynomial of degree p , if and only if for all $\mu \in ba_c(\mathbb{R})$ such that $C_\mu f = 0$ we have that, if we denote the Fourier-transform of μ by $\mathcal{F}_\mu$, $(\mathcal{F}_\mu)(\omega)$ has a zero for $\omega = \lambda$ with multiplicity at least $p + 1$. (If one does not want to relate it to μ , but directly to f it is equivalent to saying that λ is a pole of at least order $p + 1$ of the Fourier-Carleman transform of f).

Proof: See Kahane ([9],p.42-44). $\square$

The following theorem is the main result of the theory of mean-periodic functions.

Theorem 2.6.7 *If $f \in C(\mathbb{R})$ is a mean-periodic function then f belongs to the closed subspace spanned by all Bohl-functions in $\tau(f)$.*

Proof: See, for instance, Kahane ([9],th.3,p.43). $\square$

Two other important lemmas are due to Schwartz:

Lemma 2.6.8 *If W is a linear, shift-invariant and closed subspace of $C(\mathbb{R})$, then there exists an $f \in C(\mathbb{R})$, such that $W = \tau(f)$. Define such an f (which is not unique) as $\gamma[W]$.*

Proof: See Schwartz ([16],th.12,p.904). $\square$

Lemma 2.6.9 *Let W be a linear, shift-invariant and closed subspace of $C(\mathbb{R})$. Then we need at most two (but in general not one) elements of the ring $(ba_c(\mathbb{R}), +, *)$ to describe W . I.e.*

$$W = \{f \in C(\mathbb{R}) \mid C_{\mu_1} f = 0 \wedge C_{\mu_2} f = 0\}$$

for some $\mu_1, \mu_2 \in ba_c(\mathbb{R})$.

Proof: See Schwartz ([16],th.13,p.914). $\square$

Remark 2.6.10 It is clear that, if W is a finite-dimensional, linear, shift-invariant and closed subspace of $C(\mathbb{R})$, we need at least two elements in $ba_c(\mathbb{R})$ to describe W , because the cospectrum of a $\mu \in ba_c(\mathbb{R})$ is either a nonfinite (countable) set in $\mathbb{C}$ or it is empty (see lemma 2.2.7 and lemma 2.6.6). $\square$

From the theorem 2.6.7 we can derive the following lemma:

Lemma 2.6.11 *W is a finite-dimensional, linear, shift-invariant and closed subspace of $C(\mathbb{R})$ if and only if W has a finite-dimensional basis of Bohl-functions with the restriction that $t^n e^{-i\lambda t} \in W$ implies $t^j e^{-i\lambda t} \in W$ for all $j \in \{0, 1, 2, \dots, n\}$.*

Proof: ($\Rightarrow$): Since W is finite-dimensional it has a basis, say $\{f_1, f_2, \dots, f_n\}$. Because W is shift-invariant and closed $\tau(f_i) \subseteq W$ for $i = 1, 2, \dots, n$. Hence f_i is a mean-periodic function for all i . Clearly $\tau(f_i)$ is also finite-dimensional and using theorem 2.6.7, we find that every $\tau(f_i)$ consists of Bohl-functions satisfying the restriction in the lemma. Thus $W = \bigoplus_{i=1}^n \tau(f_i)$ also satisfies the requirements.

($\Leftarrow$): Obvious. □

Corollary 2.6.12 *If W is a finite-dimensional (say $\dim(W)=n$), linear, shift-invariant and closed subspace of $C(\mathbb{R})$ then it can be characterized by a linear differential operator. Namely, if $t^k e^{-i\lambda_j t} \in W$ for all $k \in \{0, 1, 2, \dots, r_j\}$ (with r_j the highest exponent such that $t^k e^{-i\lambda_j t} \in W$), then*

$$W = \mathcal{N}(S), \text{ where } S := \prod_{j=1}^k (D + i\lambda_j I)^{r_j} \text{ with } \sum_{j=1}^k r_j = n$$

Chapter 3

AR-descriptions for systems consisting of $C^\infty(\mathbb{R})$ -signals

3.1 Introduction

With the mathematical foundation, as exposed in chapter 2, at hand, we make the first step towards finding an AR-relation for a continuous-time dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^q, C(\mathbb{R}, \mathbb{R}^q), \mathcal{B})$. The important property of linear, time-invariant and closed behaviors $\mathcal{B}$ in $C(\mathbb{R}, \mathbb{R}^q)$, namely that $\mathcal{B}^\infty = \mathcal{B} \cap C^\infty(\mathbb{R}, \mathbb{R}^q)$ is dense in $\mathcal{B}$, has as a consequence that $\mathcal{B}^\infty$ determines completely and uniquely the AR-relation for the original behavior $\mathcal{B}$ (upto unimodular premultiplication). This is the motivation to study $\mathcal{B}^\infty$ instead of $\mathcal{B}$, in first instance. So, the first step is the study of behaviors with signals in $C^\infty(\mathbb{R})$. The advantage is that, in this way, interpretation problems for the AR-relation are avoided. The second step will be the transition from $C^\infty(\mathbb{R})$ -signals to $C(\mathbb{R})$ -signals. This is done in chapter 4.

Going from observations of some dynamical system towards a description in matrix form (if possible) leads to the question: which basis to choose? In fact, observations are always done with respect to some well-specified basis. But the choice of the basis has still some arbitrariness. This introduces the following notion of *equivalence of behaviors*:

Definition 3.1.1 Two behaviors $\mathcal{B}_1$ and $\mathcal{B}_2$ are called *equivalent* if there exists an invertible constant matrix V such that $u \in \mathcal{B}_1 \Leftrightarrow Vu \in \mathcal{B}_2$.

Remark 3.1.2 In fact, an equivalence relation can be introduced replacing the constant matrix V by a unimodular matrix V . $\square$

In this chapter we show that the question whether an equivalence-class of behaviors can be represented by an AR-relation can be dealt with on the level of one representation of that class. Hence it does not matter what choice of basis we make. We come back to this point later on. From now on we choose the standard basis $\{e_1, \dots, e_q\}$ for $\mathbb{R}^q$.

In spite of the fact that in $C^\infty(\mathbb{R})$ a lot of technicalities do not occur, the full treatment for multivariable dynamical systems is still quite complex and would lead to messy notation. We have tried to find a presentation which shows our ideas as clearly as possible. One part of this attempt is the treatment of a simple case first (i.e. a dynamical system with two external variables). Another part is the introduction of some notation.

Definition 3.1.3 Let $q \in \mathbb{N}$, $w \in \mathbb{R}^q$. Then we call Π a *permutation* Π if there exists a bijection $\pi : \{1, \dots, q\} \rightarrow \{1, \dots, q\}$ such that $\Pi(w) = (w_{\pi(1)}, \dots, w_{\pi(q)})$.

Definition 3.1.4 Let $\Sigma = (\mathbb{R}, \mathbb{R}^q, [\mathcal{F}]^q, \mathcal{B})$ be a continuous-time dynamical system and let $p \in \mathbb{N}$ and $0 \leq p \leq q$. If $(w_1, \dots, w_q)$ represent the external signals then we call Σ a *standard p -system* if

$$\{(w_1, \dots, w_p) \in [\mathcal{F}]^p \mid (w_1, \dots, w_p, 0, \dots, 0) \in \mathcal{B}\}$$

is finite dimensional, and there is no permutation Π and $p' > p$ such that

$$\{(w_1, \dots, w_{p'}) \in [\mathcal{F}]^{p'} \mid (w_1, \dots, w_{p'}, 0, \dots, 0) \in \Pi \mathcal{B}\}$$

is finite dimensional.

Definition 3.1.5 Let $\Sigma = (\mathbb{R}, \mathbb{R}^q, [\mathcal{F}]^q, \mathcal{B})$ be a continuous-time dynamical system. We call Σ an *p -system* if there exists a permutation Π such that $\Pi \mathcal{B} := \{\Pi(w) \mid w \in \mathcal{B}\}$ is a standard p -system.

Definition 3.1.6 Let $\Sigma = (\mathbb{R}, \mathbb{R}^q, [\mathcal{F}]^q, \mathcal{B})$ be a standard p -system. We call Σ a *standard $[p, q-p]$ -system* if

$$\left\{ \sum_{j=p+1}^q w_j e_j \mid w \in \mathcal{B}^\infty \right\} = [\mathcal{F}]^{q-p}$$

In words: there are $q - p$ free variables.

Definition 3.1.7 Let $\Sigma = (\mathcal{R}, \mathcal{R}^q, [\mathcal{F}]^q, \mathcal{B})$ be a continuous-time dynamical system. We call Σ an $[p, q-p]$ -system if there exists a permutation Π such that $\Pi\mathcal{B} := \{\Pi(w) \mid w \in \mathcal{B}\}$ is a standard $[p, n]$ -system.

Note that, given a p -system in $C^\infty(\mathcal{R}, \mathcal{R}^q)$ satisfying an AR-relation with representation-matrix R in $LDO(p, q, \mathcal{R})$ of full row rank, it is always possible to reduce the matrix R to a full row rank matrix in $LDO(p, q, \mathcal{R})$ that is upper triangular, and $R_{i,i} \neq 0$ for $i = 1, \dots, p$ and $\text{degree}(R_{i,k}) \leq \text{degree}(R_{i,i})$ for $k = i + 1, \dots, q$. We call this the *standard upper triangular form*.

I.e. a standard p -system can be obtained by premultiplying with some unimodular matrix and a basis transformation. This basis transformation is equivalent to postmultiplying the representation matrix with a permutation matrix (this is a unimodular operation).

In the sequel, if we talk about $[p, q-p]$ -behaviors we mean the behaviors corresponding to an $[p, q-p]$ -dynamical system Σ .

We will proceed as follows: start with a behavior $\mathcal{B}^\infty$ described by an AR-relation and derive properties for $\mathcal{B}^\infty$. These properties yield necessary conditions on a standard $[p, q-p]$ -behavior $\mathcal{B}^\infty$ such that an AR-relation is satisfied by this behavior. We intend to find such a set of conditions that is sufficient, too. To keep a clear view we treat a simple special case first.

Consider a standard $[p, 2-p]$ -continuous-time dynamical system with two external signals $\Sigma^\infty = (\mathcal{R}, \mathcal{R}^2, C^\infty(\mathcal{R}, \mathcal{R}^2), \mathcal{B}^\infty)$. We derive necessary conditions for $\mathcal{B}^\infty$ by starting from a general standard upper triangular AR-description.

$$\begin{aligned} S_{1,1}w_1 + S_{1,2}w_2 &= 0 \\ S_{2,2}w_2 &= 0 \end{aligned}$$

where $S_{1,1}, S_{1,2}$ and $S_{2,2}$ are in $LDO(\mathcal{R})$. The following is immediate:

Condition 1: $\mathcal{B}^\infty$ is linear, shift invariant and closed in $C^\infty(\mathcal{R}, \mathcal{R}^2)$ -sense.

If $p = 0$ (i.e. $S_{1,1} = S_{1,2} = S_{2,2} = 0$) there are no restrictions on the external signals and the behavior is the whole space. Another simple case is $p = 2$ (i.e. $S_{1,1} \neq 0 \neq S_{2,2}$). This implies that the behavior is finite dimensional. Later on we prove that a subspace is linear, shift-invariant and finite dimensional

in $C^\infty(\mathbb{R}, \mathbb{R}^q)$ (or $C(\mathbb{R}, \mathbb{R}^q)$) if and only if this subspace equals the kernel of an element in $LDO(q, q, \mathbb{R})$. Hence, if $p = 2$, there is nothing left to be proved. The case $p = 1$ remains to be treated. In this situation

$$(w_1, w_2) \in B \Leftrightarrow S_{1,1}w_1 + S_{1,2}w_2 = 0$$

with $S_{1,1}$ and $S_{1,2}$ in $LDO(\mathbb{R})$, $S_{1,1} \neq 0$.

Condition 2: B^∞ is a standard $[1,1]$ -system (or, in general, a $[p, q - p]$ -system).

If B^∞ is a standard $[0,2]$ - or $[2,0]$ -system it readily follows that B^∞ satisfies an AR-relation. The case of a standard $[1,1]$ -behavior is not fully characterized by the conditions 1 and 2. From now on $p=1$.

Consider the subspaces

$$N_1 := \{w_1 \in C^\infty(\mathbb{R}) \mid (w_1, 0) \in B^\infty\} = \ker(S_{1,1})$$

$$N_2 := \{w_2 \in C^\infty(\mathbb{R}) \mid (0, w_2) \in B^\infty\} = \ker(S_{1,2})$$

From the AR-relation above we see that N_1 is a finite-dimensional subspace. Moreover, using $\{e_1, e_2\}$ as the standard basis for $\mathbb{R}^2$, we define the projection on the i -th component

$$\mathcal{E}_i B^\infty := \{w_i \in C^\infty(\mathbb{R}) \mid w \in B^\infty\} \quad i = 1, 2.$$

Then $\mathcal{E}_1 B^\infty$ is finite dimensional ($S_{1,2} = 0$) or $\mathcal{E}_1 B^\infty = C^\infty(\mathbb{R})$ ($S_{1,2} \neq 0$). If $S_{1,2} = 0$ there is no relation between the external variables:

$$(w_1, w_2) \in B^\infty \Leftrightarrow S_{1,1}w_1 = 0 \quad \text{and } w_2 \text{ is free}$$

However, if $S_{1,2} \neq 0$ then N_2 is finite dimensional, too. Whatever $S_{1,2}$ may be, the behavior is related to the graph of the operator $-S_{1,2}$, since to every w_2 corresponds exactly one $S_{1,1}w_1$ such that $(w_1, w_2) \in B^\infty$. Only if $S_{1,2} \neq 0$, B^∞ is also related to the graph of the operator, $-S_{1,1}$. Namely, to every $w_1 \in C^\infty(\mathbb{R})$ corresponds exactly one $S_{1,2}w_2$ and thus $(w_1, w_2) \in B^\infty$ if and only if $-S_{1,2}w_2 = S_{1,1}w_1$. What we see is that, if $S_{1,2} \neq 0$ the roles of w_1 and w_2 can be exchanged.

Condition 3: $\mathcal{E}_1\mathcal{B}^\infty$ or N_2 is finite dimensional.

Finally it is clear that a behavior described by an AR-relation is locally specified.

Condition 4: $\mathcal{B}^\infty$ is locally specified.

Now we show that all these conditions together are also sufficient to derive an AR-relation for behaviors in $C^\infty(\mathbb{R}, \mathbb{R}^2)$. Suppose we have a continuous-time dynamical standard [p,2-p]-system $\Sigma^\infty = (\mathbb{R}, \mathbb{R}^2, C^\infty(\mathbb{R}, \mathbb{R}^2), \mathcal{B}^\infty)$ satisfying the conditions 1,2,3 and 4. The only interesting case is $p = 1$. Hence we assume that $\mathcal{B}^\infty$ is a standard [1,1]-system. This means N_1 is finite dimensional, and consequently $N_1 = \ker(S)$ for some $S \in LDO(\mathbb{R})$. To every $w_2 \in C^\infty(\mathbb{R})$ there exists exactly one Sw_1 such that $(w_1, w_2) \in \mathcal{B}^\infty$. In fact, if (w_1, w_2) and $(\bar{w}_1, w_2)$ are in $\mathcal{B}^\infty$ then $(w_1 - \bar{w}_1, 0) \in \mathcal{B}^\infty$ and hence $w_1 - \bar{w}_1 \in N_1$. Thus $S(w_1 - \bar{w}_1) = 0$. Now we can view $\tilde{\mathcal{B}}^\infty := \{Sw_1, w_2 | (w_1, w_2) \in \mathcal{B}^\infty\}$ as the graph of an operator, say V , $V : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$, $Vw_2 = Sw_1$. So we have

$$(w_1, w_2) \in \mathcal{B}^\infty \Leftrightarrow Vw_2 = Sw_1$$

Since $\mathcal{B}^\infty$ is shift-invariant the operator V commutes with all shift-operators σ^t , $t \in \mathbb{R}$. In fact,

$$Vw_2 = Sw_1 \Leftrightarrow \sigma^t Vw_2 = \sigma^t Sw_1 = S\sigma^t w_1 = V\sigma^t w_2$$

Moreover, since it can be shown that $\tilde{\mathcal{B}}^\infty$ is closed, the operator V is also closed. The closed-graph theorem (theorem 2.1.8) yields that V is continuous. Consequently $V = TC_\mu$ for some $T \in LDO(\mathbb{R})$ and $\mu \in ba_c(\mathbb{R})$ (see lemma 2.4.25) and

$$(w_1, w_2) \in \mathcal{B}^\infty \Leftrightarrow TC_\mu w_2 = Sw_1$$

Now we use condition 3: if $\mathcal{E}_1\mathcal{B}^\infty$ is finite dimensional, then $TC_\mu = 0$. For, if TC_μ be nonzero, then we infer from corollary 2.4.28 that $\text{Range}(TC_\mu)$ is dense in $C^\infty(\mathbb{R})$. Since $S \neq 0$ and hence surjective this is a contradiction. The description for the behavior $\mathcal{B}^\infty$ would be

$$(w_1, w_2) \in \mathcal{B}^\infty \Leftrightarrow Sw_1 = 0 \text{ and } w_2 \text{ is free in } C^\infty(\mathbb{R})$$

If N_2 is finite dimensional, the convolution operator TC_μ can only be $T\sigma^a$, for some $a \in \mathbb{R}$, because the shift operator is the only convolution operator that has not an infinite-dimensional kernel (see lemma 2.2.7). The behavior is now represented by

$$(w_1, w_2) \in B^\infty \Leftrightarrow T\sigma^a w_2 = Sw_1$$

Lemma 3.1.8 *For all $a \in \mathbb{R}$, $a \neq 0$ the behavior described by*

$$B^\infty = \{(w_1, w_2) \in C^\infty(\mathbb{R}) \mid T\sigma^a w_2 = Sw_1\}$$

is not locally specified.

Proof: Let w_1 and w_2 be arbitrary elements in $C^\infty(\mathbb{R})$ and suppose B^∞ is locally specified. Take $\varepsilon > 0$ with $0 < \varepsilon < |a|$ (without loss of generality we assume $a > 0$). We show that for all $t \in \mathbb{R}$ (define $I(t, \varepsilon) = [t - \varepsilon, t + \varepsilon]$)

$$(w_1, w_2)|_{I(t, \varepsilon)} \in B^\infty|_{I(t, \varepsilon)}$$

I.e. there are functions u_1 and u_2 , both in $C^\infty(\mathbb{R})$, such that $u_1|_{I(t, \varepsilon)} = w_1|_{I(t, \varepsilon)}$ and $u_2|_{I(t, \varepsilon)} = w_2|_{I(t, \varepsilon)}$. Fix $t_0 \in \mathbb{R}$ and define the functions u_1 and u_2 such that $u_1|_{I(t_0, \varepsilon)} = w_1|_{I(t_0, \varepsilon)}$ and $u_2|_{I(t_0, \varepsilon)} = w_2|_{I(t_0, \varepsilon)}$. Furthermore we determine u_1 on $I(t_0 - a, \varepsilon)$ satisfying $(T\sigma^a u_2)(t) = (Su_1)(t)$ on $I(t_0, \varepsilon)$ and thereafter we extend u_1 to a C^∞ -function on $(-\infty, t_0 + \varepsilon]$. Solve the equation $Su_1 = T\sigma^a u_2$ for u_2 on the interval $(-\infty, t_0 + a + \varepsilon]$ and outside the intervals $I(t_0 - a, \varepsilon)$ and $I(t_0, \varepsilon)$. We do not know if u_2 is a C^∞ -function on that interval. Solve the equation $T\sigma^a \tilde{u}_2 = Su_1$ on the interval $(-\infty, t_0 + \varepsilon]$. Then $\tilde{u}_2$ is a C^∞ -function on $(-\infty, t_0 + \varepsilon]$ and $\tilde{u}_2 - u_2$ satisfies the equation $T(\tilde{u}_2 - u_2) = 0$ on $I(t_0 + a, \varepsilon)$. So we can choose $\tilde{u}_2$ such that

$$\tilde{u}_2|_{I(t_0 + a, \varepsilon)} = u_2|_{I(t_0 + a, \varepsilon)}$$

This yields a C^∞ -function u_2 on $(-\infty, t_0 + a + \varepsilon]$ satisfying $Su_1 = T\sigma^a u_2$ and clearly $u_2|_{I(t, \varepsilon)} = w_2|_{I(t, \varepsilon)}$.

Now, we extend u_2 to a C^∞ -function on $\mathbb{R}$. Determine u_1 on $[t_0 + \varepsilon, \infty)$ by solving the equation $(Su_1)(t) = (T\sigma^a u_2)(t)$ for $u_1(t), t \in [t_0, \infty)$. And because of the overlap we can prove in the same way as before that $u_1 \in C^\infty(\mathbb{R})$. Hence we found u_1 and u_2 in $C^\infty(\mathbb{R})$ satisfying the equation for the behavior. Because, we assumed B^∞ to be locally specified we obtain

$(w_1, w_2) \in \mathcal{B}^\infty$. Since w_1 and w_2 were arbitrary we have a contradiction. $\square$

We see from this lemma that locally specifiedness implies $a = 0$.

In the forthcoming sections we generalize the idea above to continuous-time dynamical systems with an arbitrary (finite) number of external variables.

3.2 Necessary conditions

In this section we derive necessary conditions for a behavior $\mathcal{B}^\infty$ (i.e. a behavior with $C^\infty(\mathbb{R})$ -signals) to satisfy an AR-relation. In spite of the fact that these conditions look rather mathematical they have a system-theoretical interpretation, which we will try to clarify. Unfortunately, for multivariable systems, it is quite hard to find a transparent way to present these conditions. Therefore we started this chapter with a simple behavior in $C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$, where the conditions are clear and comprehensible. In this section we will extend these conditions to the general multivariable case, but the interpretation will be the same as before.

Let us start with deriving some properties of one-free behaviors. Consider an AR-relation for a *one-free* behavior $\mathcal{B}^\infty$ in $C^\infty(\mathbb{R}, \mathbb{R}^{p+1})$. So

$$\mathcal{B}^\infty = \{(w_1, \dots, w_{p+1}) \in C^\infty(\mathbb{R}, \mathbb{R}^{p+1}) \mid R\left(\frac{d}{dt}\right)w = 0\}$$

for some polynomial matrix $R \in LDO(p, p+1, \mathbb{R})$ of full row rank. There exists a unimodular matrix U and a permutation Π such that

$$UR\Pi^{-1} = \left(\begin{array}{ccc|c} S_{1,1} & \cdots & S_{1,p} & -T_{1,1} \\ & \ddots & \vdots & \vdots \\ & & S_{p,p} & -T_{p,1} \end{array} \right) \quad (3.1)$$

where T_{j1} ($j = 1, \dots, p$) and S_{kl} ($k = 1, \dots, p; l = k, \dots, p$) are in $LDO(\mathbb{R})$. Moreover $S_{kk} \neq 0$ ($k = 1, \dots, p$). I.e., if we define S as the matrix corresponding to the first p columns, S is surjective.

Then $\Pi\mathcal{B}^\infty$ is characterized as the kernel of the matrix $UR\Pi^{-1}$. Hence $\Pi\mathcal{B}^\infty$ is a standard $[p, 1]$ -system. We take $\Pi = I$ in the sequel.

Most of the proofs in this chapter are just manipulations with the equations describing a behavior. Part of these manipulations is the 'elimination-procedure'. For convenience we put the description in (3.1) in a more general context by replacing T_j by $T_j C_{\mu_j}$.

Definition 3.2.1 Let B^∞ be the $[p,1]$ -behavior satisfying

$$Rw = \begin{pmatrix} S_{1,1} & \cdots & S_{1,p} & -T_1 C_{\mu_1} \\ & \ddots & \vdots & \vdots \\ & & S_{p,p} & -T_p C_{\mu_p} \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_{p+1} \end{pmatrix} = 0 \quad (3.2)$$

then we define the *elimination-procedure* as follows: define

$$S(p+1-j) := \left(\frac{S_{p+1-j,p+1-j} \cdot I_{p-j}}{I_j} \right)$$

where I_{p-j} is the identity-matrix on $\mathbb{R}^{p-j}$. At step j ($1 \leq j \leq p$) the representation matrix R has been premultiplied by $\prod_{k=1}^{j-1} S(p+1-k)$ and the external variables $w_p, \dots, w_{p+2-j}$ have been eliminated already. Now premultiply $\prod_{k=1}^{j-1} S(p+1-k)R$ by $S(p+1-j)$ and eliminate w_{p+1-j} . Hereafter, the $(p+j-1)^{th}$ equation (which is a relation between w_{p+j-1} and w_{p+1} only) can be used to loose w_{p+j-1} in the first $p-j$ equations. The matrix $\prod_{k=1}^{j-1} S(p+1-k)R$ will be denoted by $R[p-j]$ and the corresponding behavior by $B^\infty[p-j]$.

To show what happens we carry out the first step. No variable has been eliminated and we premultiply R by

$$S(p) := \left(\frac{S_{p,p} \cdot I_{p-1}}{I_1} \right)$$

leading to

$$S(p)R = \begin{pmatrix} S_{p,p}S_{1,1} & \cdots & S_{p,p}S_{1,p-1} & S_{p,p}S_{1,p} & -S_{p,p}T_1 C_{\mu_1} \\ & \ddots & \vdots & \vdots & \vdots \\ & & S_{p,p}S_{p-1,p-1} & S_{p,p}S_{p-1,p} & -S_{p,p}T_p C_{\mu_p} \\ & & & S_{p,p} & -T_p C_{\mu_p} \end{pmatrix}$$

which is equivalent to (substituting the last equation into the previous ones)

$$S(p)R = \begin{pmatrix} S_{p,p}S_{1,1} & \cdots & S_{p,p}S_{1,p-1} & 0 & -S_{p,p}T_1C_{\mu_1} + S_{1,p}T_pC_{\mu_p} \\ & \ddots & \vdots & \vdots & \vdots \\ & & S_{p,p}S_{p-1,p-1} & 0 & -S_{p,p}T_pC_{\mu_p} + S_{p-1,p}T_pC_{\mu_p} \\ & & & S_{p,p} & -T_pC_{\mu_p} \end{pmatrix}$$

Note that, in general, the behaviors represented by these new equations $\prod_{k=1}^{j-1} S(p+1-k)Rw = 0$ are larger than the one we started with, because premultiplication by an element in $LDO(\mathcal{R})$ with a nonzero kernel leads to additional solutions.

Definition 3.2.2 Given a behavior $\mathcal{B}^\infty$ described by the matrix in (3.2), we define the operator A_i in the following way:

$$A_i(w_{i+1}, \dots, w_{p+1}) = T_i C_{\mu_i} w_{p+1} - \sum_{j=i+1}^p S_{i,j} w_j \quad (3.3)$$

Definition 3.2.3 The solution set of the last $p-i$ equations in the description for the behavior in equation (3.2)

$$\begin{pmatrix} S_{i+1,i+1} & \cdots & S_{i+1,p} & -T_{i+1}C_{\mu_{i+1}} \\ & \ddots & \vdots & \vdots \\ & & S_{p,p} & -T_pC_{\mu_p} \end{pmatrix} \begin{pmatrix} w_{i+1} \\ \vdots \\ w_{p+1} \end{pmatrix} = 0$$

is denoted by $\mathcal{O}_i$.

Note that the operator A_i applied to $(w_{i+1}, \dots, w_{p+1})$ is in fact a part of the i^{th} equation for the behavior and equals $S_{i,i}w_i$ if $(w_{i+1}, \dots, w_{p+1}) \in \mathcal{O}_i$ and $w_i \in \mathcal{E}_i \mathcal{B}^\infty$.

Lemma 3.2.4 Let A and B be linear operators with A and B have a finite-dimensional kernel. Then AB has a finite-dimensional kernel, too.

Proof: Follows from

$$\ker(AB) = \{w | Bw \in \ker A\}$$

□

Definition 3.2.5 Given a behavior B then $N_i(B) := \{w \in B | w_i = 0\}$.

If it is clear what behavior we mean we just say N_i .

Definition 3.2.6 The projection on the i^{th} component of a behavior is defined by

$$\mathcal{E}_i B := \{w_i \in \mathcal{F} | w \in B\}$$

After this short intermezzo we return to the behavior described in equation (3.1). Replacing ΠB^∞ by B^∞ we can make the following observation:

Lemma 3.2.7 The following conditions are necessary for standard $[p, 1]$ -behaviors B^∞ in $C^\infty(\mathbb{R}, \mathbb{R}^{p+1})$ to satisfy an AR-relation:

1. B^∞ is a linear, shift-invariant subspace of $C^\infty(\mathbb{R}, \mathbb{R}^{p+1})$.
2. B^∞ is locally specified.
3. For all $i \in \{1, \dots, p\}$: $\mathcal{E}_i B^\infty$ or N_i is finite dimensional.

Proof: We only prove the last condition. Suppose $\mathcal{E}_i B^\infty$ is infinite dimensional. Since the equation (i^{th} row of the representation matrix)

$$S_{i,i} w_i = T_i w_{p+1} - \sum_{j=i+1}^p S_{i,j} w_j = A_i(w_{i+1}, \dots, w_{p+1})$$

for $w_i \in \mathcal{E}_i B^\infty$ and $w_{i+1}, \dots, w_{p+1} \in \mathcal{O}_i$ is satisfied, $A_i(\mathcal{O}_i)$ (see definition 3.2.2 with $C_{\mu_i} = I$) is infinite dimensional, too. Define

$$S_p := \prod_{j=i+1}^p S_{j,j}$$

Then $S_p A_i(\mathcal{O}_i)$ is infinite dimensional also.

Consider the subspace

$$W_i := \{(w_{i+1}, \dots, w_{p+1}) | (0 | (w_{i+1}, \dots, w_{p+1})) \in B^\infty\} \quad (3.4)$$

I.e. the subspace governed by the equations

$$\begin{pmatrix} S_{i,i+1} & \cdots & S_{i,p} & -T_i \\ S_{i+1,i+1} & & \vdots & \vdots \\ & \ddots & \vdots & \vdots \\ & & S_{p,p} & -T_p \end{pmatrix} \begin{pmatrix} w_{i+1} \\ \vdots \\ w_{p+1} \end{pmatrix} = 0 \quad (3.5)$$

Premultiply the first row with S_p . Then we can eliminate $w_{i+1}, \dots, w_p$ from this row. For $(w_{i+1}, \dots, w_{p+1}) \in \mathcal{O}_i$ we find

$$S_p A_i(w_{i+1}, \dots, w_{p+1}) = S_p \left(\sum_{j=i+1}^p S_{i,j} w_j - T_i w_{p+1} \right) = -\tilde{T}_i w_{p+1}$$

Hence $S_p A_i(w_{i+1}, \dots, w_{p+1})$ for $(w_{i+1}, \dots, w_{p+1}) \in \mathcal{O}_i$ equals $-\tilde{T}_i w_{p+1}$, for some $\tilde{T}_i \in C^\infty(\mathbb{R})$. Since $S_p A_i(\mathcal{O}_i)$ is infinite dimensional, we have $-\tilde{T}_i \neq 0$, and from

$$\begin{pmatrix} S_p S_{i,i+1} & \cdots & S_p S_{i,p} & -S_p T_i \\ S_{i+1,i+1} & & \vdots & -T_{i+1} \\ & \ddots & \vdots & \vdots \\ & & S_{p,p} & -T_p \end{pmatrix} \sim \begin{pmatrix} 0 & \cdots & 0 & -\tilde{T}_i \\ S_{i+1,i+1} & \cdots & S_{i+1,p} & -T_{i+1} \\ & \ddots & \vdots & \vdots \\ & & S_{p,p} & -T_p \end{pmatrix}$$

it follows that the subspace W_i is finite dimensional. Consequently, N_i is finite dimensional, too. In fact, N_i is determined by

$$\begin{pmatrix} S_{1,1} & \cdots & S_{1,i-1} \\ & \ddots & \vdots \\ & & S_{i-1,i-1} \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_{i-1} \end{pmatrix} = Z \begin{pmatrix} w_{i+1} \\ \vdots \\ w_{p+1} \end{pmatrix}$$

for some $Z \in \text{LDO}(i-1, p+1-i, \mathbb{R})$ and where $(w_{i+1}, \dots, w_{p+1})$ is an element in the finite dimensional space W_i . □

Remark 3.2.8 If in equation (3.5) $T_{i+1} = \dots = T_p = 0$, then clearly W_i is finite dimensional and $S_p A_i(\mathcal{O}_i)$ is finite dimensional, too. Hence, if $S_p A_i(\mathcal{O}_i)$ is infinite dimensional, then $\mathcal{O}_i$ is infinite dimensional and for some $k \in$

$\{i+1, \dots, p\} T_k \neq 0$. For every $w_{p+1} \in C^\infty(\mathbb{R})$ we can find $w_{i+1}, \dots, w_p$ such that

$$\begin{pmatrix} S_{i+1,i+1} & \cdots & S_{i+1,p} \\ & \ddots & \vdots \\ & & S_{p,p} \end{pmatrix} \begin{pmatrix} w_{i+1} \\ \vdots \\ w_p \end{pmatrix} = \begin{pmatrix} -T_{i+1} \\ \vdots \\ -T_p \end{pmatrix} w_{p+1}$$

Note that, in this situation, $-\tilde{T}_i \neq 0$, and because w_{p+1} can be chosen arbitrary in $C^\infty(\mathbb{R})$ we find $S_p A_i(\mathcal{O}_i) = C^\infty(\mathbb{R})$. Hence,

$$C^\infty(\mathbb{R}) = S_p A_i(\mathcal{O}_i) = A_i(S_p(\mathcal{O}_i)) \subseteq A_i(\mathcal{O}_i)$$

Consequently

$$\{w_i \in C^\infty(\mathbb{R}) | S_{i,i} w_i \in A_i(\mathcal{O}_i)\} = C^\infty(\mathbb{R}) = \mathcal{E}_i \mathcal{B}^\infty$$

□

Let us comment on the conditions in lemma 3.2.7. The first two are quite standard when dealing with continuous-time dynamical systems. The last deals with the relationship between w_i and w_{p+1} . If $\dim(\mathcal{E}_i \mathcal{B}^\infty) < \infty$, w_i and w_{p+1} do not depend on one another. On the other hand if $\dim(\mathcal{E}_i \mathcal{B}^\infty) = \infty$, then $\mathcal{E}_i \mathcal{B}^\infty = C^\infty(\mathbb{R})$ and the role of the free variable w_{p+1} can be taken over by w_i . Or, equivalently $(i, p+1) \mathcal{B}^\infty$ is a standard $[p, 1]$ -system.

There is an important difference between the subspace E in definition 3.1.4 and the subspace N_i because the fact that E is finite dimensional is *no condition* on $\mathcal{B}^\infty$, but a consequence of the definition of p , whereas N_i being finite dimensional is a condition in case $\dim(\mathcal{E}_i \mathcal{B}^\infty) = \infty$.

Later on we show that this condition 3 of lemma 3.2.7 implies that either $\mathcal{E}_i \mathcal{B}^\infty$ is not dense in $C^\infty(\mathbb{R})$ or $\mathcal{E}_i \mathcal{B}^\infty = C^\infty(\mathbb{R})$. The consequence is that if the spectrum corresponding to w_i is bounded we know that $\mathcal{E}_i \mathcal{B}^\infty$ is finite dimensional and if not $\mathcal{E}_i \mathcal{B}^\infty = C^\infty(\mathbb{R})$.

Now consider a behavior $\mathcal{B}^\infty$ in $C^\infty(\mathbb{R}, \mathbb{R}^q)$ (i.e. with more than one free variables) satisfying an AR-relation, which is in standard upper triangular form. Then, for some $p \in \mathbb{N}$, $\mathcal{B}^\infty$ is an $[p, q-p]$ -behavior and

$$S(w_1, \dots, w_p)^t = T(w_{p+1}, \dots, w_q)^t$$

where S is in $\text{LDO}(p, p, \mathbb{R})$ is upper triangular with nonzero diagonal elements and T in $\text{LDO}(p, q-p, \mathbb{R})$. Then B^∞ is linear, shift-invariant and closed in $C^\infty(\mathbb{R}, \mathbb{R}^q)$ -sense. Because of the superposition-principle B^∞ is built up by $q - p$ subsystems. For all $j \in \{p + 1, \dots, q\}$ we have: for every $(0, \dots, 0, w_j, 0, \dots, 0)$ ($q - p$ -vector) there exists a vector $(w_1, \dots, w_p) \in C^\infty(\mathbb{R}, \mathbb{R}^p)$ such that

$$(w_1, \dots, w_p, 0, \dots, 0, w_j, 0, \dots, 0)$$

is in B^∞ . Hence

$$S \begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix} = \begin{pmatrix} T_{1,j} \\ \vdots \\ T_{p,j} \end{pmatrix} w_j \quad (j = p + 1, \dots, q) \quad (3.6)$$

$$\begin{aligned} B_j^\infty &:= \{w \in B^\infty \mid w_i = 0, p < i \leq q, i \neq j\} \\ &= \{(w_1, \dots, w_p, 0, \dots, 0, w_j, 0, \dots, 0) \in B^\infty \mid \text{equation (3.6) is satisfied}\} \end{aligned}$$

It is our idea to formulate several necessary conditions for the standard p -behaviors to be expressible in terms of conditions on the standard $[p, 1]$ -behaviors B_j^∞ , that build up B^∞ . In fact, we have

Lemma 3.2.9 *Let $\Sigma = (\mathbb{R}, \mathbb{R}^q, C^\infty(\mathbb{R}, \mathbb{R}^q), B^\infty)$ be a continuous-time dynamical system. If B^∞ satisfies an AR-relation then B^∞ satisfies*

Necessary conditions

c1 B^∞ is a linear, shift-invariant subspace, that is closed with respect to the topology of $C^\infty(\mathbb{R}, \mathbb{R}^q)$

c2 B^∞ is locally specified

There exists a permutation Π and $p \in \mathbb{N}$ such that

c3 ΠB^∞ is a standard $[p, q - p]$ -behavior

c4 For all $j \in \{p + 1, \dots, q\}$ and $i \in \{1, \dots, p\}$: $\mathcal{E}_i[(\Pi B^\infty)_j]$ or $N_i[(\Pi B^\infty)_j]$ is finite dimensional.

(3.7)

At the beginning of this section we claimed that the question whether a class of behaviors can be represented by an AR-relation can be reduced to the question whether an element in that class can be characterized by such a relation. Hence the conditions we derived for a standard p-system B^∞ are crucial for the class of behaviors containing B^∞ .

Definition 3.2.10 If a behavior B satisfies an AR-relation with representation matrix R , we will denote it by B_R .

Hence, if we are talking about a behavior B_R^∞ we implicitly assume it to satisfy an AR-relation. At this moment we have the following implication

$$B^\infty = B_R^\infty \Rightarrow B^\infty \text{ satisfies the conditions c1-c4}$$

In the next section, we prove that this implication can be reversed.

3.3 Sufficiency

In this section we prove that for a continuous-time dynamical system Σ the conditions which have been proven to be necessary, are sufficient for the corresponding behavior to satisfy an AR-relation.

Consider a continuous-time dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^q, C^\infty(\mathbb{R}, \mathbb{R}^q), B^\infty)$ satisfying the conditions stated at the end of the previous section:

c1 B^∞ is a linear, shift-invariant subspace, that is closed with respect to the topology of $C^\infty(\mathbb{R}, \mathbb{R}^q)$

c2 B^∞ is locally specified

There exists a permutation Π and $p \in \mathbb{N}$ such that

c3 ΠB^∞ is a standard $[p, q - p]$ -behavior

c4 For all $j \in \{p + 1, \dots, q\}$ and $i \in \{1, \dots, p\}$: $\mathcal{E}_i[(\Pi B^\infty)_j]$ or $N_i[(\Pi B^\infty)_j]$ is finite dimensional.

Without loss of generality we take $\Pi = I$. Hence, from now on we talk about B^∞ instead of ΠB^∞ .

As in chapter 1 we consider the subspace

$$E := \{(w_1, \dots, w_p) \in C^\infty(\mathbb{R}, \mathbb{R}^p) \mid (w_1, \dots, w_p, 0, \dots, 0) \in \mathcal{B}^\infty\} \quad (3.8)$$

which is finite dimensional (see definition 3.1.4).

Theorem 3.3.1 *A subspace V in $C^\infty(\mathbb{R}, \mathbb{R}^p)$ is finite dimensional and shift invariant if and only if V is the kernel of some $S \in \text{LDO}(p, p, \mathbb{R})$ with nonzero diagonal elements. It is always possible to choose S upper triangular.*

Proof: See appendix A.1. $\square$

From this theorem we know that $E = \ker S$, for some $S \in \text{LDO}(p, p, \mathbb{R})$. We use the idea of the graph again. Note that to every $(q-p)$ -tuple $(w_{p+1}, \dots, w_q)$ in $C^\infty(\mathbb{R}, \mathbb{R}^{q-p})$ corresponds at most one $S(w_1, \dots, w_p)$ with $(w_1, \dots, w_p) \in C^\infty(\mathbb{R}, \mathbb{R}^p)$ such that $(w_1, \dots, w_q) \in \mathcal{B}^\infty$. So by the condition that Σ is a standard $[p, q-p]$ system we can characterize $\mathcal{B}^\infty$ with the aid of the graph of an operator $K : C^\infty(\mathbb{R}, \mathbb{R}^{q-p}) \rightarrow C^\infty(\mathbb{R}, \mathbb{R}^p)$ defined as follows:

$$(w_1, \dots, w_q) \in \mathcal{B}^\infty \Leftrightarrow K(w_{p+1}, \dots, w_q)^t = S(w_1, \dots, w_p)^t$$

Since $\mathcal{B}^\infty$ is closed, and S has a continuous right-inverse the operator K is closed (also cf. lemma 5.6.1), and the closed-graph theorem yields that, K is a continuous operator from $C^\infty(\mathbb{R}, \mathbb{R}^{q-p}) \rightarrow C^\infty(\mathbb{R}, \mathbb{R}^p)$.

Definition 3.3.2 Let V be a function space in which the shift-operators σ^t , $t \in \mathbb{R}$ are well defined. If $f = (f_1, \dots, f_n) \in V^q$ then we define

$$\otimes_q \sigma^t f := (\sigma^t f_1, \dots, \sigma^t f_q)$$

Lemma 3.3.3 *Let $p, n \in \mathbb{N}$. If A is a linear operator from $C^\infty(\mathbb{R}, \mathbb{R}^n)$ into $C^\infty(\mathbb{R}, \mathbb{R}^p)$ that is continuous and moreover for all $f \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ we have $A(\otimes_n \sigma^t f) = \otimes_p \sigma^t (Af)$ then A is a $(n \times p)$ -matrix with entries $A_{ij} = T_{i,j} C_{\mu_{i,j}}$ where for all $i=1, \dots, n$; $j=1, \dots, p$ $T_{i,j} \in \text{LDO}(\mathbb{R})$ and $C_{\mu_{i,j}}$ is a convolution operator corresponding to some $\mu_{i,j} \in \text{ba}_c(\mathbb{R})$.*

Proof: Clearly $\mathcal{L} : C^\infty(\mathbb{R}, \mathbb{R}^n) \rightarrow \mathbb{R}^p$ defined by

$$\mathcal{L}(f_1, \dots, f_n) := (A(f_1, \dots, f_n))(0, \dots, 0) = \sum_{i=1}^p \sum_{j=1}^n (A_{ij} f_j)(0) e_i$$

is a continuous linear mapping. Hence, applying lemma 2.1.5 we know that

$$L_{ij}f = (\mathcal{A}_{ij}f)(0)$$

where L_{ij} is a continuous linear functional on $C^\infty(\mathbb{R})$ ($i=1,\dots,p$; $j=1,\dots,n$). Corollary 2.4.27 implies $L_{ij} = \mu_{i,j} \circ T_{i,j}$ for some $T_{i,j} \in LDO(\mathbb{R})$ and $\mu_{i,j} \in ba_c(\mathbb{R})$. Now use the shift-invariance property to obtain

$$(\sigma^t \mathcal{A}_{ij}f)(0) = (\mathcal{A}_{ij}f)(t) = L_{ij}(\sigma^t f)$$

$$\text{Hence } \mathcal{A}_{i,j} = \mu_{i,j} \circ T_{i,j} \sigma^t = T_{i,j} C_{\mu_{i,j}}.$$

□

Since the operator K satisfies the requirements in lemma 3.3.3 the characterization of B^∞ is of the following form:

$$\begin{pmatrix} T_{1,1}C_{\mu_{1,1}} & \cdots & T_{1,q-p}C_{\mu_{1,q-p}} \\ \vdots & \ddots & \vdots \\ T_{p,1}C_{\mu_{p,1}} & \cdots & T_{p,q-p}C_{\mu_{p,q-p}} \end{pmatrix} \begin{pmatrix} w_{p+1} \\ \vdots \\ w_q \end{pmatrix} = \begin{pmatrix} S_{1,1} & \cdots & S_{1,p} \\ & \ddots & \vdots \\ & & S_{p,p} \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix} \quad (3.9)$$

This relation is already close to an AR-relation. We need to prove that the remaining conditions ensure that the convolution operators are identities upto constants. Note that, until now, we used the conditions c1 and c3 only.

The proof of the main result is rather technical and therefore we split off some partial results, that will prove their value later on.

Recall that we defined the subsystems B_j^∞ , for $j = p+1, \dots, q$, from the behavior B^∞ as

$$B_j^\infty := \{w \in B^\infty \mid w_i = 0, p < i \leq q, i \neq j\}$$

As we can see from the conditions, these subsystems play an important part. In fact, using the superposition principle, we will see that the behavior B^∞ satisfies an AR-relation if and only if all B_j^∞ do.

Lemma 3.3.4 *Let $\Sigma = (\mathbb{R}, \mathbb{R}^q, [\mathcal{F}]^q, B)$ be a continuous-time dynamical p-system and let B_i^∞ represent a subsystem as defined in section 3.2. If B^∞ satisfies the conditions c1, c2, c3 and c4 then $B^\infty = B_{p+1}^\infty + \dots + B_q^\infty$.*

Proof: Because $B_j^\infty \subseteq B^\infty$ for $j = p+1, \dots, q$ ' $\supseteq$ ' is trivial. On the other hand if $w = (w_1, \dots, w_q) \in B^\infty$ then for $j = p+1, \dots, q$ we have that to w_j corresponds a vector $\sum_{i=1}^p w_i^{p+j} e_i$ such that $\sum_{i=1}^p w_i^{p+j} e_i + w_{p+j} e_j$ is in B_j^∞ .

$$w = \sum_{j=1}^p (w_j - \sum_{i=p+1}^q w_j^{p+i} e_i) e_j + \sum_{j=p+1}^q (\sum_{i=1}^p w_i^{p+j} e_i + w_{p+j} e_j)$$

Since w and $\sum_{j=p+1}^q (\sum_{i=1}^p w_i^{p+j} e_i + w_{p+j} e_j)$ are in B^∞ their difference $\sum_{j=1}^p (w_j - \sum_{i=p+1}^q w_j^{p+i} e_i) e_j$ is also in B^∞ . This last term is even in E , where E is defined in 3.8. Since $B_j^\infty = B_j^\infty + E$ for all $j \in \{p+1, \dots, q\}$ we are done. $\square$

Lemma 3.3.5 *If B^∞ is locally specified, then for all $j \in \{p+1, \dots, q\}$ B_j^∞ is locally specified.*

Proof: Obvious, since

$$(w_1, \dots, w_p, 0, \dots, 0, w_j, 0, \dots, 0) \in B_j^\infty \Rightarrow (w_1, \dots, w_p, 0, \dots, 0, w_j, 0, \dots, 0) \in B^\infty$$

$\square$

Consequently, if the behavior B^∞ satisfies the conditions c1-c4, then B_j^∞ satisfies these conditions, too.

From now on we deal with p -behaviors in $C^\infty(\mathbb{R}, \mathbb{R}^{p+1})$, i.e. one-free behaviors.

Lemma 3.3.6 *Given a p -behavior B^∞ (in fact some B_j^∞) in $C^\infty(\mathbb{R}, \mathbb{R}^{p+1})$, satisfying c4, with representation-matrix*

$$\begin{pmatrix} S_{1,1} & \cdots & S_{1,p} & -T_1 C_{\mu_1} \\ & \ddots & \vdots & \vdots \\ & & S_{p,p} & -T_p C_{\mu_p} \end{pmatrix} \quad (3.10)$$

then $C_{\mu_p} = \sigma^{ap}$ and for $j=1, \dots, p$, $i=0, \dots, p-j$ we can find $\bar{T}_{p-j}$ and $M_{p-j,p-i}$ in $LDO(\mathbb{R})$ such that

$$\bar{T}_{p-j} C_{\mu_{p-j}} = \sum_{i=0}^{j-1} M_{p-j,p-i} \sigma^{ap-i}$$

and moreover after step j in the elimination-procedure (see definition 3.2.1) we obtain

$$\begin{pmatrix} \bar{s}_{1,1} & \cdots & \bar{s}_{1,p-j} & 0 & \cdots & 0 & -\bar{T}_1 C_{\mu_1} + \sum_{i=0}^{j-1} M_{1,p-i} \sigma^{a_{p-i}} \\ & \ddots & \vdots & \vdots & & \vdots & \vdots \\ & & \bar{s}_{p-j,p-j} & 0 & & \vdots & -\bar{T}_{p-j} C_{\mu_{p-j}} + \sum_{i=0}^{j-1} M_{p-j,p-i} \sigma^{a_{p-i}} \\ & & & \bar{s}_{p+1-j,p+1-j} & \ddots & \vdots & -\bar{T}_{p+1-j} \sigma^{a_{p+1-j}} \\ & & & & \ddots & 0 & \vdots \\ & & & & & s_{p,p} & -\bar{T}_p \sigma^{a_p} \end{pmatrix}$$

where $-\bar{T}_{p-j} C_{\mu_{p-j}} + \sum_{i=0}^{j-1} M_{p-j,p-i} \sigma^{a_{p-i}} = \tilde{T}_{p-j} \sigma^{a_{p-j}}$.

Proof: See appendix, lemma A.2. □

Definition 3.3.7 The behavior $\mathcal{B}_f^\infty[p-j]$ is defined as the solution set to the first $p-j$ equations from the representation-matrix after j steps of elimination. I.e. the first $p-j$ equations of $\mathcal{B}^\infty[p-j]$ (see definition 3.2.1)

Hence, the (eliminated) variables $w_{p+1-j}, \dots, w_p$ do not play any role in $\mathcal{B}_f^\infty[p-j]$. If we leave out the entries in the matrix in equation 3.10, then we are left with a similar matrix. The following lemma is related herewith.

Lemma 3.3.8 Given a standard $[k,1]$ -behavior $\mathcal{B}^\infty$ described by

$$R_K = \begin{pmatrix} S_{1,1} & \cdots & \cdots & S_{1,k} & -T_1 C_{\mu_1} \\ & \ddots & & \vdots & \vdots \\ & & \ddots & \vdots & -T_{k-1} C_{\mu_{k-1}} \\ & & & S_{k,k} & -T_k \sigma^{a_k} \end{pmatrix}.$$

Let $\mathcal{B}^\infty$ be locally specified. Then $a_k = 0$ and $\mathcal{B}_f^\infty[k-1]$ (i.e. the behavior after 1 step in the elimination-procedure) is locally specified, too.

Proof: Suppose $a_k \neq 0$ and let $0 < 2\varepsilon < |a_k|$. Take external signals w_k and w_{k+1} in $C^\infty(\mathbb{R})$ arbitrarily. Then there are $w_1, \dots, w_{k-1}$ such that for all $t \in \mathbb{R}$

$$(w_1, \dots, w_k, w_{k+1})|_{[t-\varepsilon, t+\varepsilon]} \in \ker R_K$$

Define $I(t, \varepsilon) = [t - \varepsilon, t + \varepsilon]$. Let $t \in \mathbb{R}$ and for $\tau \in I(t, \varepsilon)$ define $u_k(\tau) = w_k(\tau)$, $u_{k+1}(\tau) = w_{k+1}(\tau)$. For $\tau \notin I(t, \varepsilon)$ define $u_k(\tau)$ and $u_{k+1}(\tau)$ in

such a way that $u_k, u_{k+1} \in C^\infty(\mathbb{R})$ and they satisfy the equation $S_{k,k}u_k = T_k\sigma^{a_k}u_{k+1}$ (see lemma 3.1.8). It is possible to solve $u_1, \dots, u_{k-1}$ out of equation (3.13) (now $u_i(\tau) = w_i(\tau)$ for $\tau \in I(t, \varepsilon)$ and $i \in \{1, \dots, k+1\}$). Note that the equation $S_{k,k}w_k = T_k\sigma^{a_k}w_{k+1}$ implies that w_k and w_{k+1} do not influence one another on the interval $I(t, \varepsilon)$, because $0 < 2\varepsilon < |a_k|$. Hence, since B^∞ is locally specified, B^∞ contains the element $(w_1, \dots, w_k, w_{k+1})$. However w_k and w_{k+1} are arbitrary which means that B^∞ is not an k -system anymore but at most an $(k-1)$ -system. This is a contradiction, so $a_k = 0$.

Introduce the behavior $\tilde{B}^\infty$ obtained from B^∞ by multiplying all the equations with $S_{k,k}$. This yields

$$\tilde{B}^\infty = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^{p+1}) \mid S_{k,k}w \in B^\infty\} \quad (3.11)$$

The following relation is obvious

$$\left. \begin{array}{l} (w^{(1)}|w_{k+1}) \in B_f^\infty[k-1] \\ S_{k,k}w_k = T_kw_{k+1} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} (w^{(1)}|w_k|w_{k+1}) \in \tilde{B}^\infty \\ S_{k,k}w_k = T_kw_{k+1} \end{array} \right. \quad (3.12)$$

Let $(u^{(1)}|u_{k+1}) \in C^\infty(\mathbb{R}, \mathbb{R}^k)$ such that $(u^{(1)}|u_{k+1})|_{I(t, \varepsilon)} \in B_f^\infty[k-1]|_{I(t, \varepsilon)}$ for all $t \in \mathbb{R}$. Define $u_k \in C^\infty(\mathbb{R})$ as a solution to $S_{k,k}u_k = T_ku_{k+1}$. Now take any $t \in \mathbb{R}$, then there exists a $(v^{(1)}|v_{k+1}) \in B_f^\infty[k-1]$ such that $(v^{(1)}|v_{k+1})|_{I(t, \varepsilon)} = (u^{(1)}|u_{k+1})|_{I(t, \varepsilon)}$. If v_k is a solution to $S_{k,k}v_k = T_kv_{k+1}$, then $(v^{(1)}|v_k|v_{k+1}) \in \tilde{B}^\infty$ and

$$\begin{aligned} S_{k,k}(u^{(1)}|u_k|u_{k+1})|_{I(t, \varepsilon)} &= S_{k,k}(v^{(1)}|v_k|v_{k+1})|_{I(t, \varepsilon)} \\ &= (S_{k,k}v^{(1)}|S_{k,k}v_k|S_{k,k}v_{k+1})|_{I(t, \varepsilon)} \in B^\infty|_{I(t, \varepsilon)} \end{aligned}$$

since $\tilde{B}^\infty = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^{k+1}) \mid S_{k,k}w \in B^\infty\}$. Hence for $t \in \mathbb{R}$

$$(S_{k,k}u^{(1)}|S_{k,k}u_k|S_{k,k}u_{k+1})|_{I(t, \varepsilon)} \in B^\infty|_{I(t, \varepsilon)}$$

and since B^∞ is locally specified $(S_{k,k}u^{(1)}|S_{k,k}u_k|S_{k,k}u_{k+1}) \in B^\infty$. This yields $(u^{(1)}|u_k|u_{k+1}) \in \tilde{B}^\infty$ and we already have $S_{k,k}u_k = T_ku_{k+1}$. Consequently (see 3.12), $(u^{(1)}|u_{k+1}) \in B_f^\infty[k-1]$ implying that $B_f^\infty[k-1]$ is locally specified. $\square$

Lemma 3.3.9 *Let B^∞ be a standard $[p, 1]$ -behavior satisfying the conditions c1-c4. Then B^∞ satisfies an AR-relation.*

Proof: The proof will be given by induction. The induction hypothesis will be that in the j^{th} step we have proven that the convolution operators $C_{\mu_p}, \dots, C_{\mu_{p+1-j}}$ are constants times the identity, $a_p = \dots a_{p+1-j} = 0$ and $B_f^\infty[p-j]$ is locally specified.

$j=1$: From equation (3.9) we know that B^∞ can be characterized by

$$w \in B^\infty \Leftrightarrow \left(\begin{array}{c|c} S_{11} & S_{12} \\ \hline & S_{p,p} \end{array} \right) \left(\begin{array}{c} (w^{(1)})^t \\ w_p \end{array} \right) = \left(\begin{array}{c} T^{(1)}C_\mu^{(1)} \\ T_p C_{\mu_p} \end{array} \right) w_{p+1} \quad (3.13)$$

where $w = (w^{(1)}|w_p|w_{p+1})$ and hence $S_{11} \in LDO(p-1, p-1, \mathbb{R})$, $S_{12} \in LDO(p-1, 1, \mathbb{R})$, $S_{p,p} \in LDO(\mathbb{R})$ and

$$T^{(1)}C_\mu^{(1)} = [T_1 C_{\mu_1}, \dots, T_{p-1} C_{\mu_{p-1}}]^t$$

Using lemma 3.3.8 we obtain $a_p = 0$. Hence we can describe B^∞ by

$$w \in B^\infty \Leftrightarrow \left(\begin{array}{c|c} S_{11} & S_{12} \\ \hline & S_{p,p} \end{array} \right) \left(\begin{array}{c} (w^{(1)})^t \\ w_p \end{array} \right) = \left(\begin{array}{c} T^{(1)}C_\mu^{(1)} \\ T_p \end{array} \right) w_{p+1} \quad (3.14)$$

where T_p can be zero. The same lemma 3.3.8 also implies that the behavior $B_f^\infty[p-1]$ is locally specified.

Induction step:

Suppose we have proven already that $C_{\mu_p}, \dots, C_{\mu_{p+1-j}}$ are constants times identities, and $B_f^\infty[p-j]$ is locally specified. Then, by lemma 3.3.6 $B_f^\infty[p-j]$ is described by the following representation-matrix

$$\left(\begin{array}{ccc|c} \bar{S}_{1,1} & \cdots & \bar{S}_{1,p-j} & -\bar{T}_1 C_{\mu_1} + \sum_{i=0}^{j-1} M_{1,p-i} \\ & \ddots & \vdots & \vdots \\ & & \bar{S}_{p-j,p-j} & -\bar{T}_{p-j} C_{\mu_{p-j}} + \sum_{i=0}^{j-1} M_{p-j,p-i} \end{array} \right)$$

where $-\bar{T}_{p-j} C_{\mu_{p-j}} + \sum_{i=0}^{j-1} M_{p-j,p-i} = \tilde{T}_{p-j} \sigma^{a_{p-j}}$. Applying lemma 3.3.8 yields $a_{p-j} = 0$ and $B_f^\infty[p-j-1]$ is locally specified. Moreover

$$\bar{T}_{p-j} C_{\mu_{p-j}} = \sum_{i=0}^{j-1} M_{p-j,p-i} + \tilde{T}_{p-j}$$

yielding $C_{\mu_{p-j}} = c_{p-j} \cdot I$ for some $c_{p-j} \in \mathbb{R}$.

Since these C_{μ_j} are the same convolution operators as in the description for the original behavior B^∞ we are done. $\square$

Now we can give the main result in this thesis:

Theorem 3.3.10 *Let $\Sigma = (\mathbb{R}, \mathbb{R}^q, C^\infty(\mathbb{R}, \mathbb{R}^q), B^\infty)$ be a continuous-time dynamical system. Then B^∞ satisfies an AR-relation if and only if B^∞ satisfies*

c1 B^∞ is a linear, shift-invariant subspace, that is closed with respect to the topology of $C^\infty(\mathbb{R}, \mathbb{R}^q)$

c2 B^∞ is locally specified

There exists a permutation Π and $p \in \mathbb{N}$ such that

c3 ΠB^∞ is a standard $[p, q - p]$ -behavior

c4 For all $j \in \{p + 1, \dots, q\}$ and $i \in \{1, \dots, p\}$: $\mathcal{E}_i[(\Pi B^\infty)_j]$ or $N_i[(\Pi B^\infty)_j]$ is finite dimensional.

(3.15)

Now the result is there it is easy to see that it does not matter what basis one chooses to describe B^∞ . Namely, for any invertible matrix V , define $B^\infty(V) = \{Vw \mid w \in B^\infty\}$ then we have

$$B^\infty \text{ satisfies c1-c4} \Leftrightarrow B^\infty = B_R^\infty \Leftrightarrow B^\infty(V) = B_{RV^{-1}}^\infty$$

Chapter 4

AR-descriptions for systems consisting of continuous signals

4.1 Introduction

At this moment, the essential parts to solve the problem, as formulated in chapter 1, have been proven. In this chapter, we pass from behaviors with signals in $C^\infty(\mathbb{R})$ to behaviors with continuous signals. We present the mathematical details that are necessary to prove that the AR-relation we found in chapter 3 for $B^\infty = B \cap C^\infty(\mathbb{R}, \mathbb{R}^q)$ is actually the one we are looking for if we want to describe the original behavior. We treated behaviors in $C^\infty(\mathbb{R}, \mathbb{R}^q)$ and found necessary and sufficient conditions for them to satisfy an AR-relation. It is our aim to extend these conditions in a natural way to behaviors in $C(\mathbb{R}, \mathbb{R}^q)$ (or perhaps other Fréchet spaces $\mathcal{F}^q$). Of course, the interpretation of the AR-relation is an important issue here. We start with a typical example:

Example: consider a behavior B described by

$$B = \{(w_1, w_2) \in C(\mathbb{R}) \times C(\mathbb{R}) \mid \exists c \in \mathbb{R} [w_1(t) = w_2(t) + \int_0^t w_2(\tau) d\tau + c]\} \quad (4.1)$$

This behavior is linear, shift invariant and closed in $C(\mathbb{R}) \times C(\mathbb{R})$. However it is not characterized by a classical AR-equation. It is of course related to the AR-relation $\frac{d}{dt} w_1 = w_2 + \frac{d}{dt} w_2$. In fact, restricted to $C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$ (or $C^1(\mathbb{R}) \times C^1(\mathbb{R})$), the behavior described by the latter AR-equation is exactly the one described in equation (4.1) and the AR-relation can be interpreted classically, whenever w_1 and w_2 are in $C^1(\mathbb{R})$. Hence a part of

the behavior represented by equation (4.1) is not included in the description with the differential equation. Another problem deals with closedness. The natural extension of the closedness-condition for $C^\infty(\mathbb{R}, \mathbb{R}^q)$ -behaviors is to require closedness in $C(\mathbb{R}, \mathbb{R}^q)$ -sense. However, $\frac{d}{dt}w_1 = w_2 + \frac{d}{dt}w_2$ corresponds to a behavior that is not closed in $C(\mathbb{R}) \times C(\mathbb{R})$. If we assume $\mathcal{B}$ to be closed in $C(\mathbb{R}) \times C(\mathbb{R})$, then the solution set to the differential equation is not allowed to be a behavior and hence the system described by equation (4.1) cannot be represented by an AR-relation. This explains the problems around the closedness-condition and it is not straightforward how to deal with it.

What we see from this example is that we must find another way to achieve that a behavior, as described in equation (4.1), can be represented by an AR-relation.

4.2 The weak interpretation of an AR-relation

In this section we prove that there is a strong relationship between linear, shift-invariant and closed behaviors in $C(\mathbb{R}, \mathbb{R}^q)$ and their restrictions $\mathcal{B}^\infty$ to $C^\infty(\mathbb{R}, \mathbb{R}^q)$. We start with one of the most important lemmas in this thesis.

Lemma 4.2.1 *If $\mathcal{B}$ is a linear, time-invariant behavior that is closed in $C(\mathbb{R}, \mathbb{R}^q)$, then $\mathcal{B}^\infty := \mathcal{B} \cap C^\infty(\mathbb{R}, \mathbb{R}^q)$ is dense in $\mathcal{B}$.*

Proof: Let $w \in \mathcal{B}$. Since $\mathcal{B}$ is linear, shift invariant and closed in $C(\mathbb{R}, \mathbb{R}^q)$ we can apply theorem 2.5.1 to obtain $C_\mu w \in \mathcal{B}$ for all $\mu \in ba_c(\mathbb{R})$. According to lemma 2.4.22 there exists a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$ such that $C_{\mu_n} w \rightarrow w$ in $C(\mathbb{R}, \mathbb{R}^q)$ -sense (i.e. μ_n is an approximate identity). Because $C_{\mu_n} w \in C^\infty(\mathbb{R}, \mathbb{R}^q)$ (corollary 2.4.18), we thus find a sequence in $\mathcal{B}^\infty$ that converges to w . Since $w \in \mathcal{B}$ was arbitrary, we are done. $\square$

In our work, convolution operators are the link between the spaces $C(\mathbb{R})$ and $C^\infty(\mathbb{R})$. The relation between these two possible signal-spaces can be made also concerning some fundamental concepts in system theory, again using convolution operators. One of these concepts is ‘locally specified’, being one of the necessary and sufficient conditions c1-c4 on behaviors in $C^\infty(\mathbb{R}, \mathbb{R}^q)$, derived in chapter 3. Hence the transition of this property from $\mathcal{B}$ to $\mathcal{B}^\infty$ and vice versa is of interest to us. First, we need an auxiliary result.

Lemma 4.2.2 *Let B be a linear, time-invariant behavior that is closed in $C(\mathbb{R}, \mathbb{R}^q)$ and let $a, b \in \mathbb{R}$. Then for $\varepsilon > 0$, and $0 \leq \delta < \frac{\varepsilon}{2}$ and $\mu \in ba_c(\mathbb{R})$ with $\text{supp}(\mu) \subseteq [-\delta, \delta]$ we have*

$$w|_{[a-\varepsilon, b+\varepsilon]} \in B|_{[a-\varepsilon, b+\varepsilon]} \Rightarrow (C_\mu w)|_{[a-\delta, b+\delta]} \in B|_{[a-\delta, b+\delta]}$$

Proof: Let $\varepsilon > 0$, and $0 \leq \delta < \frac{\varepsilon}{2}$ and $w|_{[a-\varepsilon, b+\varepsilon]} \in B|_{[a-\varepsilon, b+\varepsilon]}$. Then there exists a $g \in B$ such that $w|_{[a-\varepsilon, b+\varepsilon]} = g|_{[a-\varepsilon, b+\varepsilon]}$. Take a $\mu \in ba_c(\mathbb{R})$ with $\text{supp}(\mu) \subseteq [-\delta, \delta]$. Then (componentwise, $i = 1, \dots, q$)

$$(C_\mu w_i)(t) = \int_{\mathbb{R}} w_i(\tau + t) d\mu(\tau), \quad t \in [a - \delta, b + \delta]$$

Since $\text{supp}(\mu) \subseteq [-\delta, \delta]$ we have $a - \varepsilon \leq \tau + t \leq b + \varepsilon$ and thus on this interval $w(\tau + t) = g(\tau + t)$. Hence

$$C_\mu w|_{[a-\delta, b+\delta]} = C_\mu g|_{[a-\delta, b+\delta]} \subseteq B|_{[a-\delta, b+\delta]}$$

□

Lemma 4.2.3 *Let B be a linear, time-invariant behavior that is closed in $C(\mathbb{R}, \mathbb{R}^q)$ then the following equivalence holds. Let $L \in \mathbb{R}$, $L \geq 0$*

$$\forall \varepsilon > 0 [B^\infty \text{ is } (L + \varepsilon)\text{-complete}] \Leftrightarrow \forall \varepsilon > 0 [B \text{ is } (L + \varepsilon)\text{-complete}]$$

Proof: ($\Rightarrow$): Let B^∞ be $(L + \varepsilon)$ -complete for all $\varepsilon > 0$. Take an arbitrary $\varepsilon > 0$ and suppose we have for all $t \in \mathbb{R}$:

$$w|_{[t-\frac{\varepsilon}{2}, t+L+\frac{\varepsilon}{2}]} \in B|_{[t-\frac{\varepsilon}{2}, t+L+\frac{\varepsilon}{2}]}$$

Choose $\delta = \frac{\varepsilon}{4}$, $\mu \in [ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})]$ with $\text{supp}(\mu) \subseteq [-\delta, \delta]$, then using lemma 4.2.2 we know that, for all $t \in \mathbb{R}$

$$(C_\mu w)|_{[t-\delta, t+L+\delta]} \in B|_{[t-\delta, t+L+\delta]} \quad (4.2)$$

Clearly, $C_\mu w \in C^\infty(\mathbb{R}, \mathbb{R}^q)$ and because B^∞ is $(L + \frac{\varepsilon}{2})$ -complete we obtain $C_\mu w \in B^\infty$. This can be done for all $\mu_\alpha \in [ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})]$, where $0 < \alpha \leq \delta$, with $\text{supp}(\mu) = [-\alpha, \alpha]$. Hence we can use the approximate identity $\alpha \rightarrow 0$ to achieve $C_{\mu_\alpha} w \rightarrow w$. Since B is closed it follows that $w \in B$, yielding that B is $(L + \varepsilon)$ -complete.

($\Leftarrow$): Obvious. □

Remark 4.2.4 Of course, it would be nicer to have the equivalence

$$B^\infty \text{ is L-complete} \Leftrightarrow B \text{ is L-complete}$$

that we can not prove. However, looking at the proof of lemma 4.2.2 we see that we need the approximate identity to obtain a sequence in B^∞ . But $\mu \in [ba_c(\mathbb{R}) \cap C^\infty(\mathbb{R})]$ implies that the support of μ is an interval and not a point. Consequently, in equation (4.2) we need information about w on an interval with length larger than L . This leads to the following implication

$$B \text{ is L-complete} \Rightarrow B^\infty \text{ is L-complete}$$

and we argued why the converse might not be true. □

Corollary 4.2.5 *Let B be a linear, time-invariant behavior that is closed in $C(\mathbb{R}, \mathbb{R}^q)$. Then the following equivalence holds*

$$B^\infty \text{ is locally specified} \Leftrightarrow B \text{ is locally specified}$$

Proof: Take $L = 0$ in lemma 4.2.3. □

More concepts for linear, shift-invariant and closed behaviors can be linked in the way we showed here, using lemma 4.2.2.

Consider a behavior B in $C(\mathbb{R}, \mathbb{R}^q)$ and suppose its restriction to $C^\infty(\mathbb{R}, \mathbb{R}^q)$ can be described by an AR-relation

$$w \in B \Leftrightarrow R\left(\frac{d}{dt}\right)w = 0$$

where $R\left(\frac{d}{dt}\right) \in \text{LDO}(m, q, \mathbb{R})$, i.e. $R\left(\frac{d}{dt}\right) = A_0 + A_1 \frac{d}{dt} + \dots + A_n \left(\frac{d}{dt}\right)^n$, with $A_0, \dots, A_n$ are $p \times q$ matrices. Corresponding to R , we have a polynomial matrix $R(s) = A_0 + A_1 s + \dots + A_n s^n$, for $s \in \mathbb{C}$.

For a classical interpretation going beyond B^∞ , we need $w \in B$ to be sufficiently differentiable and for all $t \in \mathbb{R}$: $R\left(\frac{d}{dt}\right)w = 0$. The example in the introduction showed that a lot of behaviors are excluded if we use this *classical* interpretation. Therefore we introduce a *weak* interpretation.

For $f \in C(\mathbb{R}, \mathbb{R}^q)$ and $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q)$, we introduce the bilinear form

$$\langle f, \varphi \rangle = \int_{\mathbb{R}} (f(t), \varphi(t)) dt$$

where $(\cdot, \cdot)$ denotes the Euclidean inner product in $\mathbb{R}^q$.

Lemma 4.2.6 *Let $f \in C(\mathbb{R}, \mathbb{R}^q)$. If we have $\langle f, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q)$ then $f \equiv 0$.*

Proof: See Friedlander, theorem 1.3.1, p.7 ([6]). □

Now let w be sufficiently smooth and satisfy $R(\frac{d}{dt})w = 0$ classically. Then

$$\forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^p) : \langle R(\frac{d}{dt})w, \varphi \rangle = 0. \quad (4.3)$$

Applying integration by parts, this is found to be

$$\forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^p) : \langle w, \check{R}^*(\frac{d}{dt})\varphi \rangle = 0, \quad (4.4)$$

where the polynomial matrices R^* and $\check{R}$ are defined by

$$\begin{aligned} R^*(s) &:= R(-s)^t \quad (s \in \mathbb{C}) \\ \check{R}(s) &:= R(-s) \quad (s \in \mathbb{C}) \end{aligned}$$

This allows us to interpret an AR-relation in case w is not so smooth that the equation can be read in a classical sense. If $w \in C(\mathbb{R}, \mathbb{R}^q)$ satisfies (4.4), then we call w a *weak solution* to the AR-relation.

Shift operators play a crucial role in the theory. We mention the following obvious relation

Definition 4.2.7 For all $f \in [C(\mathbb{R})^q]$ and $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q)$

$$\langle \sigma^t f, \varphi \rangle = \langle f, \sigma^{-t} \varphi \rangle$$

Lemma 4.2.8 *The behavior described by equation (4.4) is linear, shift invariant and closed in $C(\mathbb{R}, \mathbb{R}^q)$.*

Proof: Linearity is obvious. Since translation commutes with differentiation, shift invariance is also clear. Let $(w_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{B}$ converging to w in $C(\mathbb{R}, \mathbb{R}^q)$ -sense. This implies

$$\forall \psi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q) [\langle w_n - w, \psi \rangle \rightarrow 0 \text{ (} n \rightarrow \infty \text{)}]$$

We have

$$\langle w_n - w, \psi \rangle \leq \sup_{t \in \text{supp}(\psi)} |w_n(t) - w(t)| \cdot \int_{\mathbb{R}} \|\psi\| dt$$

Hence for all $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^p)$

$$0 = \lim_{n \rightarrow \infty} \langle w_n, \check{R}^*\left(\frac{d}{dt}\right)\varphi \rangle = \langle w, \check{R}^*\left(\frac{d}{dt}\right)\varphi \rangle$$

and hence $w \in \mathcal{B}$. □

4.3 Necessary conditions for behaviors in $C(\mathbb{R}, \mathbb{R}^q)$

In this section we derive necessary conditions for a behavior $\mathcal{B} \subseteq C(\mathbb{R}, \mathbb{R}^q)$ to satisfy an AR-relation in a weak sense. Therefore assume a behavior $\mathcal{B}$ in $C(\mathbb{R}, \mathbb{R}^q)$ is described by such a relation

$$\mathcal{B} = \{w \in C(\mathbb{R}, \mathbb{R}^q) \mid \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^p) [\langle w, \check{R}^*\left(\frac{d}{dt}\right)\varphi \rangle = 0]\}, \quad (4.5)$$

where $R \in \mathbb{R}^{p \times q}[s]$ is a $p \times q$ polynomial matrix and consequently $\check{R}^*(s)$ a $q \times p$ polynomial matrix. Then

$$\mathcal{B}^\infty = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid R\left(\frac{d}{dt}\right)w = 0\} \quad (4.6)$$

But also vice versa, if $\mathcal{B}^\infty$ is described by equation (4.6), then, because $\mathcal{B}^\infty$ is dense in $\mathcal{B}$ and the set described in (4.5) is closed in $C(\mathbb{R}, \mathbb{R}^q)$ we obtain

$$\mathcal{B} = \overline{\mathcal{B}^\infty} = \{w \in C(\mathbb{R}, \mathbb{R}^q) \mid \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^p) [\langle w, \check{R}^*\left(\frac{d}{dt}\right)\varphi \rangle = 0]\}$$

To derive necessary conditions for $\mathcal{B}$ to satisfy such a weak AR-relation we shall extend the conditions c1-c4 on behaviors in $C^\infty(\mathbb{R}, \mathbb{R}^q)$ to conditions C1-C4 for behaviors in $C(\mathbb{R}, \mathbb{R}^q)$. Let us recall the conditions c1-c4 (see

section 3.2)

- c1 B^∞ is a linear shift-invariant subspace, that is closed with respect to the topology of $C^\infty(\mathbb{R}, \mathbb{R}^q)$.
- c2 B^∞ is locally specified.

There exists a permutation Π and $p \in \mathbb{N}$ such that

- c3 I is a standard $[p, q - p]$ -behavior.
- c4 For all $j \in \{p + 1, \dots, q\}$ and $i \in \{1, \dots, p\}$: $\mathcal{E}_i[(\Pi B^\infty)_j]$ or $N_i[(\Pi B^\infty)_j]$ is finite dimensional.

The conditions c1 and c2 do not give rise to problems and can be translated into conditions C1, C2, replacing B^∞ by B and $C^\infty(\mathbb{R}, \mathbb{R}^q)$ by $C(\mathbb{R}, \mathbb{R}^q)$. And because, a linear, shift-invariant, finite-dimensional subspace belongs to $C^\infty(\mathbb{R}, \mathbb{R}^q)$, c4 can be extended easily, too. The hard part is in c3. From the representation in (4.5) it is not clear whether or not each $(q - p)$ -tuple $(w_{p+1}, \dots, w_q)$ can be chosen in $C(\mathbb{R}, \mathbb{R}^{q-p})$ arbitrarily, as in the case B^∞ in $C^\infty(\mathbb{R}, \mathbb{R}^q)$. Nevertheless, if we leave c3 as it is, then we obtain exactly the result we are looking for. In fact, the conditions C1, ..., C4 imply c1, ..., c4, yielding that $B^\infty = B \cap C^\infty(\mathbb{R}, \mathbb{R}^q)$ is represented by

$$B^\infty = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid R\left(\frac{d}{dt}\right)w = 0\}$$

Applying lemma 4.2.8, we obtain that (4.5) is the desired representation for a behavior $B \subseteq C(\mathbb{R}, \mathbb{R}^q)$.

So, we have found the following necessary conditions for behaviors B in $C(\mathbb{R}, \mathbb{R}^q)$:

- C1 B is a linear shift-invariant subspace, that is closed with respect to the topology of $C(\mathbb{R}, \mathbb{R}^q)$.
- C2 B is locally specified.

There exists a permutation Π and $p \in \mathbb{N}$ such that

C3 I is a standard $[p, q - p]$ -behavior.

C4 For all $j \in \{p+1, \dots, q\}$ and $i \in \{1, \dots, p\}$: $\mathcal{E}_i[(\Pi\mathcal{B})_j]$ or $N_i[(\Pi\mathcal{B})_j]$ is finite dimensional.

All conditions but one are expressed in terms of the original behavior. It is a question of taste whether or not this particular condition C3 leads to a lack of elegance. In order to 'improve' our result, by expressing C3 in terms of $\mathcal{B}$ some effort is needed.

We show that the weak AR-representation of a behavior $\mathcal{B}$ in $C(\mathbb{R}, \mathbb{R}^q)$ can be replaced by an integral representation, in which case everything can be interpreted classically. The use of the integral representation is twofold. On the one hand it enables us to find a condition C3, directly related to the original behavior $\mathcal{B}$. On the other hand, we can prove the necessity of C1-C4 a lot easier than what we did in section 3.2 for behaviors $\mathcal{B}^\infty$. This will be clear after we have given the explicit integral description for $\mathcal{B}$.

Definition 4.3.1 The real rational functions in the indeterminate s are denoted by $\mathbb{R}(s)$. I.e. the fraction field of $\mathbb{R}[s]$.

The extension to matrices is denoted by $\mathbb{R}^{m \times n}(s)$ for some $m, n \in \mathbb{N}$.

Definition 4.3.2 If $f(s) \in \mathbb{R}(s)$, $f(s) = \frac{q(s)}{p(s)}$ with $p(s), q(s) \in \mathbb{R}[s]$ and $\text{degree}(p) \geq \text{degree}(q)$, then we call f *proper*.

An element $\mathbb{R}^{m \times n}(\mathbb{R})$ is called *proper* if all its entries are proper.

Suppose we have an AR-relation for a behavior $\mathcal{B}$ in $C(\mathbb{R}, \mathbb{R}^q)$, governed by the $p \times q$ representation-matrix $R(\frac{d}{dt})$, $R \in \mathbb{R}^{p \times q}[s]$, of full row rank. Then there exists a permutation matrix Π^{-1} such that

$$R\Pi^{-1} = [\tilde{R}_1 | -\tilde{R}_2]$$

where $\tilde{R}_1 \in \mathbb{R}^{p \times p}[s]$ and $\tilde{R}_2 \in \mathbb{R}^{p \times (q-p)}[s]$ with $\tilde{R}_1$ invertible as a rational matrix and $\tilde{R}_1^{-1}\tilde{R}_2$ proper (see [19], theorem 4.1, p.243). Moreover, we can find a unimodular matrix U and permutation matrix $\tilde{\Pi}^{-1}$ such that

$U\tilde{R}_1\tilde{\Pi}^{-1} = R_1$, where R_1 is upper triangular. Hence, up to a permutation, the behavior $\mathcal{B}$ is represented by

$$U[\tilde{R}_1 | -\tilde{R}_2] \left(\begin{array}{c|c} \tilde{\Pi}^{-1} & 0 \\ \hline 0 & I_{q-p} \end{array} \right) = [U\tilde{R}_1\tilde{\Pi}^{-1} | -U\tilde{R}_2] = [R_1 | -R_2]$$

Partitioning $w = (w_1 | w_2)$, $w_1 \in C(\mathbb{R}, \mathbb{R}^p)$, $w_2 \in C(\mathbb{R}, \mathbb{R}^{q-p})$ we prove that for all $w_2 \in C(\mathbb{R}, \mathbb{R}^{q-p})$, there exists $w_1 \in C(\mathbb{R}, \mathbb{R}^p)$ such that $(w_1, w_2) \in \mathcal{B}$, something that cannot be seen immediately from the equation

$$\forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q) [\langle w_1, \tilde{R}_1^* \varphi \rangle = \langle w_2, \tilde{R}_2^* \varphi \rangle]$$

In fact, we prove that there exists an integral operator $\mathcal{K}$ such that

$$\begin{aligned} \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q) [\langle w_1, \tilde{R}_1^* \varphi \rangle &= \langle w_2, \tilde{R}_2^* \varphi \rangle] \\ &\Leftrightarrow \\ w_1 - \mathcal{K}w_2 &\in \ker(P) \end{aligned} \quad (4.7)$$

We consider first the one dimensional situation, first. I.e. $p = 1, q = 2$. Note that we are dealing with functions defined on $\mathbb{R}$ and hence we cannot apply Laplace transformation 'just like that' to solve the differential equation.

Let p be a polynomial. Then associated to p , the polynomial $\check{p}$ is defined by $\check{p}(t) = p(-t)$, $t \in \mathbb{R}$. The subspace V of $C(\mathbb{R})$ is defined by

$$V = \{v \in C(\mathbb{R}) \mid \forall \varphi \in \mathcal{D}(\mathbb{R}) : \int_{\mathbb{R}} v(\check{p}(\frac{d}{dt})\varphi) dt = 0\} \quad (4.8)$$

From lemma 4.2.8 we know that V is a closed, translation-invariant subspace of $C(\mathbb{R})$. A consequence of earlier results says that $V \cap C^\infty(\mathbb{R})$ is dense in V . Applying integration by parts yields

$$\forall v \in V \cap C^\infty(\mathbb{R}) : 0 = \int_{\mathbb{R}} v(\check{p}(\frac{d}{dt})\varphi) dt = \int_{\mathbb{R}} (p(\frac{d}{dt})v)\varphi dt \quad (4.9)$$

Using lemma 4.2.6, the equation (4.9) gives

$$p(\frac{d}{dt})v = 0 \Leftrightarrow v \in V \cap C^\infty(\mathbb{R})$$

Thus we get

$$V = V \cap C^\infty(\mathbb{R}) = \ker p(\frac{d}{dt})$$

since $V \cap C^\infty(\mathbb{R})$ is finite dimensional. Hence we have proved

Lemma 4.3.3 *The conditions $\forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q) \langle f, \check{p}(\frac{d}{dt})\varphi \rangle = 0$ and $f \in \ker p(\frac{d}{dt})$ are equivalent for $f \in C(\mathbb{R}, \mathbb{R}^q)$.*

We intend to connect the notion of *properness* to the notion of integral operator of (finite) convolution type. Therefore we introduce the *two-sided Laplace transform* formally by

$$(\mathcal{L}f)(s) = \int_{-\infty}^{\infty} f(t)e^{-st} dt$$

For $\varphi \in \mathcal{D}(\mathbb{R})$ the integral defining $\mathcal{L}\varphi$ converges for all $s \in \mathbb{C}$ and $\mathcal{L}\varphi$ is an entire function. Moreover there is the following relation with the Fourier transformation

$$(\mathcal{F}\varphi)(z) = (\mathcal{L}\varphi)(-iz), \quad z \in \mathbb{C}$$

Lemma 4.3.4 *Let $\varphi \in \mathcal{D}(\mathbb{R})$ and let ψ be a continuous function on $\mathbb{R}$ such that $e^{-bt}\psi \in L_1(\mathbb{R})$ for some $b \in \mathbb{R}$. Suppose*

$$(\mathcal{L}\varphi)(s) = \int_{-\infty}^{\infty} \psi(t)e^{-st} dt$$

for all $s \in \mathbb{C}$ with $\operatorname{Re}(s)=b$. Then $\varphi = \psi$.

Proof: Observe that $e^{-bt}\psi$ is Fourier transformable. The assumption in the statement of the lemma assures that

$$\mathcal{F}(e^{-bt}\varphi)(x) = \mathcal{F}(e^{-bt}\psi)(x), \quad x \in \mathbb{R}$$

Since $\mathcal{F}(e^{-bt}\varphi) \in L_2(\mathbb{R})$ it follows that $\mathcal{F}(e^{-bt}\psi) \in L_2(\mathbb{R})$ whence ($\mathcal{F}$ is a unitary operator from $L_2(\mathbb{R})$ onto $L_2(\mathbb{R})$) $e^{-bt}\varphi = e^{-bt}\psi$, a.e. Continuity of ψ yields that $\varphi = \psi$. $\square$

Let p and q be polynomials, such that q/p is proper. Then integration by parts yields

$$\mathcal{L}(\check{q}(\frac{d}{dt})\varphi)(s) = \frac{\check{q}(s)}{\check{p}(s)} \mathcal{L}(\check{p}(\frac{d}{dt})\varphi)(s), \quad \varphi \in \mathcal{D}(\mathbb{R}), s \in \mathbb{C} \quad (4.10)$$

In first instance, we suppose that $\operatorname{degree}(p) > \operatorname{degree}(q)$. Then we have

$$\frac{q(s)}{p(s)} = \sum_{j=0}^r \sum_{i=0}^{m_j} \frac{a_i^{(j)} \cdot (i!)}{(s - \lambda_j)^{i+1}}$$

and

$$\frac{\check{q}(s)}{\check{p}(s)} = \frac{q(-s)}{p(-s)} = \sum_{j=0}^r \sum_{i=0}^{m_j} \frac{(-1)^{i+1} a_i^{(j)} \cdot (i!)}{(s + \lambda_j)^{i+1}}$$

Introduce the Bohl-function k on $\mathbb{R}$ by

$$k(t) = \sum_{j=0}^r p_j(t) e^{\lambda_j t}, \quad p_j(t) = \sum_{i=0}^{m_j} a_i^{(j)} t^i \quad t \in \mathbb{R}$$

where the coefficients are such that $k(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$. Then there exists $a > 0$ (take a larger than $\max |\lambda_j|$) such that

$$\frac{q(s)}{p(s)} = \int_0^\infty e^{-st} k(t) dt, \quad \operatorname{Re}(s) > a$$

and

$$\frac{\check{q}(s)}{\check{p}(s)} = - \int_0^\infty e^{-st} k(-t) dt, \quad \operatorname{Re}(s) > a \quad (4.11)$$

For convenience we set $h(t) = -k(-t)$, $t \in \mathbb{R}$. Now define the functions h_+ and h_- on $\mathbb{R}$ by

$$h_+(t) = \begin{cases} h(t) & t > 0 \\ 0 & t \leq 0 \end{cases}$$

and

$$h_-(t) = \begin{cases} 0 & t \geq 0 \\ -h(t) & t < 0 \end{cases}$$

Then

$$(\mathcal{L}h_+)(s) = \frac{\check{q}(s)}{\check{p}(s)}, \quad \operatorname{Re}(s) > a \quad (4.12)$$

and

$$(\mathcal{L}h_-)(s) = \frac{\check{q}(s)}{\check{p}(s)}, \quad \operatorname{Re}(s) < -a \quad (4.13)$$

We come to the following result (using the equations 4.10, 4.12, 4.13)

$$\mathcal{L}\left(\check{q}\left(\frac{d}{dt}\right)\varphi\right)(s) = \begin{cases} (\mathcal{L}h_+)(s) \mathcal{L}\left(\check{p}\left(\frac{d}{dt}\right)\varphi\right)(s) & \operatorname{Re}(s) > a \\ (\mathcal{L}h_-)(s) \mathcal{L}\left(\check{p}\left(\frac{d}{dt}\right)\varphi\right)(s) & \operatorname{Re}(s) < -a \end{cases}$$

For all $t \in \mathbb{R}$, the integral

$$h_+ * \check{p}\left(\frac{d}{dt}\right)\varphi := \int_{-\infty}^{\infty} h_+(t-\tau)(\check{p}\left(\frac{d}{dt}\right)\varphi)(\tau)d\tau$$

exists and $h_+ * \check{p}\left(\frac{d}{dt}\right)\varphi$ is a continuous function (even a $C^\infty(\mathbb{R})$ -function, namely substitute $s = t - \tau$ and use the fact that φ has compact support). Moreover for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > a$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |e^{-st} h_+(t-\tau)(\check{p}\left(\frac{d}{dt}\right)\varphi)(\tau)| d\tau dt < \infty \quad (4.14)$$

From this it can be checked that

$$(\mathcal{L}h_+)(s)\mathcal{L}(\check{p}\left(\frac{d}{dt}\right)\varphi)(s) = \mathcal{L}(h_+ * \check{p}\left(\frac{d}{dt}\right)\varphi)(s), \quad \operatorname{Re}(s) > a$$

Similarly,

$$(\mathcal{L}h_-)(s)\mathcal{L}(\check{p}\left(\frac{d}{dt}\right)\varphi)(s) = \mathcal{L}(h_- * \check{p}\left(\frac{d}{dt}\right)\varphi)(s), \quad \operatorname{Re}(s) < -a$$

Thus we obtain,

$$\mathcal{L}(\check{q}\left(\frac{d}{dt}\right)\varphi)(s) = \mathcal{L}(h_+ * \check{p}\left(\frac{d}{dt}\right)\varphi)(s), \quad \operatorname{Re}(s) > a$$

and

$$\mathcal{L}(\check{q}\left(\frac{d}{dt}\right)\varphi)(s) = \mathcal{L}(h_- * \check{p}\left(\frac{d}{dt}\right)\varphi)(s), \quad \operatorname{Re}(s) < -a$$

Using (4.14) we can apply lemma 4.3.4, and obtain

$$\begin{aligned} (\check{q}\left(\frac{d}{dt}\right)\varphi)(t) &= \int_{-\infty}^{\infty} h_+(t-\tau)(\check{p}\left(\frac{d}{dt}\right)\varphi)(\tau)d\tau \\ &= \int_{-\infty}^t h(t-\tau)(\check{p}\left(\frac{d}{dt}\right)\varphi)(\tau)d\tau, \quad t \in \mathbb{R} \end{aligned}$$

and

$$\begin{aligned} (\check{q}\left(\frac{d}{dt}\right)\varphi)(t) &= \int_{-\infty}^{\infty} h_-(t-\tau)(\check{p}\left(\frac{d}{dt}\right)\varphi)(\tau)d\tau \\ &= - \int_t^{\infty} h(t-\tau)(\check{p}\left(\frac{d}{dt}\right)\varphi)(\tau)d\tau, \quad t \in \mathbb{R} \end{aligned}$$

We want to explore these relations in the following expression

$$\int_{-\infty}^{\infty} f(t) (\tilde{q}(\frac{d}{dt})\varphi)(t) dt$$

where $f \in C(\mathbb{R})$ and $\varphi \in \mathcal{D}(\mathbb{R})$. Inserting both relations as derived, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} f(t) (\tilde{q}(\frac{d}{dt})\varphi)(t) dt &= \int_{-\infty}^0 f(t) (\int_{-\infty}^t h(t-\tau) (\tilde{p}(\frac{d}{dt})\varphi)(\tau) d\tau) dt + \\ &\quad - \int_0^{\infty} f(t) (\int_t^{\infty} h(t-\tau) (\tilde{p}(\frac{d}{dt})\varphi)(\tau) d\tau) dt \end{aligned}$$

Having checked that the order of integration can be changed in both expressions we obtain

$$\int_{-\infty}^{\infty} f(t) (\tilde{q}(\frac{d}{dt})\varphi)(t) dt = \int_{-\infty}^{\infty} (\int_0^t k(t-\tau) f(\tau) d\tau) (\tilde{p}(\frac{d}{dt})\varphi)(t) dt$$

Now the convolution integral operator $\mathcal{K}$ on $C(\mathbb{R})$ is defined by

$$(\mathcal{K}f)(t) = \int_0^t k(t-\tau) f(\tau) d\tau, \quad t \in \mathbb{R} \quad (4.15)$$

Then we see that

$$\int_{-\infty}^{\infty} f(t) (\tilde{q}(\frac{d}{dt})\varphi)(t) dt = \int_{-\infty}^{\infty} (\mathcal{K}f)(t) (\tilde{p}(\frac{d}{dt})\varphi)(t) dt \quad (4.16)$$

In case $\text{degree}(p) \geq \text{degree}(q)$, with $\lambda = \lim_{s \rightarrow \infty} \frac{q(s)}{p(s)}$

$$\frac{q(s)}{p(s)} = \lambda + \frac{q_1(s)}{p(s)} \quad \text{where } \text{degree}(p) > \text{degree}(q_1)$$

and

$$\mathcal{L}(\tilde{q}(\frac{d}{dt})\varphi) - \lambda \tilde{p}(\frac{d}{dt})\varphi = \frac{\tilde{q}_1(s)}{\tilde{p}(s)} \mathcal{L}(\tilde{p}(\frac{d}{dt})\varphi)$$

which gives

$$\int_{-\infty}^{\infty} f(t) (\tilde{q}(\frac{d}{dt})\varphi)(t) dt = \int_{-\infty}^{\infty} ((\lambda I + \mathcal{K}_1)f)(t) (\tilde{p}(\frac{d}{dt})\varphi)(t) dt$$

where the integral operator $\mathcal{K}_1$ is defined by

$$(\mathcal{K}_1 f)(t) = \int_0^t k_1(t-\tau) f(\tau) d\tau$$

with (similar as before)

$$\int_0^\infty k_1(t)e^{-st}dt = \frac{q_1(s)}{p(s)}, \operatorname{Re}(s) > a_1$$

Now consider the following AR-relation

$$\forall_{\varphi \in \mathcal{D}(\mathbb{R})} \left[\int_{\mathbb{R}} \left[w_1 \left(\check{p} \left(\frac{d}{dt} \right) \varphi \right) - w_2 \left(\check{q} \left(\frac{d}{dt} \right) \varphi \right) \right] = 0 \right]$$

Then we know this is equivalent to

$$\forall_{\varphi \in \mathcal{D}(\mathbb{R})} \left[\int_{\mathbb{R}} (w_1 - \mathcal{K}w_2) \left(\check{p} \left(\frac{d}{dt} \right) \varphi \right) = 0 \right]$$

and we conclude that

$$w_1 - \mathcal{K}w_2 \in \ker p \left(\frac{d}{dt} \right)$$

where $\mathcal{K} = \lambda I + \mathcal{K}_1$ as indicated.

Remark 4.3.5 Knowledge of the AR-relation

$$p \left(\frac{d}{dt} \right) w_1 = q \left(\frac{d}{dt} \right) w_2$$

restricted to $C^\infty(\mathbb{R})$ -variables w_1 and w_2 yields directly its interpretation for $C(\mathbb{R})$ -variables. Namely, we have to find the Bohl function k_1 on $\mathbb{R}$ such that

$$\int_0^\infty k_1(t)e^{-st}dt = \frac{q_1(s)}{p(s)}$$

where $q(s) = \lambda p(s) + q_1(s)$. □

We summarize everything in the following main theorem:

Theorem 4.3.6 For polynomials p and q with $\operatorname{degree}(p) \geq \operatorname{degree}(q)$ define

$$\mathcal{B} = \{ (w_1, w_2) \in C(\mathbb{R}, \mathbb{R}^2) \mid \int_{\mathbb{R}} (w_1 \check{p} \left(\frac{d}{dt} \right) \varphi - w_2 \check{q} \left(\frac{d}{dt} \right) \varphi) = 0 \}$$

Then $(w_1, w_2) \in \mathcal{B}$ if and only if

$$w_1 - \mathcal{K}w_2 \in \ker p \left(\frac{d}{dt} \right)$$

where

$$(\mathcal{K}u)(t) = \int_0^t k_1(t-\tau)u(\tau)d\tau + (\lambda u)(t)$$

with

$$\lambda = \lim_{s \rightarrow \infty} \frac{q(s)}{p(s)}, \quad q(s) = \lambda p(s) + q_1(s)$$

and k_1 the Bohl function satisfying

$$\int_0^\infty k_1(t)e^{-st}dt = \frac{q_1(s)}{p(s)}$$

Note that

$$B^\infty = \{(w_1, w_2) \in C^\infty(\mathbb{R}, \mathbb{R}^2) \mid (\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2\}$$

Remark 4.3.7 For behaviors in $C(\mathbb{R}, \mathbb{R}^2)$ we can always find a representation of the form

$$w_1 - \mathcal{K}w_2 \in \ker p(\frac{d}{dt})$$

for some integral operator $\mathcal{K}$. This description immediately shows that w_2 can be chosen to be the free variable. Only if $\text{degree}(p) = \text{degree}(q)$ the roles can be exchanged, because the roles of p and q can be exchanged. What we see from the description for behaviors in $C^\infty(\mathbb{R}, \mathbb{R}^2)$ is that both variables can be taken free in $C^\infty(\mathbb{R})$, since $p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2$ is solvable for any choice of either w_1 or w_2 . $\square$

The finite-dimensional case is a straightforward generalization of the one-dimensional case.

For R a polynomial matrix, $R(s) = A_0 + sA_1 + \dots + s^nA_n$, $s \in \mathbb{C}$, the differential matrix $R(\frac{d}{dt})$ is defined by

$$R(\frac{d}{dt}) = A_0 + A_1 \frac{d}{dt} + \dots + A_n (\frac{d}{dt})^n$$

Let R be a square $p \times p$ polynomial matrix such that

$$\det R(s) = 0 \tag{4.17}$$

has only a finite number of solutions. Then the corresponding differential matrix $R(\frac{d}{dt})$ has the property that the equation

$$R(\frac{d}{dt})u = 0, \quad u \in C^\infty(\mathbb{R}, \mathbb{R}^p)$$

admits a finite-dimensional solution space. There is a simple proof of this assertion, which runs as follows. Under the assumption, there exists a rational matrix function $L(s)$ such that

$$L(s)R(s) = I$$

for all $s \in \mathbb{C}$ with $\det R(s) \neq 0$. Let p be the least common multiple of the denominators in the entries of $L(s)$. Then $p(s)L(s) = K(s)$ for $s \in \mathbb{C}$ is a polynomial matrix and

$$K(s)R(s) = p(s)I$$

We conclude that

$$R(\frac{d}{dt})u = 0 \Rightarrow p(\frac{d}{dt})u = 0$$

Let $\ker p(\frac{d}{dt}) = \text{span}\{b_1, \dots, b_r\}$ where each of the b_j are Bohl functions. Then

$$\{u \in C^\infty(\mathbb{R}, \mathbb{R}^p) \mid R(\frac{d}{dt})u = 0\} \subseteq \text{span}\{\alpha b_j \mid j = 1, \dots, r, \alpha \in \mathbb{R}^p\}$$

It follows that $\ker R(\frac{d}{dt})$ is spanned by Bohl functions, i.e. functions of the form $p(t)e^{\lambda t}$ where $p: \mathbb{R} \rightarrow \mathbb{R}^n$ is a polynomial.

Remark 4.3.8 The function $s \mapsto \det R(s)$, $s \in \mathbb{C}$, is a polynomial of finite degree, so that $\det R(s) = 0$ has a finite number of solutions if and only if $\det(R(s))$ is not identically zero. $\square$

Let R be an $(p \times p)$ -polynomial matrix with $\det R(s)$ not identically vanishing on $\mathbb{C}$. Consider the subspace $\mathcal{B}$ of $C(\mathbb{R}, \mathbb{R}^p)$ defined by

$$\mathcal{B} = \{u \in C(\mathbb{R}, \mathbb{R}^p) \mid \forall_{\varphi \in \mathcal{D}(R, \mathbb{R}^p)} \langle u, \check{R}^*(\frac{d}{dt})\varphi \rangle = 0\}$$

Then $\mathcal{B}$ is a closed, shift-invariant subspace of $C(\mathbb{R}, \mathbb{R}^p)$. Using that $\mathcal{D}(\mathbb{R}, \mathbb{R}^p)$ is dense in $C^\infty(\mathbb{R}, \mathbb{R}^p)$ in the sense that

$$\forall_{w \in C^\infty(\mathbb{R}, \mathbb{R}^p)} [\forall_{\varphi \in \mathcal{D}(R, \mathbb{R}^p)} \langle w, \varphi \rangle = 0 \Rightarrow w = 0]$$

We obtain for $w \in C^\infty(\mathbb{R}, \mathbb{R}^p)$

$$\begin{aligned} w \in B \cap C(\mathbb{R}, \mathbb{R}^p) &\Leftrightarrow \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^p) \langle w, \check{R}^* \left(\frac{d}{dt} \right) \varphi \rangle = 0 \\ &\Leftrightarrow \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^p) \langle R \left(\frac{d}{dt} \right) w, \varphi \rangle = 0 \\ &\Leftrightarrow R \left(\frac{d}{dt} \right) w = 0 \end{aligned}$$

So $B \cap C(\mathbb{R}, \mathbb{R}^p)$ is finite dimensional and

$$B = \overline{B \cap C(\mathbb{R}, \mathbb{R}^p)} = B \cap C(\mathbb{R}, \mathbb{R}^p) = \ker R \left(\frac{d}{dt} \right)$$

As for functions from $\mathbb{R}$ to $\mathbb{R}$, the two-sided Laplace transform for functions from $\mathbb{R}$ to $\mathbb{R}^q$ is defined by

$$(\mathcal{L}f)(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt$$

and similarly for matrices.

For $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^n)$ the integral defining $(\mathcal{L}\varphi)(s)$ converges for all $s \in \mathbb{C}$ and $\mathcal{L}\varphi$ is an entire function from $\mathbb{C}$ to $\mathbb{C}^n$. The following result is straightforward from lemma 4.3.4.

Corollary 4.3.9 *Let $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^n)$ and let $\psi \in C(\mathbb{R}, \mathbb{R}^n)$ with for all $j=1, \dots, n$, $e^{-b_j t} \psi \in L_1(\mathbb{R}, \mathbb{R}^n)$ for certain $b_1, \dots, b_n \in \mathbb{R}$. Suppose $(\mathcal{L}\varphi)_j(s) = \int_{-\infty}^{\infty} \psi_j(t) e^{-st} dt$ for all $s \in \mathbb{C}$ with $\operatorname{Re}(s) = b_j$, $j=1, \dots, n$. Then $\varphi = \psi$.*

Now take $R \in \mathbb{R}^{p \times q}[s]$ ($0 \leq p \leq q$), then we know that (up to a permutation) R can be partitioned into $R = [R_1 | -R_2]$ and $(R_1)^{-1} R_2$ is a proper rational matrix. Then R_1 and R_2 be polynomial matrices with R_1 an $p \times p$ matrix and R_2 a $p \times (q - p)$ matrix. Of course, substituting $s = -s$ and taking the transpose does not influence properness and thus the rational matrix function

$$\check{R}_2^*(s)(\check{R}_1^*(s))^{-1} = (\check{R}_1^{-1}(s)\check{R}_2(s))^t, \quad s \in \mathbb{C}$$

is proper, too. This means that each of its entries is a rational function for which the degree of the denominator is not smaller than the degree of the numerator. Integration by parts yields for $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^p)$

$$\mathcal{L}(\check{R}_1^* \left(\frac{d}{dt} \right) \varphi) = \check{R}_1^*(s)(\mathcal{L}\varphi)(s), \quad s \in \mathbb{C}$$

and whenever $\det(R_1(s))$ is not identically vanishing

$$\mathcal{L}(\check{R}_2^*(\frac{d}{dt})\varphi)(s) = \check{R}_2^*(s)(\check{R}_1^*(s))^{-1}\mathcal{L}(\check{R}_1^*(\frac{d}{dt})\varphi)(s)$$

for all $s \in \mathbb{C}$ with $\det(\check{R}_1(s)) \neq 0$. It is not hard to see that

$$R_1^{-1}(s)R_2(s) = A_0 + \sum_{j=0}^r \sum_{i=0}^{m_j} \frac{A_i^{(j)} \cdot i!}{(s - \lambda_j)^{i+1}}$$

where A_0 and $A_i^{(j)}$, $i = 0, \dots, m_j$, $j = 0, \dots, r$ are $p \times (q - p)$ -matrices. We take $A_0 = 0$ for a moment (strictly proper). Moreover,

$$\check{R}_2^*(s)(\check{R}_1^*(s))^{-1} = - \sum_{j=0}^r \sum_{i=0}^{m_j} \frac{(-1)^i (A_i^{(j)})^t \cdot i!}{(s + \lambda_j)^{i+1}}$$

Introduce the Bohl matrix function

$$K(t) = \sum_{j=0}^r P_j(t) e^{\lambda_j t}$$

where $P_j(t) = \sum_{i=0}^{m_j} t^i A_i^{(j)}$.

Then for $a > 0$, sufficiently large

$$R_1^{-1}(s)R_2(s) = \int_0^\infty e^{-st} K(t) dt, \quad \operatorname{Re}(s) > a$$

and

$$\check{R}_2^*(s)(\check{R}_1^*(s))^{-1} = - \int_0^\infty e^{-st} \check{K}^*(t) dt, \quad \operatorname{Re}(s) > a$$

where $\check{K}^*(t) = (K(-t))^t$. Set $H(t) = -\check{K}^*(t)$, $t \in \mathbb{R}$, and define the matrix functions H_+ and H_- by

$$H_+(t) = \begin{cases} H(t) & t > 0 \\ 0 & t \leq 0 \end{cases}$$

and

$$H_-(t) = \begin{cases} 0 & t \geq 0 \\ -H(t) & t < 0 \end{cases}$$

For $\text{Re}(s) > a$ we obtain

$$(\mathcal{L}H_+)(s) = \check{R}_2^*(s)(\check{R}_1^*(s))^{-1}$$

and for $\text{Re}(s) < a$

$$(\mathcal{L}H_-)(s) = \check{R}_2^*(s)(\check{R}_1^*(s))^{-1}$$

where $(\mathcal{L}H_{\pm})(s) = \int_{-\infty}^{\infty} e^{-st} H_{\pm}(t) dt$.

We come to the result that

$$\mathcal{L}(\check{R}_2^*\left(\frac{d}{dt}\right)\varphi)(s) = \begin{cases} (\mathcal{L}H_+)(s)\mathcal{L}(\check{R}_1^*\left(\frac{d}{dt}\right)\varphi)(s) & \text{Re}(s) > a \\ (\mathcal{L}H_-)(s)\mathcal{L}(\check{R}_1^*\left(\frac{d}{dt}\right)\varphi)(s) & \text{Re}(s) < -a \end{cases}$$

and with a similar line of arguments as in the two-dimensional case, conclude that

$$(\check{R}_2^*\left(\frac{d}{dt}\right)\varphi)(t) = \int_{-\infty}^{\infty} H_{\pm}(t-\tau)(\check{R}_1^*\left(\frac{d}{dt}\right)\varphi)(\tau) d\tau$$

Now let $u \in C(\mathbb{R}, \mathbb{R}^p)$, then

$$\langle u, \check{R}_2^*\left(\frac{d}{dt}\right)\varphi \rangle = \langle \mathcal{K}u, \check{R}_1^*\left(\frac{d}{dt}\right)\varphi \rangle$$

If the matrix A_0 is nonzero then we obtain $\mathcal{K} = A_0 + \mathcal{K}_1$ where

$$(\mathcal{K}_1 u)(t) := \int_0^t K_1(t-\tau)u(\tau) d\tau$$

where K_1 satisfies

$$K_1(t) = \sum_j P_j(t)e^{\lambda_j t}, \quad P_j(t) = \sum_i A_i^{(j)} t^i$$

Now assume the matrix polynomials R_1 and R_2 satisfy the given conditions. Let $(w_1, w_2) \in \mathcal{B} \subseteq C(\mathbb{R}, \mathbb{R}^p) \times C(\mathbb{R}, \mathbb{R}^{q-p})$, satisfying

$$\forall_{\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^p)} : \langle w_1, \check{R}_1^*\left(\frac{d}{dt}\right)\varphi \rangle = \langle w_2, \check{R}_2^*\left(\frac{d}{dt}\right)\varphi \rangle \quad (4.18)$$

Then inserting

$$\langle w_2, \check{R}_2^*\left(\frac{d}{dt}\right)\varphi \rangle = \langle \mathcal{K}w_2, \check{R}_1^*\left(\frac{d}{dt}\right)\varphi \rangle$$

yields

$$\forall_{\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^p)} : \langle w_1 - \mathcal{K}w_2, \check{R}_1^* \left(\frac{d}{dt} \right) \varphi \rangle = 0$$

and so

$$w_1 - \mathcal{K}w_2 \in \ker(R_1 \left(\frac{d}{dt} \right))$$

Theorem 4.3.10 *Let a behavior $\mathcal{B} \subseteq C(\mathbb{R}, \mathbb{R}^p) \times C(\mathbb{R}, \mathbb{R}^{q-p})$, be described by*

$$\forall_{\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^p)} : \langle w_1, \check{R}_1^* \left(\frac{d}{dt} \right) \varphi \rangle = \langle w_2, \check{R}_2^* \left(\frac{d}{dt} \right) \varphi \rangle$$

where $(R_1)^{-1}R_2$ is proper, then there exists a permutation Π such that $\Pi\mathcal{B}$ is described by

$$(w_1, w_2) \in \Pi\mathcal{B} \Leftrightarrow w_1 - \mathcal{K}w_2 \in \ker(R_1 \left(\frac{d}{dt} \right))$$

where $\mathcal{K}$ is an integral operator as derived above, and vice versa.

Remark 4.3.11 Consequently, a behavior $\mathcal{B}$ in $C(\mathbb{R}, \mathbb{R}^q)$, as described by (4.5) is also characterized by

$$\mathcal{B} = \underbrace{\ker(R_1 \left(\frac{d}{dt} \right) | I)}_{\text{autonomous}} \oplus \underbrace{\ker(I - \mathcal{K})}_{\text{controllable}}$$

□

Because of the integral operator, an immediate consequence of theorem 4.3.10 is that $\Pi\mathcal{B}$ is a standard $[p, q-p]$ -behavior in $C(\mathbb{R}, \mathbb{R}^q)$.

We conclude this section with showing that the necessity of these conditions is rather easy in comparison with the proof in chapter 3.

The conditions c1, c2 are obviously necessary and c3 follows from the integral-representation also. Only the condition c4, that is the condition that lead to the most difficult proof in chapter 3, remains to be checked. Assume correct splitting with properness, i.e. $\Pi = I$.

Consider the $[p,1]$ -behaviors $\mathcal{B}_j$ ($j > p$) again, defined by

$$\mathcal{B}_j := \{w \in \mathcal{B} \mid w_l = 0, p+1 \leq l \leq q, l \neq j\}$$

The description we obtain with the integral representation reads

$$(w_1, \dots, w_p, 0, \dots, 0, w_j, 0, \dots, 0) \in \mathcal{B}_j$$

$$\Leftrightarrow$$

$$(w_1 - K_{j1}w_j, \dots, w_p - K_{jp}w_j) \in \ker R_1\left(\frac{d}{dt}\right)$$

Note that $R_1\left(\frac{d}{dt}\right)$ is invertible as a polynomial matrix and hence its kernel is finite dimensional. Let us take a look at the subspace $\mathcal{E}_i(\mathcal{B}_j)$ in C4. If it is infinite dimensional, then

$$\dim(\mathcal{E}_i(\mathcal{B}_j)) = \dim\{w_i \in C(\mathbb{R}) \mid \exists w_j \in C(\mathbb{R}) [w_i - K_{ji}w_j \in W_i]\} = \infty$$

for the finite-dimensional subspace $W_i = \mathcal{E}_i(\ker R_1\left(\frac{d}{dt}\right))$. This clearly implies $K_{ji} \neq 0$. The operator K_{ji} is of the form

$$K_{ji}w_j(t) = \int_0^t (k_{ji})(t-\tau)w_j(\tau)d\tau$$

and K_{ji} is a Bohl-function.

Note that $\dim(\mathcal{E}_i(\mathcal{B}_j)) < \infty$ implies $K_{ji} = 0$ because $\text{Range}(K_{ji})$ is dense on $C(\mathbb{R})$.

Lemma 4.3.12 *An operator $\mathcal{M} : C(\mathbb{R}) \rightarrow C(\mathbb{R})$ of the form*

$$(\mathcal{M}f)(t) = \int_0^t k(t-\tau)f(\tau)d\tau$$

with $k(t) \in C(\mathbb{R})$ is injective.

Proof: This is Titchmarsh-theorem (see [24], theorem 16, p.47). □

Hence, if $w_i = 0$ $K_{ji}w_j \in W_i$. Consequently $\mathcal{E}_j[N_i(\mathcal{B}_j)]$ is finite dimensional and for $l \in \{1, \dots, p\}$ $\mathcal{E}_p[N_i(\mathcal{B}_j)]$ is finite dimensional, too, implying $\dim(N_i(\mathcal{B}_j)) < \infty$, since

$$w_l = K_{jl}w_j + \mathcal{E}_l(\ker R_1\left(\frac{d}{dt}\right))$$

So, we have found the following necessary conditions for behaviors $\mathcal{B}$ in $C(\mathbb{R}, \mathbb{R}^q)$ to satisfy an AR-relation:

- C1 $\mathcal{B}$ is a linear, shift-invariant subspace, that is closed with respect to the topology of $C(\mathbb{R}, \mathbb{R}^q)$.
- C2 $\mathcal{B}$ is locally specified.

There exists a permutation Π and $p \in \mathbb{N}$ such that

- C3 $\Pi\mathcal{B}$ is a standard $[p, q - p]$ -behavior.
- C4 For all $j \in \{p+1, \dots, q\}$ and $i \in \{1, \dots, p\}$: $\mathcal{E}_i[(\Pi\mathcal{B})_j]$ or $N_i[(\Pi\mathcal{B})_j]$ is finite dimensional.

These conditions are the natural extensions of the conditions we found for behaviors in $C^\infty(\mathbb{R}, \mathbb{R}^q)$ in chapter 3.

4.4 Sufficient conditions for behaviors in $C(\mathbb{R}, \mathbb{R}^q)$

In this section we prove sufficiency of the conditions C1-C4 and answer the question as in the problem formulation in chapter 1.

Lemma 4.4.1 *If $\Pi\mathcal{B}$ is a standard $[p, q - p]$ -behavior, then $\mathcal{B}$ is a standard $[p, q - p]$ -behavior.*

Proof: Suppose we have

$$\{w_2 \in C(\mathbb{R}, \mathbb{R}^{q-p}) \mid \exists_{w_1 \in C(\mathbb{R}, \mathbb{R}^p)} [(w_1|w_2) \in \mathcal{B}]\} = C(\mathbb{R}, \mathbb{R}^p)$$

and let $w_2 \in C^\infty(\mathbb{R}, \mathbb{R}^{q-p})$. By assumption, there exists a $w_1 \in C(\mathbb{R}, \mathbb{R}^p)$ such that $(w_1|w_2) \in \mathcal{B}$. However, we are looking for a $w_1 \in C^\infty(\mathbb{R}, \mathbb{R}^p)$ with that property. For that purpose the following result can be applied:

$$C^\infty(\mathbb{R}, \mathbb{R}^q) = \text{span} \left(\bigcup_{\varphi \in D(\mathbb{R})} \sigma(\varphi)[C(\mathbb{R}, \mathbb{R}^q)] \right)$$

where, for all $f \in C(\mathbb{R}, \mathbb{R}^q)$

$$\sigma(\varphi)f = \int_{\mathbb{R}} \varphi(t) \sigma^t f dt = \int_{\mathbb{R}} \sigma^t f d\psi(t) = C_\psi f$$

with $\psi(t) = \int_{-\infty}^t \varphi(t) dt$, is a convolution operator (see appendix A.3). Hence

$$w_2 = \sum_{j=1}^n \sigma(\varphi_j) v_j, \text{ for some } v_j \in C(\mathbb{R}, \mathbb{R}^{q-p})$$

To every $v_j \in C(\mathbb{R}, \mathbb{R}^{q-p})$ there exists a $w_{1j} \in C(\mathbb{R}, \mathbb{R}^p)$ such that $(w_{1j}|v_j) \in \mathcal{B}$ and thus

$$\sum_{j=1}^n \sigma(\varphi_j)(w_{1j}|v_j) = \left(\sum_{j=1}^n \sigma(\varphi_j) w_{1j} \middle| \sum_{j=1}^n \sigma(\varphi_j) v_j \right) = \left(\sum_{j=1}^n \sigma(\varphi_j) w_{1j} \middle| w_2 \right) \in \mathcal{B}$$

Since $\varphi_j \in D(\mathbb{R})$, $\sum_{j=1}^n \sigma(\varphi_j) w_{1j} \in C^\infty(\mathbb{R}, \mathbb{R}^p)$. □

Lemma 4.4.2 *The conditions C1, C2, C3 and C4 are sufficient for a behavior $\mathcal{B}$ in $C(\mathbb{R}, \mathbb{R}^q)$ to satisfy an AR-relation (interpreted in a weak sense).*

Proof: First, we show that these conditions imply the conditions c1, c2, c3 and c4 on the restricted behavior, respectively. Condition c1 is trivially satisfied, since the topology on $C^\infty(\mathbb{R}, \mathbb{R}^q)$ is stronger than the one on $C(\mathbb{R}, \mathbb{R}^q)$. Condition c2 is a consequence of corollary 4.2.5. Condition c3 is proven in lemma 4.4.1 and condition c4 is a consequence of theorem 3.3.1. In fact, if $\mathcal{E}_i[(\Pi\mathcal{B})_j]$ is finite dimensional, then it is a linear, shift-invariant and closed subspace in $C(\mathbb{R})$ and consists of Bohl-functions only. This implies $\mathcal{E}_i[(\Pi\mathcal{B})_j] = \mathcal{E}_i[(\Pi\mathcal{B}^\infty)_j]$. Similarly, in case $N_i[(\Pi\mathcal{B})_j]$ is finite dimensional. Consequently, the restriction of $\mathcal{B}$ to $C^\infty(\mathbb{R}, \mathbb{R}^q)$ satisfies an AR-relation (see theorem 3.3.10), say

$$w \in \mathcal{B}^\infty \Leftrightarrow R\left(\frac{d}{dt}\right)w = 0$$

for some polynomial matrix R . If we interpret this in a weak sense (i.e. in $\mathcal{D}'(\mathbb{R}, \mathbb{R}^q)$) we obtain

$$w \in \mathcal{B}^\infty \Leftrightarrow \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q) \left[\langle w, R^*\left(\frac{d}{dt}\right)\varphi \rangle = 0 \right] \quad (4.19)$$

Define

$$\mathcal{B}_{\mathcal{D}'} := \{w \in C(\mathbb{R}, \mathbb{R}^q) \mid \text{equation 4.19 is satisfied}\}$$

Then (see lemma 4.2.8) $\mathcal{B}_{\mathcal{D}'}$ is closed in $C(\mathbb{R}, \mathbb{R}^q)$ and moreover, its restriction to $C^\infty(\mathbb{R}, \mathbb{R}^q)$ equals $\mathcal{B}^\infty$. Since $\mathcal{B}^\infty$ is dense in $\mathcal{B}$ we must have

$$\mathcal{B}_{\mathcal{D}'} = \mathcal{B}. \quad \square$$

So, at this moment we can answer the question as formulated in chapter 1 in a main theorem

Theorem 4.4.3 *A behavior $\mathcal{B}$ in $C(\mathbb{R}, \mathbb{R}^q)$ satisfies an AR-relation (in a weak sense) if and only if $\mathcal{B}$ satisfies*

C1 $\mathcal{B}$ is a linear, shift-invariant subspace, that is closed with respect to the topology of $C(\mathbb{R}, \mathbb{R}^q)$

C2 $\mathcal{B}$ is locally specified

There exists a permutation Π and $m \in \mathbb{N}$ such that

C3 $\Pi\mathcal{B}$ is a standard $[p, q - p]$ -behavior

C4 For all $j \in \{p + 1, \dots, q\}$ and $i \in \{1, \dots, p\}$: $\mathcal{E}_i[(\Pi\mathcal{B})_j]$ or $N_i[(\Pi\mathcal{B})_j]$ is finite dimensional.

where

$$\mathcal{E}_i[(\Pi\mathcal{B})_j] = \{e_i w^t \in C(\mathbb{R}) | w \in [\Pi\mathcal{B}]_j\}$$

$$N_i[(\Pi\mathcal{B})_j] = \{w \in \Pi\mathcal{B}_j | w_i = 0\}$$

Chapter 5

The Fréchet spaces $C^{-k}(\mathbb{R})$ of k -irregular distributions

5.1 Introduction

When dealing with continuous-time dynamical systems the representation by a set of differential equations leads to interpretation problems. Usually, to avoid smoothness-assumptions, weak (distributional) interpretations are considered. This is what we did in chapter 4. At the end of that chapter we concluded that the problems, in fact, are not present. However, the description we found there, is not in AR-form and if we stick to this form we do need another interpretation of differentiability. This was the author's idea when this research was started. Afterwards it was observed that a linear, shift-invariant and closed behavior is totally determined by its $C^\infty(\mathbb{R})$ -signals and we could drop the distributional approach. In that sense the significance of this chapter for the main result of this thesis is small. However there are a lot of functional analytical results included that (we think) are worthwhile to present.

Many distribution spaces have been proposed in literature, but to include all signals that occur in system theory it is close to necessary to work with the distribution space of Schwartz, $\mathcal{D}'(\mathbb{R})$, where the topological structure is complex and not easy to deal with, since it has no Fréchet topology. For our particular problem as formulated in chapter 1 we could restrict ourselves to behaviors with $C^\infty(\mathbb{R})$ -signals only, to achieve what we wanted. Hence, we did not need to prove any result in the context of distributions. However,

when not dealing with linear, time-invariant closed behaviors, the introduction of distributions might be unavoidable. This is the reason why we looked for another set up, where more topological structure is present and generalized differentiability is possible. Therefore we construct new spaces, that have the same structure as the Fréchet space $C(\mathbb{R})$. With the help of these spaces, we can introduce (in some sense) generalized differentiability such that we can get rid of the smoothness-conditions, too. This will be the main result in this chapter.

We would like to stress that we do not introduce these new spaces as some fundamental structure to work with. It is merely a tool to derive results in case generalized differentiability is needed to avoid smoothness-conditions.

5.2 Introduction of the spaces $C^{-k}(\mathbb{R})$ ($k \in \mathbb{N}$)

In this section we introduce the spaces $C^{-k}(\mathbb{R})$ in which we can define generalized integral and differential operators and in which the structure of a Fréchet space is maintained. This turns out to be related to a problem of the decomposition of the space $C(\mathbb{R})$ into the kernel of a differential operator and a complement.

Definition 5.2.1 The linear space spanned by the polynomials of degree less than or equal to k will be denoted by $\mathbf{P}^k$ ($k = 0, 1, \dots$).

Definition 5.2.2 By $P_j, j = 1, 2, \dots$ we define a sequence of projections from $C(\mathbb{R})$ to $C(\mathbb{R})$ which map $C(\mathbb{R})$ onto $\mathbf{P}^{j-1}$. We assume that the maps P_j are chosen in such a way that $P_j P_k = P_{\min\{j,k\}}$. Furthermore, we define $Q_j := I - P_j, j = 1, 2, \dots$

We come back to this choice later on. Note that such a sequence of projections exists. Take as an example the Lagrange interpolation polynomials: choose a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of mutually distinct real numbers. Let L_k denote the Lagrange interpolation polynomial for $\lambda_0, \lambda_1, \dots, \lambda_k$ (see, [1], p.31). Then, for $f \in C(\mathbb{R})$, $P_k f := \sum_{i=0}^k f(\lambda_i) L_i$ satisfies the requirements since $P_k = \langle \{L_j \mid j = 0, 1, \dots, k\} \rangle$.

Remark 5.2.3 Clearly each Q_j is an operator from $C^k(\mathbb{R})$ into $C^k(\mathbb{R})$ for $k \in \mathbb{N} \cup \{\infty\}$. $\square$

Consider the Volterra-integral operators $I_k, k \in \mathbb{N}$, on $C(\mathbb{R})$ again

$$I_0 f = f \wedge (I_k f)(t) = \int_0^t \frac{(t-\tau)^{(k-1)}}{(k-1)!} f(\tau) d\tau \quad k = 1, 2, \dots; f \in C(\mathbb{R})$$

On $C(\mathbb{R})$ we define the following operators J_k for $k \in \mathbb{N}$:

$$J_0 := I, J_k := Q_k I_k \quad (k = 1, 2, \dots)$$

The interpretation of this operator J_k is the following: the inhomogeneous differential equation $D^k f = g$ with boundary condition $P_k f = 0$ has $J_k g$ as its unique solution.

Lemma 5.2.4 For all $l \geq 0$ J_k is a continuous injection from $C^l(\mathbb{R})$ to $C^{k+l}(\mathbb{R})$.

Proof: Continuity of J_k is a consequence of the fact that Q_k is continuous on $C^{k+l}(\mathbb{R})$ and I_k is a continuous operator from $C^l(\mathbb{R})$ to $C^{k+l}(\mathbb{R})$ (see lemma 2.1.17). Let $f \in C^l(\mathbb{R})$ and $J_k f = 0$. Then $Q_k I_k f = 0$, which implies that $I_k f$ is a polynomial of degree less than k . Hence $f = 0$. $\square$

Lemma 5.2.5 On $C^k(\mathbb{R})$ we have $Q_k = J_k D^k$.

Proof: Let $f \in C^k(\mathbb{R})$. Then f can be written as

$$f(t) = \sum_{i=0}^{k-1} \frac{f^{(i)}(0)}{i!} t^i + (I_k D^k f)(t)$$

and hence

$$\begin{aligned} J_k D^k f &= Q_k I_k D^k f \\ &= Q_k \left(f - \sum_{i=0}^{k-1} \frac{f^{(i)}(0)}{i!} t^i \right) \\ &= Q_k f \end{aligned}$$

$\square$

Lemma 5.2.6 *If $0 \leq l \leq k$, then we have (where all operators are considered on $C(\mathbb{R})$):*

1. $J_k = Q_k I_{k-l} Q_l I_l$
2. $Q_k Q_{k-l} = Q_k$
3. $J_k = Q_k J_{k-l} J_l$
4. $D^k J_k = I$
5. $J_k D^l = Q_k J_{k-l}$

Proof:

$$\begin{aligned} 1. J_k &= Q_k I_{k-l} I_l = \\ &= Q_k I_{k-l} (P_l I_l + (I - P_l) I_l) = \\ &= Q_k I_{k-l} Q_l I_l \end{aligned}$$

$$2. Q_k Q_{k-l} = (I - P_k)(I - P_{k-l}) = I - P_k = Q_k$$

$$3. J_{k-l} J_l = Q_{k-l} I_{k-l} Q_l I_l \text{ and hence}$$

$$\begin{aligned} Q_k J_{k-l} J_l &= Q_k Q_{k-l} I_{k-l} Q_l I_l = \\ &= Q_k I_{k-l} Q_l I_l = \\ &= J_k \end{aligned}$$

$$4. D^k J_k = D^k Q_k I_k = D^k (Q_k I_k + P_k I_k) = D^k I_k = I$$

$$5. J_k D^l = Q_k J_{k-l} J_l D^l = Q_k J_{k-l} - Q_k J_{k-l} P_l = Q_k J_{k-l}$$

□

We emphasize that due to the pre-multiplication with Q_k the operators J_k do not have the semigroup-property. I.e. $I_{k+l} = I_k I_l$ but $J_{k+l} \neq J_k J_l$.

In chapter 2 we introduced the topological vector spaces $C^k(\mathbb{R})$ ($k \in \mathbb{N}$) as the vector space of k -times differentiable functions on which the topology is brought about by system of seminorms

$$p_m^{(k)}(f) = \sum_{j=0}^k p_m(D^j f), \quad m \in \mathbb{N}$$

or the equivalent system of seminorms

$$\tilde{p}_m^{(k)}(f) = \sum_{j=0}^{k-1} |(D^j f)(0)| + p_m(D^k f), \quad m \in \mathbb{N}$$

Define

$$\bar{p}_m^{(k)} = p_m(D^k f) + p_m(P_k f), \quad m \in \mathbb{N}$$

Lemma 5.2.7 *For all $k \in \mathbb{N}$ we have that $(\bar{p}_m^{(k)})_{m \in \mathbb{N}}$ is a complete family of seminorms on $C^k(\mathbb{R})$.*

Proof: We will only prove the completeness. If for all $m \in \mathbb{N}$ we have $\bar{p}_m^{(k)}(f) = 0$, then also $p_m(D^k f) = 0, p_m(P_k f) = 0$ for all $m \in \mathbb{N}$. From the first equality we obtain that $D^k f = 0$, which implies $f \in \mathbf{P}^{k-1}$ and the second one yields $P_k f = 0$. Hence $f = 0$. $\square$

The topologies generated by $(\bar{p}_m^{(k)})_{m \in \mathbb{N}}$, $(p_m^{(k)})_{m \in \mathbb{N}}$ and $(\tilde{p}_m^{(k)})_{m \in \mathbb{N}}$ are equivalent. For the equivalence of the first two we refer to section 2.1 (after equation 2.4). Here we show the equivalence of the latter two.

$$\begin{aligned} \bar{p}_m^{(k)} &= p_m(D^k f) + p_m(P_k f) \\ &\leq p_m(D^k f) + C \cdot p_r(f) \\ &\leq \tilde{C} \cdot p_{\max\{m, r\}}^{(k)}(f) \end{aligned}$$

On the other hand

$$\begin{aligned} p_m^{(k)}(f) &= p_m^{(k)}(J_k D^k f + P_k f) \\ &\leq \sum_{i=0}^k (p_m(D^i J_k D^k f) + p_m(D^i P_k f)) \\ &= \sum_{i=0}^k (p_m(J_{k-i} D^k f) + p_m(D^i P_k f)) \\ &\leq \sum_{i=0}^k (B_i \cdot p_{r_i}(D^k f) + C_i \cdot p_{r_{i+k}}(P_k f)) \\ &\leq C \cdot \bar{p}_n^{(k)}(f) \end{aligned}$$

where $C := \sum_{i=0}^k (B_i + C_i)$ and $n = \max_{i=1,2,\dots,2k} \{r_i\}$.

With each of the presented families of seminorms, $C^k(\mathbb{R})$ becomes a Fréchet space. We will show that $C^k(\mathbb{R})$ is the completion of $C^\infty(\mathbb{R})$ with respect to all the families of seminorms by showing it for one of the collections.

Lemma 5.2.8 *For all $k \in \mathbb{N}$, the completion of $C^\infty(\mathbb{R})$ with respect to $(\bar{p}_m^{(k)})_{m \in \mathbb{N}}$ is $C^k(\mathbb{R})$.*

Proof: If $(f_j)_{j \in \mathbb{N}}$ is a sequence in $C^\infty(\mathbb{R})$ that is a Cauchy-sequence with respect to the seminorms $(\bar{p}_m^{(k)})_{m \in \mathbb{N}}$ and f is its limit then for all $m \in \mathbb{N}$

$$\bar{p}_m^{(k)}(f_j - f) \rightarrow 0 \quad (j \rightarrow \infty).$$

Consequently

$$\begin{cases} D^k f_j & \rightarrow g \\ P_k f_j & \rightarrow h \end{cases} \quad \text{in } C(\mathbb{R})\text{-sense}$$

Hence $f = J_k g + h$ and $P_k f = h$. Clearly $f \in C^k(\mathbb{R})$. So every Cauchy-sequence with respect to the seminorms $(\bar{p}_m^{(k)})_{m \in \mathbb{N}}$ converges to an element of $C^k(\mathbb{R})$.

Now let $f \in C^k(\mathbb{R})$. We already know that $C(\mathbb{R})$ is the completion of $C^\infty(\mathbb{R})$ with respect to the seminorms $(p_m)_{m \in \mathbb{N}}$. So if $(g_j)_{j \in \mathbb{N}}$ is a sequence in $C^\infty(\mathbb{R})$ converging to $D^k f$ in $C(\mathbb{R})$ -sense, then $(J_k g_j + P_k f)_{j \in \mathbb{N}}$ converges to f , namely

$$\begin{aligned} \bar{p}_m^{(k)}(J_k g_j + P_k f - f) &= \bar{p}_m^{(k)}(J_k g_j - Q_k f) \\ &= p_m(g_j - D^k f) \rightarrow 0 \quad (j \rightarrow \infty) \end{aligned}$$

This completes the proof. $\square$

In a similar way, i.e. as the completion of $C^\infty(\mathbb{R})$ with respect to a family of seminorms, we will introduce new spaces with the same structure.

Definition 5.2.9

$$\forall k \in \mathbb{N} \forall f \in C^\infty(\mathbb{R}) \forall m \in \mathbb{N} [\bar{p}_m^{(-k)}(f) := p_m(J_k f)]$$

Since J_k is injective and continuous we have the following lemma.

Lemma 5.2.10 For all $k \in \mathbb{N}$ $(\bar{p}_m^{(-k)})_{m \in \mathbb{N}}$ is a family of seminorms on $C^\infty(\mathbb{R})$.

Proof: If for all $m \in \mathbb{N}$ $\bar{p}_m^{(-k)}(f) = 0$ then also $p_m(Q_k I_k f) = 0$ for all $m \in \mathbb{N}$. Since $(p_m)_{m \in \mathbb{N}}$ is a family of seminorms on $C^\infty(\mathbb{R})$ we obtain $Q_k I_k f \equiv 0$. However

$$D^k(Q_k I_k f) = D^k(Q_k I_k f + P_k I_k f) = D^k I_k f = f$$

Thus $f \equiv 0$. The rest is obvious. $\square$

Since J_k , $k \in \mathbb{N}$ is a continuous operator from $C(\mathbb{R})$ to $C(\mathbb{R})$ it can be seen easily that the topology on $C^\infty(\mathbb{R})$ generated by the seminorms $(\bar{p}_m^{(-k)})_{m \in \mathbb{N}}$, ($k \in \mathbb{N}$) is weaker than the one generated by $(p_m)_{m \in \mathbb{N}}$ and so its completion

contains $C(\mathbb{R})$. This new topology is still metrizable but of course $C^\infty(\mathbb{R})$ is not complete anymore with respect to this topology. The following sequence shows that the completion of $C^\infty(\mathbb{R})$ properly contains $C(\mathbb{R})$:

$$f_k(t) = \begin{cases} 0 & t < 0 \\ h(t) & 0 \leq t \leq \frac{1}{k} \\ 1 & t > \frac{1}{k} \end{cases}$$

where $h_k(t) \in C^\infty(\mathbb{R})$ is such that $f_k(t) \in C^\infty(\mathbb{R})$ for all $k \in \mathbb{N}$.

Then $p_m^{-1}(f_k(t) - H(t)) \rightarrow 0$ ($k \rightarrow \infty$) for all $m \in \mathbb{N}$, where $H(t)$ is the Heaviside-function, which obviously is not in $C(\mathbb{R})$.

To maintain the Fréchet structure we define the spaces $C^{-k}(\mathbb{R})$ as follows:

Definition 5.2.11 $C^{-k}(\mathbb{R})$, $k \in \mathbb{N}$ is defined to be the completion of the space $C^\infty(\mathbb{R})$ with respect to the family of seminorms $(\bar{p}_m^{(-k)})_{m \in \mathbb{N}}$.

Remark 5.2.12 We can see immediately that these spaces $C^{-k}(\mathbb{R})$ depend on the choice of P_k . This is in contrast with the known spaces $C^k(\mathbb{R})$. Of course it is a consequence of taking the definite integral J_k . However, in the author's opinion, there is no other logical option for these seminorms. $\square$

An important lemma is the following:

Lemma 5.2.13 For all $k, l \in \mathbb{Z}$, $k \geq l$ we have

$$\forall_{n \in \mathbb{N}} \exists_{C \in \mathbb{R}} \exists_{m \in \mathbb{N}} \forall_{f \in C^\infty(\mathbb{R})} [\bar{p}_n^{(l)}(f) \leq C \cdot \bar{p}_m^{(k)}(f)]$$

Proof:

$l \geq 0$: This follows from the fact that the families of seminorms $(p_m^{(k)})_{m \in \mathbb{N}}$ and $(\bar{p}_m^{(k)})_{m \in \mathbb{N}}$ generate the same topologies.

$k \leq 0$:

$$\begin{aligned} \bar{p}_n^{(l)}(f) &= p_n(J_{-l}f) \\ &= p_n(Q_{-l}J_{k-l}J_{-k}f) \\ &\leq C \cdot p_m(J_{-k}f) = \bar{p}_m^{(k)}(f) \end{aligned}$$

$k \geq 0, l < 0$:

$$\begin{aligned}
 \bar{p}_n^{(l)}(f) &= p_n(J_{-l}f) \\
 &\leq p_n(J_{-l}J_k D^k f) + p_n(J_{-l}P_k f) \\
 &\leq C_1 \cdot p_{n_1}(D^k f) + C_2 \cdot p_{n_2}(P_k f) \\
 &\leq C \cdot \bar{p}_m^{(k)}(f)
 \end{aligned}$$

□

Corollary 5.2.14 $C^\infty(\mathbb{R}) \hookrightarrow \dots C^k(\mathbb{R}) \hookrightarrow C^{k-1}(\mathbb{R}) \hookrightarrow \dots C(\mathbb{R}) \hookrightarrow C^{-1}(\mathbb{R}) \hookrightarrow \dots \hookrightarrow C^{-l}(\mathbb{R}) \hookrightarrow C^{-l-1}(\mathbb{R}) \hookrightarrow \dots$ ($k, l \in \mathbb{N}$).

Proof: This chain of continuous embeddings is an immediate consequence of $C^k(\mathbb{R})$, $k \in \mathbb{Z}$, being the completion of $C^\infty(\mathbb{R})$ with respect to $(\bar{p}_n^{(k)})_{n \in \mathbb{N}}$ and the inequalities in lemma 5.2.13. □

Hence we have a nested sequence of Fréchet spaces $(C^k(\mathbb{R}))_{k \in \mathbb{Z}}$, that can be considered as equivalence-classes of limits of Cauchy-sequences (with respect to the different topologies) in $C^\infty(\mathbb{R})$.

At this point it becomes clear why we took P_j such that $P_j P_k = P_{\min\{j,k\}}$. Namely, in lemma 5.2.13 we used the equality $J_{k+l} = Q_{k+l} J_l J_k$ which is derived on the basis of this property of all P_j . Hence, in order to achieve nesting, this property is imposed on the set of projections P_j .

5.3 The extension of the integral and differential operator

In this section we introduce generalized notions of integration and differentiation. It turns out that they fit in our construction of the spaces $C^{-l}(\mathbb{R})$ ($l \in \mathbb{N}$) in a natural way.

The definite integral $J_k = Q_k I_k$ plays an important role in our theory and we like to generalize this operator to the space $C^{-l}(\mathbb{R})$, ($l \in \mathbb{N}$).

Since $C^\infty(\mathbb{R})$ is dense in $C^l(\mathbb{R})$ for all $l \in \mathbb{Z}$, we hope that J_k can be extended in a continuous way to, say $\tilde{J}_k$ on $C^{-l}(\mathbb{R})$, $l \in \mathbb{N}$.

To this extent, we take a sequence $(f_n)_{n \in \mathbb{N}}$ in $C^\infty(\mathbb{R})$ that is a Cauchy-sequence in $C^{-l}(\mathbb{R})$. I.e., for all $m \in \mathbb{N}$

$$\bar{p}_m^{(-l)}(f_j - f_n) = p_m(J_l(f_j - f_n)) \rightarrow 0 \quad (j, n \rightarrow \infty)$$

If $l \leq k$ then $J_k = Q_k J_{k-l} J_l$ and since $Q_k J_{k-l}$ is a continuous operator from $C(\mathbb{R})$ to $C^{k-l}(\mathbb{R})$ we know that $(J_k f_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence in $C^{k-l}(\mathbb{R})$ and hence the extension can be defined as the limit of this sequence. (Note that this is consistent with the J_k 's on $C^{-l}(\mathbb{R})$ for $l \leq 0$).

We have some difficulties introducing $\tilde{J}_k$ on $C^{-l}(\mathbb{R})$ in case $l > k$, because $(J_k f_n)_{n \in \mathbb{N}}$ is not necessarily a Cauchy-sequence anymore. This is a consequence of the projection-operator Q_l . Namely, $J_l = Q_l J_{l-k} J_k$. Hence $(f_n)_{n \in \mathbb{N}}$ being a Cauchy-sequence in $C^{-l}(\mathbb{R})$ is equivalent to $(J_l f_n)_{n \in \mathbb{N}}$ being a Cauchy-sequence in $C(\mathbb{R})$. This does not imply the convergence of the sequence $(J_k f_n)_{n \in \mathbb{N}}$. One might think that introducing the spaces $C^{-l}(\mathbb{R})$ in case $l \in \mathbb{N}$ using the seminorms $\bar{p}_m^{(-l)}(f) := p_m(I_l f)$ would be a better idea, because then these problems would not occur. However, this is the set up we worked out in the first place and we found out that it is not possible to introduce a generalization of the differential operator in that case. In fact, it turns out to be a decomposition-problem. We will come back to this later on in this section. As regards the extension of J_k on $C^{-l}(\mathbb{R})$ in case $l > k$ we will find another way later on. We do not need this in our work but we will do it for completeness sake only.

Definition 5.3.1 The continuous extension of the operator J_k can be defined on $C^{-l}(\mathbb{R})$ for $l \leq k$ in the following way: if $F \in C^{-l}(\mathbb{R})$ and $(f_n)_{n \in \mathbb{N}}$ a sequence in $C^\infty(\mathbb{R})$ such that $f_n \rightarrow F$ in $C^{-l}(\mathbb{R})$ -sense, then

$$\tilde{J}_k F := \lim_{n \rightarrow \infty} J_k f_n \quad (\text{in } C^{k-l}(\mathbb{R})\text{-sense})$$

Lemma 5.3.2 $\tilde{J}_k$ is properly defined.

Proof: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $C^\infty(\mathbb{R})$ and suppose $f_n \rightarrow 0$ in $C^{-l}(\mathbb{R})$ -sense and $J_k f_n \rightarrow g$ in $C^{k-l}(\mathbb{R})$ -sense. Then $\bar{p}_m^{(-l)}(f_n) = p_m(J_l f_n) \rightarrow 0$ ($n \rightarrow \infty$). Since I_{k-l} is continuous from $C(\mathbb{R})$ into $C^{k-l}(\mathbb{R})$ and since Q_k is continuous from $C^{k-l}(\mathbb{R})$ into $C^{k-l}(\mathbb{R})$, we obtain

$$p_m(Q_k J_{k-l} J_l f_n) \leq C \cdot p_r(J_l f_n)$$

Since $p_r(J_l f_n) \rightarrow 0$ ($n \rightarrow \infty$), also $p_m(J_k f_n) \rightarrow 0$ ($n \rightarrow \infty$). Because $J_k f_n \rightarrow g$ in $C(\mathbb{R})$ -sense, $g = 0$. $\square$

By construction $\tilde{J}_k$ is a continuous operator from $C^{-k}(\mathbb{R})$ to $C(\mathbb{R})$. In fact, it is easy to check that for all $l \leq k$ $\tilde{J}_k$ is a continuous operator from $C^{-l}(\mathbb{R}) \rightarrow C^{k-l}(\mathbb{R})$.

Lemma 5.3.3 *On the space $C^{-l}(\mathbb{R})$, $\tilde{J}_k = Q_k J_{k-l} \tilde{J}_l$ ($0 \leq l \leq k$).*

Proof: Obvious. □

Lemma 5.3.4 $\forall l \in \mathbb{N} \forall m \in \mathbb{N} \forall F \in C^{-l}(\mathbb{R})$ [$\bar{p}_m^{(-l)}(F) = p_m(\tilde{J}_l F)$]

Proof: Let $m \in \mathbb{N}$, $F \in C^{-l}(\mathbb{R})$, then

$$\bar{p}_m^{(-l)}(F) = \lim_{n \rightarrow \infty} \bar{p}_m^{(-l)}(f_n) = \lim_{n \rightarrow \infty} p_m(J_l f_n) = p_m(\tilde{J}_l F)$$

□

Considering the interpretation of J_k , the following lemma should come as no surprise.

Lemma 5.3.5 *Let $k, l \in \mathbb{N}$ with $0 \leq l \leq k$, then*

$$\tilde{J}_k(C^{-l}(\mathbb{R})) = \{f \in C^{k-l}(\mathbb{R}) \mid P_k f = 0\}$$

Proof:

⊆: Then $f \in C^{k-l}(\mathbb{R})$ and $f = \tilde{J}_k F$ for some $F \in C^{-l}(\mathbb{R})$.

$$\begin{aligned} \tilde{J}_k F &= \lim_{n \rightarrow \infty} J_k f_n \\ &= \lim_{n \rightarrow \infty} Q_k J_{k-l} J_l f_n \\ &= \lim_{n \rightarrow \infty} Q_k Q_k J_{k-l} J_l f_n \\ &= Q_k \left(\lim_{n \rightarrow \infty} Q_k J_{k-l} J_l f_n \right) \\ &= Q_k \tilde{J}_k F \end{aligned}$$

Hence $P_k \tilde{J}_k F = P_k Q_k \tilde{J}_k F = 0$.

⊇: Let $f \in C^{k-l}(\mathbb{R})$ with $P_k f = 0$. There exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $C^\infty(\mathbb{R})$ such that $P_k f_n = 0$ and $f_n \rightarrow f$ (in $C^{k-l}(\mathbb{R})$ -sense). This implies $p_m(D^{k-l}(f_n - f)) \rightarrow 0$ for all $m \in \mathbb{N}$. Since $J_l(D^k f_n) = J_l D^l D^{k-l} f_n = Q_l D^{k-l} f_n \rightarrow Q_l D^{k-l} f$ in $C(\mathbb{R})$ -sense, $(D^k f_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence in $C^{-l}(\mathbb{R})$ -sense, converging to an element F in $C^{-l}(\mathbb{R})$, and

$$\tilde{J}_k F = \lim_{n \rightarrow \infty} J_k D^k f_n = \lim_{n \rightarrow \infty} f_n = f$$

□

Lemma 5.3.6 For all $k \geq l \geq 0$ $\tilde{J}_k$ is an injection from $C^{-l}(\mathbb{R})$ in $C^{k-l}(\mathbb{R})$.

Proof:

$$\begin{aligned}
 \tilde{J}_k F = 0 &\iff \forall_{m \in \mathbb{N}} [p_m(\tilde{J}_k F) = 0] \\
 &\iff \forall_{m \in \mathbb{N}} [p_m(Q_k J_{k-l} \tilde{J}_l F) = 0] \\
 &\iff \forall_{m \in \mathbb{N}} [p_m(\tilde{J}_l F) = 0] \\
 &\iff \forall_{m \in \mathbb{N}} [\bar{p}_m^{(-l)}(F) = 0] \\
 &\iff F = 0
 \end{aligned}$$

□

In fact, it is our aim to introduce a generalized differential operator to be able to differentiate elements in $C^{-l}(\mathbb{R})$, ($l \in \mathbb{N}$). Contrary to the integral operator J_k , this generalization turns out much easier and nicer to work with. For instance, the semigroup-property of the usual differential operator will be maintained in the extension.

We do not introduce a differential operator $\tilde{D}$ on $C(\mathbb{R})$ only, but also on an arbitrary $C^{-l}(\mathbb{R})$ ($l \in \mathbb{N}$). Furthermore we are able to 'differentiate' any element of any space $C^l(\mathbb{R})$ ($l \in \mathbb{Z}$) arbitrarily many times.

Because of the fact that $C^\infty(\mathbb{R})$ is dense in $C^l(\mathbb{R})$ we could define the extension of J_k to $C^l(\mathbb{R})$, for $k \geq l$, with the aid of Cauchy-sequences. In a similar way we will extend the differential operator D^k ($k \in \mathbb{N}$, arbitrary) to all the spaces $C^l(\mathbb{R})$, $l \in \mathbb{Z}$. Note that $k \geq 0$ and if $0 \leq k \leq l$ we are in a standard situation. Let $F \in C^l(\mathbb{R})$, $l \in \mathbb{Z}$. Take a sequence $(f_n)_{n \in \mathbb{N}}$ in $C^\infty(\mathbb{R})$ such that $f_n \rightarrow F$ in $C^l(\mathbb{R})$ -sense, i.e for all $m \in \mathbb{N}$: $\bar{p}_m^{(l)}(f_n - f_j) \rightarrow 0$ ($n, j \rightarrow \infty$). We consider two cases:

$k > l \geq 0$: In this case for all $m \in \mathbb{N}$, $p_m(D^l(f_n - f_j))$ and $p_m(P_l(f_n - f_j))$ tend to zero if $n, j \rightarrow \infty$.

$$\begin{aligned}
 p_m^{(\overline{l-k})}(D^k(f_n - f_j)) &= p_m(J_{k-l} D^l(f_n - f_j)) \\
 &= p_m(Q_{k-l} D^l(f_n - f_j)) \\
 &\leq C \cdot p_r(D^l(f_n - f_j))
 \end{aligned}$$

$l < 0$: In this case we have for all $m \in \mathbb{N}$, $p_m(J_{-l}(f_n - f_j))$ tends to zero if $n, j \rightarrow \infty$. Hence

$$p_m(J_{k-l}D^k(f_n - f_j)) = p_m(Q_{k-l}J_{-l}(f_n - f_j)) \rightarrow 0 \quad (n, j \rightarrow \infty)$$

where we used 5. from lemma 5.2.6.

Definition 5.3.7 For all $F \in C^l(\mathbb{R})$, ($l \in \mathbb{Z}$), $\tilde{D}^k F = \lim_{m \rightarrow \infty} D^k f_n$ (in $C^{l-k}(\mathbb{R})$ -sense), where $(f_n)_{n \in \mathbb{N}}$ is a sequence in $C^\infty(\mathbb{R})$ such that $f_n \rightarrow F$ in $C^{-l}(\mathbb{R})$ -sense.

Again we will show that this is a proper extension.

Lemma 5.3.8 $\tilde{D}^k$ is properly defined.

Proof: Let $f \in C^{-l}(\mathbb{R})$ and $f_n \rightarrow f, g_n \rightarrow f$ (in $C^{-l}(\mathbb{R})$ -sense) where f_n and g_n are in $C^\infty(\mathbb{R})$. As before we obtain that $D^k f_n$ and $D^k g_n$ are Cauchy-sequences in $C^{l-k}(\mathbb{R})$ and therefore they have a limit. Say $\lim_{n \rightarrow \infty} D^k f_n = F$ and $\lim_{n \rightarrow \infty} D^k g_n = G$, where F and G are in $C^{l-k}(\mathbb{R})$. Note that $D^k(f_n - g_n) \rightarrow 0$ ($n \rightarrow \infty$) in $C^{l-k}(\mathbb{R})$ -sense. For all $n \in \mathbb{N}$

$$\bar{p}_m^{(l-k)}(F - G) \leq \bar{p}_m^{(l-k)}(F - D^k f_n) + \bar{p}_m^{(l-k)}(G - D^k g_n) + \bar{p}_m^{(l-k)}(D^k g_n - D^k f_n)$$

Let $n \rightarrow \infty$, then it is clear that $\bar{p}_m^{(l-k)}(F - G) = 0$ for all $m \in \mathbb{N}$ and hence $F = G$. $\square$

So $\tilde{D}^k$ is a continuous operator from $C^{-l}(\mathbb{R})$ to $C^{l-k}(\mathbb{R})$.

That introduction of $\tilde{D}$ is consistent with the usual differential operator, i.e. $\tilde{D}^k = D^k$ on $C^k(\mathbb{R})$ is obvious again, as it was for $\tilde{J}_k$ and J_k . Let us come back to the generalization of J_k on $C^{-l}(\mathbb{R})$ in case $0 \leq k \leq l$. We could not prove that $(J_k f_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence in $C^{l-k}(\mathbb{R})$ if $(f_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence in $C^{-l}(\mathbb{R})$. We can avoid this problem by defining $\tilde{J}_k F := \tilde{D}^{l-k} \tilde{J}_l F$ for all $F \in C^{-l}(\mathbb{R})$. This is a proper definition, since $\tilde{J}_l F$ is well-defined.

Lemma 5.3.9 For all $k \in \mathbb{N}, l \in \mathbb{Z}$, $\{\tilde{D}^k f \mid f \in C^l(\mathbb{R})\} = C^{l-k}(\mathbb{R})$

Proof: It is clear for $k \leq l$, so assume $k > l$. Take $f \in C^l(\mathbb{R})$, then it is obvious that $\tilde{D}^k f \in C^{l-k}(\mathbb{R})$, since it is defined as a limit of a Cauchy-sequence $C^{l-k}(\mathbb{R})$ -sense in the complete space $C^{l-k}(\mathbb{R})$.

On the other hand if $F \in C^{l-k}(\mathbb{R})$ then there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $C^\infty(\mathbb{R})$ such that $f_n \rightarrow F$ in $C^{l-k}(\mathbb{R})$ -sense. For $k > l \geq 0$, $(J_{k-l}f_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence in $C(\mathbb{R})$ and converges to some $h \in C(\mathbb{R})$. Clearly $Q_k J_l h =: g \in C^l(\mathbb{R})$, $J_k f_n = Q_k J_l J_{k-l} f_n \rightarrow g$ in $C^l(\mathbb{R})$ -sense, and $\tilde{D}^k g = F$. And for $l < 0$ we know that $(J_{k-l}f_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence in $C(\mathbb{R})$ and thus $(Q_{-l} J_{k-l} f_n)_{n \in \mathbb{N}}$ is also a Cauchy-sequence in $C(\mathbb{R})$. But $Q_{-l} = J_{-l} D^{-l}$ and hence $(D^{-l} J_{k-l} f_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence in $C^l(\mathbb{R})$ and converges to some $h \in C^l(\mathbb{R})$ and clearly $\tilde{D}^k h = F$. $\square$

Remark 5.3.10 It is easy to check that another characterization of $C^{-l}(\mathbb{R})$, $l \in \mathbb{N}$ is the following:

$$C^{-l}(\mathbb{R}) = \{ \lim_{n \rightarrow \infty} D^l f_n \text{ in } C^{-l}(\mathbb{R})\text{-sense} \mid (f_n)_{n \in \mathbb{N}} \text{ a sequence in } C^\infty(\mathbb{R})$$

that is a Cauchy-sequence in $C(\mathbb{R})$ -sense}

Furthermore the operators $\tilde{D}^k$ on $C^{-l}(\mathbb{R})$ all are compatible with the operator D^k on $C^k(\mathbb{R})$. Moreover it is easy to see that they satisfy the semigroup-relation: for all $k, m \in \mathbb{N}$: $\tilde{D}^{k+m} = \tilde{D}^k \circ \tilde{D}^m = \tilde{D}^m \circ \tilde{D}^k$. $\square$

At the beginning of this section we had some trouble introducing $\tilde{J}_k$ on $C^{-l}(\mathbb{R})$ in case $k > l \geq 0$, because of the operator Q_l . For the seminorms $(p_m(I_l f))_{m \in \mathbb{N}}$ these problems would not occur. However, these families of seminorms lead to other difficulties. Let us start with a small lemma on a decomposition of $C(\mathbb{R})$.

Lemma 5.3.11 $C(\mathbb{R}) = \langle \{1\} \rangle \oplus \{f \in C(\mathbb{R}) \mid f(0) = 0\}$

Proof: This is trivial if we consider the continuous projection $P : C(\mathbb{R}) \rightarrow C(\mathbb{R})$, $Pf = f(0) \cdot 1$ and write $f = Pf + (I - P)f$. $\square$

If we define $C^{-1}(\mathbb{R})$ as the completion of $C^\infty(\mathbb{R})$ with respect to the seminorms $(p_m(I_1 f))_{m \in \mathbb{N}}$ then we can introduce the generalized differential operator $\tilde{D}$. It is based on

$$\forall f \in C(\mathbb{R}) [f = f(0) \cdot 1 + I_1 Df]$$

where $I_1 D$ is the projection onto $\{f \in C(\mathbb{R}) \mid f(0) = 0\}$. We use that if $f_n \rightarrow f$ in $C(\mathbb{R})$ -sense, then also $f_n(0)$ converges and hence $I_1 D f_n$. This

implies that Df_n is a Cauchy sequence in $C^{-1}(\mathbb{R})$. So $I_1 D (= J_1 D)$ has a continuous extension to $C(\mathbb{R})$. We will show now that for $I_2 D^2$ it is not possible to extend it continuously to $C(\mathbb{R})$, whereas for $J_2 D^2$ it is. Since, in this context, this extension is necessary to introduce generalized differential operators we have to take other seminorms than $p_m(I_k f)_{m \in \mathbb{N}}$.

What would happen if we would extend the idea above to $C^{-2}(\mathbb{R})$?

For $C^{-1}(\mathbb{R})$ we used

$$C(\mathbb{R}) = \langle \{1\} \rangle \oplus \overline{\{f \in C^1(\mathbb{R}) \mid f(0) = 0\}}$$

In remark 2.3.7, we had the same decomposition problem and we showed that $\{t\}$ is in $\{f \in C^2(\mathbb{R}) \mid f(0) = \dot{f}(0) = 0\}$ and hence the 'logical' second step

$$C(\mathbb{R}) = \langle \{1\}, \{t\} \rangle \oplus \overline{\{f \in C^2(\mathbb{R}) \mid f(0) = \dot{f}(0) = 0\}}$$

where $I_2 D^2$ is the projection on $\{f \in C^2(\mathbb{R}) \mid f(0) = \dot{f}(0) = 0\}$ is not possible.

Lemma 5.3.12 For all $k \in \mathbb{N}$, $Q_k = \tilde{J}_k \tilde{D}^k$ on $C(\mathbb{R})$.

Proof: Let $f \in C(\mathbb{R})$, then $\tilde{D}^k f = \lim_{m \rightarrow \infty} D^k f_n$ in $C^{-k}(\mathbb{R})$ -sense, if $(f_n)_{n \in \mathbb{N}}$ is a sequence in $C^\infty(\mathbb{R})$ and $f_n \rightarrow f$ in $C(\mathbb{R})$ -sense. $\tilde{J}_k(\tilde{D}^k f) = \lim_{n \rightarrow \infty} J_k g_n$ where $(g_n)_{n \in \mathbb{N}}$ a sequence in $C(\mathbb{R})$ such that $g_n \rightarrow \tilde{D}^k f$ in $C^{-k}(\mathbb{R})$ -sense. Take $g_n = D^k f_n$, then

$$\tilde{J}_k \tilde{D}^k f = \lim_{n \rightarrow \infty} J_k D^k f_n = \lim_{n \rightarrow \infty} Q_k f_n = Q_k f$$

□

Remark 5.3.13 There is not a nice way of generalizing the projection-operators P_k and Q_k . The best way to show this is by the example we gave earlier:

$$C^1(\mathbb{R}) = \langle \{1\}, \{t\} \rangle \oplus \{f \in C^1(\mathbb{R}) \mid f(0) = \dot{f}(0) = 0\}$$

This decomposition does not extend to $C(\mathbb{R})$.

The only way is to choose a fixed space $C^{-L}(\mathbb{R})$, ($L \in \mathbb{N}$, L large), and then define the projections on $C^{-L}(\mathbb{R})$ as on $C(\mathbb{R})$. Doing so we would have P_k and Q_k on all the spaces $C^l(\mathbb{R})$, $l \geq -L$. However we are restricted to the space $C^{-L}(\mathbb{R})$ now. □

Lemma 5.3.14 For all $k, l \in \mathbb{N}$ the operator $\tilde{D}^k$ from $C^{-l}(\mathbb{R})$ to $C^{-l}(\mathbb{R})$ with $\text{Dom}(\tilde{D}^k) = C^{k-l}(\mathbb{R})$ is closed.

Proof: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $C^{k-l}(\mathbb{R})$ such that

$$\begin{cases} f_n \rightarrow F \quad (n \rightarrow \infty) \\ \tilde{D}^k f_n \rightarrow G \quad (n \rightarrow \infty) \end{cases} \quad \text{both in } C^{-l}(\mathbb{R})\text{-sense}$$

$k \geq l$: Then this convergence is also in $C^{-k}(\mathbb{R})$ -sense, which implies $\tilde{J}_k \tilde{D}^k f_n \rightarrow \tilde{J}_k G$ in $C(\mathbb{R})$ -sense. Since $(f_n)_{n \in \mathbb{N}}$ is a sequence in $C^{k-l}(\mathbb{R})$, $\tilde{J}_k \tilde{D}^k f_n = Q_k f_n = f_n - P_k f_n$. Using $f_n \rightarrow F$ together with $Q_k f_n \rightarrow \tilde{J}_k G$ we obtain $P_k f_n \rightarrow h$ in $C^{-l}(\mathbb{R})$ -sense, for some $h \in \mathbf{P}^{k-1}$ (because $(P_k f_n)_{n \in \mathbb{N}}$ is a convergent sequence in the finite-dimensional (in $C^{-l}(\mathbb{R})$ -sense) subspace $\mathbf{P}^{k-1}$). This leads to

$$F = \tilde{J}_k G + h \quad (5.1)$$

Since $\tilde{J}_k G \in C^{k-l}(\mathbb{R})$ and $h \in C^\infty(\mathbb{R})$, we have $F \in C^{k-l}(\mathbb{R}) = \text{Dom}(\tilde{D}^k)$. Moreover $\tilde{D}^k F = G$ is immediate from equation (5.1). Hence, in case $k \geq l$, $\tilde{D}^k$ is closed.

$k < l$: There exists a sequence $(g_n)_{n \in \mathbb{N}}$ in $C(\mathbb{R})$ such that for all $n \in \mathbb{N}$ $f_n = \tilde{D}^{k-l} g_n$. Consequently $\tilde{J}_l \tilde{D}^l g_n \rightarrow \tilde{J}^l G$ in $C(\mathbb{R})$ -sense. But $(g_n)_{n \in \mathbb{N}}$ is also a sequence in $C^{k-l}(\mathbb{R})$ and so

$$\begin{aligned} \tilde{J}_l \tilde{D}^l g_n &= Q_l g_n \\ &= g_n - P_l g_n \\ &= Q_{l-k} g_n + P_{l-k} g_n - P_l g_n \\ &= \tilde{J}_{l-k} \tilde{D}^{l-k} g_n + (P_{l-k} - P_l) g_n \end{aligned}$$

Using $\tilde{D}^{k-l} g_n = f_n \rightarrow F$ ($n \rightarrow \infty$) and the continuity of the operator $\tilde{J}_{k-l}$ implies $\tilde{J}_{k-l} \tilde{D}^{k-l} g_n \rightarrow \tilde{J}_{k-l} F$ and hence (with the same arguments as for $k \geq l$):

$$\tilde{J}_{l-k} F = \tilde{J}_l G + h \quad (5.2)$$

where $h \in \mathbf{P}^{l-1}$. Since $\tilde{J}_l G \in C(\mathbb{R})$ and $h \in C^\infty(\mathbb{R})$, we obtain $\tilde{J}_{l-k} F \in C(\mathbb{R})$ and thus $F \in C^{k-l}(\mathbb{R}) = \text{Dom}(\tilde{D}^k)$. Applying $\tilde{D}^l$ to equation (5.2) yields $\tilde{D}^k F = G$. This completes the proof. $\square$

Lemma 5.3.15 For all $k \in \mathbb{N}$: $\tilde{D}^k \tilde{J}_k = I$ as an operator from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$.

Proof: Let $F \in C^{-k}(\mathbb{R})$ and $(f_n)_{n \in \mathbb{N}}$ a sequence in $C(\mathbb{R})$ such that $f_n \rightarrow F$ in $C^{-k}(\mathbb{R})$ -sense. Then $J_k f_n \rightarrow \tilde{J}_k F$ in $C(\mathbb{R})$ -sense. Define $g_n = J_k f_n$ ($n \in \mathbb{N}$) and $g = \tilde{J}_k F$, which means that $g_n \rightarrow g$ in $C(\mathbb{R})$ -sense and $(g_n)_{n \in \mathbb{N}}$ is a sequence in $C^k(\mathbb{R})$. Thus $(D^k g_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence in $C^{-k}(\mathbb{R})$.

$$\lim_{n \rightarrow \infty} D^k g_n = \lim_{n \rightarrow \infty} D^k (J_k f_n) = \lim_{n \rightarrow \infty} f_n = F \text{ (in } C^{-k}(\mathbb{R})\text{-sense)}$$

and

$$\lim_{n \rightarrow \infty} D^k g_n = \tilde{D}^k g = \tilde{D}^k \tilde{J}_k F \text{ (in } C^{-k}(\mathbb{R})\text{-sense)}$$

Hence $\tilde{D}^k \tilde{J}_k F = F$. □

Corollary 5.3.16 For all $l \geq k \geq 0$ $\tilde{D}^l \tilde{J}_l = I$ as an operator from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$.

Proof: From lemma 5.3.3 we know that, since $l \geq k$, $\tilde{J}_l = Q_l J_{l-k} \tilde{J}_k$. Thus

$$\tilde{D}^l \tilde{J}_l = \tilde{D}^l Q_l J_{l-k} \tilde{J}_k = \tilde{D}^k D^{l-k} J_{l-k} \tilde{J}_k = \tilde{D}^k \tilde{J}_k = I$$

□

Lemma 5.3.17 Let $\tilde{S} := p(\tilde{D})$ be an operator from $C^{-l}(\mathbb{R})$ to $C^{-l}(\mathbb{R})$, ($l \in \mathbb{Z}$) with domain $C^{m-l}(\mathbb{R})$, $m \in \mathbb{N}$. Then $\tilde{S}$ is closed.

Proof: Exactly the same proof as lemma 2.3.3. □

Lemma 5.3.18 Let $\tilde{S} := p(\tilde{D})$ be an operator from $C^{-l}(\mathbb{R})$ to $C^{-l}(\mathbb{R})$, ($l \in \mathbb{Z}$) with domain $C^{m-l}(\mathbb{R})$, $m \in \mathbb{N}$. Then $\tilde{S}$ is surjective.

Proof: It is sufficient to prove that $\tilde{D} + \lambda_j I$ is surjective from $C^{-l}(\mathbb{R})$ to $C^{-l-1}(\mathbb{R})$, $l \in \mathbb{N}$.

Suppose we have to solve the equation $(\tilde{D} + \lambda_j I)F = G$, where $F \in C^{-l}(\mathbb{R})$ and $G \in C^{-l-1}(\mathbb{R})$. Then $F = \tilde{D}^{l+1} f$, $G = \tilde{D}^{l+1} g$ for some $f \in C^1(\mathbb{R})$, $g \in C(\mathbb{R})$. So the equation becomes

$$(\tilde{D} + \lambda_j I) \tilde{D}^{l+1} f = \tilde{D}^{l+1} (\tilde{D} + \lambda_j I) f = \tilde{D}^{l+1} g$$

From the variation of constants-formula we know that $(D + \lambda_j I)f = g$ is solvable for some $f \in C^1(\mathbb{R})$. Clearly $F = \tilde{D}^{l+1}f$ is a solution. $\square$

Lemma 5.3.19 *The kernel of the operator $S := \prod_{j=1}^k (D + \lambda_j I)^{r_j}$ equals the kernel of $\tilde{S} := \prod_{j=1}^k (\tilde{D} + \lambda_j I)^{r_j} f$ where $\sum_{j=1}^k r_j = m$ and $\tilde{S}$ is to be considered as an operator with $\text{Dom}(\tilde{S}) = C(\mathbb{R})$.*

Proof: Let $f \in C^m(\mathbb{R}) = \text{Dom}(S)$ and $Sf = 0$, then $\tilde{D}^j f = D^j f$ for all $j \in \{0, 1, 2, \dots, m\}$. This implies $Sf = \tilde{S}f = 0$. On the other hand if $f \in \text{Dom}(\tilde{S})$ then $0 = \tilde{S}f = \sum_{j=0}^m \alpha_j \tilde{D}^j f = \alpha_0 f + \tilde{D}(\sum_{j=1}^m \alpha_j \tilde{D}^{j-1} f)$ for some $\alpha_j \in \mathbb{R}$. Since $f \in C(\mathbb{R})$ we have that $g := \sum_{j=1}^m \alpha_j \tilde{D}^{j-1} f \in C(\mathbb{R})$ and $\tilde{D}g = -\alpha_0 f \in C(\mathbb{R})$. We obtain that $g \in C^1(\mathbb{R})$ and $g := \sum_{j=1}^m \alpha_j D^{j-1} f$ hence $\tilde{D}g = Dg$. This immediately implies $\sum_{j=1}^m (D + \lambda_j I)^{r_j} f = Sf = 0$. $\square$

At this moment we have laid the foundation of an integral and differential calculus in $C^{-k}(\mathbb{R})$.

Very often the external signals are taken in $L_{p,loc}(\mathbb{R})$ for $p = 1$ or $p = 2$. We show that these spaces $L_{p,loc}(\mathbb{R})$ are contained in $C^{-1}(\mathbb{R})$. Therefore it is sufficient to prove that $L_{1,loc}(\mathbb{R})$ is in $C^{-1}(\mathbb{R})$.

Lemma 5.3.20 $L_{1,loc}(\mathbb{R}) \subset C^{-1}(\mathbb{R})$

Proof: Let $f \in L_{1,loc}(\mathbb{R})$, then $F(t) = \int_0^t f(\tau) d\tau$ is a continuous function and $F(0) = 0$. Because $J_1(C(\mathbb{R})) = \{f \in C^1(\mathbb{R}) \mid f(0) = 0\}$ it is immediate that the $C(\mathbb{R})$ -closure $\overline{J_1(C(\mathbb{R}))} = \{f \in C(\mathbb{R}) \mid f(0) = 0\}$. Hence $F(t)$ is a limit of a sequence $J_1 f_n$ where $f_n \in C(\mathbb{R})$. But $J_1 f_n \rightarrow F$ (in $C(\mathbb{R})$ -sense) implies $f_n \rightarrow f$ (in $C^{-1}(\mathbb{R})$ -sense). Thus $f \in C^{-1}(\mathbb{R})$. Conclusion: $L_{1,loc}(\mathbb{R}) \subseteq C^{-1}(\mathbb{R})$. $\square$

Remark 5.3.21 Clearly $C^{-1}(\mathbb{R})$ contains properly $L_{1,loc}(\mathbb{R})$, because there is a one-to-one relation between $L_{1,loc}(\mathbb{R})$ -functions and functions that are absolutely continuous (which forms a large subclass of $C(\mathbb{R})$). A function f is absolutely continuous if and only if there exists a $g \in L_1^{loc}(\mathbb{R})$ such that $f = f(0) \cdot 1 + J_1 g$. Since $L_{1,loc}(\mathbb{R})$ is a Fréchet space too, with about the same structure as $C(\mathbb{R})$, many results for $C(\mathbb{R})$ can be translated into

results for $L_{1,loc}(\mathbb{R})$. □

We conclude this section by showing the relation between the spaces $C^{-k}(\mathbb{R})$ and the distribution space of Schwartz. We show how the elements of $C^{-k}(\mathbb{R})$ can be regarded as distributions. Therefore we embed $C^{-k}(\mathbb{R})$ in $D'(\mathbb{R})$ (i.e. the distribution-space of Schwartz), via the natural embedding of $C(\mathbb{R})$ in $D'(\mathbb{R})$. Associate with every continuous function f the continuous linear functional

$$L_f(\varphi) := \int_{\mathbb{R}} f(t)\varphi(t)dt$$

where $\varphi \in C_c^\infty(\mathbb{R})$, the dual space of $D'(\mathbb{R})$. Starting from here and using the fact that for all $\psi \in C^{-k}(\mathbb{R})$ there exists exactly one $f \in C(\mathbb{R})$ such that $\tilde{D}^k f = \psi$ and $P_k f = 0$ we associate with every $\psi \in C^{-k}(\mathbb{R})$ the continuous linear functional

$$L_\psi(\varphi) := (-1)^k \int_{\mathbb{R}} f(t)\varphi^{(k)}(t)dt$$

where $\tilde{D}^k f = \psi$ and $P_k f = 0$. Since $C(\mathbb{R}) \subset C^{-k}(\mathbb{R})$ this embedding has to be an extension of the previous one. Indeed if we assume f to be in $C^k(\mathbb{R})$, then $\psi = \tilde{D}^k f = D^k f$ and after k times of partial integration

$$(-1)^k \int_{\mathbb{R}} f(t)\varphi^{(k)}(t)dt = \int_{\mathbb{R}} \psi(t)\varphi(t)dt$$

which is exactly the functional that corresponds to the continuous function $\psi(t)$. Thus this is a proper embedding of $C^{-k}(\mathbb{R})$ in $D'(\mathbb{R})$.

This way $C^{-k}(\mathbb{R})$ is embedded in $D'(\mathbb{R})$, the space of all continuous linear functionals on $D(\mathbb{R})$. Let $D'_{[k]}(\mathbb{R})$ denote the subspace of $D'(\mathbb{R})$ consisting of all $L \in D'(\mathbb{R})$ for which there exists $f \in C(\mathbb{R})$ such that

$$L(\varphi) = \int_{\mathbb{R}} f(t)\varphi^{(k)}(t)dt$$

The elements of $D'_{[k]}(\mathbb{R})$ are called *k-irregular distributions*. Then $\mathcal{L} : \psi \rightarrow L_\psi$ maps $C^{-k}(\mathbb{R})$ one-one onto $D'_{[k]}(\mathbb{R})$. Now let $\hat{C}^{-k}(\mathbb{R})$ be the space generated by the seminorms corresponding to another sequence of projections, say $(\hat{P}_j)_{j \in \mathbb{N}}$. Then there is a one-one map of $C^{-k}(\mathbb{R})$ onto $\hat{C}^{-k}(\mathbb{R})$, namely $\hat{\mathcal{L}}^{-1}\mathcal{L}$. In fact this relation can be seen from the following equality:

$$\begin{aligned} L_F(\varphi) &:= \int_{\mathbb{R}} (\tilde{J}_k F)(t)\varphi^{(k)}(t)dt \\ &= \int_{\mathbb{R}} [(\tilde{J}_k F)(t) + p(t)]\varphi^{(k)}(t)dt \end{aligned}$$

where $p(t) \in \mathcal{P}^{k-1}$.

Lemma 5.3.22 *The Dirac-distribution δ is not in $C^{-1}(\mathbb{R})$.*

Proof: The first question to be posed is how can we check whether an element of $D'(\mathbb{R})$ is in $C^{-1}(\mathbb{R})$ or not. It is natural to do this via the construction of $C^{-1}(\mathbb{R})$ as a distribution-space, like we just made. To find out whether or not the Dirac-distribution $\delta(\varphi)$ is in $C^{-1}(\mathbb{R})$ we must find out if there exists an element in the distribution-space $C^{-1}(\mathbb{R})$ that "acts like δ ". Or equivalently, is there an element ψ in $C^{-1}(\mathbb{R})$ such that (if $\psi = \tilde{D}f$)

$$\langle \delta, \varphi \rangle = L_\psi(\varphi) = - \int_{\mathbb{R}} f(t) \dot{\varphi}(t) dt = \varphi(0)$$

If we denote the Heaviside-function by $H(t)$ then

$$\forall \varphi \in C_c^\infty(\mathbb{R}) \quad \left[- \int_{\mathbb{R}} H(t) \dot{\varphi}(t) dt = \varphi(0) \right]$$

Hence for all $\varphi \in C_c^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} (f(t) - H(t)) \dot{\varphi}(t) dt = 0$$

This is equivalent to: for all $\omega \in C_c^\infty(\mathbb{R})$ ($\omega = \dot{\varphi}$) with $\int_{\mathbb{R}} \omega(t) dt = 0$

$$\int_{\mathbb{R}} (f(t) - H(t)) \omega(t) dt = 0 \quad (5.3)$$

Take any compact interval, say $[a, b]$ and define

$$\lambda := \int_a^b (f(t) - H(t)) dt$$

Then we have for all $\omega(t) \in C_{[a,b]}^\infty(\mathbb{R})$ with $\langle \omega, 1 \rangle = 0$ (where $\langle f, g \rangle$ denotes the usual inner product, i.e. $\langle f, g \rangle := \int_a^b f(t)g(t)dt$) that

$$\langle f - H - \lambda \cdot 1, \omega \rangle = 0$$

Define $\xi := f - H - \lambda \cdot 1$. Clearly there exists a sequence ω_n in $C_{[a,b]}^\infty(\mathbb{R})$ that converges in L_2 -sense to ξ on $[a, b]$. At the moment it is not necessarily true that $\langle \omega_n, 1 \rangle = 0$ for all $n \in \mathbb{N}$. Because we want to take advantage

of equation 5.3 we will construct another sequence $\bar{\omega}_n := \omega_n - \lambda_n \cdot 1$, where $\lambda_n := \langle \omega_n, 1 \rangle$. Now we have $\langle \bar{\omega}_n, 1 \rangle = 0$. Does the sequence $(\bar{\omega}_n)_{n \in \mathbb{N}}$ converge to ξ ?

$\langle \bar{\omega}_n, 1 \rangle = \langle \omega_n, 1 \rangle - \langle \lambda_n, 1 \rangle$. Since $\langle \bar{\omega}_n, 1 \rangle = 0$ for all $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \langle \omega_n, 1 \rangle = \lim_{n \rightarrow \infty} \langle \omega_n, 1 \rangle = \langle \xi, 1 \rangle = 0$$

we obtain that $\lim_{n \rightarrow \infty} \lambda_n = 0$. Hence $\lim_{n \rightarrow \infty} \bar{\omega}_n = \xi$ indeed. Herewith

$$0 = \langle \xi, \bar{\omega}_n \rangle \longrightarrow \langle \xi, \xi \rangle \quad (n \rightarrow \infty)$$

and hence $\xi = f - H - \lambda \cdot 1 = 0$ a.e. on $[a, b]$. Thus $f - \lambda \cdot 1 = H$ a.e. on $[a, b]$. The interval $[a, b]$ was arbitrary so take a and b such that $0 \in [a, b]$. This yields a contradiction. So there is no delta in $C^{-1}(\mathbb{R})$. $\square$

Heuristically one could expect δ to be in $C^{-2}(\mathbb{R})$, because integrating the Heaviside-function $H(t)$ leads to a continuous function and "differentiating" $H(t)$ in some sense gives us δ . Indeed δ is the second 'derivative' of the continuous function $q(t)$ defined by

$$q(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } t \geq 0 \end{cases}$$

5.4 The shift-operator on $C^{-k}(\mathbb{R})$, $k \in \mathbb{N}$

The one parameter group of shift-operators is closely related to the differentiation operator. Since we are going to deal with differential equations in some generalized sense we defined a generalized differential operator.

It will turn out that we need a kind of shift-operator on $C^{-k}(\mathbb{R})$ too. However $C^{-k}(\mathbb{R})$ is not a function-space and hence it is not clear, in advance, what $\sigma^t F$ means for $F \in C^{-k}(\mathbb{R})$, $t \in \mathbb{R}$. It seems natural to relate the definition of the shift-operator on $C^{-k}(\mathbb{R})$ to the shift-operator on a space where it is well-defined. That is what we intend to do now.

From lemma 5.3.9 we obtain that for every $F \in C^{-k}(\mathbb{R})$, $k \in \mathbb{N}$ there exists $f \in C(\mathbb{R})$ such that $\tilde{D}^k f = F$, namely take $f = \tilde{J}_k F$. On $C(\mathbb{R})$ the shift-operator σ^t is well-defined:

$$\sigma^t f(s) := f(s + t)$$

Now we define the shift operator on $C^{-k}(\mathbb{R})$ in the following way:

Definition 5.4.1 The (generalized) shift operators $\tilde{\sigma}^t$ on $C^{-k}(\mathbb{R})$, $k \in \mathbb{N}$ are defined as follows:

$$\tilde{\sigma}^t := \tilde{D}^k \sigma^t \tilde{J}_k$$

This is clearly a natural generalization of σ^t on $C(\mathbb{R})$. Namely, for $f \in C(\mathbb{R})$ $\sigma^t \tilde{J}_k f \in C^k(\mathbb{R})$ (see lemma 5.3.6) and hence $\tilde{D}^k \sigma^t = D^k \sigma^t = \sigma^t D^k$.

Definition 5.4.2 $\mathcal{M}$ is a shift-invariant subspace in $C^{-k}(\mathbb{R})$ if and only if

$$\forall t \in \mathbb{R} [F \in \mathcal{M} \Rightarrow \tilde{\sigma}^t F \in \mathcal{M}]$$

Lemma 5.4.3 $\mathcal{M}$ is a shift-invariant subspace in $C^{-k}(\mathbb{R})$ if and only if $\mathcal{M}^{(0)} := \{f \in C(\mathbb{R}) \mid \tilde{D}^k f \in \mathcal{M}\}$ is a shift-invariant subspace in $C(\mathbb{R})$.

Proof: ($\Rightarrow$): Let $f \in \mathcal{M}^{(0)}$, then $\tilde{D}^k f \in \mathcal{M}$. Since $\mathcal{M}$ is shift-invariant we also have $\tilde{\sigma}^t \tilde{D}^k f \in \mathcal{M}$. By definition $\tilde{\sigma}^t \tilde{D}^k f = \tilde{D}^k \sigma^t f \in \mathcal{M}$. Clearly $\sigma^t f \in C(\mathbb{R})$ and hence $\sigma^t f \in \mathcal{M}^{(0)}$.

($\Leftarrow$): Let $F \in \mathcal{M}$ then we have for all $f \in C(\mathbb{R})$ such that $\tilde{D}^k f = F$ that $\tilde{\sigma}^t F = \tilde{\sigma}^t \tilde{D}^k f = \tilde{D}^k \sigma^t f$ and because $\sigma^t f \in \mathcal{M}^{(0)}$ we are done. $\square$

Definition 5.4.4 The set of generalized differential operators on a field $\mathbb{K}$ will be denoted by $LDO^g(\mathbb{K})$.

Remark 5.4.5 The previous lemma also holds true if we replace $\tilde{D}^k$ by some $\tilde{S} \in LDO^g(\mathbb{R})$ of order k . $\square$

Lemma 5.4.6 If $\mathcal{M}$ is a subspace in $C^{-k}(\mathbb{R})$ and $\mathcal{M}^{(0)}$ is a subspace of $C(\mathbb{R})$ defined by $\mathcal{M}^{(0)} := \{f \in C(\mathbb{R}) \mid \tilde{D}^k f \in \mathcal{M}\}$ then

$$\mathcal{M}^{(0)} = P^{k-1} \oplus \tilde{J}_k \mathcal{M}$$

Proof: This is a consequence of lemma 5.3.5 and the fact that all $f \in C(\mathbb{R})$ can be written as $f = P_k f + \tilde{J}_k \tilde{D}^k f$. $\square$

Lemma 5.4.7 Let $\mathcal{M}$ be a subspace in $C^{-k}(\mathbb{R})$. Then $\mathcal{M}$ is closed in $C^{-k}(\mathbb{R})$ if and only if $\mathcal{M}^{(0)} = \{f \in C(\mathbb{R}) \mid \tilde{D}^k f \in \mathcal{M}\}$ is closed in $C(\mathbb{R})$.

Proof: ($\Rightarrow$): Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}^{(0)}$ converging to f in $C(\mathbb{R})$ -sense. Then clearly $P_k f_n \rightarrow P_k f$ in $C(\mathbb{R})$ -sense. Using lemma 5.4.6 we obtain that for all $n \in \mathbb{N}$ $f_n = P_k f_n + \tilde{J}_k G_n$ for some $G_n \in \mathcal{M}$ and $\tilde{J}_k G_n$ converges in $C(\mathbb{R})$ -sense. Consequently $(G_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence in $C^{-k}(\mathbb{R})$ and converges to some $G \in C^{-k}(\mathbb{R})$. Since $(G_n)_{n \in \mathbb{N}}$ is also a sequence in the closed subspace $\mathcal{M}$ we know that $G \in \mathcal{M}$. Hence we have $f_n \rightarrow f$ and

$$f_n = P_k f_n + \tilde{J}_k G_n \rightarrow P_k f + \tilde{J}_k G \quad (n \rightarrow \infty)$$

Thus $f = P_k f + \tilde{J}_k G \in \mathcal{M}^{(0)}$.

($\Leftarrow$): Let $(F_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}$ converging to F in $C^{-k}(\mathbb{R})$ -sense. Then clearly $\tilde{J}_k F_n \rightarrow \tilde{J}_k F$ in $C(\mathbb{R})$ -sense. Since $\tilde{D}^k(\tilde{J}_k F_n) = F_n$ is in $\mathcal{M}$ for all $n \in \mathbb{N}$ we have that $\tilde{J}_k F_n$ is in $\mathcal{M}^{(0)}$. $\mathcal{M}^{(0)}$ being closed in $C(\mathbb{R})$ yields $\tilde{J}_k F \in \mathcal{M}^{(0)}$ and hence $\tilde{D}^k(\tilde{J}_k F) = F \in \mathcal{M}$. $\square$

Corollary 5.4.8 $\mathcal{M}$ is a closed shift-invariant subspace in $C^{-k}(\mathbb{R})$ if and only if $\mathcal{M}$ is the closure in $C^{-k}(\mathbb{R})$ -sense of the Bohl-functions in $\mathcal{M}^{(0)}$ (as defined in lemma 5.4.6).

Proof: This is an immediate consequence of the fact that $\tilde{D}^k$ maps $C(\mathbb{R})$ continuously into $C^{-k}(\mathbb{R})$ and of lemma 2.6.8 and the theory in section 2.6. $\square$

The relation between the shift-operator and the differentiation operator is important for us and the following lemmas express this relation. Furthermore they show that we extended the differential operator D in a natural way.

Lemma 5.4.9 For all $f \in C(\mathbb{R})$, $\lim_{t \rightarrow 0} \frac{\sigma^t f - f}{t}$ exists in $C^{-1}(\mathbb{R})$ -sense and moreover the limit equals $\tilde{D}f$.

Proof: Let $f \in C(\mathbb{R})$, $n \in \mathbb{N}$, then

$$\begin{aligned} p_n^{-1}\left(\frac{\sigma^t f - f}{t} - \tilde{D}f\right) &= p_n(I_1\left(\frac{\sigma^t f - f}{t}\right) - f + f(0) \cdot 1) \\ &= \max_{x \in [-n, n]} \left(\frac{1}{t} \int_0^x f(\xi + t) d\xi - \frac{1}{t} \int_0^x f(\xi) d\xi - f(x) + f(0) \right) \\ &= \max_{x \in [-n, n]} \left(\frac{1}{t} \int_x^{x+t} f(\xi) d\xi + \frac{1}{t} \int_t^x f(\xi) d\xi - \frac{1}{t} \int_0^x f(\xi) d\xi + \right. \end{aligned}$$

$$\begin{aligned}
&= \max_{x \in [-n, n]} \left(\frac{1}{t} \int_x^{x+t} f(\xi) d\xi - f(x) + f(0) - \frac{1}{t} \int_0^t f(\xi) d\xi \right) \\
&\leq 2 \cdot \max_{x \in [-n, n]} \frac{1}{t} \int_x^{x+t} |f(\xi) - f(x)| d\xi
\end{aligned}$$

Hence it remains to prove that

$$\lim_{t \rightarrow 0} \max_{x \in [-n, n]} \frac{1}{t} \int_x^{x+t} |f(\xi) - f(x)| d\xi = 0$$

but this follows from the fact that f is uniformly continuous on any compact interval. Hence

$$\forall n \in \mathbb{N} \left[\lim_{t \rightarrow 0} p_n^{-1} \left(\frac{\sigma^t f - f}{t} - \tilde{D}f \right) = 0 \right]$$

which completes the proof. $\square$

Lemma 5.4.10 For all $F \in C^{-k}(\mathbb{R})$, $\lim_{t \rightarrow 0} \frac{\tilde{\sigma}^t F - F}{t}$ exists in $C^{-k-1}(\mathbb{R})$ -sense and equals $\tilde{D}F$.

Proof: Let $F \in C^{-k}(\mathbb{R})$, $m \in \mathbb{N}$, then if $F = \tilde{D}^k f$ for some $f \in C(\mathbb{R})$

$$\begin{aligned}
p_m^{-k-1} \left(\frac{\tilde{\sigma}^t F - F}{t} - \tilde{D}F \right) &= p_m^{-k-1} \left(\frac{\tilde{\sigma}^t \tilde{D}^k f - \tilde{D}^k f}{t} - \tilde{D}^{k+1} f \right) \\
&= p_m^{-k-1} \left(\tilde{D}^k \left(\frac{\sigma^t f - f}{t} \right) - \tilde{D}^{k+1} f \right) \\
&= p_m(\tilde{J}_{k+1} \tilde{D}^k \left(\frac{\sigma^t f - f}{t} \right) - \tilde{J}_{k+1} \tilde{D}^{k+1} f) \\
&= p_m(Q_{k+1} J_1 \tilde{J}_k \tilde{D}^k \left(\frac{\sigma^t f - f}{t} \right) - \tilde{J}_{k+1} \tilde{D}^{k+1} f) \\
&= p_m(Q_{k+1} I_1 \left(\frac{\sigma^t f - f}{t} - P_k \frac{\sigma^t f - f}{t} \right) - Q_{k+1} f) \\
&= p_m(Q_{k+1} [I_1 \left(\frac{\sigma^t f - f}{t} \right) - f]) \\
&= p_m(Q_{k+1} [I_1 \left(\frac{\sigma^t f - f}{t} \right) - f + f(0) \cdot 1])
\end{aligned}$$

From the proof in the previous lemma we know that

$$\lim_{t \rightarrow 0} p_m([I_1(\frac{\sigma^t f - f}{t}) - f + f(0) \cdot 1]) = 0$$

and since Q_{k+1} is a continuous operator from $C(\mathbb{R})$ to $C(\mathbb{R})$ we obtain

$$\lim_{t \rightarrow 0} p_m^{-k-1}(\frac{\sigma^t F - F}{t} - \tilde{D}F) = 0$$

Since m was arbitrary we have:

$$\tilde{D}F = \lim_{t \rightarrow 0} \frac{\tilde{\sigma}^t F - F}{t} \text{ in } C^{-k-1}(\mathbb{R})\text{-sense}$$

□

Lemma 5.4.11 *For all $k \in \mathbb{N}$ the following is true:*

Let $F \in C^{-k}(\mathbb{R})$. If $\lim_{t \rightarrow 0} \frac{\tilde{\sigma}^t F - F}{t}$ exists in $C^{-k}(\mathbb{R})$ -sense, then $F \in C^{-k+1}(\mathbb{R})$ and $\lim_{t \rightarrow 0} \frac{\tilde{\sigma}^t F - F}{t} = \tilde{D}F$, where $\tilde{D}$ is the generalized differential operator from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ with $\text{Dom}(\tilde{D}) = C^{-k+1}(\mathbb{R})$.

Proof: Because $F \in C^{-k}(\mathbb{R})$ there exists $f \in C(\mathbb{R})$ such that $\tilde{D}^k f = F$. From lemma 5.4.10 we obtain that $\lim_{t \rightarrow 0} \frac{\sigma^t f - f}{t} = \tilde{D}f$. Now let

$$\begin{cases} \varphi_m := \frac{\sigma^{\frac{1}{m}} f - f}{\frac{1}{m}} & \rightarrow \tilde{D}f \text{ (} m \rightarrow \infty \text{)} \\ \tilde{D}^k \varphi_m = \tilde{D}^k \left(\frac{\sigma^{\frac{1}{m}} f - f}{\frac{1}{m}} \right) & \rightarrow \psi \text{ (} m \rightarrow \infty \text{)} \end{cases}$$

From lemma 5.3.14 we know that $\tilde{D}^k$ as an operator from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ with domain $\text{Dom}(\tilde{D}^k) = C(\mathbb{R})$, is a closed operator and so $\tilde{D}f \in \text{Dom}(\tilde{D}^k) = C(\mathbb{R})$ and

$$\tilde{D}^k(\tilde{D}f) = \lim_{m \rightarrow \infty} \tilde{D}^k \left(\frac{\sigma^{\frac{1}{m}} f - f}{\frac{1}{m}} \right) = \lim_{m \rightarrow \infty} \frac{\tilde{\sigma}^{\frac{1}{m}} F - F}{\frac{1}{m}} = \tilde{D}F$$

Hence $f \in C^1(\mathbb{R})$ and $\tilde{D}^k f = F \in C^{-k+1}(\mathbb{R})$, $\tilde{D}F = \lim_{t \rightarrow 0} \frac{\sigma^t F - F}{t}$. □

Lemma 5.4.12 *A continuous linear functional $\mathcal{L}$ on $C^{-k}(\mathbb{R})$, $k \in \mathbb{N}$ is of the form: $\mathcal{L}(\psi) = \mu(\tilde{J}_k \psi)$, $\psi \in C^{-k}(\mathbb{R})$, for some $\mu \in ba_c(\mathbb{R})$ such that $\mu(\mathbf{P}^{k-1}) = 0$.*

Proof: Since $\mathcal{L}$ is continuous we have, for all $\psi \in C^{-k}(\mathbb{R})$

$$|\mathcal{L}(\psi)| \leq C \cdot \bar{p}_m^{(-k)}(\psi) = C \cdot p_m(\tilde{J}_k \psi) \leq C_1 \cdot p_r(f)$$

for some $C_1 \in \mathbb{R}$, $r \in \mathbb{N}$ and $f \in C(\mathbb{R})$ such that $\tilde{D}^k f = \psi$. Note that this is independent of the choice of f .

Hence $|(\mathcal{L} \circ \tilde{D}^k)(f)| \leq C_1 \cdot p_r(f)$ for all $f \in C(\mathbb{R})$. Consequently $\mathcal{L} \circ \tilde{D}^k$ is a continuous linear functional on $C(\mathbb{R})$, and clearly $(\mathcal{L} \circ \tilde{D}^k)|_{\mathbf{P}^{k-1}} = 0$. Thus $\mathcal{L} \circ \tilde{D}^k = \mu$ for some $\mu \in ba_c(\mathbb{R})$ with $\mu|_{\mathbf{P}^{k-1}} = 0$, and herewith $\mathcal{L}(\psi) = \mu(\tilde{J}_k \psi)$ for all $\psi \in C^{-k}(\mathbb{R})$. $\square$

We conclude this section with two commutation-relations.

Lemma 5.4.13 *For all $t \in \mathbb{R}$, $k \in \mathbb{N}$: $\sigma^t Q_k - Q_k \sigma^t = P_k \sigma^t - \sigma^t P_k$ on $C(\mathbb{R})$.*

Proof: Obvious. $\square$

Lemma 5.4.14 *For all $t \in \mathbb{R}$, $k \in \mathbb{N}$: $\tilde{\sigma}^t \tilde{J}_k - \tilde{J}_k \tilde{\sigma}^t = P_k \sigma^t \tilde{J}_k$ on $C^{-k}(\mathbb{R})$.*

Proof: Let $t \in \mathbb{R}$, $k \in \mathbb{N}$, then

$$\begin{aligned} \tilde{\sigma}^t \tilde{J}_k - \tilde{J}_k \tilde{\sigma}^t &= (\tilde{\sigma}^t \tilde{J}_k - \tilde{J}_k \tilde{\sigma}^t) \tilde{D}^k \tilde{J}_k \\ &= (\tilde{\sigma}^t \tilde{J}_k \tilde{D}^k - \tilde{J}_k \tilde{D}^k \tilde{\sigma}^t) \tilde{J}_k \\ &= (P_k \sigma^t - \sigma^t P_k) \tilde{J}_k \\ &= P_k \sigma^t \tilde{J}_k \end{aligned}$$

since $\tilde{J}_k \tilde{D}^k = Q_k$ on $C(\mathbb{R})$ and $\tilde{J}_k(C(\mathbb{R})) = \{f \in C(\mathbb{R}) | P_k f = 0\}$ (see lemma 5.3.5). $\square$

5.5 Convolution-operators on $C^{-k}(\mathbb{R})$, $k \in \mathbb{N}$

Definition 5.5.1 *A generalized convolution operator on $C^{-k}(\mathbb{R})$ is an operator of the form $\tilde{C}_\mu = \tilde{D}^k C_\mu \tilde{J}_k$, where C_μ is a convolution operator on $C(\mathbb{R})$.*

It is clear that $\tilde{C}_\mu$ is a continuous operator from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ for all $k \in \mathbb{N}$. The generalized convolution operator also commutes with all (generalized) shift-operators. Namely, on $C^{-k}(\mathbb{R})$

$$\begin{aligned}\tilde{C}_\mu \tilde{\sigma}^t &= \tilde{D}^k C_\mu \tilde{J}_k \tilde{D}^k \sigma^t \tilde{J}_k \\ &= \tilde{D}^k C_\mu \sigma^t \tilde{J}_k \\ &= \tilde{D}^k \sigma^t C_\mu \tilde{J}_k \\ &= \tilde{D}^k \sigma^t \tilde{J}_k \tilde{D}^k C_\mu \tilde{J}_k = \tilde{\sigma}^t \tilde{C}_\mu\end{aligned}$$

Lemma 5.5.2 Every convolution-operator $\tilde{C}_\mu$ on $C^{-k}(\mathbb{R})$, $k \in \mathbb{N}$ is the strong (=pointwise) limit of a sequence in the linear span $\langle \{\tilde{\sigma}^t | t \in \mathbb{R}\} \rangle$.

Proof: Let $F \in C^{-k}(\mathbb{R})$, then $\tilde{J}_k F =: f \in C(\mathbb{R})$ and hence there exists a sequence $(C_n)_{n \in \mathbb{N}}$ in $\text{span} \langle \{\sigma^t | t \in \mathbb{R}\} \rangle$ such that $C_n f \rightarrow C_\mu f$ in $C(\mathbb{R})$ -sense (see theorem 2.5.1). But also $(\tilde{C}_n)_{n \in \mathbb{N}}$ (where $\tilde{C}_n := \tilde{D}^k C_n \tilde{J}_k$) converges in $C^{-k}(\mathbb{R})$ -sense to $\tilde{C}_\mu F$. Namely,

$$\begin{aligned}\bar{p}_m^{(-k)}(\tilde{C}_n F - \tilde{C}_\mu F) &= p_m(\tilde{J}_k \tilde{D}^k C_n f - \tilde{J}_k \tilde{D}^k C_\mu f) \\ &= p_m(Q_k(C_n f - C_\mu f)) \\ &\leq C \cdot p_r(C_n f - C_\mu f) \rightarrow 0 \quad (n \rightarrow \infty)\end{aligned}$$

□

Lemma 5.5.3 For all $k \in \mathbb{N}$: W is a continuous operator from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ such that $\tilde{\sigma}^t W - W \tilde{\sigma}^t = 0$ for all $t \in \mathbb{R}$ if and only if for all $l \in \mathbb{N}$, $l \leq k$ W is continuous from $C^{-l}(\mathbb{R})$ to $C^{-l}(\mathbb{R})$ and $\tilde{D}W = W\tilde{D}$ on $C^{-k+1}(\mathbb{R})$.

Proof: Let $k \in \mathbb{N}$. Take $f \in C^{-k+1}(\mathbb{R})$, then for all $t \in \mathbb{R}$, $t \neq 0$

$$0 = (\tilde{\sigma}^t W - W \tilde{\sigma}^t)f = \frac{(\tilde{\sigma}^t - I)}{t} W f - W \frac{(\tilde{\sigma}^t - I)}{t} f \quad (5.4)$$

From lemma 5.4.10 we know that $\lim_{t \rightarrow 0} \frac{(\tilde{\sigma}^t - I)}{t} f$ exists in $C^{-k}(\mathbb{R})$ -sense and equals $\tilde{D}f$. Since W is continuous from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ we obtain:

$$\lim_{t \rightarrow 0} W\left(\frac{\tilde{\sigma}^t f - f}{t}\right) = W\tilde{D}f$$

Therefore $\lim_{t \rightarrow 0} \frac{\tilde{\sigma}^t W f - W f}{t}$ exists in $C^{-k}(\mathbb{R})$ -sense too.
Apply lemma 5.4.11 to obtain:

$$\lim_{t \rightarrow 0} \frac{\tilde{\sigma}^t W f - W f}{t} = \tilde{D} W f$$

and $W f \in C^{-k+1}(\mathbb{R})$.

So, $\tilde{D} W f = W \tilde{D} f$ and W maps $C^{-k+1}(\mathbb{R})$ into itself.

Now, let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $C^{-k+1}(\mathbb{R})$ such that

$$\begin{cases} f_n & \rightarrow 0 \\ W f_n & \rightarrow g \end{cases} \text{ in } C^{-k+1}(\mathbb{R})\text{-sense}$$

This implies $f_n \rightarrow f$ in $C^{-k}(\mathbb{R})$ -sense and, since W is continuous from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$, $W f_n \rightarrow W f$ in $C^{-k}(\mathbb{R})$ -sense. But also $W f_n \rightarrow g$ in $C^{-k}(\mathbb{R})$ -sense and thus $W f = g$. Because $f \in C^{-k+1}(\mathbb{R})$, $W f \in C^{-k+1}(\mathbb{R})$. Hence W is a closed operator from $C^{-k+1}(\mathbb{R})$ into itself. The closed-graph-theorem yields that W is continuous from $C^{-k+1}(\mathbb{R})$ into itself. We can repeat this argument as often as we want. So we are done. $\square$

At this point we can generalize the result on the characterization of convolution-operators on $C(\mathbb{R})$.

Lemma 5.5.4 *W is a continuous operator from $C^k(\mathbb{R})$ to $C^k(\mathbb{R})$, $k \in \mathbb{Z}$ such that $\tilde{\sigma}^t W - W \tilde{\sigma}^t = 0$ for all $t \in \mathbb{R}$ if and only $W = \tilde{C}_\mu$ for some $\mu \in ba_c(\mathbb{R})$.*

Proof: For $k \in \mathbb{N}$ it is clear (see lemma 2.4.32). So we only consider $C^{-k}(\mathbb{R})$ for $k \in \mathbb{N}$.

$\Rightarrow$: From lemma 2.4.4 we know that $W|_{C(\mathbb{R})} = C_\mu$ for some $\mu \in ba_c(\mathbb{R})$, because $W|_{C(\mathbb{R})}$ maps $C(\mathbb{R})$ into itself (see lemma 2.4.13). Let $F \in C^{-k}(\mathbb{R})$, then, for some $f \in C(\mathbb{R})$, $\tilde{D}^k f = F$ and

$$\begin{aligned} W F &= W \tilde{D}^k f = \tilde{D}^k W f \\ &= \tilde{D}^k C_\mu f \\ &= \tilde{D}^k C_\mu (\tilde{J}_k F + P_k f) \\ &= \tilde{D}^k C_\mu \tilde{J}_k F + C_\mu \tilde{D}^k P_k f \\ &= \tilde{D}^k C_\mu \tilde{J}_k F \end{aligned}$$

$\Leftarrow$: The continuity of W is a consequence of the closed-graph-theorem, since W is defined on the whole of $C^{-k}(\mathbb{R})$. Furthermore, using the commutation-relation from lemma 5.4.14 we arrive at

$$\begin{aligned}\tilde{\sigma}^t \tilde{D}^k C_\mu \tilde{J}_k &= \tilde{D}^k C_\mu \tilde{\sigma}^t \tilde{J}_k \\ &= \tilde{D}^k C_\mu (\tilde{J}_k \tilde{\sigma}^t + P_k \sigma^t \tilde{J}_k) \\ &= \tilde{D}^k C_\mu \tilde{J}_k \tilde{\sigma}^t\end{aligned}$$

where we used that C_μ maps $\mathbf{P}^{k-1}$ into itself. □

Lemma 5.5.5 *K is a continuous linear operator from $C^{-k}(\mathbb{R})$ ($k \in \mathbb{N}$) to $C(\mathbb{R})$ such that $\sigma^t K - K \tilde{\sigma}^t = 0$ for all $t \in \mathbb{R}$ if and only if $K = C_\mu \tilde{J}_k$ for some $\mu \in ba_c(\mathbb{R})$ with $\mu(\mathbf{P}^{k-1}) = 0$.*

Proof: Since K maps $C^{-k}(\mathbb{R})$ into $C(\mathbb{R})$ we can evaluate in one point. Hence $(KF)(0)$ defines for all $F \in C^k(\mathbb{R})$ a continuous linear functional on $C^{-k}(\mathbb{R})$. Thus $(KF)(0) = \mathcal{L}(F) = \mu(\tilde{J}_k F)$ for all $F \in C^{-k}(\mathbb{R})$, where $\mu \in ba_c(\mathbb{R})$ and $\mu(\mathbf{P}^{k-1}) = 0$.

Then we have

$$\begin{aligned}(KF)(t) &= (\sigma^t KF)(0) = (K \tilde{\sigma}^t F)(0) \\ &= \mu(\tilde{J}_k \tilde{\sigma}^t F) \\ &= \mu(\sigma^t \tilde{J}_k F) \\ &= (C_\mu \tilde{J}_k F)(t)\end{aligned}$$

where we used lemma 5.4.14 and $\mu(\mathbf{P}^{k-1}) = 0$. □

Corollary 5.5.6 *K is a continuous operator from $C^{-k}(\mathbb{R})$ to $C^l(\mathbb{R})$ for ($l \geq 0$) such that $\sigma^t K - K \tilde{\sigma}^t = 0$ for all $t \in \mathbb{R}$ if and only if $K = C_\mu \tilde{J}_k$ for some $\mu \in ba_c(\mathbb{R})$ with $\mu(\mathbf{P}^{k-1}) = 0$ and for every S in $LDO_l(\mathbb{R})$ there exists a $\nu \in ba_c(\mathbb{R})$ such that $SC_\mu = C_\nu$.*

Lemma 5.5.7 *Let $k \in \mathbb{N}$. The following statements are equivalent*

- *K a closed operator from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ with $\mathcal{D}(K) = C(\mathbb{R})$ and $\forall t \in \mathbb{R} [\tilde{\sigma}^t K - K \tilde{\sigma}^t = 0]$*

- $K = \tilde{S}\tilde{C}_\mu$ where $\tilde{S}$ is in $LDO^g(\mathbb{R})$ of order k and μ is not weakly differentiable.

Proof: ($\Rightarrow$): $\text{Ker}(K)$ is a closed subspace in $C^{-k}(\mathbb{R})$. Corollary 5.4.8 yields that $\text{ker}(K)$ is the closure, in $C^{-k}(\mathbb{R})$ -sense of a countable set of Bohl-functions, satisfying the restriction-property. Similar as in the proof of lemma 2.4.23 $\text{ker}(K)$ can be decomposed into

$$\text{ker } K = M_n \oplus R$$

where M_n consists of n linearly independent Bohl-functions with the restriction that $t^l e^{-i\lambda t} \in M_n$ ($\lambda \in \mathbb{C}, l \in \mathbb{N}$) implies $t^j e^{-i\lambda t} \in M_n$ for $j = 0, 1, \dots, l-1$. R is a complementary (not necessarily shift-invariant) subspace in $C^{-k}(\mathbb{R})$.

We have chosen M_n in such a way that it equals the kernel of a linear differential operator with constant coefficients, say S . From lemma 5.3.19 we know that $M_n = \text{ker}(\tilde{S})$ too. There exists a continuous operator $\tilde{J} : C^{-k}(\mathbb{R}) \rightarrow C(\mathbb{R})$ such that $\tilde{S}\tilde{J} = I$ on $C^{-k}(\mathbb{R})$ and $(I - \tilde{J}\tilde{S}) =: P$ is the projection from $C(\mathbb{R})$ onto M_n . The rest of the ($\Rightarrow$)-part of the proof is the same as in lemma 2.4.23. However everything is generalized now. So $K\tilde{J}$ is continuous from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ and hence $K\tilde{J} = \tilde{C}_\mu$ for some $\mu \in ba_c(\mathbb{R})$, and $\tilde{C}_\mu = \tilde{D}^k C_\mu \tilde{J}_k$, etc.

($\Leftarrow$): $K = \tilde{C}_\mu \tilde{S} = \tilde{S}\tilde{C}_\mu = \tilde{S}\tilde{D}^k C_\mu \tilde{J}_k$ and hence $\text{Dom}(K) = C(\mathbb{R})$. Moreover $\tilde{C}_\mu$ maps $C(\mathbb{R})$ into itself and $\tilde{S}$ with $\text{Dom}(\tilde{S}) = C(\mathbb{R})$ is a closed operator from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ (lemma 5.3.17). This completes the proof. $\square$

Remark 5.5.8 As in the non-generalized case the representation of the operator K is far from unique. $\square$

Now we will give generalizations of some lemmas from section 2.5.

Lemma 5.5.9 Let $\tilde{C} \in \text{span} < \{\tilde{\sigma}^t | t \in \mathbb{R}\} >$ and let K be a closed linear mapping from $C(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ with $\sigma^t(\text{Dom}(K)) = \text{Dom}(K)$ and $K\sigma^t f = \tilde{\sigma}^t Kf$, $f \in \text{Dom}(K)$ and $t \in \mathbb{R}$. Then $\tilde{C}(\text{Dom}(K)) \subseteq \text{Dom}(K)$ and for all $f \in \text{Dom}(K)$, $\tilde{C}Kf = K\tilde{C}f$.

Proof: Obvious. $\square$

Theorem 5.5.10 *Let K be a closed linear mapping from $C(\mathbb{R})$ to $C^{-k}(\mathbb{R})$. Suppose for all $t \in \mathbb{R}$, $\sigma^t \text{Dom}(K) \subseteq \text{Dom}(K)$ and $K\sigma^t f = \tilde{\sigma}^t Kf$, $f \in \text{Dom}(K)$. Then for all $\mu \in \text{ba}_c(\mathbb{R})$ and for all $f \in \text{Dom}(K)$, $C_\mu f \in \text{Dom}(K)$ and $KC_\mu f = \tilde{C}_\mu Kf$. Moreover if S is some element in $\text{LDO}(\mathbb{R})$ then if f, Sf are in $\text{Dom}(K)$, then $KSf = \tilde{S}Kf$.*

Proof: The first part is similar to the proof of lemma 2.5.3, using the lemmas 5.5.2 and 5.5.9. Let $f, Sf \in \text{Dom}(K)$, where the operator S can be written as $\prod_{j=1}^n (D + \lambda_j I)$, and define for all $t \in \mathbb{R}$

$$\mu_t(\tau) = \begin{cases} 0 & \tau < 0 \\ \frac{\tau}{t} & 0 \leq \tau \leq t \\ 1 & \tau \geq t \end{cases}$$

then $\mu_t \in \text{ba}_c(\mathbb{R})$.

For all $f \in C(\mathbb{R})$, $C_{\mu_t} f = \int_{\mathbb{R}} \sigma^\tau f d\mu_t(\tau) = \frac{1}{t} \int_0^t \sigma^\tau f d\tau$. Consequently for all $F \in C^{-k}(\mathbb{R})$

$$\begin{aligned} \tilde{C}_{\mu_t} \tilde{D}F &= \tilde{D} \tilde{D}^{k+1} C_{\mu_t} \tilde{J}_{k+1} F \\ &= \tilde{D}^{k+1} C_{\mu_t} D \tilde{J}_{k+1} F \\ &= \tilde{D}^{k+1} \left[\frac{1}{t} \int_0^t \sigma^\tau D \tilde{J}_{k+1} F d\tau \right] \\ &= \tilde{D}^{k+1} \left[\frac{\sigma^t - I}{t} \tilde{J}_{k+1} F \right] \\ &= \frac{\tilde{\sigma}^t - I}{t} F \end{aligned}$$

We see that $\tilde{C}_{\mu_t}$ maps $C^l(\mathbb{R})$ into $C^{l+1}(\mathbb{R})$ for all $l \in \mathbb{Z}$. From the first part of the proof we obtain

$$\begin{aligned} \left(\prod_{j=1}^n \tilde{C}_{\mu_t} \right) KSf &= K \left(\prod_{j=1}^n C_{\mu_t} \right) Sf \\ &= KS \left(\prod_{j=1}^n C_{\mu_t} \right) f \end{aligned}$$

Let us evaluate $S(\prod_{j=1}^n C_{\mu_t})f$ step by step:

$$\begin{aligned} (D + \lambda_j I) C_{\mu_t} f &= C_{\mu_t} Df + \lambda_j C_{\mu_t} f \\ &= \frac{\sigma^t - I}{t} f + \lambda_j C_{\mu_t} f \end{aligned}$$

This can also be done in the generalized case:

$$(\tilde{D} + \lambda_j I) \tilde{C}_{\mu_t} F = \frac{\tilde{\sigma}^t - I}{t} F + \lambda_j \tilde{C}_{\mu_t} F$$

We can do this n times to arrive at

$$S[\prod_{j=1}^n C_{\mu_t}] f = [\prod_{j=1}^n (\frac{\sigma^t - I}{t} + \lambda_j C_{\mu_t})] f$$

Hence

$$\begin{aligned} KS(\prod_{j=1}^n C_{\mu_t}) f &= K[\prod_{j=1}^n (\frac{\sigma^t - I}{t} + \lambda_j C_{\mu_t})] f \\ &= [\prod_{j=1}^n (\frac{\tilde{\sigma}^t - I}{t} + \lambda_j \tilde{C}_{\mu_t})] K f \\ &= \tilde{S}[\prod_{j=1}^n \tilde{C}_{\mu_t}] K f \end{aligned}$$

Define $\tilde{C}_{\nu_t} = \prod_{j=1}^n \tilde{C}_{\mu_t}$ then we have

$$\tilde{C}_{\nu_t} K S f = \tilde{S} \tilde{C}_{\nu_t} K f$$

We claim that, if $t \downarrow 0$, $\tilde{C}_{\nu_t}$ converges strongly to the identity operator I on all spaces $C^l(\mathbb{R})$, $l \in \mathbb{Z}$. We will show it for $l \in \mathbb{Z}^-$ only; the other part goes similarly.

$$\begin{aligned} \tilde{C}_{\nu_t} F &= \tilde{C}_{\mu_t} \circ \dots \circ \tilde{C}_{\mu_t} F \\ &= \tilde{C}_{\mu_t} \circ \dots \circ \tilde{D}^k C_{\mu_t} \tilde{J}_k F \\ &= \tilde{D}^k C_{\mu_t} \circ \dots \circ C_{\mu_t} \tilde{J}_k F \\ &= \tilde{D}^k (\frac{1}{t^n} \int_0^t \dots \int_0^t \sigma^{\tau_1 + \dots + \tau_n} \tilde{J}_k F d\tau_n \dots d\tau_1) \end{aligned}$$

where we used that the convolution operators C_{μ_t} commute with all shift-operators σ^t .

If $f \in C(\mathbb{R})$, $0 < t < \frac{1}{n}$, then for all $m \in \mathbb{N}$

$$p_m(\frac{1}{t^n} \int_0^t \dots \int_0^t \sigma^{\tau_1 + \dots + \tau_n} f d\tau_n \dots d\tau_1 - f) =$$

$$\begin{aligned}
&= p_m \left(\frac{1}{t^n} \int_0^t \dots \int_0^t (\sigma^{\tau_1+\dots+\tau_n} f - f) d\tau_n \dots d\tau_1 \right) \\
&\leq \frac{1}{t^n} \int_0^t \dots \int_0^t p_m (\sigma^{\tau_1+\dots+\tau_n} f - f) d\tau_n \dots d\tau_1 \\
&\leq \max_{\tau_1+\dots+\tau_n \leq t} (\sigma^{\tau_1+\dots+\tau_n} f - f) \rightarrow 0 \quad (t \downarrow 0)
\end{aligned}$$

where we used uniform continuity.

Hence

$$\frac{1}{t^n} \int_0^t \dots \int_0^t \sigma^{\tau_1+\dots+\tau_n} \tilde{J}_k F d\tau_n \dots d\tau_1 \rightarrow \tilde{J}_k F$$

in $C(\mathbb{R})$ -sense.

Thus if $t \downarrow 0$ we obtain $\tilde{C}_\nu F \rightarrow F$ in $C^{-k}(\mathbb{R})$ -sense, since $\tilde{D}^k$ is continuous from $C(\mathbb{R})$ to $C^{-k}(\mathbb{R})$.

We can complete the proof by using the closedness of the operator $\tilde{S}$ from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ with $\mathcal{D}(\tilde{S}) = C^{n-k}(\mathbb{R})$. Namely,

$$\begin{cases} \tilde{C}_\nu K f & \rightarrow K f \text{ in } C^{-k}(\mathbb{R})\text{-sense} \\ \tilde{S} \tilde{C}_\nu K f = \tilde{C}_\nu K S f & \rightarrow K S f \text{ in } C^{-k}(\mathbb{R})\text{-sense} \end{cases}$$

implies $K f \in \text{Dom}(\tilde{S})$ and $\tilde{S} K f = K S f$. □

Lemma 5.5.11 *Let $\tilde{S} \in LDO^g(\mathbb{R})$ and $\mu \in \text{ba}_c(\mathbb{R})$. The operator $\tilde{S} \tilde{C}_\mu$ with $\text{Dom}(\tilde{S} \tilde{C}_\mu) = \{ f \in C^{-k}(\mathbb{R}) \mid \tilde{C}_\mu f \in \text{Dom}(\tilde{S}) \}$ is closed from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$.*

Proof: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $\text{Dom}(\tilde{S})$ such that

$$\begin{cases} f_n & \rightarrow f \\ \tilde{S} \tilde{C}_\mu f_n & \rightarrow g \end{cases} \text{ in } C^{-k}(\mathbb{R})\text{-sense}$$

Since $\tilde{C}_\mu$ is continuous from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ we obtain

$$\begin{cases} \tilde{C}_\mu f_n & \rightarrow \tilde{C}_\mu f \\ \tilde{S} \tilde{C}_\mu f_n & \rightarrow g \end{cases} \text{ in } C^{-k}(\mathbb{R})\text{-sense}$$

Because $(\tilde{C}_\mu f_n)_{n \in \mathbb{N}}$ is a sequence in $\text{Dom}(\tilde{S})$ and with this domain $\tilde{S}$ is a closed operator from $C^{-k}(\mathbb{R})$ to $C^{-k}(\mathbb{R})$, this yields $\tilde{C}_\mu f \in \text{Dom}(\tilde{S})$ and $\tilde{S} \tilde{C}_\mu f = g$. □

Theorem 5.5.12 *Let K be a closed operator from $C(\mathbb{R})$ to $C^{-k}(\mathbb{R})$, and $C^\infty(\mathbb{R}) \subseteq \text{Dom}(K)$ then $K|_{C^\infty(\mathbb{R})}$ is continuous. If in addition $\sigma^t \text{Dom}(K) = \text{Dom}(K)$ and $K\sigma^t f = \tilde{\sigma}^t Kf$ for all $t \in \mathbb{R}$ and $f \in \text{Dom}(K)$. Then there are $\mu \in ba_c(\mathbb{R})$ and $\tilde{S} \in LDO^g(\mathbb{R})$ of some finite order such that $K|_{C^\infty(\mathbb{R})} = \tilde{S}\tilde{C}_\mu|_{C^\infty(\mathbb{R})}$.*

Proof: Similar to the proof of theorem 2.5.4 we can show that $K|_{C^\infty(\mathbb{R})}$ is continuous from $C^\infty(\mathbb{R})$ to $C^{-k}(\mathbb{R})$.

Using the relation ($t \in \mathbb{R} \setminus \{0\}$, $f \in C^\infty(\mathbb{R})$)

$$\left(\frac{\tilde{\sigma}^t - I}{t}\right)Kf - K\left(\frac{\sigma^t - I}{t}\right)f = 0$$

and the fact that $K|_{C^\infty(\mathbb{R})}$ is continuous from $C^\infty(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ we obtain $\tilde{D}Kf = KDf$ and $Kf \in C^{-k+1}(\mathbb{R})$.

Let

$$\begin{cases} f_n & \rightarrow f \text{ in } C^\infty(\mathbb{R})\text{-sense} \\ Kf_n & \rightarrow g \text{ in } C^{-k+1}(\mathbb{R})\text{-sense} \end{cases}$$

Then also $Kf_n \rightarrow g$ in $C^{-k}(\mathbb{R})$ -sense and thus $Kf = g$. Hence $K|_{C^\infty(\mathbb{R})}$ is also a continuous operator from $C^\infty(\mathbb{R})$ to $C^{-k+1}(\mathbb{R})$. We can repeat this as often as we want, which implies that $K|_{C^\infty(\mathbb{R})}$ is, in fact, a continuous operator from $C^\infty(\mathbb{R})$ into itself. Applying lemma 2.4.25 yields that $K|_{C^\infty(\mathbb{R})} = SC_\mu|_{C^\infty(\mathbb{R})}$ for some $S \in LDO(\mathbb{C})$ and $\mu \in ba_c(\mathbb{R})$. □

Lemma 5.5.13 *Let K_1 and K_2 be linear operators from $C(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ with domain $\text{Dom}(K_1)$ and $\text{Dom}(K_2)$ respectively. If $\sigma^t(\text{Dom}(K_i)) \subseteq \text{Dom}(K_i)$, $C^\infty(\mathbb{R}) \subseteq \text{Dom}(K_i)$, for all $f \in \text{Dom}(K_i)$: $\tilde{\sigma}^t K_i f = K_i \sigma^t f$ ($i = 1, 2$) and $K_1|_{C^\infty(\mathbb{R})} = K_2|_{C^\infty(\mathbb{R})}$ then $K_1 = K_2$.*

Proof: Firstly we will construct a $\mu \in ba_c(\mathbb{R})$ such that C_μ maps any space $C^l(\mathbb{R})$ ($l \in \mathbb{N}$) into $C^\infty(\mathbb{R})$.

Define for all $t \in \mathbb{R}$

$$h_t(\tau) = \begin{cases} 0 & \tau < 0 \\ g(\tau) & 0 \leq \tau \leq t \\ 1 & \tau \geq t \end{cases}$$

where $g(\tau)$ is chosen in such a way that $h_t(\tau) \in C^\infty(\mathbb{R})$ and $g'(\tau) \geq 0$. Define $\mu_t(\tau) := h_t(\frac{\tau}{t})$. For all $f \in C(\mathbb{R})$

$$(C_{\mu_t}f)(s) = \int_{\mathbb{R}} (\sigma^\tau f)(s) d\mu_t(\tau) = \frac{1}{t} \int_0^t h'(\frac{\tau}{t})(\sigma^\tau f)(s) d\tau = \int_0^1 h'_t(\theta) f(\theta t + s) d\theta$$

Clearly $C_{\mu_t}f \in C^\infty(\mathbb{R})$, $C_{\mu_t}f \rightarrow f$ ($t \downarrow 0$) in $C(\mathbb{R})$ -sense.

The generalized convolution-operator $\tilde{C}_{\mu_t}$ has the same properties, namely: let $F \in C^{-k}(\mathbb{R})$, then $\tilde{C}_{\mu_t}F = \tilde{D}^k C_{\mu_t} \tilde{J}_k F$ where $\tilde{J}_k F \in C(\mathbb{R})$. Hence $C_{\mu_t} \tilde{J}_k F \rightarrow \tilde{J}_k F$ in $C(\mathbb{R})$ -sense, and since $\tilde{D}^k$ is continuous from $C(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ we showed that $\tilde{C}_{\mu_t}$ converges strongly to the identity operator on $C^{-k}(\mathbb{R})$.

Let $f \in \text{Dom}(K_1)$, then $\tilde{C}_{\mu_t}f \in \text{Dom}(K_1)$, because $\tilde{C}_{\mu_t}f \in C^\infty(\mathbb{R})$. Hence if $t \downarrow 0$ then, because $K_2|_{C^\infty(\mathbb{R})}$ is continuous

$$\begin{cases} \tilde{C}_{\mu_t}f & \rightarrow f \\ K_2 \tilde{C}_{\mu_t}f & \rightarrow g \end{cases} \text{ in } C^{-k}(\mathbb{R})\text{-sense}$$

But $K_2 \tilde{C}_{\mu_t}f = K_1 \tilde{C}_{\mu_t}f = \tilde{C}_{\mu_t} K_1 f$ and $\tilde{C}_{\mu_t} K_1 f \rightarrow K_1 f$. Since K_2 is closed we obtain $f \in \text{Dom}(K_2)$ and $K_2 f = K_1 f$. Of course we can exchange the roles of K_1 and K_2 . $\square$

Corollary 5.5.14 *If the operator K has the properties of theorem 5.5.12 then $K = \tilde{S} \tilde{C}_\mu$ for some $\tilde{S} \in LDO^q(\mathbb{R})$ and $\mu \in ba_c(\mathbb{R})$.*

Corollary 5.5.15 *Let K be a closed operator from $C(\mathbb{R})$ to $C^{-k}(\mathbb{R})$, $C^\infty(\mathbb{R}) \subseteq \text{Dom}(K)$ and $K\sigma^t|_{C^\infty(\mathbb{R})} = \tilde{\sigma}^t K|_{C^\infty(\mathbb{R})}$. Then if K is injective it equals a constant times a shift-operator.*

Corollary 5.5.16 *Let K be a closed operator from $C(\mathbb{R})$ to $C^{-k}(\mathbb{R})$, $C^\infty(\mathbb{R}) \subseteq \mathcal{D}(K)$ and $K\sigma^t|_{C^\infty(\mathbb{R})} = \sigma^t K|_{C^\infty(\mathbb{R})}$. Then if $\ker(K)$ is finite-dimensional, $K = \tilde{\sigma}^t \tilde{S}$ for some $\tilde{S} \in LDO^q(\mathbb{R})$, $a \in \mathbb{R}$.*

5.6 Behaviors in $C(\mathbb{R}) \times C(\mathbb{R})$ with one input and one output

In this section we will give a description of a behavior that contains two external signals. We have only one interesting situation. Namely, it is obvious that if $p=0$, w_1 and w_2 are free variables and if $p=2$ $\mathcal{B}$ is a subspace of

$C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$. In both cases there are no interpretation problems and the introduction of the spaces $C^{-k}(\mathbb{R})$ is superfluous. Let us therefore assume $p=1$. Let us assume the conditions we found in chapter 4 for behaviors in $C(\mathbb{R}, \mathbb{R}^q)$, applied to $q=2, p=1$. Take $\Pi = I$.

C1 B is a linear, shift-invariant subspace, that is closed with respect to the topology of $C(\mathbb{R}) \times C(\mathbb{R})$

C2 B is locally specified

C3 B is a standard $[1, 1]$ -behavior

C4 $\mathcal{E}_1(B)$ or $N_1(B)$ is finite dimensional

The first condition yields

$$\{w_1 \in C(\mathbb{R}) \mid (w_1, 0) \in B\}$$

is a finite-dimensional subspace and because of linearity and shift-invariance equal to $\ker(S_1)$ for some $S_1 \in \text{LDO}(\mathbb{R})$. Define $\text{order}(S_1) := k$, and $\tilde{B}$ as

$$\tilde{B} := \{(\tilde{S}_1 w_1, w_2) \mid (w_1, w_2) \in B\} \subseteq C^{-k}(\mathbb{R}) \times C(\mathbb{R})$$

Since to every w_2 corresponds exactly one $\tilde{S}_1 w_1$ we can consider $\tilde{B}$ as the graph of an operator, say $\tilde{V}$. Define $\tilde{V}$ as an operator from $C(\mathbb{R})$ to $C^{-k}(\mathbb{R})$ with $\text{Dom}(\tilde{V}) = C(\mathbb{R})$ (see C3) as follows

$$\tilde{V} w_2 = \tilde{S}_1 w_1$$

where $(\tilde{S}_1 w_1, w_2) \in \tilde{B}$. Hence

$$(w_1, w_2) \in B \Leftrightarrow \tilde{V} w_2 = \tilde{S}_1 w_1$$

Lemma 5.6.1

$$\tilde{B} := \{(\tilde{S}_1 w_1, w_2) \mid (w_1, w_2) \in B\} \text{ is closed in } C^{-k}(\mathbb{R}) \times C(\mathbb{R})$$

Proof: Take a sequence $(\tilde{S}_1 w_{1n}, w_{2n})_{n \in \mathbb{N}}$ in $\tilde{B}$ such that $(\tilde{S}_1 w_{1n}, w_{2n}) \rightarrow (u, v)$ in $C^{-k}(\mathbb{R}) \times C(\mathbb{R})$ -sense. Since $\tilde{S}_1$ is surjective from $C(\mathbb{R})$ onto $C^{-k}(\mathbb{R})$ (see lemma 5.3.18) and $\ker(\tilde{S}_1)$ is finite-dimensional there exists $\tilde{S}_1^-$, continuous from $C^{-k}(\mathbb{R})$ to $C(\mathbb{R})$ such that $\tilde{S}_1 \tilde{S}_1^- = I$. Hence $(\tilde{S}_1^- \tilde{S}_1 w_{1n}, w_{2n})$

$(\tilde{S}_1^- u, v)$ in $C(\mathbb{R}) \times C(\mathbb{R})$ -sense. Because $(I - \tilde{S}_1^- \tilde{S}_1)w_{1n} \in \ker(\tilde{S}_1)$ we have $((I - \tilde{S}_1^- \tilde{S}_1)w_{1n}, 0) \in B$.

$$(\tilde{S}_1^- \tilde{S}_1 w_{1n}, w_{2n}) = \underbrace{(w_{1n}, w_{2n})}_{\in B} - \underbrace{(I - \tilde{S}_1^- \tilde{S}_1)w_{1n}, 0)}_{\in B}$$

Hence $(\tilde{S}_1^- \tilde{S}_1 w_{1n}, w_{2n})_{n \in \mathbb{N}}$ is a sequence in B , that since B is closed in $C(\mathbb{R}) \times C(\mathbb{R})$, converges to $(\tilde{S}_1^- u, v) \in B$. This yields $(\tilde{S}_1 \tilde{S}_1^- u, v) = (u, v) \in \tilde{B}$. $\square$

Hence $\tilde{V}$ is a continuous operator from $C(\mathbb{R})$ into $C^{-k}(\mathbb{R})$ (closed graph theorem) and thus (using linearity and shift-invariance) $\tilde{V} = \tilde{S}_2 C_\mu$ for some $\tilde{S}_2 \in LDO_q(\mathbb{R})$ and $\mu \in ba_c(\mathbb{R})$ (see corollary 5.5.14). This leads to

$$(w_1, w_2) \in B \Leftrightarrow \tilde{S}_1 w_1 = \tilde{S}_2 C_\mu w_2$$

Along the same arguments as in the treatment of $B^\infty \subseteq C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$ in section 3.1 it can be shown that the conditions C2 and C4 imply $C_\mu = c \cdot I$ for some constant $c \in \mathbb{R}$, $c \neq 0$. Consequently

$$(w_1, w_2) \in B \Leftrightarrow \tilde{S}_1 w_1 = c \cdot \tilde{S}_2 w_2$$

where the equality is in $C^{-k}(\mathbb{R})$. Since $C^{-k}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$ this implies that for all $\varphi \in \mathcal{D}(\mathbb{R})$

$$\langle \tilde{S}_1 w_1, \varphi \rangle = c \langle \tilde{S}_2 w_2, \varphi \rangle$$

But an easy argument, using sequences in $C^\infty(\mathbb{R})$ shows that

$$\langle \tilde{S}_1 w_1, \varphi \rangle = \langle w_1, \tilde{S}_1^* \varphi \rangle \wedge \langle \tilde{S}_2 w_2, \varphi \rangle = \langle w_2, \tilde{S}_2^* \varphi \rangle$$

Hence this is exactly the same description for the behavior B as the one we found in chapter 4.

Note that from the theory in the previous section we could have taken $\text{Dom}(\tilde{V}) = \mathcal{U}$ with $C^\infty(\mathbb{R}) \subseteq \mathcal{U} \subseteq C(\mathbb{R})$. The operator $\tilde{V}$ is closed (not necessarily continuous) in this case. Nevertheless, it is possible to show that $\tilde{V} = \tilde{S}_2 C_\mu$.

Remark 5.6.2 It is important to see that, in fact, everything is determined by the restriction of B to $C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$. Namely, consider

$$B' := \{ (S_1 w_1, w_2) \mid (w_1, w_2) \in B \cap C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) \}$$

then $B' = \text{graph}(V|_{C^\infty(\mathbb{R})})$ and since B' is closed in $C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$ we know that $V|_{C^\infty(\mathbb{R})}$ is a continuous operator from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$ that commutes with all shift-operators. Hence $V|_{C^\infty(\mathbb{R})} = S_2 C_\mu|_{C^\infty(\mathbb{R})}$ and

$$(S_1 w_1, w_2) \in B' \Leftrightarrow S_1 w_1 = S_2 C_\mu w_2$$

Since V satisfies the conditions in theorem 5.5.12 we have $V = \tilde{S}_1 \tilde{C}_\mu$ and

$$\text{graph}(V) = \overline{\{(S_2 C_\mu f, f) | f \in C^\infty(\mathbb{R})\}} = \text{graph}(\tilde{S}_2 \tilde{C}_\mu)$$

where the closure is to be taken in $C^{-k}(\mathbb{R}) \times C(\mathbb{R})$. This immediately yields the result above. So, just restrict B to $C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$, determine the operator and take the closure in the right space. $\square$

Chapter 6

Conclusions, suggestions and additions

In this thesis, we have treated an open problem in the behavioral approach to system theory. Namely

Determine necessary and sufficient conditions for a continuous-time dynamical system to satisfy an AR-relation.

Contrary to the discrete-time case, where the solution to this problem can be derived using algebraic methods, the continuous-time analog requires besides algebraic methods a functional analytic approach. An important step is the re-definition of a dynamical system Σ in continuous time as a quadruple $\Sigma = (T, W, \mathcal{F}, \mathcal{B})$, where $T \subseteq \mathbb{R}$ is the time-axis, W a vector space, $\mathcal{F}$ a Fréchet space such that $C^\infty(T, W)$ is dense in $\mathcal{F}$ and the behavior $\mathcal{B} \subseteq \mathcal{F}$. Thus we force the problem to be considered in a functional analytic setting with links to harmonic analysis and complex analysis. So the crucial ingredients in the solution method come from these fields. Firstly, there is the fact that $\mathcal{B}^\infty$, the intersection from $\mathcal{B}$ with $C^\infty(\mathbb{R}, \mathbb{R}^q)$, is dense in $\mathcal{B}$. Secondly, to arrive at a description for $\mathcal{B}$, we use the concept of the graph of a linear operator. The conditions on $\mathcal{B}$ ensure that this linear operator is continuous and shift invariant. And so we need characterizations of all kinds of shift-invariant operators. Results on these matters are gathered in chapter 2 and seem interesting in themselves, not only in the context of system theory.

In fact, the set up we have used allows us to treat the same problem for be-

haviors in Fréchet spaces other than $C(\mathbb{R}, \mathbb{R}^q)$, for instance $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$. The treatment of this problem in general Fréchet spaces has not been the goal in this thesis. However, in the appendix, we will indicate globally how crucial parts in our thesis should be rephrased to be applicable in this general context. We mention the existence of an approximate identity, smoothing character of convolution operators coming from $\mathcal{D}(\mathbb{R})$ -functions, invariance of shift-invariant closed subspaces under convolution operators, and the characterization of finite-dimensional shift-invariant subspaces.

For now, as promised in chapter 1, we return to the case where the signal space $\mathcal{F} = L_{1,loc}(\mathbb{R})$. The introduction of a (generalized) shift operator on the Fréchet space $L_{1,loc}(\mathbb{R})$ is the first step in proving that for a linear, shift-invariant behavior $\mathcal{B}$ with signals in $L_{1,loc}(\mathbb{R})$ its restriction $\mathcal{B}^\infty$ to $C^\infty(\mathbb{R}, \mathbb{R}^q)$ is dense in $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$. To express the problems of defining the translation group in full, we take the pure standpoint that $L_{1,loc}(\mathbb{R})$ is the vector space of all equivalence classes of locally integrable functions on $\mathbb{R}$. Elements are denoted by $[f]$. The translation group $(\hat{\sigma}^t)_{t \in \mathbb{R}}$ on $L_{1,loc}(\mathbb{R})$ is defined as follows: take f_1 and f_2 in the class $[f]$. Then for all t in $\mathbb{R}$, $\sigma^t f_i$ ($i = 1, 2$) is of the form

$$(\sigma^t f_i)(\tau) = f_i(t + \tau), \quad \tau \in \mathbb{R}$$

It is clear that $\sigma^t f_1$ and $\sigma^t f_2$ are locally integrable and also that $(\sigma^t f_1)(\tau) = (\sigma^t f_2)(\tau)$ except for a set of measure zero. So $[\sigma^t f_1] = [\sigma^t f_2]$ and we can define the shift-operator $\hat{\sigma}^t$ on $L_{1,loc}(\mathbb{R})$ by: $\hat{\sigma}^t[f] = [\sigma^t f]$ where σ^t is defined on the arbitrary function $f \in [f]$. It can be shown that $(\hat{\sigma}^t)_{t \in \mathbb{R}}$ is a co-semigroup on $L_{1,loc}(\mathbb{R})$ and we define its infinitesimal generator by $\hat{D}$. On the space $L_{1,loc}(\mathbb{R})$ we have $\text{Dom}(\hat{D}) = \text{AC}(\mathbb{R})$ the space of absolute continuous functions on $\mathbb{R}$ (see theorem, [10], p.101). Clearly, $\text{Dom}^\infty(\hat{D}) = C^\infty(\mathbb{R})$.

Now we can apply theorem A.3 from the appendix:

$$C^\infty(\mathbb{R}) = \text{span}(\bigcup_{\varphi \in \mathcal{D}(\mathbb{R})} \hat{\sigma}(\varphi)[L_{1,loc}(\mathbb{R})])$$

Moreover $\hat{\sigma}(\varphi) \in \overline{\text{span}\{\hat{\sigma}^t | t \in \mathbb{R}\}}$ (strong closure) and hence $\hat{\sigma}(\varphi)(\mathcal{B}) \subseteq \mathcal{B}$. If we take an approximate identity $\{\varphi_n\}_{n \in \mathbb{N}}$ in $\mathcal{D}(\mathbb{R})$, then $\hat{\sigma}(\varphi_n)$ tends pointwise to the identity. This clearly implies that $\mathcal{B}^\infty$ is dense in $\mathcal{B}$ in $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$ -sense, for a linear, shift-invariant behavior $\mathcal{B}$ that is closed in

$L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$.

The extension to other Fréchet spaces leads to still more problems. For instance, the notion of 'locally specified' needs some modification, too. In case $\mathcal{F} = L_{1,loc}(\mathbb{R})$ we have to define what we mean by the restriction of an equivalence-class to some interval. Let χ_I denote the *indicator function of the interval I*. Then the mapping

$$\chi(t, \varepsilon) : [f] \mapsto [\chi_{[t-\varepsilon, t+\varepsilon]} f]$$

is continuous from $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$ into $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$. We call a behavior *locally specified in $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$* if

$$\forall \varepsilon > 0 \quad [\forall t \in \mathbb{R} \quad [\chi(t, \varepsilon)[f] \in \chi(t, \varepsilon)\mathcal{B}] \Rightarrow [f] \in \mathcal{B}]$$

Note that on $C(\mathbb{R}, \mathbb{R}^q)$, and hence on $C^\infty(\mathbb{R}, \mathbb{R}^q)$, this definition coincides with the standard one. Similarly as for $C(\mathbb{R}, \mathbb{R}^q)$, it can be shown that for a linear, shift-invariant and closed behavior $\mathcal{B}$ in $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$, $\mathcal{B}$ is locally specified if and only if $\mathcal{B}^\infty$ is. However, we only need that locally specifiedness in $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$ implies locally specifiedness in $C^\infty(\mathbb{R}, \mathbb{R}^q)$ to achieve the same results as for $C(\mathbb{R}, \mathbb{R}^q)$. We can come up with similar necessary and sufficient conditions for behaviors in $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$ as for those in $C(\mathbb{R}, \mathbb{R}^q)$. Namely

C1 $\mathcal{B}$ is a linear, shift-invariant subspace, that is closed with respect to the product topology of $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$

C2 $\mathcal{B}$ is $(L_{1,loc}(\mathbb{R}, \mathbb{R}^q))$ -locally specified

There exists a permutation Π and $p \in \mathbb{N}$ such that

C3 $\Pi(\mathcal{B})$ is a standard $[p, q - p]$ -behavior

C4 For all $j \in \{p + 1, \dots, q\}$ and $i \in \{1, \dots, p\}$: $\mathcal{E}_i[\Pi(\mathcal{B})]_j$ or $N_i(\Pi(\mathcal{B}))_j$ is finite dimensional.

Using the convolution operator $\sigma(\varphi)$, $\varphi \in \mathcal{D}(\mathbb{R})$ as for $C(\mathbb{R}, \mathbb{R}^q)$, we know that the restriction of $\mathcal{B}$ to $C^\infty(\mathbb{R}, \mathbb{R}^q)$ satisfies the conditions c1-c4, and hence

$$w \in \mathcal{B}^\infty \Leftrightarrow R\left(\frac{d}{dt}\right)w = 0$$

for some polynomial matrix R . If we interpret this in a weak sense (i.e. in $\mathcal{D}'(\mathbb{R}, \mathbb{R}^q)$) we obtain

$$w \in \mathcal{B}^\infty \Leftrightarrow \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q) [\langle w, R^* \left(\frac{d}{dt} \varphi \right) \rangle = 0] \quad (6.1)$$

Define

$$\mathcal{B}_{\mathcal{D}'} := \{w \in L_{1,loc}(\mathbb{R}, \mathbb{R}^q) \mid \text{equation 6.1 is satisfied}\}$$

Then $\mathcal{B}_{\mathcal{D}'}$ is closed in $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$ and moreover, its restriction to $C^\infty(\mathbb{R}, \mathbb{R}^q)$ equals $\mathcal{B}^\infty$. Since $\mathcal{B}^\infty$ is dense in $\mathcal{B}$ we must have $\mathcal{B}_{\mathcal{D}'} = \mathcal{B}$. Note that we can find a similar description for behaviors in $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$ as derived for behaviors in $C(\mathbb{R}, \mathbb{R}^q)$ in theorem 4.3.10, using an integral operator $\mathcal{K}$.

When we look at the derivation of the AR-relation for $\mathcal{B}^\infty$ in section 3.3 we see that the conditions c1 and c3 yield a description of the form (we focus on $[1,1]$ -behaviors, just to get the idea)

$$(w_1, w_2) \in \mathcal{B}^\infty \Leftrightarrow T_1 C_{\mu_{1,1}} w_2 - S_{1,1} w_1 = 0$$

For a long time the author thought that the condition ‘locally specified’ would imply that the support of $\mu_{1,1}$ consists of one point. In fact, we thought it would be in the point zero, which would lead to an AR-equation for $\mathcal{B}^\infty$. This idea of relating the notion of locally specified to the support of elements that are describing the behavior was already used in the conjecture we mentioned in chapter 1. We return to that conjecture later on. We will introduce a definition that is strongly related to the one in the conjecture:

Definition 6.0.3 Let $\mathcal{B} \subseteq C(\mathbb{R}, \mathbb{R}^q)$ be a behavior. Then we define

$$\mathcal{B}_\varepsilon^\perp := \{\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q) \mid \text{supp}(\varphi) \subseteq [-\varepsilon, \varepsilon], \forall w \in \mathcal{B} : \langle \varphi, w \rangle = 0\}$$

We call $\mathcal{B}$ *dually locally specified* if the following equivalence holds

$$w \in \mathcal{B} \Leftrightarrow \forall \varepsilon > 0 \forall \varphi \in \mathcal{B}_\varepsilon^\perp [\varphi * w = 0]$$

Lemma 6.0.4 For a shift-invariant behavior $\mathcal{B}$ in $C(\mathbb{R}, \mathbb{R}^q)$ ‘dually locally specified’ implies ‘locally specified’.

Proof: Assume $\mathcal{B}$ to be dually locally specified and moreover for all $\varepsilon > 0$

$$\forall t \in \mathbb{R} : w|_{[t-\varepsilon, t+\varepsilon]} \in \mathcal{B}|_{[t-\varepsilon, t+\varepsilon]}$$

Take an arbitrary $\varepsilon > 0$ and $t_0 \in \mathbb{R}$. Then $w|_{[t_0-\varepsilon, t_0+\varepsilon]} \in \mathcal{B}|_{[t_0-\varepsilon, t_0+\varepsilon]}$ implies $\sigma^{-t_0} w|_{[-\varepsilon, \varepsilon]} = \sigma^{-t_0} v|_{[-\varepsilon, \varepsilon]}$ for some $v \in \mathcal{B}$. Then for any $\varphi \in \mathcal{B}_\varepsilon^\perp$

$$(\varphi * w)(t_0) = (\varphi * \sigma^{-t_0} v)(0) = \int_{-\varepsilon}^{\varepsilon} \varphi(\tau)(\sigma^{-t_0} v)(\tau) d\tau = 0$$

Hence, since our choice of ε and t_0 was arbitrary we have proved the right-hand side of the dually locally specified-equivalence. Hence $w \in \mathcal{B}$ and thus $\mathcal{B}$ is locally specified. $\square$

The advantage of defining locally specified via $\mathcal{B}^\perp$ is that it might be possible to talk about this notion in case of an arbitrary Fréchet space where the restriction of an element in $\mathcal{F}$ to an interval does not make sense.

Let us return to the conjecture we referred to earlier in this section. The following question was posed to Prof. Hörmander:

Suppose $\mathcal{B} \subseteq \mathcal{D}'(\mathbb{R}, \mathbb{R}^q)$ is linear, shift-invariant and closed and let $\mathcal{B}^\perp$ denote the set of all distributions with compact support such that

$$\{w \in \mathcal{B} \text{ and } v \in \mathcal{B}^\perp\} \Leftrightarrow \{v * w = 0\}$$

Locally specified has been transformed into: for all $t > 0$

$$\{w \in \mathcal{B}\} \Leftrightarrow \{v * w = 0 \text{ for all } v \in \mathcal{B}^\perp \text{ having compact support on } [0, t]\}$$

(The difference with the definition of dually locally specified above is that $\mathcal{B}^\perp$ is contained in $\mathcal{D}(\mathbb{R})$ instead of $\mathcal{D}'(\mathbb{R})$).

Question: is it true that a continuous-time dynamical system can be represented by an AR-relation if and only if it is linear, shift-invariant, closed and locally specified?

A counterexample to the conjecture (Hörmander)

(We will consider a behavior in $C(\mathbb{R})$ and take $\mathcal{B}^\perp \subseteq \mathcal{D}(\mathbb{R})$).

Let $\mathcal{B}$ be the closure in $C(\mathbb{R})$ of the span($\{e^{ik^2 t} \mid k \in \mathbb{N}\}$). Then clearly, $\mathcal{B}$ is linear, shift invariant and closed. It is not that easy to see that $\mathcal{B}$ is locally specified. Let $\mathcal{B}^\perp$ be the space of all $v \in \mathcal{D}(\mathbb{R})$ with $v * \mathcal{B} = 0$. Then its

Fourier transform is zero in the points k^2 , $k \in \mathbb{N}$. Define the Weierstrass product

$$F(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{k^2}\right)$$

This is an entire function of order $\frac{1}{2}$, i.e.

$$|F(z)| \leq C e^{\pi|z|^{\frac{1}{2}}}$$

(see [17], theorem 8.25, p.251; [25], th.6, p.69). In fact, we have (see [25], p.76)

$$F(z) = \frac{\sin(\pi\sqrt{z})}{\pi\sqrt{z}}$$

To obtain an element in $\mathcal{D}(\mathbb{R})$ with arbitrary small support we multiply $F(z)$ with a Fourier transform of, for instance, an element θ in the Gel'fand-Shilov space S_0^α .

Definition 6.0.5 The Schwartz space $S(\mathbb{R})$ consists of all functions $f \in C^\infty(\mathbb{R})$ for which

$$t^k \left(\frac{d}{dt}\right)^l f \in L_\infty(\mathbb{R}), \quad (k, l \in \mathbb{N} \cup \{0\}).$$

Definition 6.0.6 The space $S_\alpha^\beta(\mathbb{R})$ consists of all functions $f \in S(\mathbb{R})$ for which there exist $A, B, C > 0$ such that

$$\sup_{t \in \mathbb{R}} |t^k \left(\frac{d}{dt}\right)^l f| < C A^k B^l (k!)^\alpha (l!)^\beta \quad (k, l \in \mathbb{N} \cup \{0\})$$

The following properties can be deduced from ([7], pp. 226-207-222, 175):

1. S_0^α is nontrivial for $\alpha > 1$.
2. $\mathcal{F}(S_0^\alpha) = S_\alpha^0$.
3. The conditions $\alpha > 1$ and $a > 0$ are sufficient to have the equivalence

$$\forall b' > b \quad [|f(z)| \leq c e^{-a|\operatorname{Re} z|^{\frac{1}{\alpha}} + b|\operatorname{Im} z|}]$$

$\Leftrightarrow$

$$\forall b' > b \quad [f \in S_\alpha^0(\mathbb{R}) \text{ and } \operatorname{supp}(f) \subseteq (-b', b')]$$

Let $1 < \alpha < 2$ and $\theta \in S_0^\alpha$. Then

$$|\hat{\theta}(z)| \leq c_1 e^{-a_1 |\operatorname{Re} z|^{\frac{1}{\alpha} + b_1} |\operatorname{Im} z|}$$

and hence

$$\begin{aligned} |F(z)\hat{\theta}(z)| &\leq c_2 e^{-a_1 |\operatorname{Re} z|^{\frac{1}{\alpha} + b_1} |\operatorname{Im} z| + a_0 |\operatorname{Re} z|^{\frac{1}{2} + b_0} |\operatorname{Im} z|^{\frac{1}{2}}} \\ &\leq c_2 e^{-a_2 |\operatorname{Re} z|^{\frac{1}{\alpha} + b_2} |\operatorname{Im} z|} \end{aligned}$$

for all a_2, b_2 with $a_2 < a_1$ and $b_2 > b_1$. According to property 2 and 3 above $F(z)\hat{\theta}(z)$ is the Fourier transform of an element in S_0^α with support in $[-b_2, b_2]$, where b_2 is arbitrarily close to b_1 . If $u \in B^\perp$, then $\operatorname{supp}(\mu) \subseteq [-a, a]$ implies

$$|\hat{u}(z)| = \left| \int_{\mathbb{R}} e^{itz} u(t) dt \right| \leq C \cdot e^{a |\operatorname{Im} z|}$$

Hence $\hat{u}(z)$ is an entire function of exponential type. Since $F(z)$ is an entire function of order $\frac{1}{2}$ it is of exponential type, too. Applying proposition 3 in [12] (Malgrange, p.305) yields $\hat{u}(z)/F(z)$ is also of exponential type. We prove that $|\hat{u}(x)/F(x)| = \mathcal{O}(|x|^N)$ for some $N \in \mathbb{N}$, then we obtain from a refinement of the Paley-Wiener theorem (see Kahane, [9], p.16) that $\hat{u}(z)/F(z)$ is the Fourier transform of an element in $\mathcal{E}'$.

Take $z = x + iy$, $x, y \in \mathbb{R}$, and determine the sequence $(a_n)_{n \in \mathbb{N}}$ such that

$$\begin{cases} |(\sin(\pi\sqrt{x}))| \geq \frac{1}{2} & x \notin [n^2 - a_n, n^2 + a_n] \\ |(\sin(\pi\sqrt{x}))| < \frac{1}{2} & x \in [n^2 - a_n, n^2 + a_n] \end{cases}$$

Define $\tilde{F}(z) = \sin(\pi\sqrt{z}) = \pi\sqrt{z} \cdot F(z)$ and for some $n \in \mathbb{N}$ let $x \in [n^2 - a_n, n^2 + a_n] =: I(a_n)$. Then there exist x_n and $\tilde{x}_n$ in $I(a_n)$ such that (by the mean value theorem)

$$\hat{u}(x) = (x - n^2)\hat{u}'(x_n) \wedge \tilde{F}(x) = (x - n^2)\tilde{F}'(\tilde{x}_n)$$

Since $\tilde{F}'(x) = \frac{\pi}{2\sqrt{x}} \cdot \cos(\pi\sqrt{x})$ we have

$$\begin{aligned} \left| \frac{\hat{u}(x)}{F(x)} \right| &= \left| \frac{\hat{u}(x)}{\tilde{F}(x)} \cdot \pi\sqrt{x} \right| = \left| \frac{(x - n^2)\hat{u}'(x_n)}{(x - n^2)\tilde{F}'(\tilde{x}_n)} \cdot \pi\sqrt{x} \right| \\ &= \left| \frac{\hat{u}'(x_n)}{\cos(\pi\sqrt{\tilde{x}_n})} \cdot \pi\sqrt{x} \cdot \frac{2\sqrt{\tilde{x}_n}}{\pi} \right| \leq |\hat{u}'(x_n)| \cdot C|x| \end{aligned}$$

for some $C \in \mathbb{R}$. Note that we used that $\tilde{x}_n \in I(a_n)$ implies

$$|\cos(\pi\sqrt{\tilde{x}_n})| \geq \frac{1}{2}\sqrt{3}$$

From the Paley-Wiener theorem we know that $\hat{u}'$ (being the Fourier transform of $xu \in \mathcal{D}(\mathbb{R})$) is certainly bounded on $\mathbb{R}$. Consequently $\hat{u}(z)/F(z)$ is an entire function and equals the Fourier transform of an element in $\mathcal{E}'$. Moreover define $\hat{S}(z) = F(z)\hat{\theta}(z)$, such that $S \in \mathcal{B}^\perp$. This way we connect an arbitrary element in $\mathcal{B}^\perp$ to an element in $\mathcal{B}^\perp$ with arbitrary small support. Now we can obtain the equality

$$\hat{u}(z)\hat{\theta}(z) = \frac{\hat{u}(z)}{F(z)} \cdot F(z)\hat{\theta}(z)$$

and hence for all $w \in \mathcal{B}$

$$w * u * \theta = (w * S) * T = 0$$

Letting $\theta \rightarrow \delta$ we are done.

All the ideas we used for $C(\mathbb{R})$ suggest further research in generalizing the results in this thesis to arbitrary Fréchet spaces. What are the essential properties of an arbitrary Fréchet space such that only small modifications are needed to extend the theory as exposed in our work.

Probably another interesting point is the link to delay-systems. Namely, a description of the form (say $\mathcal{B} \subseteq C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$)

$$TC_\mu w_1 = w_2$$

($T \in \text{LDO}(\mathbb{R})$, $\mu \in ba_c(\mathbb{R})$) includes delay-systems, since C_μ can be a finite sum of translation operators. Until now, (in the context of this thesis) no research is done or planned in this direction.

In our work we have used the characterization of linear shift-invariant finite-dimensional subspaces, which has enabled us to use the concept of the graph of an operator. However, a linear shift-invariant closed subspace of $C(\mathbb{R})$ that is infinite dimensional can also be characterized using the theory of mean-periodic functions (see lemma 2.6.9). If one is interested in larger classes of systems, it might be interesting to investigate if a similar technique (using the graph) is applicable to derive some kind of description for these 'systems'.

Appendix

The proofs of theorem 3.3.1 and lemma 3.3.6

Theorem A.1 *A subspace V in $C^\infty(\mathbb{R}, \mathbb{R}^p)$ is finite dimensional and shift invariant if and only if V is the kernel of some $S \in LDO(p, p, \mathbb{R})$. It is always possible to choose S uppertriangular and with nonzero diagonal elements. (I.e. theorem 3.3.1)*

Proof: ($\Leftarrow$): Obvious.

($\Rightarrow$): The proof will use the fact that such a subspace V consists of Bohl functions only. They are linear combinations of the form $b \cdot t^a e^{\lambda t}$ with $b \in \mathbb{R}^p, a \in \mathbb{N}, \lambda \in \mathbb{C}$. Because of the shift invariance and closedness we know that if $b \cdot t^a e^{\lambda t}$ is in V then $b \cdot t^l e^{\lambda t}$ is also in V for $0 \leq l < a, l \in \mathbb{N}$. The proof is by induction with respect to the dimension.

$\dim(V)=1$: then $V = \langle b^1 \cdot e^{\lambda t} \rangle$ for certain b^1 and λ . If $b^1 = (b_1^1, \dots, b_p^1)$ then define j as the greatest integer in $\{1, \dots, p\}$ such that $b_j^1 \neq 0$. Then define $S \in LDO(p, p, \mathbb{R})$ as follows: all entries are zero, except

- for $i = 1, \dots, j-1$ we have $S_{i,i} = b_i^1$;
- $S_{j,j} = \frac{d}{dt} - \lambda$;
- for $i = j+1, \dots, p$ we have $S_{i,i} = 1$;
- for $i = 1, \dots, j-1$ we have $S_{i,j} = -b_i^1$;

This leads to the following matrix

$$\left(\begin{array}{ccccc|ccc} b_j^1 & 0 & \cdots & 0 & -b_1^1 & 0 & \cdots & 0 \\ 0 & \ddots & & 0 & \vdots & \vdots & & \vdots \\ \vdots & & & b_j^1 & -b_{j-1}^1 & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & \frac{d}{dt} - \lambda & 0 & \cdots & 0 \\ \hline 0 & \cdots & \cdots & \cdots & 0 & 1 & & 0 \\ \vdots & & & & \vdots & & \ddots & \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & & 1 \end{array} \right)$$

Note that this matrix is upper triangular indeed and has nonzero diagonal elements.

Induction step:

Suppose we can find such a matrix S for all $(r-1)$ -dimensional subspaces V that are linear and shift invariant. Now take a subspace V with the same properties that is r -dimensional. Choose a basis for V with an element of the form $b \cdot e^{\lambda t}$ (i.e. $a = 0$ in $t^a e^{\lambda t}$). This is possible because of the shift-invariance. Calculate S_1 such that $\langle b \cdot e^{\lambda t} \rangle = \ker S_1$ and apply S_1 to all the $r-1$ remaining elements in the basis for V . The result is an $(r-1)$ -dimensional subspace V_{r-1} . Clearly the subspace V_{r-1} is linear and shift-invariant too. Because of the induction-assumption, $V_{r-1} = \ker S_{r-1}$ for some S_{r-1} satisfying the requirements. It remains to prove that $V = \ker S_{r-1} S_1$, $S_{r-1} S_1$ is uppertriangular and has nonzero diagonal elements.

We have the following equivalence relations

$$S_{r-1} S_1 f = 0 \Leftrightarrow S_1 f \in \ker(S_{r-1}) \Leftrightarrow S_1 f \in S_1(V) \Leftrightarrow f \in V + \ker(S_1) = V$$

Since S and S_{r-1} are uppertriangular their product has that same property. Furthermore the diagonal elements of this product are always products of diagonal elements themselves. $\square$

Lemma A.2 *Given a p -behavior in $C^\infty(\mathbb{R}, \mathbb{R}^{p+1})$, satisfying the condition c4 at the beginning of section 3.3, with representation-matrix*

$$\left(\begin{array}{cccc} S_{1,1} & \cdots & S_{1,p} & -T_1 C_{\mu_1} \\ & \ddots & \vdots & \vdots \\ & & S_{p,p} & -T_p C_{\mu_p} \end{array} \right)$$

then for $j=1, \dots, p$ we can find $\tilde{T}_{p-j}$ and $M_{p-j, p-i}$ in $LDO(\mathbb{R})$ such that

$$\tilde{T}_{p-j} C_{\mu_{p-j}} = \sum_{i=0}^{j-1} M_{p-j, p-i} \sigma^{a_{p-i}}$$

and moreover after step j in the elimination-procedure we obtain

$$\begin{pmatrix} \bar{s}_{1,1} & \dots & \bar{s}_{1,p-j} & 0 & \dots & 0 & -\bar{T}_1 C_{\mu_1} + \sum_{i=0}^{j-1} M_{1,p-i} \sigma^{a_{p-i}} \\ & \ddots & \vdots & \vdots & & \vdots & \vdots \\ & & \bar{s}_{p-j,p-j} & 0 & & -\bar{T}_{p-j} C_{\mu_{p-j}} + \sum_{i=0}^{j-1} M_{p-j,p-i} \sigma^{a_{p-i}} \\ & & & \bar{s}_{p+1-j,p+1-j} & \ddots & & -\bar{T}_{p+1-j} \sigma^{a_{p+1-j}} \\ & & & & \ddots & 0 & \vdots \\ & & & & & s_{p,p} & -\bar{T}_p \sigma^{a_p} \end{pmatrix}$$

where $-\bar{T}_{p-j} C_{\mu_{p-j}} + \sum_{i=0}^{j-1} M_{p-j, p-i} \sigma^{a_{p-i}} = \tilde{T}_{p-j} \sigma^{a_{p-j}}$. (I.e. lemma 3.3.6)

Proof: The proof will be by induction to j . Because $\text{Range}(T_p C_{\mu_p})$ is dense if $T_p C_{\mu_p} \neq 0$, the finite-dimensionality of $\mathcal{E}_p \mathcal{B}^\infty$ implies $T_p C_{\mu_p} = 0$. It remains to consider the case that N_p is finite dimensional. Hence $\ker(T_p C_{\mu_p})$ is finite dimensional and thus $T_p C_{\mu_p} = T_p \sigma^{a_p}$ (for some $a_p \in \mathbb{R}$). Elimination of w_p can be carried out by premultiplying the equations with

$$\left(\frac{S_{p,p} \cdot I_{p-1}}{I_1} \right)$$

leading to

$$\begin{pmatrix} S_{p,p} S_{1,1} & \dots & S_{p,p} S_{1,p-1} & 0 & -S_{p,p} T_1 C_{\mu_1} + S_{1,p} \tilde{T}_p \sigma^{a_p} \\ & \ddots & \vdots & \vdots & \vdots \\ & & S_{p,p} S_{p-1,p-1} & 0 & -S_{p,p} T_{p-1} C_{\mu_{p-1}} + S_{p-1,p} \tilde{T}_p \sigma^{a_p} \\ & & & S_{p,p} & -\tilde{T}_p \sigma^{a_p} \end{pmatrix}$$

(A.2)

We are left to prove that

$$-S_{p,p} T_{p-1} C_{\mu_{p-1}} + S_{p-1,p} \tilde{T}_p \sigma^{a_p} = \tilde{T}_{p-1} \sigma^{a_{p-1}}$$

for some $\tilde{T}_{p-1} \in \text{LDO}(\mathbb{R})$, $a_{p-1} \in \mathbb{R}$. Let N_p be finite dimensional and so $\dim(W_p) < \infty$ with

$$\{w \in \mathcal{B}^\infty \mid w_1 = \dots = w_{p-1} = 0\} = W_p$$

The subspace W_p is the solution set to

$$\begin{pmatrix} S_{p-1,p} & -T_{p-1}C_{\mu_{p-1}} \\ S_{p,p} & -\tilde{T}_p\sigma^{a_p} \end{pmatrix} \begin{pmatrix} w_p \\ w_{p+1} \end{pmatrix} = 0$$

The first step in the elimination applied to this part of the representation-matrix comes down to

$$\left(\begin{array}{c|c} S_{p,p} & \\ \hline I_1 \end{array} \right) \begin{pmatrix} S_{p-1,p} & -T_{p-1}C_{\mu_{p-1}} \\ S_{p,p} & -\tilde{T}_p\sigma^{a_p} \end{pmatrix} \quad (\text{A.3})$$

According to lemma 3.2.4 the solution set remains finite dimensional. The matrix in equation (A.3) is equivalent with

$$\begin{pmatrix} 0 & -S_{p,p}T_{p-1}C_{\mu_{p-1}} + S_{p-1,p}\tilde{T}_p\sigma^{a_p} \\ S_{p,p} & -\tilde{T}_p\sigma^{a_p} \end{pmatrix}$$

Clearly, since $S_{p,p} \neq 0$ this solution set is finite dimensional if and only if

$$-S_{p,p}T_{p-1}C_{\mu_{p-1}} + S_{p-1,p}\tilde{T}_p\sigma^{a_p} = \tilde{T}_{p-1}\sigma^{a_{p-1}}$$

for some nonzero $\tilde{T}_{p-1} \in \text{LDO}(\mathbb{R})$, $a_{p-1} \in \mathbb{R}$ and the claim holds.

On the other hand, if $\mathcal{E}_{p-1}\mathcal{B}^\infty$ is finite dimensional, then (see definition 3.2.2)

$$\dim(A_{p-1}(\mathcal{O}_{p-1})) < \infty$$

Consequently

$$\dim(S_{p,p}A_{p-1}(\mathcal{O}_{p-1})) < \infty$$

If $T_{p-1} = T_p = 0$ we are done, so assume one of them to be nonzero. It is important to see that $A_{p-1}(w_p, w_{p+1}) = [-S_{p,p}T_{p-1}C_{\mu_{p-1}} + S_{p-1,p}\tilde{T}_p\sigma^{a_p}]w_{p+1}$ with $(w_p, w_{p+1}) \in \mathcal{O}_i$ and the operator A_i commutes with all shift operators

and hence equals some $\tilde{T}_{p-1}\tilde{C}_{\nu_{p-1}}$: i.e. to any $w_{p+1} \in C^\infty(\mathbb{R})$ corresponds a w_p such that $(w_p, w_{p+1}) \in \mathcal{O}_{p-1}$ and

$$S_{p,p}A_{p-1}(w_p, w_{p+1}) = -\tilde{T}_{p-1}\tilde{C}_{\nu_{p-1}}w_{p+1}$$

We know that $\text{Range}(\tilde{T}_{p-1}\tilde{C}_{\nu_{p-1}})$ is dense in $C^\infty(\mathbb{R})$ or equals zero. Since w_{p+1} can be chosen arbitrarily and $\dim(S_{p,p}A_{p-1}(\mathcal{O}_{p-1})) < \infty$ we must have $\tilde{T}_{p-1}\tilde{C}_{\nu_{p-1}} = 0$. We find

$$-S_{p,p}T_{p-1}C_{\mu_{p-1}} + S_{p-1,p}\tilde{T}_p\sigma^{ap} = \tilde{T}_{p-1}\tilde{C}_{\nu_{p-1}} = 0$$

Hence, the claim holds with $\tilde{T}_{p-1} = 0$.

Induction step:

Suppose the claim holds for $j=1, \dots, k-1$. This assumption says that after $k-1$ ($k > 1$) steps of elimination we have the following form

$$\underbrace{\begin{pmatrix} \bar{S}_{1,1} & \dots & \bar{S}_{1,p+1-k} & 0 & \dots & 0 & -\bar{T}_1 C_{\mu_1} + \sum_{i=0}^{k-2} M_{1,p-i} \sigma^{ap-i} \\ & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & \bar{S}_{p+1-k,p+1-k} & 0 & \vdots & -\bar{T}_{p+1-k} C_{\mu_{p+1-k}} + \sum_{i=0}^{k-2} M_{p+1-k,p-i} \sigma^{ap-i} \\ & & & \ddots & \vdots & -\tilde{T}_{p+2-k} \sigma^{ap+2-k} \\ & & & & \ddots & \vdots \\ & & & & & 0 \\ & & & & & S_{p,p} & -\tilde{T}_p \sigma^{ap} \end{pmatrix}}_{R[p+1-k]} \quad (A.4)$$

where $-\bar{T}_{p+1-k} C_{\mu_{p+1-k}} + \sum_{i=0}^{k-2} M_{p+1-k,p-i} \sigma^{ap-i} = \tilde{T}_{p+1-k} \sigma^{ap+1-k}$. Premultiplying $R[p+1-k]$ with $S(p+1-k)$ leads to $R[p-k]$ and corresponding $\mathcal{B}^\infty[p-k]$. Introduce

$$V_{p-k} = \{(w_{p-k+1}, \dots, w_{p+1}) \mid (0 \mid (w_{p-k+1}, \dots, w_{p+1})) \in \mathcal{B}^\infty[p-k]\}$$

If N_{p-k} is finite dimensional, then according to lemma 3.2.4

$$\dim\{w \in \mathcal{B}^\infty[p-k] \mid w_{p-k} = 0\} < \infty$$

Hence

$$\dim(V_{p-k}) = \dim\{w \in \mathcal{B}^\infty[p-k] \mid w_1 = \dots = w_{p-k} = 0\} < \infty$$

The subspace V_{p-k} is determined by the last k equations in $R[p-k]$:

$$\begin{pmatrix} 0 & \cdots & 0 & -S_{p+1-k,p+1-k}\bar{T}_{p-k}C_{\mu_{p-k}} + \sum_{i=0}^{k-1} M_{p-k,p-i}\sigma^{a_{p-i}} \\ \bar{S}_{p+1-k,p+1-k} & \ddots & \vdots & -\tilde{T}_{p+1-k}\sigma^{a_{p+1-k}} \\ & \ddots & 0 & \vdots \\ & & S_{p,p} & -\tilde{T}_p\sigma^{a_p} \end{pmatrix} \quad (\text{A.5})$$

Since the matrix in (A.5) has a finite dimensional kernel we have

$$-S_{p+1-k,p+1-k}\bar{T}_{p-k}C_{\mu_{p-k}} + \sum_{i=0}^{k-1} M_{p-k,p-i}\sigma^{a_{p-i}} = \tilde{T}_{p-k}\sigma^{a_{p-k}}$$

On the other hand, if $\mathcal{E}_{p-k}B^\infty$ is finite dimensional, then

$$\dim(A_{p-k}(\mathcal{O}_{p-k})) < \infty$$

As before we have

$$-S_{p+1-k,p+1-k}\bar{T}_{p-k}C_{\mu_{p-k}} + \sum_{i=0}^{k-1} M_{p-k,p-i}\sigma^{a_{p-i}} = \tilde{T}_{p-k}C_{\nu_{p-k}}$$

for some $\tilde{T}_{p-k} \in \text{LDO}(\mathbb{R})$ and $\nu_{p-k} \in \text{ba}_c(\mathbb{R})$. Define $S_p := \prod_{j=p+1-k}^p S_{j,j}$, then

$$\dim(S_p A_{p-k}(\mathcal{O}_{p-k})) < \infty$$

We know for all $(w_{p+1-k}, \dots, w_{p+1}) \in \mathcal{O}_{p-k}$

$$S_p A_{p-k}(w_{p+1-k}, \dots, w_{p+1}) = \tilde{T}_{p-k}C_{\nu_{p-k}}w_{p+1}$$

Therefore $\tilde{T}_{p-k} = 0$ and

$$0 = -S_{p+1-k,p+1-k}\bar{T}_{p-k}C_{\mu_{p-k}} + \sum_{i=0}^{k-1} M_{p-k,p-i}\sigma^{a_{p-i}}$$

□

Some remarks on c_0 -groups on Fréchet spaces

Let $\mathcal{F}$ be a Fréchet space with seminorms $(p_n)_{n \in \mathbb{N}}$ and $(\alpha^t)_{t \in \mathbb{R}}$ a c_0 -semigroup on $\mathcal{F}$. I.e.

$$\lim_{t \rightarrow 0} \alpha^t x = x, \quad \alpha^{t+\tau} = \alpha^t \alpha^\tau, \quad \alpha^0 = I$$

The infinitesimal generator δ_α of $(\alpha^t)_{t \in \mathbb{R}}$ is defined on

$$\text{Dom}(\delta_\alpha) := \{x \in \mathcal{F} \mid \lim_{t \rightarrow 0} \frac{\alpha^t x - x}{t} \text{ exists}\}$$

and

$$\delta_\alpha(x) := \lim_{t \rightarrow 0} \frac{\alpha^t x - x}{t} \quad (x \in \text{Dom}(\delta_\alpha))$$

The operator δ_α is densely defined and closed (see for instance theorem 1.27, [13], p.22). The C^∞ -domain of δ_α is defined by

$$\text{Dom}^\infty(\delta_\alpha) = \cap_{k=1}^\infty \text{Dom}(\delta_\alpha^k)$$

It can be shown that $\text{Dom}^\infty(\delta_\alpha)$ is dense in $\mathcal{F}$, too (see for instance, theorem 2.12, [13], p.39). Now for $\varphi \in \mathcal{D}(\mathbb{R})$ we define $\alpha(\varphi)$ on $\mathcal{F}$ by

$$\alpha(\varphi)x = \int_{\mathbb{R}} \varphi(t) \alpha^t x dt, \quad \varphi \in \mathcal{D}(\mathbb{R}) \quad (\text{A.6})$$

For all $x \in \mathcal{F}$ this definition makes sense (see [3]). In fact, if we take partitions of $\text{supp}(\varphi) \subseteq [-N, N]$, $t_i = -N + \frac{iN}{2n}$, $i = 0, 1, \dots, 2n$ $[t_1, \dots, t_{2n}]$ of the interval in $\mathbb{R}$ where φ has its support, then

$$\lim_{N \rightarrow \infty} \sum_{i=1}^{2n} \varphi(t_i) (t_i - t_{i-1}) \alpha^{t_i} x$$

converges and is defined as $\alpha(\varphi)x$. Hence $\alpha(\varphi)$ is in the strong closure of $\text{span}\{\alpha^t \mid t \in \mathbb{R}\}$.

If we use theorem 3.3 from [3] of Dixmier and Malliavin for the group $(\mathbb{R}, +)$ and the representation of this commutative group by $(\alpha^t)_{t \in \mathbb{R}}$ on $\mathcal{F}$, we get the following result:

Theorem A.3 $\text{Dom}^\infty(\delta_\alpha) = \text{span}(\cup_{\varphi \in \mathcal{D}(\mathbb{R})} \alpha(\varphi)[\mathcal{F}])$.

Proof: See theorem 3.3, [3], p.313. □

Application: In case $\mathcal{F} = C^\infty(\mathbb{R})$, $\alpha^t = \sigma^t$, then $\delta_\alpha = \frac{d}{dt}$, $\text{Dom}(\delta_\alpha) = C^\infty(\mathbb{R})$ and $\text{Dom}^\infty(\delta_\alpha) = C^\infty(\mathbb{R})$. Applying the lemma yields

$$C^\infty(\mathbb{R}) = \text{span}(\cup_{\varphi \in \mathcal{D}(\mathbb{R})} \sigma(\varphi)[C^\infty(\mathbb{R})])$$

I.e. every $C^\infty(\mathbb{R})$ -function can be written as a finite sum of convolution of $\mathcal{D}(\mathbb{R})$ and $C^\infty(\mathbb{R})$ -functions.

Consequently

1. In case $\mathcal{F} = C(\mathbb{R})$, $\alpha^t = \sigma^t$, then

$$C^\infty(\mathbb{R}) = \text{span}(\cup_{\varphi \in \mathcal{D}(\mathbb{R})} \sigma(\varphi)[C(\mathbb{R})])$$

2. In case $\mathcal{F} = C^{-k}(\mathbb{R})$, $\alpha^t = \tilde{\sigma}^t$, then

$$C^\infty(\mathbb{R}) = \text{span}(\cup_{\varphi \in \mathcal{D}(\mathbb{R})} \sigma(\varphi)[C^{-k}(\mathbb{R})])$$

3. In case $\mathcal{F} = L_{1,loc}(\mathbb{R})$, $\alpha^t = \hat{\sigma}^t$ and $\delta_\alpha = \hat{\sigma}_\delta = \hat{D}$ with $\text{Dom}(\hat{D}) = \text{AC}(\mathbb{R})$, we obtain $\text{Dom}^\infty(\delta_\alpha) = C^\infty(\mathbb{R})$ whence

$$C^\infty(\mathbb{R}) = \text{span}(\cup_{\varphi \in \mathcal{D}(\mathbb{R})} \hat{\sigma}(\varphi)[L_{1,loc}(\mathbb{R})])$$

where

$$\hat{\sigma}(\varphi)f = \int_{\mathbb{R}} \varphi(t) \hat{\sigma}^t f dt$$

Let (φ_n) be an approximate identity in $\mathcal{D}(\mathbb{R})$, with $\text{supp}(\varphi_n) \subseteq [-\frac{1}{n}, \frac{1}{n}]$ and $\int_{\mathbb{R}} \varphi_n(t) dt = 1$ (compare with proof of lemma 2.4.22), then

$$\begin{aligned} p_k(\alpha(\varphi_n)x - x) &= p_k[\int_{\mathbb{R}} \varphi_n(t) \alpha^t x dt - x] \\ &\leq \int_{\mathbb{R}} \varphi_n(t) p_k[\alpha^t x - x] dt \\ &\leq \max_{t \in [-\frac{1}{n}, \frac{1}{n}]} p_k(\alpha^t x - x) \cdot 1 \rightarrow 0 \quad (n \rightarrow \infty) \end{aligned}$$

since $\alpha^t \rightarrow \alpha^0 = I$ for $n \rightarrow \infty$.

We conclude with the generalization of lemma 2.6.11

Lemma A.4 *Let (V, T) be a locally convex topological vector space and W a finite-dimensional subspace of V . If $(\alpha^t)_{t \in \mathbf{R}}$ is a c_0 -group on V , then the following statements are equivalent:*

- *There exist $k \in \mathbf{N}$, $\lambda_1, \dots, \lambda_k \in \mathbf{C}$ and positive integers $r_1, \dots, r_k$ and $x_1, \dots, x_k \in V$ such that*

$$(a) \quad (\delta_\alpha - \lambda_l I)^{r_l}(x_l) = 0, \quad (l = 1, \dots, k)$$

$$(b) \quad W = \text{span}\{\delta_\alpha^{j_l}(x_l) \mid l = 1, \dots, k; j_l = 0, 1, \dots, r_l - 1\}.$$

- *The finite-dimensional subspace W is invariant under $(\alpha^t)_{t \in \mathbf{R}}$. The dimension of W is less than or equal to $\sum_{i=1}^k r_i$.*

Proof: See ([13]), theorem 2.14, p.40. □

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Samenvatting

In 1978 heeft Willems een nieuwe axiomatische opzet voor de bestudering van dynamische systemen geïntroduceerd. Hij beschouwt een systeem als een triplet $\Sigma = (T, W, \mathcal{B})$ waarbij $T \subseteq \mathbb{R}$ de tijdsas is, W het signaal-alfabet en $\mathcal{B} \subseteq W^T$ het gedrag van het systeem dat de uitwendige variabelen bevat. Gebruikelijke systeemtheoretische concepten zoals 'bestuurbaarheid' en 'waarneembaarheid' krijgen in deze opzet een nieuwe definitie.

Als $\mathcal{B}$ een lineaire deelruimte is van W^T spreken we van een lineair systeem en als $(T, +)$ een ondergroep is van $(\mathbb{R}, +)$ en $\sigma^t \mathcal{B} \subseteq \mathcal{B}$ voor alle $t \in T$ dan heet Σ tijdsinvariant. Hierbij is σ^t de gebruikelijke translatie operator op W^T .

Bij tijdsinvariante lineaire systemen denkt men meestal aan een input-output systeem beschreven door een toestandsrepresentatie of een AR-relatie. De volgende vraag ligt daarom voor de hand: gegeven een tijdsinvariant lineair systeem $\Sigma = (T, W, \mathcal{B})$, genereer (nodige en voldoende) voorwaarden opdat $\mathcal{B}$ bepaald is door een AR-relatie. Dit probleem is voor $T = \mathbb{Z}$ (discrete tijd) en $W = \mathbb{R}^q$ volledig opgelost en wel met zuiver algebraïsche methoden. Als $T = \mathbb{R}$ (continue tijd) en $W = \mathbb{R}^q$, zijn puur algebraïsche overwegingen niet toereikend; ook topologische aspecten gaan een rol spelen.

Als model voor het geval van continue tijd wordt in dit proefschrift de volgende probleemstelling beschouwd: gegeven een deelruimte $\mathcal{B}$ van de vectorruimte $C^\infty(\mathbb{R}, \mathbb{R}^q)$ van alle C^∞ -functies van $\mathbb{R}$ in $\mathbb{R}^q$. Bepaal nodige en voldoende voorwaarden opdat er een p , $1 \leq p \leq q$ bestaat en een $p \times q$ polynoommatrix R met

$$\mathcal{B} = \ker(R(\frac{d}{dt})) = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid R(\frac{d}{dt})w = 0\}$$

In het derde hoofdstuk van dit proefschrift worden dergelijke voorwaarden afgeleid. Naast de voor de hand liggende eisen dat $\mathcal{B}$ een lineaire, gesloten, translatie-invariante, lokaal bepaalde deelruimte van $C^\infty(\mathbb{R}, \mathbb{R}^q)$ moet zijn, is er een wat subtielere eis die betrekking heeft op de onderlinge relaties van de uitwendige variabelen $w = (w_1, \dots, w_q)$ en $\mathcal{B}$ beschrijft als een superpositie van elementaire $\mathcal{B}_j$'s waarin één richting de overige vastlegt modulo een eindig-dimensionale deelruimte. Essentieel in de bewijsvoering is een volledige karakterisatie van de continue lineaire operatoren van $C^\infty(\mathbb{R}, \mathbb{R}^q)$ die commuteren met de groep $(\sigma^t)_{t \in \mathbb{R}}$.

De volgende stap (hoofdstuk 4 en hoofdstuk 6) is de uitbreiding van de resultaten van hoofdstuk 3 naar algemenere ruimten zoals $C(\mathbb{R}, \mathbb{R}^q)$ en $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$. Het feit dat, dat voor een gesloten translatie-invariante deelruimte $\mathcal{B}$ van $C(\mathbb{R}, \mathbb{R}^q)$ of $L_{1,loc}(\mathbb{R}, \mathbb{R}^q)$, $\mathcal{B} \cap C^\infty(\mathbb{R}, \mathbb{R}^q)$ dicht is in $\mathcal{B}$, leidt tot vergelijkbare nodige en voldoende voorwaarden, als in het C^∞ -geval, waaronder $\mathcal{B} = \ker(R(\frac{d}{dt}))$. Hierbij moet de laatstgenoemde gelijkheid in zwakke (distributionele) zin worden geïnterpreteerd.

Curriculum Vitae

30 september 1962	geboren te Maastricht
1974-1980	Gymnasium β , Jeanne d'Arc Lyceum te Maastricht
1980-1988	Wiskunde, afstudeerrichting Systeemtheorie Technische Universiteit Eindhoven
1988-1989	Toegevoegd onderzoeker, vakgroep Systeemtheorie Technische Universiteit Eindhoven
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STELLINGEN

bij het proefschrift

Introduction to a behavioral approach
for
continuous-time systems

door

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1. Zij T_{pc} de lokaal convexe topologie op W^R voortgebracht door de seminormen $p_x(w) = |w(x)|$, $x \in \mathbb{R}$ en zij $F_{pc} := (W^R, T_{pc})$. Dan geldt dat elke gesloten, verschuivings-invariante deelruimte van F_{pc} eindig geheugen heeft.

Referenties

[1] C.Praagman & J.M.Soethoudt Continuous time behaviors, Memorandum nr.401 from Institute of Economic Research, Faculty of Economics, University of Groningen (1990).

2. Elke gesloten, verschuivings-invariante deelruimte van $L_{2,loc}^+(\mathbb{R})$ heeft eindig geheugen.

Referenties

[1] C.Praagman & J.M.Soethoudt A note on finite memory in locally square integrable systems, Memorandum nr.422 from Institute of Economic Research, Faculty of Economics, University of Groningen (1991).

3. Stel W is een operator van $C^{-k}(\mathbb{R})$ naar $C^{-k}(\mathbb{R})$ ($k \in \mathbb{N}$) en Z een operator van $C(\mathbb{R})$ naar $C(\mathbb{R})$ die voldoet aan $Z := \tilde{J}_k W \tilde{D}^k$. (Voor de definities van $C^{-k}(\mathbb{R})$, $\tilde{J}$ en $\tilde{D}$ verwijs ik naar hoofdstuk 5 van dit proefschrift). Dan zijn de volgende beweringen equivalent:

- W commuteert met de verschuivingsoperatoren $\tilde{\sigma}^t$, i.e. $\tilde{\sigma}^t W - W \tilde{\sigma}^t = 0$, voor alle $t \in \mathbb{R}$.
- Voor alle $f \in C(\mathbb{R})$ voldoet de operator Z aan

$$(\sigma^t Z - Z \sigma^t)f = P_k \sigma^t Z f \text{ en } Z(P^{k-1}) = \{0\}$$

(de definities van P_k en P^{k-1} kunnen worden gevonden aan het begin van paragraaf 5.2 van dit proefschrift).

4. Stel dat voor een verzameling $\mu_t \in ba_c(\mathbb{R})$, $t \in \mathbb{R}$ de volgende relatie geldt: $\mu_{t+s} = \sigma^{-s} \mu_t + \mu_s$. Voor de Fourier getransformeerde $\mathcal{F}_{\mu_t}$ van μ_t hangt de uitdrukking $\mathcal{F}_{\mu_t}(\omega)/(e^{-i\omega t} - 1)$ niet van t af en er geldt $\mu_t = (\sigma^{-t} - I)\mu$ voor zekere $\mu \in ba_c(\mathbb{R})$. Zo een μ_t kan geassocieerd worden met een continue operator van $C(\mathbb{R})$ naar $C(\mathbb{R})$, V , op de volgende manier: als voor alle $f \in C(\mathbb{R})$ geldt

$$(\sigma^t V - V \sigma^t)f = (Vf)(t) \cdot 1$$

dan is $(Vf)(t) = \int_{\mathbb{R}} f d\mu_t$

(Voor de definitie van $ba_c(\mathbb{R})$ verwijs ik naar definitie 2.1.12 van het proefschrift).

5. Neem μ en ν uit $ba_c(\mathbb{R})$ zodanig dat voor de Fouriergetransformeerden $\mathcal{F}_\mu$ en $\mathcal{F}_\nu$, geldt dat $\mathcal{F}_\nu/\mathcal{F}_\mu$ een eigenlijke rationale functie is. Definieer

$$\mathcal{B} = \{(w_1, w_2) \in C(\mathbb{R}, \mathbb{R}^2) \mid C_\mu w_1 = C_\nu w_2\}$$

Dan is $(w_1, w_2) \in \mathcal{B}$ dan en slechts dan als

$$w_1 - \mathcal{K}w_2 \in \ker C_\mu$$

waarbij

$$(\mathcal{K}u)(t) = \int_0^t k_1(t-\tau)u(\tau)d\tau + (\lambda u)(t), \quad t \in \mathbb{R}$$

met

$$\lambda = \lim_{s \rightarrow \infty} \frac{\mathcal{F}_\nu(s)}{\mathcal{F}_\mu(s)}$$

en k_1 de Bohl functie die voldoet aan

$$\int_0^\infty k_1(t)e^{-st}dt = \frac{\mathcal{F}_\nu(s)}{\mathcal{F}_\mu(s)} - \lambda$$

6. Laat $\mathcal{B}$ een lineaire, shift-invariante deelruimte in $C^\infty(\mathbb{R}, \mathbb{R}^q)$ zijn. Dan zijn de volgende beweringen equivalent:

- $\mathcal{B}$ is lokaal bepaald (locally specified)
- $\mathcal{B}^{(1)} := \{\frac{d}{dt}w \mid w \in \mathcal{B}\}$ is lokaal bepaald
- $\mathcal{B}^{(-1)} := \{w \mid \frac{d}{dt}w \in \mathcal{B}\}$ is lokaal bepaald

7. Gelukkig hoeft een vrouw niet meer te kiezen tussen een kind en een job.

8. Ongetrouwde mannen komen vaker op tijd voor een afspraak dan getrouwde mannen.

9. Zucht naar erkenning is een bekentenis van zwakte.

10. Op het einde van een promotie zorgt de tijdsdruk ervoor dat de som van het vermogen om te denken en het vermogen om dingen goed op te schrijven nagenoeg constant is. Dientengevolge is een laatste stelling vaak van een bedenkelijk niveau.