BMW algebras of simply laced type

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BMW algebras
of
Simply Laced Type

Proefschrift

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CHAPTER 1

Introduction

1.1. Knot theory

The main topic of this thesis is BMW algebras of simply laced type. However, to understand the background for the definitions and results it is important to have some understanding of knots and links and some of the algebras which have been introduced for them.

In this first chapter an overview is given of some basic results and ideas on knots and links. This will lead to the definition of the BMW algebras of simply laced type in Chapter 2. The ability to visualize the theory by pictures of links and knots makes knot theory a very accessible subject of research.

We start with the mathematical definition of a knot.

Definition 1.1.1. A knot $K$ is a tamely embedded closed curve in a compact part of $\mathbb{R}^3$.

An embedding of a closed curve is considered to be tame if it can be extended to an embedding of a tube around the curve. Equivalently, a tame embedding is a knot that can be represented as a polygonal path in 3-dimensional space.

Two knots are said to be isotopic when one can be continuously deformed into the other without intersecting itself.

Knots are generally studied using 2-dimensional projections, called knot diagrams. Such a diagram consists of edges and crossings and uniquely defines a knot. The part of the knot which is drawn continuously in the diagram is considered to cross over, or in front of, the part of the knot which is interrupted.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{knots.png}
\caption{Knot diagrams of the unknot, the trefoil knot, and the oriented figure-8 knot.}
\end{figure}

The simplest knot is the knot with a diagram without crossings. It is called the unknot. Another small knot is the trefoil knot which has a diagram where the knot crosses itself three times.

It is possible to give an orientation to a knot, a direction in which the knot will be travelled. Such a knot is called oriented.
One of the main goals in knot theory is to determine whether two knots are isotopic. One way to study this is by examining their diagrams. To determine equivalence between knot diagrams, a number of deformations of the plane, called *Reidemeister moves*, were introduced.

\[ \text{Figure 2. Reidemeister moves I, II and III.} \]

These moves, shown in Figure 2, define equivalence relations on knot diagrams.

**Definition 1.1.2.** Two knot diagrams are said to be ambient isotopic if they are related by a sequence of Reidemeister moves I, II and III together with planar isotopy.

Two knot diagrams are said to be regularly isotopic when they are related by a sequence of only Reidemeister moves II and III together with planar isotopy.

Reidemeister showed that equivalence of knot diagrams can be used to determine equivalence of the corresponding knots.

**Theorem 1.1.3. (Reidemeister)** Two knots are isotopic if and only if they possess ambient isotopic diagrams.

The definition of a link involves multiple knots in one space.

**Definition 1.1.4.** A link is a finite disjoint union of knots. Each knot in a link is called a component of the link. The knots are disjoint but can be interconnected as displayed in Figure 3.

The 2-dimensional projection of a link is called a *link diagram*. An oriented link is obtained by assigning an orientation to every component of the link. The notions of isotopy for knots and ambient and regular isotopy for knot diagrams carry over to links and link diagrams in a natural way.
Figure 3. Two diagrams of unoriented links: the Borromean rings with 3 components and the Whitehead link with 2 components.

One of the main tools in the research of knots and links are the knot invariants and link invariants.

**Definition 1.1.5.** A knot invariant under ambient isotopy is a quantity of the knot diagram which does not change by deformation of the plane nor by any sequence of Reidemeister moves I, II and III and isotopies of the plane.

A knot invariant under regular isotopy is a quantity of the knot diagram which does not change by deformation of the plane nor by any sequence of Reidemeister moves II and III and isotopies of the plane.

Link invariants are defined accordingly. There are many invariants of knots and links. The number of components of a link, called its multiplicity, is an example of a link invariant. This number will never change by applying Reidemeister moves to the link diagram.

An example of a knot invariant is its unknotting number, the minimal number of crossings in the knot diagram which needs to be changed to make the knot isotopic to the unknot.

In the next sections some other, polynomial-valued, invariants are introduced which will lead to the introduction of the BMW algebras.

### 1.2. Braids, the Hecke algebra and the HOMFLY polynomial

A way to represent link diagrams is by braids. One of the interesting things about this approach is the fact that it gives some obvious connections between knot theory and other mathematical areas. One of the main advantages of working with braids is the fact that they form a group, the braid group.

We start with the definition of an $n$-strand braid. Fix points $b_i$ on a line in $\mathbb{R}^2$ with $b_1 < b_2 < \ldots < b_{n-1} < b_n$. Let $I$ denote the unit interval $[0,1]$.

**Definition 1.2.1.** An $n$-strand braid is a disjoint union of $n$ smooth curves in $\mathbb{R}^2 \times I$, called strands, connecting each of the points $\{(b_1,1), \ldots, (b_n,1)\}$ with a distinct point $\{(b_1,0), \ldots, (b_n,0)\}$ in such a way they intersect each intermediate plane precisely once.

Braids are depicted by a braid diagram, that is, a 2-dimensional projection of the braid. As in link diagrams, we distinguish between over and under crossings of the strands. When a strand connects two points $(b_i, 1)$ and $(b_i, 0)$ directly below each other, we say the strand goes monotonically down.
Two braids are considered to be isotopic if there is a continuous set of braids deforming one braid into the other. From now on we will consider braids only up to isotopy.

The braids form a group, the braid group on \( n \) strands. The unity element is the braid with a diagram with no crossings where all strands go monotonically down. Multiplication of two braids \( b_1 \) and \( b_2 \) is done by placing them on top of each other and connecting the strands by removing the end points in the middle. The new braid is now obtained after rescaling. The \textit{inverse of a braid} is its reflection with respect to a horizontal line. Composition of a braid and its inverse results in a braid isotopic to the unity element.

The braid group on \( n \) strands is generated by \( n-1 \) so called \textit{simple braids} \( s_i \), for \( i = 1, \ldots, n-1 \). The braid \( s_i \) is the braid which interchanges the points \( b_i \) and \( b_{i+1} \) and has lines going monotonically down connecting all other points as displayed in Figure 5. The strand going down from north east to south west crosses over the strand that connects the points at the north west and south east. The braid where the north west to south east strand crosses over the other strand is the inverse of this braid. Verification that these braids are indeed each other’s inverses can be done using Reidemeister move II.

Defined in an algebraic way, the braid group is part of a larger family of groups, namely the Artin groups. We restrict to the braid groups in this section. A definition of the Artin groups is given in § 1.4.

\begin{definition}
\textit{The braid group} \( B_n \) \textit{on} \( n \) \textit{strands is given by the presentation with} \( n-1 \) \textit{generators} \( s_1, \ldots, s_{n-1} \) \textit{and relations}
\begin{align*}
\text{(B1)} & \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \\
\text{(B2)} & \quad s_is_j = s_js_i \quad \text{when} \ |i-j| \geq 2.
\end{align*}
\end{definition}
The two types of relations defining $B_n$ are called the braid relations. Artin proved in [Art] that the geometric and the algebraic definition of the braid group are equivalent. Two generators $s_i$ and $s_j$ with $|i - j| \geq 2$ interchange distinct points $b_i, b_{i+1}$ and $b_j, b_{j+1}$, so they obviously commute. This braid relation is therefore also known as far commutativity. The other braid relation is shown in Figure 6.

![Figure 6. The braid relation $s_1 s_2 s_1 = s_2 s_1 s_2$ in $B_3$.](image)

The natural orientation of a braid is downwards for every strand. This makes it possible to relate the braids to oriented links.

**Definition 1.2.3.** The closure of a braid $b$ is the link diagram $L_b$ obtained from the braid diagram of $b$ by connecting every end point at the top of the diagram with the corresponding end point at the bottom of the diagram by $n$ strands in such a way that do not cross each other.

Consider the braid diagram in Figure 4. Its closure is the diagram of the Whitehead link shown in Figure 3. The following theorem gives an important relation between braids and links.

**Theorem 1.2.4. (Alexander)** Every isotopic, oriented link can be represented by the closure of a braid.

This braid (diagram) of a link is not unique. Markov formulated an equivalence relation on braid diagrams which determines whether two closures $L_b$ and $L_{b'}$ of two distinct braids $b$ and $b'$ are isotopic.

**Theorem 1.2.5. (Markov)** Two braids have ambient isotopic closures if and only if one can be obtained from the other by a sequence of the following two moves.

(i) $b \mapsto aba^{-1}$ with $a, b \in B_n$,

(ii) $b \mapsto bs_n^\pm 1$ with $b \in B_n$ and $s_n \in B_{n+1}$.

The two moves are called the Markov moves. A closure of a braid diagram $b$ and the closure of $aba^{-1}$ are easily to be seen equivalent using the Reidemeister moves. Markov’s second move adds a strand to a braid diagram $b$ as $s_n^\pm 1 \notin B_n$. Here Reidemeister move I shows $L_b = L_{bs_n^\pm 1}$.

If, for all $i$, the relation $s_i^2 = 1$ is added to the algebraic definition of the braid group, we get $\Sigma_n$, the symmetric group on $n$ letters. Adding a different quadratic relation for the $s_i$ gives the Hecke algebra. This algebra occurs in several settings. Here we introduce a two parameter definition of this algebra.
Definition 1.2.6. The Hecke algebra $H_n$ is the associative algebra over a two parameter ring $\mathbb{Z}[|\pm 1, m]$ with $n$ generators $g_1, \ldots, g_n$ and relations

\begin{align*}
(H1) & \quad l^{-1}g_i - lg_i^{-1} = -m & \text{for all } i \in M, \\
(B1) & \quad g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} & \text{for all } i, \\
(B2) & \quad g_j g_i = g_i g_j & \text{when } |i - j| \geq 2.
\end{align*}

Remarks 1.2.7. (i). Usually, the Hecke algebra is defined as an algebra over the ring $\mathbb{Z}[|q^{\pm 1}]$ with the quadratic relation $\tilde{g}_i^2 = (q-1)\tilde{g}_i + q$. This original presentation can be obtained by substitution of the scaled generators $g_i = lq^{-\frac{1}{2}}\tilde{g}_i$ and by substitution of $m = q^{-2} - q^2$.

(ii). Notice the shift in indexing. The braid group $B_n$ on $n$ strands is related to the Hecke algebra $H_{n-1}$. The indexing of the Hecke algebras will be used later to relate the algebras with Coxeter graphs. In Section 1.4 we will redefine the braid group $B_n$, as the Artin group $A_{\Sigma_n}$.

(iii). Like the symmetric group $\Sigma_n$, the braid group $B_n$ has order $n!$. The Hecke algebra $H_{n-1}$ is a free $\mathbb{Z}[|l^{\pm 1}, m]$-module of dimension $n!$.

The Hecke algebras can be constructed recursively as shown in the next lemma.

Lemma 1.2.8. Every element $x \in H_n$ can be written as a linear combination of elements $y$ and $y_1 g_n y_2$ with $y, y_1, y_2 \in H_{n-1}$.

Proof. We prove the lemma by induction on the rank $n$. The lemma is trivial for $H_1$. Assume the lemma holds for all $k < n$. Let $x \in H_n$. We need to show any monomial in $H_n$ can be written as a product of generators with at most one $g_n$.

Assume we have $w_1 g_n w_2 g_n w_3$ in $H_n$ with $w_i \in H_{n-1}$ for all $i$. By the induction hypothesis, we have $w_2 \in H_{n-2}$ or $w_2 = v_1 g_{n-1} v_2$ with $v_1, v_2 \in H_{n-2}$.

By far commutativity, the first case is straightforward as we get

$$w_1 g_n w_2 g_n w_3 = w_1 w_2 g_n^2 w_3 = l^2 w_1 w_2 w_3 - ml w_1 w_2 g_n w_3.$$ 

In the second case, far commutativity gives

$$w_1 g_n v_1 g_{n-1} v_2 g_n w_3 = w_1 v_1 g_n g_{n-1} g_n v_2 w_3.$$ 

By the other braid relation this results in

$$w_1 v_1 g_n g_{n-1} g_n v_2 w_3 = w_1 v_1 g_{n-1} g_n v_2 w_3.$$ 

Hence, we proved the induction step which completes the proof of the lemma. \qed

There exists a trace function on $H_n$ which is invariant under the Markov moves. So, in view of Theorem 1.2.5, it gives a polynomial-valued link invariant for oriented links.

Proposition 1.2.9. Each Hecke algebra $H_n$ for $n \in \mathbb{N}$ supports a linear trace function $\text{tr}: H_n \rightarrow \mathbb{Z}[|l^{\pm 1}, m]$, characterized by

(i) $\text{tr}(xy) = \text{tr}(yx),$

(ii) $\text{tr}(1) = 1,$

(iii) $\text{tr}(x g_n) = m(l - l^{-1})^{-1} \text{tr}(x)$ when $x \in H_{n-1}$. 
Proof. This proof is taken from [M]. The lemma is proven by induction on $n$. To this end an initial definition of the trace is taken, $\text{tr}(1) = 1$ and $\text{tr}(xg_ny) = (m^{-1}(l - l^{-1}))^{-1} \text{tr}(xy)$ for $x, y \in H_{n-1}$.

We want to show $\text{tr}(xy) = \text{tr}(yx)$. By the initial definition and the induction step, it is only necessary to check cases where $x = g_n$ and $y = v_1 g_n v_2$ with $v_1, v_2 \in H_{n-1}$.

We need to show $\text{tr}(g_n v_1 g_n v_2) = \text{tr}(v_1 g_n v_2 g_n)$. This leaves three cases to check,

(i). $v_1, v_2 \in H_{n-2}$,

(ii). $v_1 = w_1 g_{n-1} w_2$ with $w_1, w_2 \in H_{n-2}$ and $v_2 \in H_{n-2}$,

(iii). $v_1 = w_1 g_{n-1} w_2$ and $v_2 = w_3 g_{n-1} w_4$ with $w_1, w_2, w_3, w_4 \in H_{n-2}$.

The first case is satisfied by the induction hypothesis and far commutativity. When $v_1 = w_1 g_{n-1} w_2$ with $w_1, w_2 \in H_{n-2}$ and $v_2 \in H_{n-2}$, we use both braid relations to find

$$g_n w_1 g_{n-1} w_2 g_n v_2 = w_1 g_n g_{n-1} g_n w_2 v_2 = w_1 g_n g_{n-1} g_n w_2 v_2.$$ 

Now

$$\text{tr}(g_n v_1 g_n v_2) = \text{tr}(w_1 g_n g_{n-1} g_n g_{n-1} w_2 v_2)$$

$$= m(l - l^{-1})^{-1} \text{tr}(w_1 g_n g_{n-1} g_n w_2 v_2)$$

$$= m(l - l^{-1})^{-1}(l^2 \text{tr}(w_1 w_2 v_2) - ml \text{tr}(w_1 g_n g_{n-1} w_2 v_2)),$$

and the other way around we find

$$\text{tr}(v_1 g_n v_2 g_n) = \text{tr}(w_1 g_n g_{n-1} w_2 v_2 g_n)$$

$$= (l^2 \text{tr}(w_1 g_n g_{n-1} w_2 v_2) - ml \text{tr}(w_1 g_n g_{n-1} w_2 v_2 g_n))$$

$$= m(l - l^{-1})^{-1}(l^2 \text{tr}(w_1 w_2 v_2) - ml \text{tr}(w_1 g_n g_{n-1} w_2 v_2)).$$

Notice the opposite case, where $v_2 = w_1 g_{n-1} w_2$ with $w_1, w_2 \in H_{n-2}$ and $v_1 \in H_{n-2}$, is completely analogous to this case.

Finally, in the third case we find

$$\text{tr}(xy) = \text{tr}(w_1 g_n g_{n-1} g_n g_{n-1} w_2 v_2)$$

$$= m(l - l^{-1})^{-1} \text{tr}(w_1 g_n g_{n-1} g_n g_{n-1} w_2 v_2)$$

$$= m(l - l^{-1})^{-1}(l^2 \text{tr}(w_1 w_2 v_3 g_{n-1} w_4) - ml \text{tr}(w_1 g_n g_{n-1} w_2 v_3 g_{n-1} w_4))$$

$$= m^2 l^2(l - l^{-1})^{-2} \text{tr}(w_1 w_2 v_3 w_4)$$

$$- m^2 l(l - l^{-1})^{-1} \text{tr}(w_1 g_n g_{n-1} w_2 v_3 g_{n-1} w_4),$$

and

$$\text{tr}(yx) = \text{tr}(w_1 w_3 g_{n-1} g_n w_4)$$

$$= \text{tr}(v_1 w_3 g_{n-1} g_n g_{n-1} w_4)$$

$$= m(l - l^{-1})^{-1} \text{tr}(v_1 w_3 g_{n-1} g_n g_{n-1} w_4)$$

$$= m(l - l^{-1})^{-1}(l^2 \text{tr}(v_1 g_n g_{n-1} w_2 v_3 w_4) - ml \text{tr}(w_1 g_n g_{n-1} w_2 v_3 g_{n-1} w_4))$$

$$= m^2 l^2(l - l^{-1})^{-2} \text{tr}(v_1 w_2 v_3 w_4)$$

$$- m^2 l(l - l^{-1})^{-1} \text{tr}(w_1 g_n g_{n-1} w_2 v_3 g_{n-1} w_4).$$
which completes the induction step.

When two braid diagrams are related by the first Markov move, they have the same trace. The traces of two diagrams equivalent by Markov’s second move differ by a scalar multiple. Hence this trace function defines a link invariant, by the normalization

\[ P_{L_0}(l, m) = (m^{-1}(l - l^{-1}))^{n-1} \text{tr}(b) \]

and by setting \( P_{\text{unknot}}(l, m) = 1 \).

This polynomial-valued link invariant for oriented knots is known as the HOMFLY polynomial, named after the six authors of the paper [Hom]. This polynomial can also be calculated directly from the diagram of a link. This is done with a skein relation defined on almost identical link diagrams. These diagrams only differ in a small region of the diagram containing precisely one crossing. We distinguish between the three diagrams \( L_+, L_- \) and \( L_0 \) shown in Figure 7.

\[
\begin{array}{ccc}
L_+ & L_- & L_0 \\
\includegraphics[width=0.3\textwidth]{figure7}
\end{array}
\]

**Figure 7.** Three links which are the same everywhere except for the small piece inside the circle.

**Lemma 1.2.10.** The following skein relation holds.

\[ l^{-1}P_{L_+}(l, m) - lP_{L_-}(l, m) = -mP_{L_0}(l, m) \]

where \( L_+, L_- \) and \( L_0 \) are three identical diagrams only differing at one crossing, as depicted in Figure 7.

**Proof.** Consider three such diagrams only differing in one crossing. Then \( L_+ \) has \( s_i \), \( L_- \) has \( s_i^{-1} \) and \( L_0 \) has one crossing less. So \( P_{L_+}(l, m) = \text{tr}(g_i)M \), \( P_{L_-}(l, m) = \text{tr}(g_i^{-1})M \) and \( P_{L_0}(l, m) = m^{-1}(l - l^{-1})M \) with \( M \) the common part of the three diagrams. Applying the Hecke algebra relation \( l^{-1}g_i - lg_i^{-1} = -m \) gives the desired result. \( \square \)

We end this section by illustrating how the HOMFLY polynomial is computed using the two ways we described.

**Example 1.2.11.** Consider again the link diagram \( L \) of the Whitehead link shown in Figure 3. We assign an orientation to both components such that it corresponds to the braid diagram in Figure 4. The corresponding Hecke algebra element is \( g_1g_2^{-1}g_1g_2^{-2} \) and the orientation of \( L \) is chosen such that the top left crossing is travelled in the directions top to bottom and left to right.

We start with the calculation of \( \text{tr}(g_1g_2^{-1}g_1g_2^{-2}) \). First rewrite it to a linear combination of elements with at most one \( g_2^{-1}g_2 \).
\[ g_1 g_2^{-1} g_1 g_2^{-2} = g_1 g_2^{-1} g_1 (l^{-2} + l^{-1} m g_2^{-1}) = l^{-2} g_1 g_2^{-1} g_1 + l^{-1} m g_1 g_2^{-1} g_1 g_2^{-1} = l^{-2} g_1 g_2^{-1} g_1 + l^{-1} m g_1 (l^{-2} g_2 + m l^{-1}) g_1 g_2^{-1} = l^{-2} g_1 g_2^{-1} g_1 + l^{-3} m g_1 g_2 g_1 g_2^{-1} + l^{-2} m^2 g_1 g_2^{-1} g_1 g_2^{-1} = l^{-2} g_1 g_2^{-1} g_1 + l^{-3} m g_2 g_1 + l^{-2} m^2 g_1 g_2^{-1} g_1 g_2^{-1}. \]

As
\[ t r(g_1 g_2^{-1} g_1) = t r(g_1 g_2^{-1} g_1) = m(l - l^{-1})^{-1} (l^2 - m l (l - l^{-1})^{-1}) \]
and
\[ t r(g_2 g_1) = m(l - l^{-1})^{-2} \]
this results in
\[ t r(g_1 g_2^{-1} g_1 g_2^{-2}) = m^2 (l - l^{-1})^{-2} (m l^3 - m^3 l^{-1} - m l^{-1}) + m(l - l^{-1})^{-1} (1 + m^2). \]

As \( g_1 g_2^{-1} g_1 g_2^{-2} \in B_3 \), this gives for the HOMFLY polynomial,
\[ P_L(l, m) = (m^{-1} l - l^{-1})^2 t r(g_1 g_2^{-1} g_1 g_2^{-2}) = m l^3 - m^3 l^{-1} - m l^{-1} + (m + m^{-1})(l - l^{-1}). \]
In Figure 8 the picture calculation using the skein relation is illustrated. This leads to the same result.

1.3. Tangles and the BMW Algebra

Another way to classify links is by tangles. Again this leads to an algebra with a trace function resulting in another link invariant, although here a link invariant for unoriented links.

\[
D_+ \quad D_- \quad D_0 \quad D_\infty
\]

**Figure 9.** The four diagrams are the same everywhere except for a small piece inside the circle.

In the previous section we used a triple of link diagrams which differed only in a small region containing one crossing, obtaining the diagrams \(L_+, L_-\) and \(L_0\). However, when links without orientation are considered, we can alter a particular crossing also in a fourth way. Hence, for unoriented links we define four link diagrams \(D_+, D_-, D_0\) and \(D_\infty\) as shown in Figure 9.

Fix again points \(b_i\) on a line in \(\mathbb{R}\) with \(b_1 < b_2 < \ldots < b_{n-1} < b_n\). Let \(I\) denote the unit interval \([0,1]\).

**Definition 1.3.1.** An \((n,n)\)-tangle is a piece of a link diagram in \(\mathbb{R} \times I\), that is, a 2-dimensional projection of a disjoint union of \(n\) piecewise linear curves connecting two distinct points from \(\{(b_1,1), \ldots, (b_n,1)\} \cup \{(b_1,0), \ldots, (b_n,0)\}\) where the crossings are preserved.

\[
\text{Figure 10. Example of a (4,4)-tangle.}
\]

The notion of ambient and regular isotopy carries over to tangles. Also composition of tangles is done in the same way as composition of link diagrams.

Let \(U(n)\) be the monoid of \((n,n)\)-tangles modulo regular isotopy. We define an algebra of tangles, known as the Kauffman tangle algebra. We define the algebra over the ring:

\[
E = \mathbb{Q}(\delta)[t^{\pm 1}].
\]
Definition 1.3.2. The Kauffman tangle algebra of type \( \Lambda \), denoted \( KT(\Lambda)_n \), over \( E \), is the algebra constructed from \( E[U(n)] \) by factoring out the following relations:

(i) The Kauffman skein relation

\[
+m = +m
\]

Here, the diagrams indicate tangles which differ only in the crossing shown.

(ii) The self-intersection relation

\[
= l^{-1} \quad \text{and} \quad = l
\]

(iii) The idempotent relation

\[
T \cup O = \delta T,
\]

where \( T \cup O \) is the union of a tangle \( T \) and a closed loop \( O \) having no crossings with \( T \) nor self-intersections.

The algebra \( KT(\Lambda)_n \) is generated by \( 2n - 2 \) tangles we denote by \( G_i \) and \( E_i \) for \( i = 1, \ldots, n \). The tangle \( G_i \) interchanges the \( i \)-th and \((i+1)\)-st boundary points vertically and the tangle \( E_i \) connects the \( i \)-th and \((i+1)\)-st node by two horizontal strands as shown in Figure 11.

![Figure 11. The 2n - 2 generators of KT(\Lambda)_n.](image)

As for the braid group we have an algebraic description of the tangle algebra. This definition was first given by Birman and Wenzl in [BW] and independently by Murakami in [M]. We refer to these algebras as the BMW algebras of type \( \Lambda \).

Definition 1.3.3. The Birman-Murakami-Wenzl algebra of type \( \Lambda \), or BMW algebra of type \( \Lambda \), is the algebra, denoted by \( B(\Lambda)_n \), over \( \mathbb{Q}(l, \delta) \), whose presentation is given on generators \( g_i \) and \( e_i \) (\( i = 1, \ldots, n \)) by the following defining relations.
1. INTRODUCTION

\( g_i g_j = g_j g_i \quad \text{when} \quad |i - j| \geq 2. \)  
\( g_i g_{i+1} g_i = g_i g_{i+1} g_{i+1} \)

\( m e_i = l(g_i^2 + mg_i - 1) \quad \text{for all} \quad i. \)

\( g_i e_i = l^{-1} e_i \quad \text{for all} \quad i. \)

\( e_i g_{i+1} e_i = l e_i \)

where \( m = (l - l^{-1})/(1 - \delta). \)

**Remarks 1.3.4.**
(i). This definition of the BMW algebra slightly differs from the original definition in [BW]. In Theorem 2.3.8 we show the two definitions coincide.

(ii). The defining relation \( (D1) \) shows the algebra is actually defined by only \( n \) generators. As \( g_i \) is invertible, we can multiply this relation by \( g_i^{-1} \) and obtain the variation

\( g_i + m = g_i^{-1} + me_i.\)

This altered relation is the algebraic version of the Kauffman skein relation. It also shows \( B(A_n) \) is related to the Hecke algebra \( H_n \). The Hecke algebra \( H_n \) is isomorphic to the quotient algebra of \( B(A_n) \) obtained by taking \( e_i = 0 \) for all \( i \).

(iii). The ring \( E \) over which we defined \( KT(A) \) is inside our field \( \mathbb{Q}(l, \delta) \) by the definition \( m = (l - l^{-1})/(1 - \delta). \)

We will study the properties and relations of the BMW algebras in more detail in Chapter 2.

**Lemma 1.3.5. (Birman and Wenzl)** Let \( B(A_n) \) be the BMW algebra of type \( A_n \). Then every element \( x \in B(A_n) \) can be written as a linear combination of elements \( y_1 \chi y_2 \) where \( y_1, y_2 \in B(A_{n-1}) \) and with \( \chi \in \{1, g_n, e_n\} \).

**Proof.** We omit the proof here as it involves relations between the generators \( g_i \) and \( e_i \) which will be developed later. The proof is given near the end of §2.3. \( \square \)

There exists a homomorphism \( \varphi \) from \( B(A_n) \) to \( KT(A)_{n+1} \) defined by \( \varphi(g_i) = G_i \) and \( \varphi(e_i) = E_i \). Morton and Wasserman proved in [MW] that \( \varphi \) is in fact an isomorphism between \( B(A_n) \) and \( KT(A)_{n+1} \). We omit the proof, but in Chapter 5 we follow their ideas to show the existence of a surjective homomorphism between the BMW algebra \( B(D_n) \) and another tangle algebra \( KT(D)_n \).

Birman and Wenzl showed that the BMW algebras of type \( A \) are finite dimensional. They established the following result.

**Theorem 1.3.6. (Birman and Wenzl)** The BMW algebra \( B(A_n) \) has dimension \( (n + 1)!! = (2n + 1)(2n - 1) \cdots 5 \cdot 3 \cdot 1 \) over \( \mathbb{Q}(l, \delta) \).

Morton and Wasserman determined this dimension using a third algebra, the Brauer algebra \( C(A)_n \). This monoid algebra was introduced by Brauer in [Bra] as an algebra over \( \mathbb{Z}[\delta^{\pm 1}] \). The main idea behind this algebra is the notion of an \( n \)-connector.

**Definition 1.3.7.** An \( n \)-connector is a pairing of \( 2n \) points into \( n \) pairs.
1.3. TANGLES AND THE BMW ALGEBRA

Such an $n$-connector can be pictured by a so-called Brauer diagram. The $2n$ points are divided in two sets $t = 1, \ldots, n$ and $b = \bar{1}, \ldots, \bar{n}$ of points in the plane and connected by $n$ strands as determined by the pairing. The strands are allowed to intersect and no distinction is made between over and under crossings in contrast with the tangles and braid diagrams.

Multiplication of two $n$-connectors $a_1$ and $a_2$ is defined by stacking their diagrams on top of each other, identifying the bottom set $b$ of $a_1$ and the top set $t$ of $a_2$ and connecting the strands to find a new pairing. Besides a pairing, this can result in closed loops not connected to the top or the bottom. When a composition gives, say $r$ closed loops, the new pairing has a coefficient $\delta^r$. So $a_1a_2 = \delta^r a$ with $a$ the $n$-connector obtained from concatenation of the two pairings and $r$ the number of closed loops in $a_1a_2$.

We have an obvious map $nc$ from a tangle in $\mathbf{KT}(A)_n$ to an $n$-connector. The information of over and under crossings in the tangle diagram $T$ is simply ignored. We call the picture of the $n$-connector we obtain this way the Brauer diagram of $T$, denoted by $nc(T)$.

**Definition 1.3.8.** The Brauer algebra $\mathbf{C}(A)_n$ of type $A$ over $\mathbb{Z}[\delta^{\pm 1}]$ is the free $\mathbb{Z}[\delta^{\pm 1}]$-module with basis the set of $n$-connectors with the described multiplication.

In Chapter 5 we return to the Brauer algebra when we study a Brauer algebra variant $\mathbf{C}(D)_n$ to determine the dimension of the tangle algebra $\mathbf{KT}(D)_n$.

The closure of a tangle $T$ is the link diagram obtained by connecting the opposite end points $(b_i, 0)$ and $(b_i, 1)$ of $T$ by strands which do not cross each other or intersect themselves. As we use unoriented tangles, this operation results in diagrams of unoriented links.

The definition of a trace function on the elements of $\mathbf{B}(A_n)$ leads to a polynomial-valued invariant under regular isotopy for unoriented links introduced by Kauffman in [Kau].

**Proposition 1.3.9.** Each BMW algebra $\mathbf{B}(A_n)$ for $n \in \mathbb{N}$ supports a linear trace function $\text{tr} : \mathbf{B}(A_n) \to E$, characterized by

(i) $\text{tr}(xy) = \text{tr}(yx)$,

(ii) $\text{tr}(1) = 1$,

(iii) $\text{tr}(xg_n) = \delta^{-1} \text{tr}(x)$ when $x \in \mathbf{B}(A_{n-1})$.

**Proof.** The proof is similar to the proof of Lemma 1.2.9 and uses the rewrite result of Lemma 1.3.5. $\square$

By the normalization $KP_L(\delta) = \delta^{n-1} \text{tr}(x)$ a link invariant for unoriented links is obtained which is further defined by $KP_{\text{unknot}}(\delta) = 1$.

A four term skein relation can be derived from the definition which enables the direct computation of $KP_L(\delta)$ from the link diagrams.

**Lemma 1.3.10.** The following relation holds,

$$KP_{D_+}(l,m) + mKP_{D_0}(l,m) = KP_{D_-}(l,m) + mKP_{D_\infty}(l,m)$$

for unoriented links $D_+, D_-, D_0$ and $D_\infty$, four identical link diagrams which differ in a small region containing one crossing, as shown in Figure 9.
Proof. The proof is similar to the proof of Lemma 1.2.10. See [BW] for further details.

In [MW], Morton and Wasserman gave an explicit isomorphism between $B(A_n)$ and $KT(A_{n+1})$. This isomorphism was obtained by constructing a basis in $B(A_n)$, analogous to a basis of $KT(A_{n+1})$ using the $(n+1)$-connectors of the Brauer algebra. In Chapter 5 we study a similar construction for the algebras $B(D_n)$ and $KT(D_n)$.

1.4. Artin group representations

It had been an open question for a long time whether the braid group, defined in §1.2, is linear. Burau’s construction of a matrix representation is to represent a generator $s_i$ by the $n \times n$ matrix which is the identity matrix with a $2 \times 2$ block on the $(i, i+1)$ row and column. It is straightforward to verify these matrixes satisfy far commutativity. By taking the $2 \times 2$ block equal to

$$
\begin{pmatrix}
1 - t & t \\
1 & 0
\end{pmatrix}
$$

we obtain that the other braid relation is satisfied as well. Burau proposed this representation in [Bur] in 1936. It was an open question for a long time whether this representation was faithful. Nowadays, it is known that the representation is faithful for $n = 3$ and unfaithful for $n \geq 5$. It is still an open question for $n = 4$.

In 2000, Krammer used another representation to prove $B_4$ is linear (cf. [Kra]). Shortly after, both Bigelow in [Big] and Krammer in [Kra2] extended this result to all braid groups, proving linearity for all braid groups. The representation Krammer introduced was inspired by work of Lawrence in an area of physics, [L]. Therefore we denote this representation the Lawrence-Krammer representation, or simply the LK representation, of the braid group. The following definition is taken from [Kra].

**Definition 1.4.1.** Let $R$ be a commutative ring with unit element and $q, r$ two invertible elements of $R$. Let $V$ be the $\frac{n(n-1)}{2}$ dimensional linear vector space over $R$ generated by elements $x_{k,j}$ with $1 \leq k < j \leq n$. The action of the generators $s_i$ of the braid group $B_n$ on $V$ is defined by

$$
\begin{aligned}
&x_{k,j} \\
x_{k-1,j} + (1 - q)x_{k,j} &\quad \text{if } i < k - 1 \text{ or } i > j, \\
t(q - 1)x_{k,k+1} + qx_{k+1,j} &\quad \text{if } i = k - 1, \\
t^2x_{k,j} &\quad \text{if } i = k = j - 1, \\
x_{k,j} + tq^{i-k}(q - 1)x_{k+1,j} &\quad \text{if } i < k < j - 1, \\
x_{k,j} + tq^{i-k}(q - 1)x_{j-1,j} &\quad \text{if } i < k = j - 1, \\
(1 - q)x_{k,j} + qx_{k,j+1} &\quad \text{if } i = j.
\end{aligned}
$$

The initial motivation for our study of BMW algebras of simply laced type came from two results inspired by this new representation of the braid group. In [Z], Zinno showed the LK-representation of the braid group $B_n$ factors through the BMW algebra $B(A_{n-1})$.

The second result was inspired by the fact that the braid groups are part of a larger family of groups, the Artin groups. These groups, introduced by Artin in [Art], are defined in relation to Coxeter graphs $M$. 
Definition 1.4.2. The Artin group $A(M)$ of type $M$ and rank $n$, is given by the presentation with $n$ generators $s_1, \ldots, s_n$ and relations

\begin{align*}
(B1) \quad & s_is_j = s_js_is_j \quad \text{when } i \sim j, \\
(B2) \quad & s_is_j = s_js_i \quad \text{when } i \not\sim j.
\end{align*}

Remarks 1.4.3. (i). In this thesis we will merely work with irreducible Coxeter graphs of simply laced type. These graphs are shown in Figure 12.

(ii). From the Coxeter graph of type $A_n$ it is clear the braid group $B_n$ on $n$ strands is the same as the Artin group $A(A_{n-1})$.

(iii). The definition of the Hecke algebra can also be generalized for every arbitrary Coxeter graph. From here we denote the Hecke algebras by $H(M)$, referring to the related Coxeter type. The algebras $H_n$ introduced in Definition 1.2.7 are the Hecke algebras $H(A_n)$ of type $A$.

In [CW], Cohen and Wales extended the results of Bigelow and Krammer and proved linearity of all finite type Artin groups (also [Dig]). To do so, the Lawrence-Krammer representation was generalized to a representation for all Artin groups of finite, irreducible and simply laced type. That is, the Coxeter graph $M$ is of type $A_n$, $D_n$ or $E_6$, $E_7$, or $E_8$ as shown in Figure 12. As the other finite irreducible type Artin groups can be embedded in these, linearity for all finite types was established this way.
In this thesis we restrict our studies to groups and algebras related to Coxeter graphs of finite, irreducible and simply laced type.

Cohen and Wales showed the LK-representation for an Artin group \( A(M) \) is related to the roots in the positive root system \( \Phi^+ \) of type \( M \). The next theorem is taken from that paper. The coefficients of the representation are taken over a ring \( \mathbb{Z}[t^{\pm 1},t^{\pm 1}] \) and \( V \) is the free module over that ring with generators \( x_\beta \) indexed by \( \beta \in \Phi^+ \).

**Theorem 1.4.4. (Cohen and Wales)** Let \( A(M) \) be an Artin group of type \( A, D, E \). Then, for each \( i \in \{1, \ldots, n\} \) and each \( \beta \in \Phi^+ \), there are polynomials \( T_{i,\beta} \in \mathbb{Z}[t] \) such that the following map on the generators of \( A(M) \) determines a representation of \( A(M) \) on \( V \).

\[
\sigma_i \mapsto \sigma_i = \tau_i + l^{-1}T_i,
\]

where \( \tau_i \) is determined by

\[
\tau_i(x_\beta) = \begin{cases} 
0 & \text{if } (\alpha_i, \beta) = 2 \\
x_{\beta - \alpha_i} & \text{if } (\alpha_i, \beta) = 1 \\
x_{\beta - \alpha_i} & \text{if } (\alpha_i, \beta) = 0 \\
x_{\beta - \alpha_i} & \text{if } (\alpha_i, \beta) = -1,
\end{cases}
\]

and \( T_i \) is the linear map on \( V \) determined by \( T_i x_\beta = T_{i,\beta} x_\alpha \) on the generators of \( V \). If \( r \) is specialized to a real number \( r_0 \), \( 0 < r_0 < 1 \), in \( V \otimes \mathbb{R} \), we obtain a faithful representation of \( A(M) \) on the resulting free \( \mathbb{Z}[t,t^{-1}] \)-module \( V_1 \) with basis \( x_\beta \) \( (\beta \in \Phi^+) \).

Here the \( T_{i,\beta} \) are determined by the equations in Table 1.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{i,\beta} ) = 0</td>
<td>( \beta = \alpha_j ) and ( i \neq j )</td>
</tr>
<tr>
<td>( r^i )</td>
<td>( \beta = \alpha_i )</td>
</tr>
<tr>
<td>( r^{i+n} - r^i )</td>
<td>( \beta = \alpha_i + \alpha_j )</td>
</tr>
<tr>
<td>( r T_{i,\beta - \alpha_i} )</td>
<td>( (\alpha_j, \beta) = 1 ) and ( (\alpha_i, \alpha_j) = 0 )</td>
</tr>
<tr>
<td>( T_{j,\beta - \alpha_i} + (r - r^{-1}) T_{i,\beta - \alpha_i} )</td>
<td>( (\alpha_i, \beta) = 0 ) and ( (\alpha_j, \beta) = 1 ) and ( (\alpha_i, \alpha_j) = -1 )</td>
</tr>
<tr>
<td>( r T_{j,\beta - \alpha_i} + (r - r^{-1}) T_{i,\beta - \alpha_i} )</td>
<td>( (\alpha_i, \beta) = -1 ) and ( (\alpha_j, \beta) = 1 ) and ( (\alpha_i, \alpha_j) = -1 )</td>
</tr>
<tr>
<td>( r T_{j,\beta - \alpha_i} )</td>
<td>( (\alpha_i, \beta) = 1 ) and ( (\alpha_j, \beta) = 0 ) and ( (\alpha_i, \alpha_j) = -1 )</td>
</tr>
</tbody>
</table>

Table 1. Equations for \( T_{i,\beta} \).

**Remark 1.4.5.** The LK-representation given by Definition 1.4.1 can be obtained from the representation in Theorem 1.4.4 for \( A(A_n) \) by a diagonal transformation with respect to the basis \( x_\beta \) for \( \beta \in \Phi^+ \) and by the substitution \( q = r^2 \). Now, for a root \( \beta = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \), set \( x_{i,j} = (r^{-1})^{i+j} x_\beta \). Elaborating this rewrite shows \( r \) always appears to an even power.

Together, these two results were the initial motivation for the study of BMW algebras of simply laced type. One of the main questions which arose, was whether
the LK-representations of other simply laced Artin groups also factor through an algebra like the LK-representations of Artin groups of type A do through the BMW algebra. This lead to the definition of BMW algebras of simply laced type and the study of these algebras. This thesis describes our research and the related results.

In the next chapter the BMW algebras of simply laced type are defined and some basic properties of the algebras are studied. Most importantly, finite dimensionality of the algebras is established. The main objective of our further research is the determination of the dimension of the algebras over $\mathbb{Q}(l, \delta)$.

In Chapter 3, properties of the root systems of the Weyl group $W(M)$ are discussed. Admissible $W(M)$-orbits $B$ of sets of mutually orthogonal positive roots are introduced. We show the orbits carry a useful poset structure.

In Chapter 4, a chain of ideals $I_r$ of the BMW algebra is studied extensively. The goal is to determine a spanning set for every quotient $I_r/I_{r+1}$ and to derive an upper bound for the dimension of the BMW algebras of simply laced type.

In Chapter 5, a tangle algebra $KT(D)_n$ of tangles with a pole is introduced. We show there exists a surjective homomorphism from the BMW algebras of type D to $KT(D)_n$. This leads to the determination of a lower bound for the dimension of the BMW algebra $B(D_n)$.

In the last chapter, the generalized version of the Lawrence-Krammer representation introduced in [CW] is shown to factor through the quotient ideal $I_1/I_2$ for every BMW algebra of simply laced type. A further generalization of this representation is given which factors through other quotient ideals $I_C/I_{r+1}$.
CHAPTER 2

BMW algebras of simply laced type

2.1. Introduction

In [CW], faithful representations were given for the Artin groups of spherical type, following the construction of Krammer for braid groups, [Kra]. Faithful representations for the Artin groups of type $A_n$, $D_n$, and $E_m$ for $m = 6, 7, 8$ were explicitly constructed. Since each Artin group of spherical irreducible type embeds into at least one of these, this shows each is linear. As the representations for type $A_n$ occur in earlier work of Lawrence [L], they are called Lawrence-Krammer representations.

Zinno, in [Z], observed that the Lawrence-Krammer representation of the Artin group of type $A_n$, the braid groups on $n + 1$ braids, factors through the BMW algebra, the Birman-Murakami-Wenzl algebra. (cf. [BW, M].)

In this chapter BMW algebras $B(M)$ of simply laced type are defined. They are associated with each simply laced Coxeter graph $M$ of rank $n$, as shown in Figure 12 on page 15.

The main goal of this chapter is to establish the following result.

**Theorem 2.1.1.** The BMW algebras of simply laced spherical type $A_n$, $D_n$, and $E_n$ ($6 \leq n \leq 8$) are finite dimensional.

The proof is at the end of § 2.3, where we study direct consequences of the defining relations. In § 2.4 properties of the Artin groups relevant for further study of the algebras are described.

This chapter is an adapted version of the first five sections of [CGW]. Besides textual changes to merge the article into the thesis, notice that some changes in notation have been made. For example, the indeterminate $x$ used in the paper has been replaced by $\delta$.

2.2. Definition of the algebras $B(M)$

We define the algebras by means of $2n$ generators and five kinds of relations. For each node $i$ of the diagram $M$ we define two generators $g_i$ and $e_i$ with $i = 1, \ldots, n$. If two nodes are connected in the diagram we write $i \sim j$, with $i, j$ the indices of the two nodes, and if they are not connected we write $i \not\sim j$. We let $l, \delta$ be two indeterminates.

We define the BMW algebras of simply laced type over the field $\mathbb{Q}(l, \delta)$.

**Definition 2.2.1.** Let $M$ be a simply laced Coxeter diagram of rank $n$. The BMW algebra of type $M$ is the algebra, denoted by $B(M)$, or simply $B$, with identity
element, over \( \mathbb{Q}(l, \delta) \), whose presentation is given on generators \( g_i \) and \( e_i \) \((i = 1, \ldots, n)\) by the following defining relations

\[
\begin{align*}
(B1) & \quad g_i g_j = g_j g_i \quad \text{when} \quad i \neq j, \\
(B2) & \quad g_i g_j g_i = g_j g_i g_j \quad \text{when} \quad i \sim j, \\
(D1) & \quad me_i = l(g_i^2 + mg_i - 1) \quad \text{for all} \quad i, \\
(R1) & \quad g_i e_i = l^{-1} e_i \quad \text{for all} \quad i, \\
(R2) & \quad e_i g_i e_i = le_i \quad \text{when} \quad i \sim j,
\end{align*}
\]

where \( m = (l - l^{-1})/(1 - \delta) \).

The first two relations are the braid relations commonly associated with the Coxeter diagram \( M \). Just as for Artin and Coxeter groups, if \( M \) is the disjoint union of two diagrams \( M_1 \) and \( M_2 \), then \( B \) is the direct sum of the two BMW algebras \( B(M_1) \) and \( B(M_2) \). For the solution of many problems concerning \( B \), this gives an easy reduction to the case of connected diagrams \( M \).

In (D1) the generators \( e_i \) are expressed in terms of the \( g_i \) and so \( B \) is in fact already generated by \( g_1, \ldots, g_n \). We shall show below that the \( g_i \) are invertible elements in \( B \), so that there is a group homomorphism from the Artin group \( A(M) \) to the group \( B^\times \) of invertible elements of \( B(M) \) sending the \( i \)-th generator \( s_i \) of \( A(M) \) to \( g_i \). As we shall see at the end of § 6.2, the Lawrence-Krammer representation is a constituent of the regular representation of \( B(M) \). This generalizes Zinno’s result [Z]. As a consequence of [CW], the homomorphism \( A \to B^\times \) is injective.

The fact that the BMW algebras of type \( A_n \) coincide with those defined by Birman & Wenzl [BW] and Murakami [M] is given in Theorem 2.3.8.

The Lawrence-Krammer representation of the Artin groups is based on two parameters, in [CW] denoted by \( t \) and \( r \). The two parameters \( m \) and \( l \) here are related by \( m = r - r^{-1} \) and \( l = 1/(tr^3) \).

### 2.3. Preliminaries

For the remainder of this chapter, let \( M \) be a simply laced Coxeter diagram of rank \( n \), and we let \( B(M) \) be the BMW algebra of type \( M \) over \( \mathbb{Q}(l, \delta) \).

The following proposition collects several identities that are useful for the proof of the finite dimensionality of \( B(M) \), Theorem 2.1.1. Recall that \( m \) is related to \( \delta \) and \( l \) via

\[
\begin{align*}
(1) \quad m &= (l - l^{-1})/(1 - \delta).
\end{align*}
\]

**Proposition 2.3.1.** For each node \( i \) of \( M \), the element \( g_i \) is invertible in \( B(M) \) and the following identities hold.

\[
\begin{align*}
(2) & \quad e_i g_i = l^{-1} e_i, \\
(3) & \quad g_i^{-1} = g_i + m - me_i, \\
(4) & \quad g_i^2 = 1 - mg_i + ml^{-1}e_i, \\
(5) & \quad e_i^2 = \delta e_i.
\end{align*}
\]

**Proof.** By (D1), \( e_i \) is a polynomial in \( g_i \), so \( g_i \) and \( e_i \) commute, so (2) is equivalent to (R1). From (D1) we obtain the expression \( g_i^2 + mg_i - ml^{-1}e_i = 1 \). Application of (R1) to the third monomial on the left hand side gives \( g_i(g_i + m - \]
me_i = 1. So $g_i^{-1}$ exists and is equal to $g_i + m - me_i$. This establishes (3). Also by (D1), the element $g_i^2$ can be rewritten to a linear combination of $g_i$, $e_i$ and 1, which leads to (4).

As for (5), using (D1) and (R1), we find

$$e_i^2 = e_i lm^{-1}(g_i^2 + mg_i - 1) = lm^{-1}(l^{-2}e_i + ml^{-1}e_i - e_i) = \delta e_i.$$ 

\[\square\]

**Remarks 2.3.2.** (i). There is an anti-involution on $B(M)$ determined by

$$g_1 \cdots g_n \mapsto g_n \cdots g_1$$

on products of generators $g_i$ of $B(M)$. We denote this anti-involution by $x \mapsto x^{op}$.

(ii). The inverse of $g_i$ can be used for a different definition of the $e_i$, namely

$$me_i = m + g_i - g_i^{-1}$$

for all $i$.

(iii). By (5), the element $\delta^{-1}e_i$ is an idempotent of $B(M)$ for each node $i$ of $M$.

The braid relation (B2) for $i$ and $j$ adjacent nodes of $M$ can be seen as a way to rewrite an occurrence $ijj$ of indices into $iji$. It turns out that there are more of these relations in the algebra, with some $e$’s involved.

**Proposition 2.3.3.** The following identities hold for $i \sim j$.

(6) \quad $g_jg_ie_j = e_ig_je_i$,

(7) \quad $g_je_ig_j = g_i^{-1}e_ig_i^{-1}$

\[= g_i e_j g_i + m(e_j g_i - e_ig_j + g_je_i) + m^2(e_j - e_i)\]

(8) \quad $e_je_i = e_i e_j = e_i g_je_i = e_i + m(e_i - e_ie_j)$,

(9) \quad $g_je_ig_i = g_i^{-1}e_j = g_i e_j + m(e_j - e_ie_j)$.

(10) \quad $e_ie_ige_i = e_i$.

**Proof.** By (D1) and (B2),

$$g_i g_j e_j = g_i g_j (lm^{-1}(g_j^2 + mg_j - 1)) = lm^{-1}(g_i g_j g_j + mg_j g_i - g_j g_i)$$

\[= lm^{-1}(g_j g_i g_j + mg_j g_i - g_j g_i) = lm^{-1}(g_j g_i - 1)g_j g_i = e_i g_j g_i,

proving the first equality in (6).

We next prove

(11) \quad $e_ig_j e_ig_j e_i = e_i g_j^{-1} e_i$ for $n \in \mathbb{N}$, $n \geq 1$.

Indeed, by (B2), (R1), (R2), and the first identity of (6), which we have just established,

$$e_i g_j e_ig_j e_i = e_i g_j^{-1} e_i g_j e_i = e_i g_j^{-1} e_i g_j e_i = l^{-1} e_i g_j^{-1} e_i g_j e_i = e_i g_j^{-1} e_i.$$  

The following relation is very useful for determining relations between the $e_i$.

(12) \quad $e_i e_j g_i e_j e_i = (l + m^{-1})e_i - m^{-1} e_i e_j e_i$. 
To verify it, we start rewriting one factor \( e_i \) by means of (D1), and then use (11) with \( n = 2 \) and \( n = 1 \) as well as (R1) and (R2):

\[
e_i e_j g_i e_j e_i = e_i (lm^{-1} (g_j^2 + mg_j - 1)) g_j e_j e_i = lm^{-1} (l e_i + m \delta e_i - l^{-1} e_j e_j e_i)
\]

\[
= (l + m^{-1}) e_i - m^{-1} e_i e_j e_i.
\]

We next show (10). Multiplying (R2) for \( e_j \) by the left and by the right with \( e_i \), we find \( e_i e_j g_i e_j e_i = l e_i e_j e_i \). Using (12) we obtain \( (l + m^{-1}) e_i - m^{-1} e_i e_j e_i = l e_i e_j e_i \), whence \( (l + m^{-1}) e_i e_j e_i = (l + m^{-1}) e_i \). As \( lm \neq -1 \), we find \( e_i e_j e_i = e_i \). This proves (10).

In order to prove the second equality of (6), we expand \( g_i g_j e_i \) by substituting the relation (10). We find

\[
g_i g_j e_i = g_j g_i e_j e_i = e_j g_i g_j e_j e_i = l^{-1} e_j g_i e_j e_i = e_j e_i.
\]

The first parts of the equalities of (9) and (8) are direct consequences of (6) and (10). In order to show the second part of (8), we use the second equality of (6) and (4):

\[
e_j e_i g_j = (e_j g_i g_j) g_j = e_j g_i (m l^{-1} e_j - m g_j + 1)
\]

\[
= m e_i - m e_i g_j + e_j g_i = m (e_j - e_i e_i) + e_j g_i.
\]

The second part of (9) follows from this by the anti-involution of Remark 2.3.2(i).

For the first part of (7), as the \( g_i \) and \( g_j \) are invertible this is \( g_i g_j e_i g_j = e_j \). By (6) the left side is \( e_j e_i e_j \) which is \( e_j \) by (10).

Finally we derive the second part of (7).

\[
g_j e_i g_j = g_j e_i e_j g_j = (m (e_j - e_i e_j) + g_i e_j) e_j g_j
\]

\[
= m e_i e_j g_j - m e_j e_i g_i g_j + g_i e_j e_i g_j
\]

\[
= m (m (e_j - e_i e_i) + e_j g_i) - m e_i g_j + g_i (m (e_j - e_i e_i) + e_j g_i)
\]

\[
= m^2 e_j - m^2 e_i e_i + m (e_j g_i - e_i e_i + e_j g_i) - m g_i e_j e_i + g_i e_j g_i
\]

\[
= g_i e_j g_i + m^2 e_j - m^2 e_i e_i + m (e_j g_i - e_i g_i + g_i e_j)
\]

\[
- m (m (e_i - e_j e_i) + g_i e_i)
\]

\[
= g_i e_j g_i + m^2 e_j - m^2 e_i e_i + m (e_j g_i - e_i g_i + g_i e_j - g_i e_i).
\]

□

The above identities suffice for a full determination of the BMW algebra associated with the braid group on 3 braids.

**Corollary 2.3.4.** The BMW algebra of type \( A_2 \) has dimension 15 and is spanned by the monomials

\[
1,
\]

\[
g_1, g_2, e_1, e_2,
\]

\[
g_1 g_2, g_1 e_2, g_2 g_1, g_2 e_1, e_1 g_2, e_1 e_2, e_2 g_1, e_2 e_1,
\]

\[
g_1 g_2 g_1, g_1 e_2 g_1.
\]
Proof. Let $B$ be the BMW algebra of type $A_2$. Of the sixteen possible words of length 2 the eight consisting of two elements with the same index can be reduced to words of length 1. For, by (D1) $g_i^2$ can be written as a linear combination of $g_i$, $e_i$ and 1 and by (5) $e_i^2$ is a scalar multiple of $e_i$. Finally, by relation (R1) the remaining four words reduce to $e_i$.

Now consider words of length 3. By the knowledge that $\delta^{-1}e_i$ is an idempotent and relation (10) it is clear that no words of length 3 can occur containing only $e$'s. Words containing only $g$'s can be reduced if two $g$'s with the same index occur next to each other. This leaves two possible words $g_i g_j g_i$ either of which can be rewritten to the other one by (B1).

If a word contains $e$'s and $g$'s, no $e$ and $g$ may occur next to each other having the same index as this can be reduced by relation (R1). So the only sequences of indices allowed here are $i, j, i$ and $j, i, j$. If a $g$ occurs in the middle, we can reduce the word by relation (R2) or (6). This leaves the case with an $e$ in the middle. By (8), (9), and (10) these words reduce unless both the other elements are $g$'s. Finally by (7) the two words left, viz. $g_i e_j g_i$ and $g_j e_i g_j$, are equal up to some terms of shorter length, so at most one is in the basis.

All words of length 4 that can be made by multiplication with a generator from the two words left of length 3, can be reduced. First consider $g_i g_j g_i$. Multiplication by a $g$ gives, immediately or after applying (B2), a reducible $g^2$ component. Similarly, multiplication by an $e$ will result in a reducible $e_i g_i$ word part. This leaves us with multiples of $g_i e_j g_i$. As noted above, they can be expressed as a linear combination of $g_j e_i g_i$ and terms of shorter length. Again, multiplication by $g$ leads to a $g^2$ component and the word can be reduced. Multiplication by $e$ will always enable application of relation (R2) to the constructed word and can therefore be reduced, proving that no reduced words of length 4 occur in $B$.

Finally, by use of the 15 elements as a basis, one can construct an algebra satisfying all relations of the BMW algebra, so the dimension of $B$ is indeed 15. This is done in [BW] and later in this paper.

Proposition 2.3.5. The following identities hold for $i \not= j$.

\begin{align}
  e_i g_j &= g_j e_i, \\
  e_i e_j &= e_j e_i.
\end{align}

Proof. By (D1), the $e_i$ are defined as polynomials in $g_i$ and belong to the subalgebra of $B(M)$ generated by $g_i$. By (B1) this subalgebra commutes with $g_j$.

Proposition 2.3.6. There is a unique semilinear automorphism of $B(M)$ of order 2 determined by

\begin{align*}
  g_i &\mapsto -g_i^{-1}, & e_i &\mapsto e_i, & l &\mapsto -l^{-1}, & m &\mapsto m.
\end{align*}

It commutes with the opposition involution of Remark 2.3.2(i).

Proof. Using the identities proved above, it is readily verified that the defining relations of $B(M)$ are preserved.
We recall the definition of the BMW algebra as given in [BW]; however, we take the parameters $q$, $r$ to be indeterminates over the field.

**Definition 2.3.7.** Let $q, r$ be indeterminates. The Birman-Murakami-Wenzl algebra $BMW_n$ is the algebra over $\mathbb{C}(r, q)$ generated by $1, g_1, g_2, \ldots, g_{n-1}$, which are assumed to be invertible, subject to the relations

$$

g_{i+1}g_i = g_i g_{i+1}, \\
g_i g_j = g_j g_i \text{ if } |i-j| \geq 2, \\
e_i g_i = r^{-1} e_i, \\
e_i g_i^{-1} e_i = r^{\pm 1} e_i,
$$

where $e_i$ is defined by the equation $(q - q^{-1})(1 - e_i) = g_i - g_i^{-1}$.

We now show that our definition of the BMW algebra of type $A_n$ coincides with this one.

**Theorem 2.3.8.** Let $n \geq 2$. The BMW algebra $B(A_{n-1})$ is the Birman-Murakami-Wenzl algebra $BMW_n$ where $l = r$ and $m = q^{-1} - q$.

**Proof.** To show both definitions are of the same algebra, we take our parameters $l = r$ and $m = q^{-1} - q$. The first two relations for both algebras are the same. It is evident from the definition of $e_i$ in both $BMW_n$ and $B(M)$ that $g_i$ and $e_i$ commute, so the third relation for $BMW_n$ is equivalent to (2) and (R1) for $B(M)$. Also the relation $e_i g_i^{-1} e_i = l^{-1} e_i$, the final defining relation for $BMW_n$, observe that, for $i \sim j$, by (3), (R2), (5), (10), and (1),

$$
e_i g_j^{-1} e_i = e_i (g_j + m - me_j) e_i = (l + m e_j - m) e_i = l^{-1} e_i.
$$

The definition of $e_i$ follows from Remark 2.3.2(ii). This shows that $B(M)$ is a homomorphic image of $BMW_n$. To go the other way it is shown in [BW] (4) that $e_i g_i^{\pm 1} e_i = r^{\pm 1} e_i$ and so all the relations of $B(M)$ are verified for $BMW_n$ except (D1). This follows from (10) in [BW] when corrected reads $g_i^2 = (q - q^{-1})(g_i - r^{-1} e_i) + 1$. The invertibility of the $g_i$ follows from (3). This shows the algebras are isomorphic. □

Although it is not needed for our computations, there is a cubic relation which is sometimes instructive.

**Proposition 2.3.9.** The elements $g_i$ of $B(M)$ satisfy the cubic relation

$$(g_i^2 + mg_i - 1)(g_i - l^{-1}) = 0.$$

**Proof.** By (D1) and (2), we have $(g_i^2 + mg_i - 1)(g_i - l^{-1}) = e_i (g_i - l^{-1}) = 0$. □

In [BW] Proposition 3.2, it is shown that the algebras of type $A_{n-1}$, the so-called BMW algebras, are finite dimensional. This uses in a crucial way that the symmetric group $\Sigma_n \cong W(A_{n-1})$ is doubly transitive on the cosets of $\Sigma_{n-1}$. 

Proof of Lemma 1.3.5. As \( g^{-1} = g_i + m - me_i \), we every word in \( B(M) \) can be expressed as a linear combination of words in only \( g_i \) and \( e_i \). We prove the lemma by induction on the rank \( n \). The lemma is trivial when \( A_n \) has rank 1.

Assume the lemma holds for all \( B(A_n) \) algebras with rank \( n < k \) and consider \( B(A_k) \). Let \( x \) be an element of \( B(A_k) \). We need to show any word can be written with at most one \( g_n \).

Assume we have \( x = w_1 \chi_1 w_2 \chi_2 w_3 \) in \( B(A_k) \) with \( w_i \in B(A_{n-1}) \) for all \( i \) and \( \chi_1, \chi_2 \in \{ g_n, e_n \} \). By the induction hypothesis, we have \( w_2 = v_1 \chi v_2 \) with \( v_1, v_2 \in B(A_{n-2}) \) and \( \chi' \in \{ 1, g_n-1, e_n-1 \} \).

By far commutativity, we get

\[
\begin{align*}
w_1 \chi_1 w_2 \chi_2 w_3 &= w_1 v_1 \chi v_2 w_2 w_3. \\
\end{align*}
\]

If \( \chi' = 1 \), the product \( \chi_1 \chi_2 \) is in the right shape if at least one of the terms is also equal to 1. It rewrites to \( 1 - mg_n + ml^{-1} e_i \) if \( \chi_1 = \chi_2 = g_n \) and to a scalar multiple of \( e_i \) otherwise.

When \( \chi' = g_n-1 \), the defining relations \((B2)\) and \((R2)\), together with \((6)\) cover all the possible cases.

Finally when \( \chi' = e_n-1 \), we can rewrite all possibilities using the relations \((7) - (10)\).

The proof is completed by induction on the total number of initial occurrences of \( g_n \) and \( e_n \) in the word \( x \).

This lemma does not hold for the other algebras of simply laced type. However, we provide an alternative proof of finite dimensionality which applies to the algebras of type \( A_n \) as well.

Let \((W,R)\) be the Coxeter system of type \( M \) and let \( \{ r_1, \ldots, r_n \} = R \). Assume furthermore that \( M \) is spherical. Then the number of positive roots, \( |\Phi^+| \), is the length of the longest word in the generators \( r_i \) of \( W \). This means that any product in \( B(M) \) of \( g_i \) and \( e_i \) of longer length can be rewritten by using the relations \( (B1) \) and \( (B2) \) until one of \( g_i^2, g_i e_i, e_i g_i, e_i^2 \) occurs as a subproduct for some \( i \). In the Coxeter group, \( r_i \) has order 2 so we can remove the square and obtain a word of shorter length. In our algebra, we can rewrite the four words to obtain a linear combination of words of shorter length. This leads to the following result.

Proposition 2.3.10. If the diagram \( M \) is spherical, then any word in the generators of \( B(M) \) of length greater than \( |\Phi^+| \) in \( g_i, g_i^{-1}, e_i \) can be expressed as a sum of words of smaller length by using the defining relations of \( B(M) \). In particular, \( B(M) \) is finite dimensional.

Proof. We can express \( g_i^{-1} \) by \( e_i \) and \( g_i \) to get sums of words in \( g_i \) and \( e_i \).

Suppose \( w \) is a word in \( g_i \) and \( e_i \) of length greater than \( |\Phi^+| \). Consider the word in the Coxeter group \( w' \) in \( r_i \) where each \( g_i, e_i \) in \( w \) is replaced by \( r_i \). Notice that if \( i \neq j \) that both \( r_i \) and \( r_j \) commute and that both \( e_i \) and \( g_i \) commute with both \( e_j \) and \( g_j \). In particular, the same changes can be made without changing \( w \) or \( w' \).

Suppose the relation \((B2)\) is used in \( w', r_j r_i r_j = r_i r_j r_i \). Consider the same term in \( w \) where \( r_i \) are replaced by \( g_i \), or \( e_i \) and the same for \( r_j \). We showed in the previous
Lemma 2.4.1. Let \( g \) be the inverse of the anti-involution applied to 2.3.2(i) applied to the inverse of the image of \( w \). Let \( B \) be the BMW algebra of type \( \psi \). Let \( \psi \in \mathfrak{A} \) be the image in \( \mathfrak{B}_{\mathfrak{A}} \) of \( \psi \). For a subset \( \psi \) of \( \mathfrak{A} \) and the morphism of groups \( \Phi : \mathfrak{A} \to \mathfrak{W} \) determined by \( s_i \mapsto r_i \), that is, \( \pi \circ \psi \) is the identity on \( \mathfrak{W} \).

There is a map \( \psi : \mathfrak{W} \to \mathfrak{A} \) sending \( x \) to the element \( \psi(x) = s_{i_1} \cdots s_{i_t} \) whenever \( x = r_{i_1} \cdots r_{i_t} \) is an expression for \( x \) as a product of elements of \( R \) of minimal length. For \( \beta \in \Phi \), we shall denote by \( r_\beta \) the reflection with root \( \beta \) and by \( s_\beta \) its image \( \psi(r_\beta) \) in \( \mathfrak{A} \). For a subset \( X \) of \( \mathfrak{W} \) we write \( \psi(X) \) to denote \( \{ \psi(w) \mid w \in X \} \). The map \( \psi \) is a section of the morphism of groups \( \pi : \mathfrak{A} \to \mathfrak{W} \) over \( \mathfrak{B} \) with \( l = 3 \).

Proof of Theorem 2.1.1. This is a direct consequence of the above proposition. □

2.4. Artin group properties

In this section we shall often abbreviate the fact \( M \) being simply laced by writing \( M \in \mathfrak{A} \mathfrak{D} \).

We let \( \langle \mathfrak{A}, S \rangle \) be an Artin system of type \( M \), that is, a pair consisting of an Artin group \( \mathfrak{A}(M) \) with distinguished generating set \( S = \{ s_1, \ldots, s_n \} \) corresponding to the nodes of \( M \). Similarly, we let \( (\mathfrak{W}, R) \) be the Coxeter system of type \( M \), where \( R \) is the set of fundamental reflections \( r_1, \ldots, r_n \). We shall write \( \Phi \) for the root system associated with \( (\mathfrak{W}, R) \) and \( \Phi^+ \) for the set of positive roots with respect to simple roots \( \alpha_1, \ldots, \alpha_n \) whose corresponding reflections are \( r_1, \ldots, r_n \).

There is a map \( \psi : \mathfrak{W} \to \mathfrak{A} \) sending \( x \) to \( \psi(x) = s_{i_1} \cdots s_{i_t} \) whenever \( x = r_{i_1} \cdots r_{i_t} \) is an expression for \( x \) as a product of elements of \( R \) of minimal length. For \( \beta \in \Phi \), we shall denote by \( r_\beta \) the reflection with root \( \beta \) and by \( s_\beta \) its image \( \psi(r_\beta) \) in \( \mathfrak{A} \). For a subset \( X \) of \( \mathfrak{W} \) we write \( \psi(X) \) to denote \( \{ \psi(w) \mid w \in X \} \). The map \( \psi \) is a section of the morphism of groups \( \pi : \mathfrak{A} \to \mathfrak{W} \) determined by \( s_i \mapsto r_i \), that is, \( \pi \circ \psi \) is the identity on \( \mathfrak{W} \).

Let \( \mathfrak{B}(M) \) be the BMW algebra of type \( M \over \mathbb{Q}(l, \delta) \). By means of the composition \( \psi \) and the morphism of groups \( \mathfrak{A} \to \mathfrak{B}^* \), we find a map \( \mathfrak{W} \to \mathfrak{B} \). We shall write \( \tilde{w} \) or, if \( r_{i_1} \cdots r_{i_q} \) is a reduced expression for \( w \), also \( i_1 \cdots i_q \) to denote the image in \( \mathfrak{B}(M)^* \) of \( w \) under this map. In particular, \( g_i \rightarrow \tilde{r}_i = i \).

Let \( g \in \mathfrak{A} \). By \( g^{-op} \) we denote the anti-involution \( op \) of \( \mathfrak{B}(M) \) introduced in Remark 2.3.2(i) applied to the inverse of the image of \( g \) in \( \mathfrak{B}(M) \), which is the same as the inverse of the anti-involution applied to \( g \), viewed as an element of \( \mathfrak{B}(M) \).

Lemma 2.4.1. Let \( i, j \) be nodes of \( M \). There is a unique element of minimal length in \( \mathfrak{W}(M) \), denoted by \( w_{ji} \), such that \( w_{ji} \alpha_j = \alpha_i \). It has the following properties.

(i) If \( i = i_1 \sim i_2 \sim \cdots \sim i_q = j \) is the geodesic in \( M \) from \( i \) to \( j \), then \( \tilde{w}_{ji} = i_2 i_1 i_2 \cdots i_{q-1} i_q^{-2} i_2 i_q^{-1} \).

(ii) \( w_{ji}^{-1} = w_{ij} \).

(iii) \( \tilde{w}_{ij}^{-op} = \tilde{w}_{ji} \).

(iv) \( \tilde{w}_{ij} e_i = e_j e_{i_q} \cdots e_{i_2} e_1 = e_j \tilde{w}_{ij} \).

(v) \( \tilde{w}_{ij} e_i = \tilde{w}_{ij}^{-op} e_i = \tilde{w}_{ji}^{-1} e_i \).
Proof. Consider the graph $\Gamma$ whose nodes are the elements of $\Phi^+$ and in which two nodes $\alpha, \beta$ are adjacent whenever there is a node $k$ of $M$ such that $r_k \alpha = \beta$. An expression $w = r_{i_1} \cdots r_{i_t}$ of an element $w$ of $W$ satisfying $w\alpha_j = \alpha_i$ represents a path $\alpha_j, r_i \alpha_j, \ldots, r_i \alpha_j, w\alpha_j = \alpha_i$ from $\alpha_j$ to $\alpha_i$ in $\Gamma$. Clearly, if $w$ is of minimal length then this path is a geodesic. This geometric setting readily leads to a proof of (i).

A geodesic in $\Gamma$ from $\alpha$ to $\beta$ is given by a backwards traversal of the geodesic from $\beta$ to $\alpha$. The corresponding element of $W$ is $w^{-1}$, whence (ii) and (iii).

Finally, (iv) and (v) follow by induction from (i) and, respectively, (6) and (9). \qed

For a positive root $\beta$, we write $ht(\beta)$ to denote its height, that is, the sum of its coefficients with respect to the $\alpha_i$. Furthermore, the support of $\beta$, notation $\text{Supp}(\beta)$, is the set of $k \in \{1, \ldots, n\}$ such that the coefficient of $\alpha_k$ in $\beta$ is nonzero.

**Proposition 2.4.2.** For each node $i$ of $M$ and each positive root $\beta$ there is a unique element $w \in W$ of minimal length such that $w\alpha_i = \beta$. This element satisfies the following properties:

(i) If $\beta = \alpha_j$ for some $j$, then $w = w_{ij}$.

(ii) If $j$ is the unique node of $M$ in $\text{Supp}(\beta)$ nearest to $i$, then $l(w) = ht(\beta) + l(w_{ij}) - 1$.

Proof. Suppose first that $i$ lies in the support of $\beta$. Then $\beta$ can be obtained from $\alpha_i$ by building up with addition of one fundamental root at a time, which corresponds to finding an element $w$ of $W$ by multiplication to the right of the fundamental reflection corresponding to the newly added fundamental root. This shows that there exists $w \in W$ of length at most $ht(\beta) - 1$ such that $w\alpha_i = \beta$. But the height of $\beta$ is clearly at most $l(w) + 1$, so the minimal length of any element $w$ of $W$ so that $w\alpha_i = \beta$ must be $ht(\beta) - 1$.

Next suppose that $i$ does not lie in the support of $\beta$ and let $j$ be the nearest node to $i$ in $\text{Supp}(\beta)$. Then, with $y \in W$ as in the first paragraph with respect to $\beta$ and $j$ so that $y\alpha_j = \beta$ and $l(y) = ht(\beta) - 1$, we have that $yw_{ij}\alpha_j = \beta$ and that $l(yw_{ij}) \leq l(w) + l(w_{ij}) = ht(\beta) + l(w_{ij}) - 1$. On the other hand, in order to transform $\alpha_i$ into $\beta$ by a chain of roots differing by a fundamental root, we need to apply each root but $i$ and $j$ on the geodesic in $M$ from $i$ to $j$ at least twice (once for creation of the presence of the node in the support, and one for making it vanish). We also need both $i$ and $j$ at least once. Hence, in order to make a fundamental root of $\text{Supp}(\beta)$ occur in the image $w\alpha_i$ of $\alpha_i$, of some $u \in W$, we need $l(u) \geq l(w_{ij})$, with equality only if $u = w_{ij}$ and $w\alpha_i = \alpha_j$. Notice that the fundamental reflections in $w_{ij}$ except for $\alpha_i$ do not contribute at all to the creation of the fundamental nodes in $\text{Supp}(\beta)$, so that the estimate for the fundamental roots needed to build up $\beta$ stays as before. Taking $w = yu$ we find $l(w) = l(yu) = l(y) + l(u) = l(y) + l(w_{ij}) = ht(\beta) + l(w_{ij}) - 1$.

Next we prove uniqueness of $w$ as stated. Suppose $v \in W$ also satisfies $l(v) = ht(\beta) + l(w_{ij}) - 1$. As argued above, we must have $v = v'w_{ij}$ and $l(v) = l(v') + l(w_{ij})$ so, without loss of generality, we may assume $i = j$ lies in the support of $\beta$. If $l(w) = 0$ then there is nothing to show. Suppose therefore $l(w) > 0$ and apply induction on $l(w)$. Take nodes $k, h$ of $M$ such that $l(r_k w) < l(w)$ and $l(r_k v) < l(v)$ while $r_k \beta = \beta - \alpha_k$ and $r_h \beta = \beta - \alpha_h$. Such $k$ and $h$ exist by the way $\beta$ is built up of fundamental roots via $w$ and $v$, respectively. Notice that $(\beta, \alpha_k) = (\beta, \alpha_h) = 1$.\[\Box\]
Now consider \((\beta - \alpha_k, \alpha_k)\). The value equals \(-1\) if \(k = h\); \(1\) if \(h \neq k \neq h\); and \(2\) if \(k \sim h\). In the first case, we apply induction to \((r_hw)\alpha_i = \beta - \alpha_h = (r_hv)\alpha_i\), and find \(r_hw = r_hv\), whence \(w = v\).

In the non-adjacent case, \(\beta - \alpha_h - \alpha_k\) is also a root, so there is a unique minimal \(u \in \mathbf{W}\) such that \(u \alpha_i = \beta - \alpha_h - \alpha_k\). Now \(r_hr_ku \alpha_i = \beta = u \alpha_i\), so \(r_hr_ku = r_ku\alpha_i\) and \(r_ku \alpha_i = r_hu \alpha_i\), whence, by induction, both \(r_hw = r_ku\) and \(r_hu = r_kv\). But then \(w = r_hr_ku = r_kr_hu = v\).

Finally, if \(k \sim h\), we find \((\beta - \alpha_k, \alpha_h) = 2\), whence \(\beta = \alpha_h + \alpha_k\). But then \(i\) must be either \(h\) or \(k\). Assuming (without loss of generality) \(i = h\), we find \(w = r_k\) and \(v = r_h = r_i\), a contradiction with \(v \alpha_i = \alpha_i + \alpha_h\).

This establishes that \(w\) is unique, and finishes the proof of the lemma.

**Definition 2.4.3.** For a node \(i\) of \(\mathbf{M}\) and a positive root \(\beta\) we denote by \(w_{\beta,i}\) the unique element (by the above proposition) of minimal length in \(\mathbf{W}\) for which \(w_{\beta,i} \alpha_i = \beta\). We denote by \(D_i\) the set \(\{w_{\beta,i} \mid \beta \in \Phi^+\}\).

If \(w \in D_i\) then \(wr_iw^{-1}\) is a shortest expression of the reflection corresponding to \(w \alpha_i\) as a conjugate of \(r_i\).

**Corollary 2.4.4.** For each node \(i\) of \(\mathbf{M}\), the set \(D_i\) satisfies the following properties, where \(j\) is a node of \(\mathbf{M}\).

(i) If \(r_jv \in D_i\) and \(v \in \mathbf{W}\) with \(l(r_jv) = l(v) + 1\), then \(v \in D_i\).

(ii) \(w_{ij} \in D_j\).

**Lemma 2.4.5.** If \(i\) and \(j\) are nodes of \(\mathbf{M}\), then \(\hat{w}_{\alpha_j,i}e_i = \hat{w}_{ij}e_j\).

**Proof.** Building up \(w_{\alpha_i,j}\) from the right, and letting the intermediate results act on \(\alpha_i\), we find a shortest path \(i = i_1 \sim i_2 \sim \cdots \sim i_q = j\) in \(\mathbf{M}\) from \(i\) to \(j\). The element \(\hat{w}_{ij}\) represents the corresponding element \(\hat{w}_{i_1} \hat{w}_{i_2} \hat{w}_{i_3} \hat{w}_{i_4}\) of \(\mathbf{B}(\mathbf{M})\).

**Lemma 2.4.6.** For all nodes \(i, j, k\) of \(\mathbf{M}\) we have \(\hat{w}_{ki} \hat{w}_{jk}e_j = \hat{w}_{ij}e_j\).

**Proof.** Denote by \(i = i_1 \sim i_2 \sim \cdots \sim i_q = k\) the geodesic from \(i\) to \(k\) and by \(k = k_1 \sim k_2 \sim \cdots \sim k_p = j\) the geodesic from \(k\) to \(j\). Then there is an \(m \in \{1, \ldots, q\}\) such that \(k = k_1 = k_q \sim k_2 = k_{q-1} \sim \cdots \sim k_m = k_{q-m+1} \sim k_{m+1} \neq i_{q-m}\). Then the geodesic from \(i\) to \(j\) is \(i = i_1 \sim i_2 \sim \cdots \sim i_{q-m} \sim k_m \sim k_{m+1} \sim \cdots \sim k_{p-1} \sim k_p\) and so

\[
\hat{w}_{ki} \hat{w}_{jk}e_j = \hat{w}_{ki} \hat{w}_{jk} e_k = e_{i_1} \cdots e_{i_{q-m}} \hat{w}_{k_m} \hat{w}_{k_{m+1}} e_{k_{m+1}} \cdots e_{k_p} = e_{i_1} \cdots e_{i_{q-m}} \hat{w}_{k_m} \hat{w}_{k_{m+1}} \cdots e_{k_p} = e_{i_1} \cdots e_{i_{q-m}} e_{k_m} e_{k_{m+1}} \cdots e_{k_p} = \hat{w}_{ij} e_j.
\]

For \(\alpha, \beta \in \Phi^+\) with \(\alpha \leq \beta\) (that is, for each \(i\), the difference of the coefficient of \(\alpha_i\) in \(\beta\) and the coefficient of \(\alpha_i\) in \(\alpha\) is nonnegative), let \(w_{\beta,\alpha}\) be the (unique) shortest element of \(\mathbf{W}\) mapping \(\alpha\) to \(\beta\). Clearly, \(l(w_{\beta,\alpha}) = \text{ht}(\beta) - \text{ht}(\alpha)\). Thus,
The following relations hold for elements $h_{\beta,i}$ of the Artin group $A(M)$, where we are always assuming that $s_{\alpha} = w_{\alpha}^{-1}w_{\beta}^{-1}$ is a node orthogonal to $\beta$, we shall need the following Artin group element.

\begin{equation}
    h_{\beta,i} = d_{\beta}^{-1}s_{i}d_{\beta}.
\end{equation}

**Lemma 2.4.7.** The following relations hold for elements $h_{\gamma,k}$ of the Artin group $A(M)$, where we are always assuming that $\gamma$ is a positive root and $(\alpha_{k}, \gamma) = 0$.

\begin{align}
    h_{\beta},h_{\beta,j} & = h_{\beta,j}h_{\beta,i} & \text{if } i \neq j \\
    h_{\beta,i}h_{\beta,j} & = h_{\beta,j}h_{\beta,i} & \text{if } i \sim j \\
    h_{\beta+i} & = h_{\beta,i} & \text{if } i \neq j \\
    h_{\beta+i} & = h_{\beta-i} & \text{if } i \sim j \\
    h_{\beta+\alpha} & = h_{\beta,i} & \text{if } i \sim j \\
    h_{\alpha+y} & = h_{\alpha,y} & \text{if } i \sim j.
\end{align}

**Proof.** The rules are all straightforward applications of corresponding rules for $d_{j}$. We prove (19) and (23) and leave the rest to the reader.

For rule (19), we have $d_{\beta-\alpha} = s_{j}d_{\beta}$ in the Artin group whereas $i \sim j$, $(\alpha_{i}, \beta) = -1$, and $(\alpha_{j}, \beta) = 1$, so $h_{\beta-\alpha} = h_{\beta,j}$ is the Hecke algebra element corresponding to the Artin group element $d_{\beta+\alpha} = d_{\beta-\alpha}^{-1}s_{j}d_{\beta+\alpha} = d_{\beta-\alpha}^{-1}s_{j}s_{i}d_{\beta-\alpha} = d_{\beta-\alpha}^{-1}s_{i}d_{\beta-\alpha}$, and so $h_{\beta-\alpha,i}$ coincides with $h_{\beta,i}$.

We finish with (23). It is a direct consequence of $s_{i}^{-1}d_{\alpha} = d_{\alpha+i} = s_{j}^{-1}d_{\alpha}$, and the fact that $k$ is adjacent to neither $i$ nor $j$.

\begin{equation}
    h_{\alpha+y} = h_{\alpha,y} = h_{\alpha+i,k}.
\end{equation}

As before, let $Y$ be the set of nodes $i$ of $M$ for which $\alpha_{i}$ is orthogonal to the highest root $\alpha_{0}$ of $\Phi^{+}$.

**Lemma 2.4.8.** The following properties hold for $Y$.

(i) If $i$ is a node of $M$ and $\beta \in \Phi^{+}$ satisfies $(\alpha_{i}, \beta) = 0$, then there is a node $j$ of $Y$ such that $h_{\beta,i} = s_{j}$.

(ii) For each $j$ in $Y$ there exist non-adjacent nodes $i$, $k$ with $h_{\alpha,i} = s_{j}$.

**Proof.** (i). If $\beta = \alpha_{0}$, then $i$ is a node orthogonal to $\alpha_{0}$ and so $h_{\beta,i} = s_{i}$ and $i$ belongs to $Y$ by definition of $Y$. We continue by induction with respect to the height of $\beta$. Assume $ht(\beta) < ht(\alpha_{0})$. Then there is a node $j$ such that $(\alpha_{j}, \beta) = -1$, so $\gamma = \beta + \alpha_{j}$ is a root, whence $d_{\gamma} = s_{j}d_{\gamma}$. If $i \neq j$, then, by (18), $h_{\beta,i} = h_{\gamma,i}$. Otherwise, by (21) $h_{\beta,i} = h_{\gamma,i}$. In both cases the expression found for $h_{\beta,i}$ is as required by the induction hypothesis.

(ii). Let $j$ be a node in $Y$. Then $h_{\alpha_{0},j} = \hat{j}$. Let $\beta$ be a minimal positive root for which there exists a node $k$ with $(\alpha_{k}, \beta) = 0$ and $h_{\beta,k} = \hat{j}$. If $ht(\beta) > 1$, take a node $i$ such that $(\alpha_{i}, \beta) = 1$. By Lemma 2.4.7, either $i \sim k$ and $h_{\beta-\alpha_{i},i} = \hat{j}$, or $(\alpha_{i}, \alpha_{k}) = 0$ and $h_{\beta-\alpha_{i},k} = \hat{j}$. Therefore, we may assume $ht(\beta) = 1$, and so $\beta = \alpha_{i}$ for some $i$ with $(\alpha_{i}, \alpha_{k}) = 0$. 

Lemma 2.4.9. If $i$ is a node of $M$ and $\beta$ a positive root such that $(\alpha_i, \beta) = 0$, then
\[ s_i s_\beta = s_\beta s_i. \]

Proof. We proceed by induction on $\text{ht}(\beta)$. If $\text{ht}(\beta) = 1$, then $\beta = \alpha_j$. As $(\alpha_i, \beta) = 0$, we have $i \neq j$ and so $s_i s_\beta = s_is_j = s_js_i = s_\beta s_i$ by the braid relations. Assume now that $\text{ht}(\beta) > 1$. Let $j$ be a node of $M$ such that $(\alpha_j, \beta) = 1$, so $\beta - \alpha_j$ is a positive root. Then $s_\beta = s_j s_{\beta - \alpha_j} s_j$. If $j \neq i$, then $(\alpha_i, \beta - \alpha_j) = 0$, so, by the induction hypothesis, $s_is_{\beta - \alpha_j} = s_{\beta - \alpha_j} s_i$, whence $s_is_\beta = s_is_js_{\beta - \alpha_j} s_j = s_j s_{\beta - \alpha_j} s_j s_is_j = s_j s_{\beta - \alpha_j} s_j s_j s_is_j = s_\beta s_is_j s_is_j$. Otherwise, $j = i$, and $\gamma = \beta - \alpha_i - \alpha_j$ is a positive root with $(\alpha_j, \gamma) = 0$ and $s_\beta = s_is_j s_is_j$. By the induction hypothesis, $s_is_j = s_is_j$, whence $s_is_\beta = s_is_j s_is_j s_is_j = s_is_j s_is_j$. If $j = i$, then $s_is_\beta = s_is_j s_is_j s_is_j = s_is_j s_is_j$. □

2.5. Some ideals of the BMW algebra

In this section, let $M$ be a simply laced Coxeter diagram (not necessarily spherical). In the BMW algebra $\mathcal{B}(M)$ of type $M$, the $e_i$ generate an ideal (by which we mean a 2-sided ideal). Taking products of $e_i$’s for non-adjacent nodes $i$ of $M$, we obtain further ideals.

Definition 2.5.1. Let $C$ be a coclique of $M$, that is, a subset of the nodes of $M$ in which no two nodes are adjacent. The ideal of type $C$ is the (2-sided) ideal of $\mathcal{B}(M)$ generated by $e_C$, where
\[ e_C = \prod_{y \in C} e_y. \]

The element $e_C$ is well defined as the product does not depend on the order of the $e_y$ in view of (14). The ideal $\mathcal{B}(M)e_C\mathcal{B}(M)$ is denoted by $I_C$. By $I_r$, for $r = 1, \ldots, n$, we denote the ideal generated by all $I_C$ for $C$ a coclique of size $r$.

Since the $e_i$ are scalar multiples of idempotents, so are their products $e_C$ for $C$ a coclique of $M$.

Proposition 2.5.2. Let $C, D$ be cocliques of $M$.

(i) If $|D| \subseteq |C|$ then $I_C \subseteq I_D$.

(ii) If $\{r_j \mid j \in D\}$ is in the same $W$-orbit as $\{r_j \mid j \in C\}$ then $I_D = I_C$.

(iii) The quotient algebra $\mathcal{B}(M)/I_1$ is the Hecke algebra of type $M$ over $\mathbb{Q}(l, \delta)$, with parameter $m$.

Proof. (i) is immediate from the definition of $I_C$ and the commutation of the $e_i$ for $i \in C$.

(ii). For $|D| = |C| = 1$, say $D = \{i\}$ and $C = \{j\}$, this follows from the existence of the invertible element $\tilde{w}_{ij}$ as in Lemma 2.4.1(iv). More generally, by [God], there exists $w \in W$ such that $\tilde{w}D\tilde{w}^{-1} = \tilde{C}$. This implies $\tilde{w}e_D\tilde{w}^{-1} = e_C$, whence $I_D = I_C$.

(iii). By (6), invertibility of the $g_i$ and connectedness of $M$, the ideal $I_1$ coincides with $I_{(j)}$ for any node $j$ of $M$. Consequently, the quotient ring $\mathcal{B}(M)/I_1$ is obtained by setting $e_i = 0$ for all $i$. This means that the braid relations (B1) and (B2) and (D1) are the defining relations for $\mathcal{B}(M)/I_1$ in terms of $g_i$. Now (D1) reads $g_i^2 + mg_i - 1 = 0$, so we obtain the defining relations of the Hecke algebra. □
By (i), we have the chain of ideals

\[ I_1 \supset I_2 \supset \ldots \supset I_k, \]

where \( k \) is the maximal coclique size of \( M \). This is a decreasing series of ideals. We already know from (iii) of the above proposition that \( I_1 \) is properly contained in \( B(M) \). Straightforward calculations for the Lawrence-Krammer representation, described in [CW] and in [P] for the non-spherical types, show that (D1), (R1), (R2) are also satisfied, so it is a representation of \( B(M) \). Furthermore it can be seen that \( e_i \) is not represented as 0 but \( e_i e_j \) is for any two distinct non-adjacent nodes \( i, j \) of \( M \). These calculations will be presented in a more general setting later, in § 6.2. As a consequence \( I_2 \) is properly contained in \( I_1 \).

It is also clear from the definition that \( I_r = \{0\} \) when \( r \) is bigger than the maximal coclique size of \( M \). These sizes are \( \lfloor (n + 1)/2 \rfloor \) for \( A_n \); \( \lfloor n/2 \rfloor + 1 \) for \( D_n \); 3 for \( E_6 \); and 4 for both \( E_7 \) and \( E_8 \).
CHAPTER 3

Monoidal Posets

3.1. Introduction

The beautiful properties of the high root used in § 2.4 to construct Lawrence-Krammer representations of the Artin group with non-commutative coefficients have analogues for certain sets of orthogonal roots. We study these properties here and exploit them in Chapter 4 to study the quotient ideals $I_C/I_{r+1}$, with $|C| = r$, and in Chapter 6 to construct a linear representation of the Artin monoid. In many instances, the monoid representation extends to an Artin group representation as we will also discuss there.

Let $M$ be a Coxeter diagram of simply laced type, i.e., its connected components are of type A, D or E. The Lawrence-Krammer representation ([Big, CW, Dig, Kra2]) has a basis consisting of positive roots of the root system of the Weyl group $W = W(M)$ of type $M$. Here we use instead, a $W$-orbit $B$ of sets of mutually orthogonal positive roots. Not all $W$-orbits of this kind are allowed; we call those which are allowed, admissible (cf. Definition 3.2.2; a precise list for $M$ connected is in Table 2).

In the Lawrence-Krammer representation we used the partial ordering of the positive roots given by $\beta \leq \gamma$ iff $\gamma - \beta$ is a sum of positive roots with non-negative coefficients. In Proposition 3.3.1 we generalize this ordering to an ordering $(B, <)$ on admissible $W$-orbits $B$ of mutually orthogonal roots. In the action of $w \in W$ on a set $B \in B$ the image $wB$ is the set of positive roots in $\{\pm w\beta \mid \beta \in B\}$. In the single root case, there is a unique highest element, the well-known highest root. This property extends to $(B, <)$: there is a unique maximal element $B_0$ in $B$ (cf. Corollary 3.3.6).

When labelling the nodes of an irreducible diagram $M$, we use the labelling of Figure 12 on page 15. If $M$ is disconnected, the representations are easily seen to be a direct sum of representations corresponding to the components. Since the poset construction also behaves nicely, it suffices to prove the theorem only for $M$ connected. Therefore, we will assume $M$ to be connected for the greater part of this chapter.

This chapter is an extended version of the first part of [CGW2]. The additional part of the paper is included in Chapter 6.

3.2. Admissible orbits

Let $M$ be a simply laced Coxeter diagram. Let $(W, R)$ be the Coxeter system of type $M$ with $R = \{r_1, \ldots, r_n\}$. 

By $\Phi^+$ we denote the positive root system of type $M$ and by $\alpha_i$ the fundamental root corresponding to the node $i$ of $M$. We are interested in sets $B$ of mutually commuting reflections. Since each reflection is uniquely determined by a positive root, the set $B$ corresponds bijectively to a set of mutually orthogonal roots of $\Phi^+$. We will almost always identify $B$ with this subset of $\Phi^+$. The action of $w \in W$ on $B$ is given by conjugation in case $B$ is described by reflections and by $w\{\beta_1, \ldots, \beta_p\} = \Phi^+ \cap \{\pm w\beta_1, \ldots, \pm w\beta_p\}$ in case $B$ is described by positive roots. The action of an element $w \in W$ on $B$ should not be confused with the action of $w$ on a root: in our case we have $w\{\alpha_i\} = \{\alpha_i\}$ whereas the usual action on roots implies $w\alpha_i = \alpha_i$. For example, if $r_i$ is the reflection about $\alpha_i$, $r_i\{\alpha_i\} = \{\alpha_i\}$ but $r_i\alpha_i = -\alpha_i$.

The $W$-orbit $B$ of a set $B$ of mutually orthogonal positive roots is the vertex set of a graph with edges labelled by the nodes of $M$, the edges with label $j$ being the unordered pairs $\{B, r_j B\}$ (so $r_j B \neq B$) for $B \in B$. The results of §3.3 show that if $B$ is admissible, the edges of this graph can be directed so as to obtain a partially ordered set (poset) having a unique maximal element. This section deals with the notion of admissibility.

We let $\text{ht}(\beta)$ be the usual height of a root $\beta \in \Phi^+$ which is $\sum_i a_i$ where $\beta = \sum a_i \alpha_i$.

**Proposition 3.2.1.** Let $M$ be of simply laced type. Every $W$-orbit of sets of mutually orthogonal positive roots satisfies the following properties for $B \in B$, $j \in M$ and $\beta, \gamma \in B$.

(i) There is no node $i$ for which $(\alpha_i, \beta) = 1$, $(\alpha_i, \gamma) = -1$ and $\text{ht}(\beta) = \text{ht}(\gamma) + 1$.

(ii) Suppose $(\alpha_j, \beta) = -1$ and $(\alpha_j, \gamma) = 1$ with $\text{ht}(\gamma) = \text{ht}(\beta) + 2$. Then there is no node $i$ for which $\alpha_i \in B^\perp$ and $i \sim j$.

**Proof.** Let $B$ be a set of mutually orthogonal positive roots, and $\beta, \gamma \in B$.

(i). Suppose there is a node $i$ for which $(\alpha_i, \beta) = 1$, $(\alpha_i, \gamma) = -1$ and $\text{ht}(\beta) = \text{ht}(\gamma) + 1$. As $\beta$ and $\gamma$ are orthogonal we have $(\beta, \gamma + \alpha_i) = 1$ so $\beta - \gamma - \alpha_i \notin \Phi$. This is not possible as $\text{ht}(\beta - \gamma - \alpha_i) = 0$.

(ii). Let $\beta$ and $\gamma$ be as in the hypothesis and assume there is an $i$ for which $\alpha_i \in B^\perp$ and $i \sim j$. Then $(\alpha_i, \gamma - \alpha_j) = 1$, so $\gamma - \alpha_j - \alpha_i$ is a root. As $\text{ht}(\gamma) = \text{ht}(\beta) + 2$ we have $\text{ht}(\gamma - \alpha_j - \alpha_i) = \text{ht}(\beta)$. But $(\beta, \gamma - \alpha_j - \alpha_i) = 1$, so $\beta - \gamma + \alpha_j + \alpha_i$ is a root which contradicts $\text{ht}(\beta - \gamma + \alpha_j + \alpha_i) = 0$. \hfill $\square$

**Definition 3.2.2.** Let $\mathcal{B}$ be a $W$-orbit of sets of mutually orthogonal positive roots. We say that $\mathcal{B}$ is admissible if for each $B \in \mathcal{B}$ and $i, j \in M$ with $i \neq j$ and $\gamma, \gamma - \alpha_i + \alpha_j \in B$, we have $r_i B = r_j B$.

Not all $W$-orbits on sets of mutually orthogonal positive roots are admissible. The $W$-orbit of the triple $\{\alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \cdots + \alpha_5\}$ of positive roots for $M = D_5$ is a counterexample. Suppose that $M$ is disconnected with components $M_i$. Then $\mathcal{B}$ is admissible if and only if each of the corresponding $W(M_i)$-orbits is admissible. So there is no harm in restricting our admissibility study to the case where $M$ is connected. In that case, Proposition 3.2.4 below gives a full characterization of admissible orbits.

**Lemma 3.2.3.** Let $r$ be a reflection in $W$ and let $\beta, \gamma$ be two mutually orthogonal positive roots moved by $r$. Then there exists a reflection $s$ which commutes with $r$ such that $\{\beta\} = rs\{\gamma\}$. 
3.2. ADMISSIBLE ORBITS

Proof. Let \( \delta_r \) be the positive root corresponding to the reflection \( r \). Now \( r\{\beta\} = \{\pm \beta, \pm \delta_r\} \cap \Phi^+ \) and \( (\gamma, \beta \pm \delta_r) = \pm (\gamma, \delta_r) \). Using this we can construct a new positive root \( \delta \) depending on \( (\beta, \delta_r) \), \( (\gamma, \delta_r) \) as indicated in the table below.

<table>
<thead>
<tr>
<th>( (\beta, \delta_r) )</th>
<th>( (\gamma, \delta_r) )</th>
<th>( \pm \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( \beta + \gamma - \delta_r )</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>( \beta - \gamma - \delta_r )</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>( \beta - \gamma + \delta_r )</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>( \beta + \gamma + \delta_r )</td>
</tr>
</tbody>
</table>

It is easy to check that the reflection \( s \) with root \( \delta \) commutes with \( r \) and indeed \( \{\beta\} = rs\{\gamma\} \). \( \Box \)

Proposition 3.2.4. Let \( M \) be connected. The following statements concerning a \( \mathbf{W} \)-orbit \( \mathbf{B} \) of sets of mutually orthogonal positive roots are equivalent.

(i) \( \mathbf{B} \) is admissible.

(ii) For each pair \( r, s \) of commuting reflections of \( \mathbf{W} \) and each \( B \in \mathbf{B} \) such that \( \gamma, rs\gamma \) both belong to \( B \), we have \( rB = sB \).

(iii) For each reflection \( r \) of \( \mathbf{W} \) and each \( B \in \mathbf{B} \) the size of \( rB \setminus B \) is one of \( 0, 1, 2, 4 \).

(iv) \( \mathbf{B} \) is one of the orbits listed in Table 2.

Below in the proof we show that four is the maximum possible roots in \( rB \setminus B \) which can be moved and so only three is ruled out in part (iii).

Proof. (i) \( \implies \) (ii). By (i), assertion (ii) holds when \( r \) and \( s \) are fundamental reflections. The other cases follow by conjugation since each pair of commuting reflections is conjugate to a pair of fundamental conjugating reflections. (As each reflection is conjugate to a fundamental reflection, the reflections orthogonal to it can be determined and the system of roots orthogonal to a reflection has the type obtained by removing nodes connected to the extending node of the affine diagram.)

(ii) \( \implies \) (iii). When all \( r, s \in \mathbf{W} \) move at most two mutually orthogonal roots, the implication holds trivially. If \( r \) would move five mutually orthogonal roots then the \( 6 \times 6 \) Gram matrix for these roots together with the root of \( r \) is not positive semi-definite as its determinant is \(-16\), a contradiction. Hence \( r \) moves at most 4 roots.

Assume we have a \( B \in \mathbf{B} \) such that \( r \) moves precisely three roots of \( B \), say \( \beta_1, \beta_2, \beta_3 \). By Lemma 3.2.3 we know there exists a reflection \( s \) such that \( \beta_1 = rs\beta_2 \). Now \( \beta_2 = \beta_1 \pm \delta_r \pm \delta_r \) with \( \delta_r \), \( \delta_r \) the positive roots corresponding to \( r \) and \( s \), respectively. As \( \beta_2 \) is orthogonal to \( \beta_1 \) and \( \beta_2 \), we find \( (\beta_2, \delta_r) = \pm (\beta_3, \delta_r) \), so \( s \) moves \( \beta_3 \) as well. But obviously \( r/\beta_3 \neq s/\beta_3 \), so \( rB \neq sB \), which contradicts (ii).

(iii) \( \implies \) (i). Let \( B \in \mathbf{B} \) and \( i, j \in M \) with \( i \neq j \) and \( \gamma, \gamma - \alpha_i + \alpha_j \in B \). When both \( r_i, r_j \) do not move any other root then \( r_iB = r_jB \). Without loss of generality we can assume \( r_i \) moves four roots of \( B \). Let \( \beta \) be a third root in \( B \) moved by \( r_i \). As \( \beta \) has to be orthogonal to \( \gamma, \gamma - \alpha_i + \alpha_j \) we find \( (\alpha_i, \beta) = (\alpha_j, \beta) \), so \( \beta - \alpha_i - \alpha_j \) or \( \beta + \alpha_i + \alpha_j \) is a positive root as well. This root is also moved by \( r_i \) and mutually orthogonal to \( \gamma, \gamma - \alpha_i + \alpha_j \) and \( \beta \).

So now \( \{\gamma, \gamma - \alpha_i + \alpha_j, \beta - \alpha_i - \alpha_j\} \subseteq B \). But these 4 roots are also the roots moved by \( r_j \). We know from above that 4 is the maximal number of mutually
orthogonal roots moved by \( r_i \) (or by \( r_j \) for that matter). We find \( r_j B = r_j B \) which proves \( B \) is admissible.

At this point we have achieved equivalence of (i), (ii), and (iii), a fact we will use throughout the remainder of the proof.

(iii) \( \implies \) (iv). In Table 1, we have listed all \( W \)-orbits of sets of mutually orthogonal positive roots. It is straightforward to check this (for instance by induction on the size \( t \) of such set), so we omit the details. For all orbits in Table 1 but not in Table 2 we find, for some set \( B \) in the orbit \( B \), a reflection \( r \) which moves precisely three roots.

We will use the observation that if \( B \) belongs to a non-admissible orbit for \( W \) of type \( M \), then it also does not belong to an admissible orbit for \( W \) of any larger type.

For \( M = D_n \) the sets not in Table 2 contain at least one pair of roots \( \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \) but also at least one root \( \varepsilon_0 \pm \varepsilon_q \) without the corresponding other positive root containing \( \varepsilon_0, \varepsilon_q \). (Here the \( \varepsilon_j \) are the usual orthogonal basis such that \( \Phi^+ = \{\varepsilon_i \pm \varepsilon_j \mid i < j\} \).) For these sets the reflection corresponding to a positive root \( \varepsilon_j \pm \varepsilon_q \) moves precisely three roots.
$M = E_n$. The orbit of sets of three mutually orthogonal roots which is not in Table 2 is the orbit of $\{\alpha_2, \alpha_3, \alpha_5\}$, which is not admissible in the subsystem of type $D_4$ corresponding to these three roots and $\alpha_4$, as $r_4$ moves all three roots.

The orbit of four mutually orthogonal positive roots not in Table 2 contains the set $\{\alpha_2, \alpha_3, \alpha_5, \alpha_7\}$ and $r_4$ moves exactly three of these.

The orbit of five mutually orthogonal positive roots not in Table 2 contains the set $\{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_0\}$ and $r_4$ moves again exactly three of these.

If $M = E_7$, the orbit of sets of six mutually orthogonal positive roots containing $\{\alpha_2, \alpha_5, \alpha_7, \alpha_3, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_0\}$ remains. Clearly the reflection $r_1$ moves only the last three roots. If $M = E_8$, the orbit of sets of six mutually orthogonal positive roots is not admissible as it contains the orbit of $E_7$ we just discussed.

Finally the orbit of seven mutually orthogonal positive roots in $E_8$ contains

$$\{\alpha_2, \alpha_3, \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_7, \alpha_0, \bar{\alpha_0}\}.$$ 

Here the reflection $r_8$ moves only the last three roots.

(iv) $\Rightarrow$ (iii). All orbits for type $A_n$ are admissible as here every reflection moves at most two mutually orthogonal roots.

All sets in the first collection of orbits in $D_n$ contain from every pair of roots $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ at most one element. So again as for $A_n$, every reflection moves at most two mutually orthogonal roots.

All sets in the second collection of orbits in $D_n$ contain from every pair of roots $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ both roots or none of them. So every reflection here will always move an even number of roots, so the size of $rB \setminus B$ is equal to $0, 2$ or $4$.

The orbits in $E_n$ of sets containing fewer than three roots are admissible as every reflection will never move more than two roots. For the remaining six orbits it is an easy exercise to verify for one chosen set that every reflection moves indeed $0, 1, 2$ or $4$ roots.

We finish this section with some further comments on Table 2. If $B$ is a set of mutually orthogonal positive roots as indicated in Table 2, then the type $X$ of the system of all roots orthogonal to $B$ is listed in the table. In the final column of the table we list the structure of the stabilizer in $W$ acting on $B$. If $B$ has an element of whose members are fundamental roots, this stabilizer can be found in [How]. Two distinct lines represent different classes; sometimes even more than two, in which case they fuse under an outer automorphism (so they behave identically). This happens for instance for $M = D_n$ (first line) with $n = 2t$. In the second line for $D_n$, the permutation action is not faithful.

3.3. Monoidal posets

In this section we show that admissible orbits carry a nice poset structure. An arbitrary $W$-orbit of sets of mutually orthogonal positive roots satisfies all of the properties of the proposition below except for (iii).
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A monoidal poset consists of a set $\mathcal{B}$ together with a partial order $\leq$ and a binary operation $\otimes$ that satisfies certain conditions. The concept is useful in the study of algebraic structures and their representations.

### Proposition 3.3.1

Let $M$ be a spherical simply laced diagram and $\mathcal{B}$ an admissible $\mathcal{W}$-orbit of sets of mutually orthogonal positive roots. Then there is an ordering $\leq$ on $\mathcal{B}$ with the following properties.

(i) For each node $i$ of $M$ and each $B \in \mathcal{B}$, the sets $\mathcal{B}$ and $r_i B$ are comparable. Furthermore, if $\alpha_i(\beta) = \pm 1$ for some $\beta \in B$, $r_i B \neq B$.

(ii) Suppose $i \sim j$ and $\alpha_i \in B^\perp$. If $r_j B < B$, then $r_i r_j B < r_i B$. Also, $r_j B > B$ implies $r_j r_i B > r_i B$.

(iii) If $i \not\sim j$, $r_i B < B$, $r_j B < B$, and $r_i B \neq r_j B$, then $r_i r_j B < r_j B$ and $r_j r_i B < r_i B$.

(iv) If $i \not\sim j$, $r_i B < B$, and $r_j B < B$, then either $r_i r_j B = r_j B$ or $r_i r_j B < r_j B$, $r_j r_i B < r_i B$, $r_i r_j r_i B < r_i r_j B$, and $r_j r_i r_j B < r_j B$.

It readily follows from the existence result that there is a unique minimal ordering $\leq$ satisfying the requirements of the proposition (it is the transitive closure of the pairs $(B, r_j B)$ for $B \in \mathcal{B}$ and $j$ a node of $M$ such that $r_j B > B$). This poset $(\mathcal{B}, \leq)$ with this minimal ordering is called the monoidal poset (with respect to $\mathcal{W}$) on $\mathcal{B}$ (so $\mathcal{B}$ should be admissible for the poset to be monoidal).

**Proof.** We define the relation $\leq$ on $\mathcal{B}$ as follows: for $B, C \in \mathcal{B}$ we have $B < C$ if and only if there are $\beta \in B \setminus C$ and $\gamma \in C \setminus B$, of minimal height in $B \setminus C$, respectively $C \setminus B$, such that $\text{ht}(\beta) < \text{ht}(\gamma)$. It is readily verified that $\leq$ is an ordering. We show that it also satisfies properties (i),..., (iv).
We will need various properties involving the actions of the $r_i$ on $B$. Clearly, if $\alpha_i \in B^+$, then $r_i B = B$. As described earlier, if $\alpha_i$ is in $B$ we replace $r_i \alpha_i = -\alpha_i$ with $\alpha_i$ and say $\alpha_i$ is fixed. Note that then also $r_i B = B$. If $(\alpha_i, \beta) = \pm 1$, we say $r_i$ moves $\beta$. In this case with Proposition 3.3.1(i) in mind, we see $r_i B \neq B$ and we say $r_i$ lowers $B$ if $B > r_i B$ and we say $r_i$ raises $B$ if $B < r_i B$. We also use this for a single root $\beta$: if $\beta + \alpha_i$ is a root, we say $r_i$ raises $\beta$ and if $\beta - \alpha_i$ is a root we say $r_i$ lowers $\beta$.

(i). If $\alpha_i$ is orthogonal to each member of $B$ or $\alpha_i \in B$, then $r_i B = B$, so $B$ and $r_i B$ are comparable. So we may assume that $(\alpha_i, \beta) = \pm 1$ for at least one $\beta$. Notice if $(\beta, \alpha_i) = \pm 1$, that $r_i \beta = \beta \pm \alpha_i$ is in $B$ but not in $B$ as $(\beta, \beta \pm \alpha_i) = 2 \pm 1 \neq 0$ whereas different elements of $B$ are orthogonal. In particular, $r_i B \neq B$. If $(\beta, \alpha_i) = \pm 1$ holds for exactly one member of $B$, then clearly $r_i B$ and $B$ are comparable. Suppose now $B$ and $r_i B$ are not comparable. Then there exist at least one $\beta \in B \setminus r_i B$ and one $\gamma \in B$ such that $h(\beta) = h(r_i \gamma)$ and $r_i \gamma$ is an element of minimal height in $r_i B \setminus B$. Clearly $\beta \neq r_i \gamma$ as they are in different sets by their definition. As $\gamma \in B \setminus r_i B$ we have $h(\gamma) \geq h(\beta)$ by our assumption that $\beta$ is of minimal height in $B \setminus r_i B$. As $h(r_i \gamma) = h(\beta)$ we have $(\alpha_i, \gamma) = 1$. Also $r_i \beta \in r_i B \setminus B$ and as $r_i \gamma$ with $h(r_i \gamma) = h(\beta)$ is of minimal height in $r_i B \setminus B$ we see $h(r_i \beta) \geq h(r_i \gamma)$. In particular we must have $(\alpha_i, \beta) = -1$. But according to Condition (i) Proposition 3.2.1, this never occurs.

(ii). By the assumption $r_i B < B$, there is a root $\beta \in B$ of minimal height among those moved by $r_j$ such that $\beta - \alpha_j \in r_j B$. Then $\beta - \alpha_j$ is minimal among those moved by $r_i$ in $r_i B$ and $\beta - \alpha_j \in r_i r_j B$, so $r_i r_j B < r_j B$.

The proof for the second assertion is a bit more complicated. By the assumption $r_j B > B$, there is no root $\beta \in B$ of minimal height among those moved by $r_j$ such that $\beta - \alpha_j \in r_j B$. Indeed all that are moved go to $\beta + \alpha_j$ in $r_j B$. Suppose that $\delta$ has minimal height among the roots moved by $r_i$ in $r_i B$. This implies $\delta = \gamma \pm \alpha_j$ for some $\gamma \in B$. If $\delta = \gamma + \alpha_j$ for all choices of $\delta$, then $r_i r_j B > r_j B$, as $r_i (\gamma + \alpha_j) = \gamma + \alpha_j + \alpha_i$. So assume that $\delta = \gamma - \alpha_j$ has minimal height among the roots moved by $r_i$ in $r_j B$ for some $\gamma \in B$. Let $h$ be the minimal height of all elements of $B$ moved by $r_j$. We know each of these roots is raised in height by $r_j$ and so $\gamma$ is not one of them. In particular $h(\gamma) > h$. Also $h(\gamma) - 1 = h(\delta) \leq h + 1$. It follows that $h(\gamma) \leq h + 2$. The two cases are $\gamma$ has height $h + 1$ or $h + 2$.

By Condition (i) for Proposition 3.2.1 for $\gamma$ and $\beta$ the case $h + 1$ is ruled out. But then Condition (ii) of Proposition 3.2.1 with $\alpha_i$, $\gamma$, and $\beta$ rules out the case $h + 2$.

(iii). Suppose $r_j B < B$ and $r_i B < B$ with $r_i B \neq r_j B$. Choose $\beta$ an element of smallest height in $B$ moved by $r_j$ for which $\beta - \alpha_j$ is a root. Choose $\gamma$ an element of smallest height in $B$ moved by $r_i$ with $\gamma - \alpha_i$ a root. We are assuming $\beta - \alpha_j$ is a root. This means as $(\alpha_i, \alpha_j) = 0$ that $\beta - \alpha_i$ is a root if and only if $\beta - \alpha_j \pm \alpha_i$ is a root.

To prove the result we will get a contradiction if we assume $r_i$ raises $r_j B$. Suppose then $r_i$ raises $r_j B$. In this case all elements $\zeta$ of smallest height in $r_j B$ which are moved by $r_i$ have $\zeta + \alpha_i$ as roots. We will show first that $\gamma - \alpha_j$ is not a root. If it were, $r_i$ lowers it as $\gamma - \alpha_j$ is a root. This means it is a root of smallest height moved by $r_i$ as $\gamma$ is a root of smallest height moved by $r_i$ in $B$ and in $r_j B$ this has height one smaller. But it is lowered, not raised. This means $\gamma - \alpha_j$ is not a root. Depending on $(\gamma, \alpha_j)$, either $\gamma$ or $\gamma + \alpha_j$ is a root of $r_j B$. Suppose $(\gamma, \alpha_j) = 0$
and so $γ$ is a root of $r_j B$. As $r_j$ raises $r_j B$, all elements of smallest height moved by $r_j$ must be raised. As $γ$ is lowered, there must be an $r_j δ ∈ r_j B$ with $δ − α_j$ a root of $r_j B$ and $ht(δ − α_j) < ht(γ)$. Its height must be one less than $ht(γ)$ as heights are lowered at most one by $r_j$. Now in $r_j B$, the elements $δ − α_j$ and $γ$ contradict condition (i) for Proposition 3.2.1. Suppose then $γ + α_j$ is root. The smallest height of elements for which $r_j$ moves roots in $r_j B$ is now either $ht(γ)$ or $ht(γ) − 1$. (It cannot be $ht(γ) + 1$ as $γ + α_j$ is lowered.) If it is $ht(γ)$ there is an element $δ$ of height $ht(γ)$ which is raised by $r_j$. Now $δ$ and $γ + α_j$ contradict Condition (i) of Proposition 3.2.1.

We are left with one case in which an element of height $ht(γ) − 1$ in $r_j B$ is raised by $r_j$. This means there is $δ$ in $B$ of height $ht(γ)$ which is lowered by $r_j$ and raised by $r_j$. Recalling $γ$ is lowered by $r_j$ and raised by $r_j$. As $i ≠ j$ we have $δ + α_i − α_j, γ + α_j − α_i$ in $r_j r_i B$. Now $(δ, γ + α_j − α_i) = 2$ so $δ = γ + α_j − α_i$. By admissibility of $B$ we have $r_j B = r_j B$ contradicting the starting assumptions.

(iv). We shall use the results of the following computations throughout the proof.

Let $ε ∈ B$ and write $ρ = (α_i, ε)$ and $σ = (α_j, ε)$. Then $ε − ρα_i = r_i(ε), ε − σα_j = r_j(ε), ε − (ρ + σ)α_j = r_j r_i(ε), ε − (ρ + σ)α_j + (α_i + α_j) = r_j r_j(ε) = r_j r_j r_i(ε)$. Note that $ρ = σ = 1$ would imply $α_i − α_j = 2$, whence $ε = α_i + α_j$. This means $r_j r_i B = r_i B$. To see this, suppose $α_i + α_j ∈ B$. Then $α_i ∈ r_j B$ as it is $r_j (α_i + α_j)$. Now all other elements of $r_j B$ are orthogonal to $α_i$ as $α_i$ is one of the elements. Now $r_j r_j B = r_j B$. So we can assume this does not occur.

By assumption, there are $β, γ ∈ B$ such that $β − α_i ∈ r_i B$ and $γ − α_j ∈ r_j B$ and such that $ht(β), ht(γ)$ are minimal with respect to being moved by $α_i, α_j$ respectively. By symmetry, we may also assume that $ht(β) ≤ ht(γ)$.

If $β = γ$, then $(α_i, β) = (α_j, γ) = 1$, a case that has been excluded. Therefore, we may assume that $β$ and $γ$ are distinct. In particular, $(α_j, β) = 0$ or $−1$.

Suppose first that a $β$ can be chosen so that $(α_j, β) = 0$. This is certainly the case if $ht(β) < ht(γ)$. Now $β − α_i − α_j ∈ r_j r_i B$, so $β − α_i ∈ r_i B$ is a root of smallest height moved by $r_j$ and so $r_j r_i B < r_i B$. Recall from our choice no root of height smaller than $ht(β)$ is moved by $r_j$.

Since $β ∈ r_j B$ and $β − α_i ∈ r_j r_i B$, we have $r_j r_j B < r_j B$ unless there is $δ ∈ B$ with $δ − α_j ∈ r_j B$, $ht(δ − α_j) = ht(β)$, and $(α_j, δ − α_j) = −1$. But then the inner products show $δ = −α_i$ is a positive root. Notice this shows $δ = −α_j + α_j$ cannot be a root. Hence, indeed, $r_j r_j B < r_j B$.

Since $β − α_i − α_j ∈ r_j r_i r_j B$ and $β − α_i ∈ r_j r_i B$, a similar argument to the previous paragraph shows that $r_j r_i r_j B < r_j r_i B$.

It remains to show $r_j r_i r_j B < r_j r_i B$. Both sides contain $β − α_i − α_j$ and $r_i$ does not lower or raise $r_j r_i β$. We need to look at the $δ$ in $B$ of height $ht(γ)$. We know that for $δ$ with $ht(δ) < ht(γ)$ that $(δ, α_j) = 0$. Looking at the equations above with $σ = 0$ we see $r_i$ does not change $r_j r_i δ = δ − ρ(α_i + α_j)$. We also know that $(γ, α_j) = 1$. This means $σ = 1$. Using the equations again with $σ = 1$ and $ρ$ we must compare $γ − ρ(α_i + α_j) − α_j$ with $γ − ρ(α_i + α_j) − α_j − α_i$ which is lower in particular, $r_i r_j r_i B < r_j r_i B$.

Suppose then that $(α_j, β) = −1$. This means in particular that $ht(β) = ht(γ)$. If $(α_i, γ) = 0$ we can use the argument above. We are left then with the case in which $(α_i, γ) = 1, (α_j, β) = 1, (α_i, β) = −1, (α_j, γ) = −1$, and of course $(α_j, α_j) = −1$. 

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This means $\gamma - \alpha_i$ and $\beta - \alpha_j$ are positive roots of height $\text{ht}(\beta) - 1$. But $(\gamma - \alpha_i, \beta - \alpha_j) = 0 + 1 + 1 - 1 = 1$ so by subtracting one root from the other, we should get another positive root. As both roots are of the same height, this would give a root of height 0 which is not possible, proving this case never arises.

We showed during the proof that if $\alpha_i + \alpha_j \in B$ and $i \sim j$, then $r_i r_j B = r_j B$. This is case (iv) of Proposition 3.3.1. The following lemma shows this is if and only if.

**Lemma 3.3.2.** Suppose that $(B, \prec)$ is a monoidal poset for $(W, R)$ for which $r_i r_j B = r_j B$ with $i \sim j$. If $r_i B < B$ and $r_j B < B$, then $\alpha_i + \alpha_j \in B$.

**Proof.** Suppose $\alpha_i + \alpha_j$ is not in $B$. Let $\beta$ be an element of smallest height moved in $B$ by $r_j$ for which $\beta - \alpha_j$ is a root. Such a root exists because $r_j B < B$. As $\alpha_i + \alpha_j$ is not in $B$, we know $\beta - \alpha_j \neq \alpha_i$, and even $\alpha_i$ is not in $r_j B$. It follows as $r_i r_j B = r_j B$ that $\alpha_i \in (r_j B)^\perp$. In particular $r_i (\beta - \alpha_j)$ is in $r_j B$. As $\beta - \alpha_j \pm \alpha_i$ is not orthogonal to $\beta - \alpha_j$ we must have $\beta + \alpha_i$ a root. Now $r_j$ lowers $\beta$ and $r_i$ raises $\beta$.

As $r_j B < B$ there exists $\gamma$, an element of smallest height in $B$ moved by $r_j$, for which $\gamma - \alpha_i$ is a root. We know $\text{ht}(\gamma) \leq \text{ht}(\beta)$ as $r_i$ moves $\beta$. Suppose $(\gamma, \alpha_j) = 0$. Then $\gamma \in r_j B$ and $r_j \gamma = \gamma - \alpha_i$ is also in $r_j B$. This contradicts the hypothesis that elements of $r_i B$ are all orthogonal. This implies $(\gamma, \alpha_j) = \pm 1$. This in turn means $\text{ht}(\beta) \leq \text{ht}(\gamma)$ as $\text{ht}(\beta)$ is the height of the smallest element moved by $r_j$.

Now we have $\text{ht}(\beta) = \text{ht}(\gamma)$. If $\gamma - \alpha_i$ and $\gamma - \alpha_j$ were both roots, an inner product computation would show $(\gamma, \alpha_i + \alpha_j) = 2$ so $\gamma = \alpha_i + \alpha_j$. This means $\gamma + \alpha_j$ is a positive root, so in $r_j B$ we have $\beta - \alpha_j$ and $\gamma + \alpha_j$ contradicting (i) of Proposition 3.2.1.

In order to address the monoid action later we will need some more properties of this action in terms of lowering and raising. We begin with the case in which two different fundamental reflections act the same on a member $B$ of $B$.

Before we begin we need to examine the case in which some $B$ has two indexes which raise it to the same $B'$. In particular we have

**Lemma 3.3.3.** Suppose $B \in B$ and $r_i B = r_k B > B$ with $k \neq i$. If $\beta$ is the element of $B$ of smallest height moved by either $r_i$ or $r_k$, then $\beta + \alpha_i + \alpha_k$ is also in $B$. Furthermore, $i \not\sim k$.

**Proof.** Let $\beta$ be an element of smallest height $B$ moved by either $r_i$ or $r_k$.

We know that all elements of smaller height are not moved by $r_i$ and $r_k$. Elements of the same height could be moved by $r_i$ or $r_k$, but then the root would have to be added. Suppose $(\alpha_i, \beta) = -1$, so $r_i \beta = \beta + \alpha_i$. If $(\alpha_k, \beta) = 0$, then $\beta \in r_k B = r_i B$ as is $\beta + \alpha_k$ and so $(\beta, \beta + \alpha_i) = 2 - 1 \neq 0$, which contradicts that elements of $r_i B$ are mutually orthogonal. In particular $(\alpha_k, \beta) = -1$ (for otherwise, $(\alpha_k, \beta) = 1$ and so $r_k B < B$).

If $i \sim k$, then $(\alpha_i, \alpha_k) = -1$, and so $(\alpha_k, \beta + \alpha_i) = -2$, which implies that $\beta + \alpha_i = -\alpha_k$, contradicting that $\beta + \alpha_i$ be a positive root. This means $i \not\sim k$ which proves the last part of the lemma.

Now by hypothesis $r_i r_k B = B$ and so $\beta + \alpha_k + \alpha_i$ is in $B$ which proves the remainder of the lemma.
Notice that if $\beta$ and $\beta + \alpha_i + \alpha_k$ are two roots in $B$ with $(\alpha_i, \alpha_k) = 0$, $(\beta, \alpha_i) = (\beta, \alpha_k) = -1$, the hypothesis of the lemma is satisfied, and $r_k$ maps $\beta$ to $\beta + \alpha_i$ and $\beta + \alpha_i + \alpha_k$ to $\beta + \alpha_k$. Acting by $r_k$ has the same effect except the order of the roots has been interchanged.

**Lemma 3.3.4.** Suppose $(B, <)$ is a monoidal poset for $(W, R)$. Let $B \in B$ and let $i, j \in M$ and $\beta, \gamma \in B$. Then the following assertions hold.

(i) If $i \neq j$ and $r_i r_j B < r_i B < B$, then $r_i r_j B < r_i B < B$.

(ii) If $i \neq j$, $B < r_i B$, $B < r_j B$, and $r_i B \neq r_j B$, then $r_i r_j B > r_i B$ and $r_i r_j B > r_j B$.

(iii) If $i \sim j$, $B < r_i B$, and $B < r_j B$, then $r_i B < r_j r_i B < r_i r_j B$, and $r_j B < r_j r_i B < r_j r_i B$.

(iv) If $i \sim j$ and $r_i r_j r_i B < r_j r_i B < r_i B$, then also $r_j r_i r_j B < r_j r_i B < r_j B < B$.

(v) If $\alpha_i \not\in B^+ \cup B$, then either $r_i B < B$ or $r_i B > B$.

**Proof.** We can refer to Proposition 3.3.1 for the properties of $(B, <)$.

(i) If $r_i B = r_i r_j B$, then also $r_i B = B$, a contradiction. Suppose $r_i r_j B > r_i B$. Then, by transitivity $B > r_j B$. Also $B > r_i B$ by hypotheses. Notice if $r_i B = r_j B$, $r_i r_j B = r_i^2 B = B$ but $r_j B < B$. Now $r_i r_j B < r_i B$ by Proposition 3.3.1(iii), a contradiction. Hence, by Proposition 3.3.1(i), $r_i r_j B < r_j B$.

If $r_j B = B$, then also $r_i B = r_i r_j B$, a contradiction. Suppose $r_j B > B$. Then, by transitivity, $r_j B > r_i r_j B$. Therefore, Proposition 3.3.1(iii) gives $r_i r_j B > r_i r_j B = r_i B$, a contradiction. Hence, by Proposition 3.3.1(i), $r_j B < B$. But also $r_i B < B$, so Proposition 3.3.1(iii) gives $r_i r_j B < r_i B$ (and $r_i r_j B < r_i B$).

(ii) If $r_i r_j B = r_i B$, then $r_i B = B$, a contradiction. If $r_i r_j B < r_i B$, then, by Proposition 3.3.1(iii) applied to $r_j B$ we have $r_j B < B$, a contradiction. Hence by Proposition 3.3.1(i), $r_i r_j B > r_i B$. The proof of $r_i r_j B > r_j B$ is similar.

(iii) Suppose $r_i r_j B = r_j B$. If $\alpha_i \in (r_j B)^c$, then, as $r_j$ lowers $r_j B$, by Proposition 3.3.1(ii) $r_i$ lowers $r_i r_j B = B$ which is a contradiction. This means $\alpha_i \in r_j B$ and so $\alpha_i + \alpha_j \in B$. Notice neither $\alpha_i$ nor $\alpha_j$ are in $B$ as they are not orthogonal to $\alpha_i + \alpha_j$. As both $r_i$ and $r_j$ raise $B$, there must be $k, \ell$ with $i \sim k$ and $j \sim \ell$ with $\alpha_k$ and $\alpha_\ell$ in $B$. Neither are orthogonal to $\alpha_i + \alpha_j$ and this is impossible. This means $r_i r_j B \neq r_j B$.

Suppose $r_i r_j B < r_j B$. We can not have $r_i r_j B = r_j r_i B = B$ by Lemma 3.3.3. Now Proposition 3.3.1(iv) gives $r_i B < B$, a contradiction. Hence $r_i r_j B > r_j B$. The roles of $i$ and $j$ are symmetric, so similarly we find $r_j r_i B > r_i B$.

If $r_j r_i r_j B = r_i r_j B$ then $B = r_i B$, a contradiction. Suppose $r_j r_i r_j B < r_i r_j B$. As also $r_j B < r_i r_j B$, Proposition 3.3.1(iv) gives $r_i B < B$, a contradiction, because $\alpha_i + \alpha_j \in r_i r_j B$ would imply $\alpha_j \in r_i B$ whence $r_i B = B$.

Similarly, it can be shown that $r_i r_j r_i B > r_j r_i B$.

(iv) If $r_i B = B$, then $r_j r_i r_i B = r_j r_i B$, a contradiction. If $r_j B < B$, then the result follows from Proposition 3.3.1(iv) because $\alpha_i + \alpha_j \in B$ would imply $\alpha_j \in r_i B$ whence $r_i r_j B = r_i B$.

Suppose therefore $r_j B > B$. If $r_j B = r_i r_j B$, then $r_j r_i B = r_j r_i r_j B$, a contradiction. If $r_j B > r_i r_j B$, then by Proposition 3.3.1(iv) $r_j r_i r_j B > r_i r_j B$, a contradiction because $\alpha_i + \alpha_j \in r_i B$ would imply $\alpha_i \in B$ whence $r_i B = B$. 


Hence $r_j B < r_i B$. But then by transitivity $r_j r_j B > r_j r_i B$, and, since $r_j r_j B = r_j B$, gives Proposition 3.3.1(iv) $r_j r_i r_j B = r_j r_i B$ (for otherwise $\alpha_i + \alpha_j \in r_i r_j B$, implying $\alpha_j \in r_j B$ so $r_j B = B$), a final contradiction.

(v). The hypotheses imply that there exists $\beta \in B$ with $(\alpha_i, \beta) = \pm 1$. Then $r_i \beta = \beta \pm \alpha_i$, which is not orthogonal to $\beta$. As the elements of $B$ are orthogonal by definition, $r_i \beta$ does not belong to $B$, so $r_i B \neq B$, and the conclusion follows from Proposition 3.3.1(i).

Proof. Pick $B_0$ a maximal element of $B$. This means $r_i B_0$ is either $B_0$ or lowers $B_0$. This is possible as $B$ is finite. We need more properties of the poset determined by $>$. To begin with this we consider certain Weyl group elements, $w$, for which $wB_0 = B$ for a fixed element $B \in B$. In particular we let $w = r_i r_{i_2} \cdots r_{i_s}$ be such that $B_0 > r_i B_0 > r_{i_{s-1}} r_{i_s} B_0 > \cdots > r_{i_2} r_{i_3} \cdots r_{i_s} B_0$. We prove this by induction on the minimal length of a chain from $B_0$ to $wB$ which satisfies the descending property of the definition of $B'$. We refer to $wB$ as $B_1$. Suppose $B = r_i r_{i_2} \cdots r_{i_s} B_0$. This means $r_i \beta = \beta \pm \alpha_i$, which is not orthogonal to $\beta$. As the elements of $B$ are orthogonal by definition, $r_i \beta$ does not belong to $B$, so $r_i B \neq B$, and the conclusion follows from Proposition 3.3.1(i).

Lemma 3.3.5. In the notation just above, $B' = B$.

Proof. Notice that $B_0$ is in $B'$ by definition using $w$ the identity. Recall that $r_i B_0$ is either $B_0$ or lower. In particular nothing raises $B_0$. We show first that if $B \in B'$ and $r_j B > B$ then $r_j B \in B'$. We prove this by induction on the minimal length of a chain from $B_0$ to $wB$ which satisfies the descending property of the definition of $B'$. In particular we let $w = r_i r_{i_2} \cdots r_{i_s}$ be such that $B_0 > r_i B_0 > r_{i_{s-1}} r_{i_s} B_0 > \cdots > r_{i_2} r_{i_3} \cdots r_{i_s} B_0$. We say this chain has length $s$, the length of $w$. We in fact show that there is a chain from $B_0$ to $r_j B$ of length less than or equal to $s - 1$. We have seen that no $r_i$ raises $B_0$. Suppose that $B = r_i B_0$ and $r_j B > B$. Suppose $r_j B = B_0$ the induction assumption is true. If $r_j B \neq B_0$ we can use Lemma 3.3.4(ii) or (iii) to see that $r_j r_i B > B_0$ a contradiction. In particular the induction assumption is true for $s \leq 1$.

We can now assume $s \geq 2$. Pick a $B$ with a chain of length $s$ and assume the result is true for any $B' \in B'$ with a shorter chain length. Suppose $r_j B$ is not in $B'$ and $r_j B > B$. Notice $r_i B = r_{i_2} r_{i_3} \cdots r_{i_s} B_0 > B$ by the hypothesis. Clearly $r_j B \neq r_i B$ as $r_i B$ is in $B'$ using the element $r_{i_2} r_{i_3} \cdots r_{i_s}$. In particular we can use Lemma 3.3.4(ii) or (iii). In either case $r_j r_i B > r_i B$ and by our choice of $s$ and the induction assumption, $r_j r_i B$ is in $B'$ and has a chain of length at most $s - 1$ from $B_0$ to it.

Suppose first $i_s \sim j$ and use Lemma 3.3.4(ii). By the induction assumption there is a chain down to $r_j r_i B$ of length at most $s - 2$ and then by multiplying by $r_{i_s}$ gives a chain down to $r_j B$ of length at most $s - 1$ and the induction gives $r_j B \in B'$. Suppose now $i_s \sim j$ and use Lemma 3.3.4(iii). Again $r_j r_i B$ is in $B'$ by the induction hypothesis and has a chain down to it of length at most $s - 2$. Using the induction again, and the hypothesis of the minimality of $s$, we see also $r_i r_j r_i B$ is in $B'$ and has a chain to it of length at most $s - 3$. Now using this as $r_j r_i r_j B$, multiplying by $r_j$ and then by $r_{i_1}$ gives a chain to $r_j B$ of length at most $s - 1$ and we are done with this part.

In particular, if $B \in B'$ and $r_j B > B$, then $r_j B$ is in $B'$. If $B \in B'$ and $r_j B = B$ of course $r_j B \in B'$. Suppose $r_j B < B$. Then the sequence to $B$ and then $r_j B$ gives a sequence to $r_j B$ and $r_j B$ is in $B'$. We see that $B'$ is closed under the action of $W$ and as $B$ is an orbit, $B' = B$. □
Corollary 3.3.6. There is a unique maximal element $B_0$ in $B$.

Proof. We have just shown that for every element $B$ in $B$ except $B_0$ there is a sequence lowering to $B$ and so $B_0$ is the only maximal element. □

This shows that each $B \in B$ has a level associated with it, namely the smallest $s$ for which $B$ can be obtained from $B_0$ as above with a Weyl group element $w$ of length $s$. Namely the smallest $s$ for which there is a reduced expression $w = r_{s_1}r_{s_2} \cdots r_{s_t}$ with $wB_0 = B$ for which $B_0 > r_{s_1}B_0 > r_{s_2-1}r_{s_1}B_0 > \cdots > r_{s_t}r_{s_{t-1}} \cdots r_{s_1}B_0 > B$. In particular $B_0$ has level 0 and if $r_iB_0 < B_0$ it has level 1. The next lemma says that this $s$ is the shortest length of any word $w$ for which $wB_0 = B$.

Lemma 3.3.7. Suppose $w$ is an element of $W$ of the smallest length for which $wB_0 = B$. Then this length, $s$, is the length of the shortest word defining $B$ as an element of $B'$. In particular if the word is $r_{s_1}r_{s_2} \cdots r_{s_t}$, then $B_0 > r_{s_1}B_0 > r_{s_2-1}r_{s_1} \cdots > r_{s_t}r_{s_{t-1}} \cdots r_{s_1}B_0 = B$ and this is the shortest which does this. It is reduced.

Proof. Suppose $w$ is an element of $W$ for which $wB_0 = B$ and for which as in the definition of $B'$, we have $w = r_{s_1}r_{s_2} \cdots r_{s_t}$ and $r_{s_1}B_0 > r_{s_2-1}r_{s_1}B_0 > \cdots > r_{s_t}r_{s_{t-1}} \cdots r_{s_1}B_0 = B$ with this the shortest possible. Suppose $w'$ is any other Weyl group element with $wB_0 = B$. If $w' = r_{j_1}r_{j_2} \cdots r_{j_t}$ is a reduced decomposition of length $t$, then $t$ is at most $s$ and we get a sequence $B_0$, $r_{j_t}B_0$, $r_{j_{t-1}}r_{j_t}B_0$, $\cdots$, $r_{j_1}r_{j_2} \cdots r_{j_t}B_0 = B$. If any of these differences do not have the relation $>$ between them, the level of $B$ would be strictly smaller than $s$, contradicting the minimality of $s$. Hence, $t = s$ and the sequence corresponding to $w'$ is also a chain. In particular, $w$ is reduced and any other reduced expression gives a descending sequence of the same length. This shows there is a reduced word with this length taking $B_0$ to $B$ and any word doing this of shorter or the same length, has to be descending at each step. This proves the lemma.

Lemma 3.3.8. Suppose that $(B, <)$ is a monoidal poset for $(W, R)$.

(i) For each $B \in B$ and each element $w \in W$ of minimal length such that $B = wB_0$ and node $i$ of $M$ such that $l(r_iw) < l(w)$, we have $r_iB > B$.

(ii) For each $B \in B$, if $w, w' \in W$ are of minimal length such that $B = wB_0 = w'B_0$, then $l(w) = l(w')$ and, for each node $i$ such that $r_iB > B$, there is $w'' \in W$ of length $l(w)$ such that $B = w''B_0$ and $l(r_iw'') < l(w'')$.

Proof. For (i) we use the characterization in Lemma 3.3.7 and realize that any of the equivalent expressions also give a descending sequence. In particular if $l(r_iw) < l(w)$, an equivalent word can be chosen to start with $r_i$ and so $r_iB$ is one step above $B$ in the chain to $B$ from this word and so $r_iB > B$.

For (ii) again use Lemma 3.3.7 and so $l(w) = l(w')$. If $r_iB > B$ for some $i$, there is a sequence from $r_iB$ to $B_0$. If $w'B_0 = r_iB$ accomplishes this in the minimal number of steps, $w'' = s_iw'$ satisfies the conclusion of the lemma. □

3.4. The positive monoid

We now turn our attention to the Artin group $A(M)$ associated with the Coxeter system $(W, R)$. We recall that the defining presentation of $A(M)$, or simply $A$, has generators $s_i$ corresponding to the fundamental reflections $r_i \in R$ and braid
relations $s_i s_j s_i = s_j s_i s_j$ if $i \sim j$ and $s_i s_j = s_j s_i$ if $i \not\sim j$. The monoid $A^+$ given by the same presentation is known ([P]) to embed in $A$.

For each admissible $W$-orbit of a set of mutually commuting reflections, we shall construct a linear representation of $A^+$. To this end, we need a special element $h_{B,i}$ of $A^+$ for each pair $(B,i)$ consisting of a set $B$ of mutually commuting reflections and a node $i$ of $M$ whose reflection $r_i$ does not belong to $B$ but commutes with each element of $B$. As in the previous section, we shall represent reflections by positive roots.

We now define the elements $h_{B,i}$. As in § 2.4 we do this by defining reduced words $v_{B,i} \in A$ and letting $h_{B,i} = v_{B,i}^{-1} s_i v_{B,i}$. Later we shall consider the image of these elements in a certain Hecke algebra.

We make definitions of $v_{B,i}$ which depend on certain chains from $B_0$ to $B$ and show in an early lemma that conjugating $s_i$ by any of them gives the same element. Furthermore, this element corresponds to a fundamental generator of $A$ commuting with every reflection having its positive root in $B_0$.

**Definition 3.4.1.** Suppose $(B,<)$ is a monoidal poset of $(W,R)$, with maximal element $B_0$. Let $(B,i)$ be a pair with $B \in B$ and $i$ a node of $M$ such that $\alpha_i \in B_i^-$. Choose a node $j$ of $M$ with $r_j B > B$. If $j \not\sim i$ let $v_{B,i} = s_j v_{r_j B,i}$ and if $i \sim j$ let $v_{B,i} = s_j s_i v_{r_j B,i}$. We define $v_{B_0,i}$ as the identity.

Furthermore, set $h_{B,i} = v_{B,i}^{-1} s_i v_{B,i}$.

Notice this definition makes sense as a nondeterministic algorithm assigning an element of $A$ to each pair $(B,i)$ as specified because

- if $\alpha_i \in B_i^+$ and $i \not\sim j$, then $\alpha_i \in (r_j B)^+$;
- if $i \sim j$, then $\alpha_i + \alpha_j \in (r_j B)^+$ and $\alpha_j \in (r_j r_i B)^+.$

By Lemma 3.3.7, $v_{B,i}$ will be a reduced expression whose length is the length of a chain from $B$ to $B_0$. The elements $v_{B,i}$ are not uniquely determined, but we will show that the elements $h_{B,i}$ are.

**Lemma 3.4.2.** For $B$ and $(B,i)$ as in Definition 3.4.1, suppose that $v_{B,i}$ and $v_{B,i}'$ both satisfy Definition 3.4.1. Then the elements $h_{B,i}$ of $A$ defined by each are the same, i.e., $h_{B,i} = v_{B,i}^{-1} s_i v_{B,i} = v_{B,i}'^{-1} s_i v_{B,i}'$.

Furthermore, each $h_{B,i}$ is a fundamental generator $s_i$ of $A$ whose root $\alpha_i$ is orthogonal to every root of $B_0$.

**Proof.** We use induction on the height from $B_0$. The case of $B_0$ is trivial.

We first dispense with the case in which $r_j B = r_j B$. We know from Lemma 3.3.3 that in $B$ there are two elements $\beta$ and $\beta + \alpha_j + \alpha_{j'}$ with $j \not\sim j'$. As $\alpha_i \in B_i^+$ we know $(\beta, \alpha_i) = 0$ and also $(\beta + \alpha_j + \alpha_{j'}, \alpha_i) = 0$. It follows that $(\alpha_j, \alpha_i) = (\alpha_{j'}, \alpha_i) = 0$ as the inner products of fundamental roots are 0 or $-1$. In particular using $r_j$ we get $v_{B,i} = s_j v_{r_j B,i}$, and $v_{B,i}^{-1} s_i v_{B,i} = v_{r_j B,i}^{-1} s_j^{-1} s_i s_j v_{B,i}$. As $s_j^{-1} s_i s_j = s_i$ this is $v_{r_j B,i}^{-1} s_i v_{B,i} = h_{r_j B,i}$. The same is true for $r_{j'}$ and we are assuming $r_j B = r_{j'} B$.

Now we can use induction.

We next suppose $r_j B > B$ and $r_{j'} B > B$ with $r_j B \not\sim r_{j'} B$. There will be two cases depending on whether $j \not\sim j'$ or $j \sim j'$. Suppose first $j \not\sim j'$. We use Lemma 3.3.4(ii) to see that $r_j r_{j'} B > r_{j'} B$ and $r_j r_{j'} B > r_j B$. Suppose first $i \not\sim j$ and $i \not\sim j'$.

For the chain starting with $r_j$ we can follow it with $r_{j'}$ and if we start with $r_{j'}$ we
We now consider the cases in which result is true again using induction at all the higher levels. Suppose next that \( i \sim j \) but \( i \not\sim j' \). Using the chain for \( r_j \) we get \( B < r_j B < r_i r_j B \) by Proposition 3.3.1(ii). As above by Lemma 3.3.4(ii) we get \( r_j r_j B > r_j B \) and now again by Proposition 3.3.1(ii) using \( \alpha_i \in (r_j B) \perp \) we get \( r_j r_j r_j B > r_j r_j B \). Now for the \( r_j' \) chain continue through \( r_j \) and then \( r_j \) to reach \( v_{B,i} = s_j s_j s_j' v_{r_j r_j r_j B,i} \). Through the \( r_j \) chain which goes through \( r_i r_j B \) add \( r_j \) for which \( j' \not\sim i \). Here we get \( v_{B,i} = s_j s_j s_j' v_{r_j r_j r_j B,i} \). Again use induction at all the levels to get the needed result. Notice \( r_i r_j B \neq r_j r_j B \) as \( r_i r_j r_j B > r_i r_j B \) as above and so \( r_i \) raises \( r_j r_j B \).

![Diagram](image.png)

**Figure 1**

The final case in which \( j \not\sim j' \) with \( j \sim i \sim j' \), see Figure 1. For this we again use Lemma 3.3.4(ii) and (iii) and Proposition 3.3.1. In particular \( r_j B > B \) and \( r_i r_j B > r_j B \). Also as \( r_j B \neq r_i B \) we have \( r_j r_j B > r_j B \). Now by Lemma 3.3.4(iii) we have \( r_j r_j r_j r_j B > r_j r_j r_j B \) as \( j \not\sim j' \). Now using Proposition 3.3.1(ii) we see \( r_i r_j r_i r_j B > r_i r_j r_i r_j B \). Notice \( r_i r_j B \neq r_j r_j B \) by Lemma 3.3.3 as \( i \sim j' \). Following the trail of \( \alpha_i \) we see it is in \( (r_j r_j B) \perp \). Following this chain after \( r_i r_j \) and using induction we see \( v_{B,i} = s_j s_j s_j s_j v_{r_j r_j r_j r_j B,i} \). Going up through \( r_i r_j \) gives the same result as \( s_j s_j s_j s_j s_j = s_j s_j s_j s_j s_j' \) is similar to \( s_j' s_j s_j s_j s_j' \). In particular the result is true again using induction at all the higher levels.

We now consider the cases in which \( j \sim j' \). As before, the easiest case is when \( i \not\sim j \) and \( i \not\sim j' \). In this case as before use Lemma 3.3.4(iii) to obtain \( r_j r_j r_j r_j B > \)
As \( \alpha_i \) is orthogonal to \( \alpha_j \) and \( \alpha_{j'} \) we obtain
\[
v_{B,i} = s_j s_j' s_i v_{r_j r_j' r_j B,i}.
\]
The same result holds for the other order and the result follows by induction.

The only other possibility is \( i \sim j \) and \( i \not\sim j' \) as \( i \) could not be adjacent to both \( j \) and \( j' \) (for otherwise there would be a triangle in the Dynkin diagram). For this we use the familiar six sides diagram generated by \( r_j \) and \( r_{j'} \) using Lemma 3.3.4(iii), see Figure 2. At \( r_j B \) we may also act by \( r_i \) which we know raises \( r_j B \).

It is clear that \( r_j r_j B \neq r_i r_j B \) as \( \alpha_j \) is in \( (r_i r_j B)^\perp \) and so \( r_j \) could not raise (or even lower) it. We can proceed by Lemma 3.3.4(ii) to \( r_j r_j r_{j'} B \). Notice \( r_j r_j r_{j'} B \neq r_i r_j B \) as \( i \sim j \). Now proceed by Lemma 3.3.4(iii) to \( r_i r_j r_{j'} r_j B = r_j r_j r_j B \). By following the perpendicularities we see \( \alpha_{j'} \in (r_i r_j r_j B)^\perp \). Using this chain which starts with \( r_j \) and continues with \( r_i \), we find \( v_{B,i} = s_j s_j' s_j s_i v_{r_j r_j' r_j B,B,j'} \). Using the other direction starting with \( r_{j'} \), then \( r_j \), then \( r_i \), we can continue with \( r_{j'} \) and \( r_j \) to get \( r_j r_j r_{j'} r_j B \) and conclude using this direction \( v_{B,i} = s_j s_j' s_j s_i v_{r_j r_j' r_j r_j' B,B,j'} \). At the juncture \( r_j r_j B \) we act by \( r_i \) or by \( r_{j'} \). These two could not be equal as again \( \alpha_j \) is in \( (r_i r_j r_j B)^\perp \) and so \( r_j \) could not move it. However, if they were equal, it lowers it to \( r_{j'} r_{j'} B \). These words are equivalent and we can use induction as usual for the last time.

![Figure 2](image_url)

This finishes all cases and shows the words have the same effect under conjugation on \( s_i \).

We finish this section by exhibiting relations that hold for the \( h_{B,i} \). Since we are actually interested in their images in the Hecke algebra \( H \) of type \( M \) under the natural morphism \( \mathbb{Q}(m)[A] \to H \), we phrase the result in terms of elements of this algebra.
Proposition 3.4.3. Suppose that \((B, \prec)\) is a monoidal poset with maximal element \(B_0\). Let \(Y\) be the set of nodes of \(M\) such that \(\alpha_i\) is orthogonal to \(B_0\) and denote \(Z\) the Hecke algebra over \(\mathbb{Q}(m)\) of the type \(Y\). Then the images of the elements \(h_{B,i} \in A\) in the Hecke algebra of type \(M\) under the natural projection from the group algebra of \(A\) over \(\mathbb{Q}(m)\) actually are fundamental generators of \(Z\) and satisfy the following properties.

(i) \(h_{B,i}^2 = 1 - mh_{B,i}\).
(ii) \(h_{B,i}h_{B,j} = h_{B,j}h_{B,i}\) if \(i \not\sim j\).
(iii) \(h_{B,i}h_{B,j}h_{B,i} = h_{B,j}h_{B,i}h_{B,j}\) if \(i \sim j\).
(iv) \(h_{r_iB,i} = h_{B,i}\) if \(i \not\sim j\).
(v) \(h_{r_ir_jB,j} = h_{B,i}\) if \(i \sim j\) and \((\alpha_i, B) \neq 0\).

Proof. By [Cri] we can identify the Hecke algebra of type \(Y\) with the subalgebra of the Hecke algebra \(H\) generated by the \(s_i\) for \(i \in Y\). As described above we define \(h_{B,i} = v_{B,i}^{-1}s_iv_{B,i}\), where we consider this element in the Hecke algebra.

By Lemma 3.4.2, it is a fundamental generator of \(Z\).

(i). This clearly follows from the quadratic Hecke algebra relations we are assuming.

(ii). Assume first \(r_iB > B\) and both \(\alpha_i\) and \(\alpha_j\) are orthogonal to \(\alpha_k\). We are assuming here \(i \not\sim j\). Then we can take \(v_{B,i} = s_kv_{r_kB,i}\) and \(v_{B,j} = s_kv_{r_kB,j}\). Now \(h_{B,i} = (v_{r_kB,i})^{-1}s_i^{-1}s_is_kv_{r_kB,i}\). This is \(h_{r_kB,i}\) and we can use induction. Suppose \(i \sim k\) but \(j \not\sim k\). Then we can take \(v_{B,i} = s_kv_{r_kB,i}k\) and we can take \(v_{B,j} = s_kv_{r_kB,j}\). Then \(h_{B,j} = d_{r_kB,i}B_j\) and \(h_{B,i} = d_{r_kB,i}B_j\). Now as above we can again use induction.

The final case with \(i \not\sim j\) is when \(i \sim j\). Now \(v_{B,i} = s_kv_{r_kB,i}k\) and \(v_{B,j} = s_kv_{r_kB,j}\). Suppose that \(r_ir_kB = r_jr_kB\). If so \(v_{B,i} = s_kv_{r_kB,i}k\) and \(v_{B,j} = s_kv_{r_kB,j}\). Now \(v_{B,i}^{-1}s_iv_{B,i} = v_{B,j}^{-1}s_jv_{B,j}\). Doing the same with \(r_j\) gives the same thing and so they commute. This means \(r_jr_kB = r_kr_jB\) and we can use Lemma 3.3.4(ii) to get \(r_ir_jr_kB = r_jr_kB\) and \(r_jr_kB > r_kr_jB\). Now applying this with \(v_{B,i}\) gives \(v_{B,i} = s_kv_{r_kB,i}k\) and \(v_{B,j} = s_kv_{r_kB,j}\). Let \(B' = r_kr_jB\). Now \(h_{B,i} = h_{r_kB,i}\) and \(h_{B,j} = h_{r_kB,j}\). Now use induction.

(iii). Suppose \(i \sim j\). We wish to show \(h_{B,i}h_{B,j}h_{B,i} = h_{B,j}h_{B,i}h_{B,j}\). Suppose first \(k \not\sim i\) and \(k \not\sim j\). In this case \(v_{B,i} = s_kv_{r_kB,i}\) and \(v_{B,j} = s_kv_{r_kB,j}\). This means \(h_{B,i} = h_{r_kB,i}\) and \(h_{B,j} = h_{r_kB,j}\). Now use induction.

We are left with the case where \(k \sim i \sim j\). Then \(j \not\sim k\) as there are no triangles in the Dynkin diagram. Notice on the chain from \(\alpha_k\) we start with \(r_k\), apply \(r_j\) and then can then if we wish add \(r_j\) provided \(r_jr_kB\). The chain from \(\alpha_k\) is \(r_k\) which fixes \(\alpha_1\), and then we can continue with \(r_i\) and then \(r_j\) which forces \(r_ir_jr_kB > r_jr_kB\) by Proposition 3.3.1(ii). Now \(v_{B,i} = s_kv_{r_kB,i}\) and \(v_{B,j} = s_kv_{r_kB,j}\). Now check that if \(B' = r_ir_jB\) that \(h_{B,i} = h_{B',i}\) and \(h_{B,j} = h_{B',j}\). Now use induction.

(iv). Suppose \(r_jB > B\). Then \(v_{B,i} = s_jv_{r_jB,i}\). Now conjugating \(r_i\) by \(v_{B,i}\) has the same effect as conjugating \(v_{r_jB,i}\) as \(s_j^{-1}s_js_i = s_j\). If \(r_jB < B\), use the same argument on \(r_jB\) which is raised by \(r_j\).

(v). Assume first that \(r_jB > B\). Then \(v_{B,i} = s_jv_{r_jB,i}\). Notice \(s_i^{-1}s_j^{-1}s_js_i = s_j\) and conjugating \(s_i\) by \(v_{B,i}\) has the same effect as conjugating \(s_j\) by \(v_{r_jB,j}\) and...
the result follows. If \( r_jB < B \), then \( r_ir_jB < jB \) by Proposition 3.3.1(ii). Now apply the above to \( r_ir_jB \). As \( (\alpha_j, B) \neq 0 \), we know \( r_jB \neq B \) by Lemma 3.3.4(v).

All cases have been completed. \( \square \)

**Remark 3.4.4.** For the definition of \( v_{B,i} \) we have used chains (and their labels) from \( B \) to \( B_0 \) depending on \( i \). In particular for \( r_jB > B \) and \( i \neq j \) we use \( s_jv_{r_jB,i} \), and for \( j = i \) we use \( s_jv_{r_jB,i} \). If we were to use just any chain we would not get this unique element without some further work. For instance, if \( M = D_3 \) and \( B = \{\varepsilon_3 + \varepsilon_4, \varepsilon_2 + \varepsilon_3\} \), both \( \alpha_1 \) and \( \alpha_3 \) are in \( B^\perp \) and \( r_2r_1 \) and \( r_2r_3 \) both take \( B \) to \( B_0 = \{\varepsilon_1 + \varepsilon_4, \varepsilon_2 + \varepsilon_3\} \). If we use the definition here, with \( v_{B,1} = s_2s_1 \), we find \( h_{B,1} = (s_2s_1)^{-1}s_1(s_2s_1) = s_2 \). However, if we would use \( v_{B,1} = s_2s_3 \), corresponding to a non-admitted chain, we find \( s_2^{-1}s_2^{-1}s_1s_2s_3 \) instead of \( s_2 \) and we would need a proper quotient of the Hecke algebra for \( h_{B,1} \) to be well defined.

**Corollary 3.4.5.** Let \((B, <)\) a monoidal poset. Retain the notation of the previous proposition. Denote \( Y \) the set of all nodes \( j \) of \( M \) such that \( (\alpha_j, B_0) = 0 \) and \( Z \) the Hecke algebra whose type is the diagram \( M \) restricted to \( Y \). Then, for each node \( j \) in \( Y \), there is a minimal element \( B \) of \((B, <)\) and a node \( k \) of \( M \) such that \((\alpha_k, B) = 0 \) and \( h_{B,k} = s_j \), the image of the fundamental generator of \( A(M) \) in \( Z \).

**Proof.** The following proof is similar to one of Lemma 2.4.8 on page 29. Let \( j \) be a node of \( Y \). Then \( h_{B_0,j} = s_j \). Let \( B \in B \) be minimal such that there exists a node \( k \) with \((\alpha_k, B) = 0 \) and \( h_{B,k} = s_j \). Suppose there is a node \( i \) such that \( r_iB < B \). If \( i \neq k \) then by Proposition 3.4.3(iv) \( h_{r_iB,k} = h_{B,k} = s_j \). If \( i \sim k \) then by Proposition 3.4.3(v) \( h_{r_ir_iB,i} = h_{B,k} = s_j \) and by Proposition 3.3.1(ii), \( r_ir_iB < B \).

Both cases contradict the minimal choice of \( B \), so \( B \) must be a minimal element of \((B, <)\). \( \square \)

**Example 3.4.6.** Suppose \( M \) is a connected simply laced diagram. Then the type of \( Y \) as defined in Corollary 3.4.5 is given in Table 2. We deal with two series in particular.

If \( M = A_{n-1} \) and \( B \) is the \( W \)-orbit of \( \{\alpha_1, \alpha_3, \ldots, \alpha_{2p-1}\} \), then
\[
B_0 = \{\varepsilon_1 - \varepsilon_{n-p+1}, \varepsilon_2 - \varepsilon_{n-p+2}, \ldots, \varepsilon_p - \varepsilon_n\} \quad \text{and} \quad Y = \{\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{n-p-1}\}.
\]

Therefore, the Hecke algebra \( Z \) is of type \( A_{n-2p-1} \).

If \( M = D_n \) and \( B \) is the \( W \)-orbit of \( \{\alpha_{n-2p+2}, \ldots, \alpha_{n-2}, \alpha_n\} \), then
\[
B_0 = \{\varepsilon_1 + \varepsilon_{2p}, \varepsilon_2 + \varepsilon_{2p-1}, \ldots, \varepsilon_p + \varepsilon_{p+1}\} \quad \text{and} \quad Y = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-2p}, \alpha_{n-p+1}\}.
\]

The Hecke algebra \( Z \) has type \( A_1D_{n-2p} \) (where \( D_1 \) is empty and \( D_2 = A_1A_1 \)).

### 3.5. Poset properties

In the previous section, we showed for each \( B \in B \), that there exists a word \( w \) of minimal length in the Weyl group \( W(M) \) such that \( B = wB_0 \). Moreover, for each node \( i \) such that \( r_iB > B \), there is \( w' \in W(M) \) of length \( l(w) \) such that \( B = w'B_0 \) and \( l(r_iw') < l(w') \). In particular, if the word is \( r_1r_2 \cdots r_{i-1}r_i \) then \( B_0 > r_1B_0 > r_2r_1B_0 > \cdots > r_{i-1}r_i \cdots r_1B_0 = B \) and this is the shortest word which does this. We denote, for each \( B \in B \), this minimal length of \( w \) by \( s_B \).
Definition 3.5.1. We call a set $C \in B$ fundamental when for all other sets $B \in B$ we have $s_B \leq s_C$.

Like for single roots, the notion of the height of a set in any orbit $B$ is introduced.

Definition 3.5.2. Let $s_C$ be the minimal length of a reduced word taking $B_0$ to a random fundamental set $C$. Then the height of a set $B \in B$, denoted by $\text{ht}(B)$, is defined as

$$\text{ht}(B) = s_C - s_B.$$ 

Besides the height of a set, we will also use the natural notion of distance in a connected graph. We denote by $d(B, B')$ the length of a shortest path in the poset from $B$ to $B'$. Notice this distance is also the length of any reduced word taking $B$ to $B'$. In particular, for any set $B \in B$, we have $s_B = d(B, B_0)$.

Definition 3.5.3. Let $B, B' \in B$. A set $B'$ with $\text{ht}(B') \geq \text{ht}(B)$ is called directly reachable from $B$ if there is a reduced word $w$ with $wB = B'$ and $l(w) = \text{ht}(B') - \text{ht}(B)$.

Besides fundamental sets, we will also work with minimal sets.

Definition 3.5.4. We call a set $B \in B$ minimal when there is no $i \in M$ with $r_iB < B$.

From the definition it is immediately clear that a minimal set is only directly reachable from itself. The fundamental sets will play a distinct role in the spanning sets we will construct. We therefore list some properties of these sets at the bottom of our poset.

Lemma 3.5.5. Let $C$ be a fundamental set in $B$. Then $C$ satisfies the following properties.

(i) $C$ is minimal,
(ii) $C$ contains at least one fundamental root $\alpha_i$,
(iii) For every single reflection $r_i$ with $\alpha_i \notin C \cup C^\perp$ we have $r_iC > C$,
(iv) When $\alpha_i \in C$, for all $j \in M$ then the set $w_{ij}C$ is also fundamental,
(v) If $i \sim j$ with $\alpha_i \in C$, then $r_jC$ is directly reachable from precisely two sets, $C$ and $r_ir_jC$.

Proof.

(i) If there is an $i \in M$ with $r_iC < C$, then $s_{r_iC} > s_C$ which is not possible when $C$ is fundamental.
(ii) If $C$ contains no fundamental roots, there is a $\beta \in C$ of minimal height which is lowered by at least one simple reflection $r_i$. As $\beta$ is of minimal height this implies $r_iC < C$, contradicting the minimality of $C$.
(iii) Directly from the minimality of $C$ and the properties of $(B, <)$.
(iv) By induction on the distance from $i$ to $j$ by showing $w_{ij_k}C$ is fundamental for all $j_k$ on the shortest path from $i$ to $j$. 
(v) It is obvious that \( r_j C \) is directly reachable from \( C \) and \( r_i r_j C \). Assume \( r_j C \) is also directly reachable from a third set \( B \in B \). So there is a simple reflection \( r_k \) with \( r_k r_j C < r_j C \). This node \( k \) is not adjacent to at least one of the nodes \( i, j \), say \( i \). Now \( r_k r_j C < r_j C \) and \( r_i r_j C < r_j C \) imply a set \( r_i r_k r_j C < r_i r_j C \), contradicting the minimality of \( r_i r_j C \).

\[ \square \]

**Lemma 3.5.6.** Let \( B \in B \) be such that, for \( i \neq j \), we have \( r_i B < B \) and \( r_j B < B \). If there is no \( B' \in B \) such that both \( r_i B \) and \( r_j B \) are directly reachable from \( B' \) then \( \alpha_j \in r_i B \) and \( \alpha_i \in r_j B \).

**Proof.** Assume \( i \not\sim j \). When \( r_i B = r_j B \), the statement is trivial. When \( r_i B \neq r_j B \), the poset relations show both sets are directly reachable from \( r_i r_j B \). Assume \( i \sim j \). If \( r_j r_i B < r_i B \), then both sets are directly reachable from \( r_j r_i r_j B = r_i r_j r_i B \). So the only poset relation which makes it possible that \( r_i B \) and \( r_j B \) are not both directly reachable from one set \( B' \), is when \( r_i r_j B = r_j B \). By Lemma 3.3.2, \( \alpha_j \in r_i B \) and \( \alpha_i \in r_j B \). \[ \square \]
CHAPTER 4

Ideals of the BMW algebra

4.1. Introduction

The objective of this chapter is to study the quotient ideals $I_C / I_{r+1}$. Recall Definition 2.5.1. In contrast with the case where $|C| = 1$, two cocliques $C$ and $D$ of the same size $r$ do not always generate the same ideal. To each coclique $C$ of size $r$, a set $C$ of mutually orthogonal positive roots in an admissible $W$-orbit is matched. Two cocliques $C$ and $D$ will generate the same ideal if and only if they are matched to sets which lie in the same admissible $W$-orbit.

In § 4.3 we define a spanning set for the quotient $I_1 / I_2$. The spanning set is proven to be closed under the action of the elements of $B(M)$. This results in an upper bound for the dimension of $I_1 / I_2$. In Chapter 6 we show the dimension actually meets this bound.

In the remaining part of this chapter we discuss a strategy for obtaining a spanning set for every quotient ideal $I_C / I_{r+1}$. Such a spanning set would provide an upper-bound for the dimension of the ideal $I_1$. As $B(M) / I_1$ is isomorphic to the Hecke algebra $H(M)$, this would lead to an upper bound for the dimension of $B(M)$ over $\mathbb{Q}(l, \delta)$.

The results for $I_1 / I_2$ are taken from [CGW].

4.2. Subalgebras centralizing the quotient ideals

Throughout this section, $M$ is a connected, simply laced spherical diagram. By $B(M)$ we denote the corresponding BMW algebra over $\mathbb{Q}(l, \delta)$, by $(A, S)$ the corresponding Artin system, and by $(W, R)$ the corresponding Coxeter system. Furthermore, $\Phi^+$ is the set of positive roots associated with $(W, R)$ and $Y$ the set of nodes $i$ of $M$ with $\alpha_i$ orthogonal to the highest set $B_0$ of an admissible $W$-orbit $B$. Existence of this is established in Corollary 3.3.6.

Let $C$ be a coclique of size $r$ of $M$, that is, a collection of $r$ mutually non-adjacent nodes of the diagram $M$. To $C$ we assign an admissible set $\mathcal{C}$ in the following way.

(i) $\alpha_i \in C$ when $i \in C$,
(ii) $C$ is contained in an admissible orbit $B$.
(iii) $C$ contains the maximum number of fundamental roots in $B$, that is, every $B \in B$ contains at most $r$ fundamental roots.

It can be shown that $C$ is uniquely determined by $C$. We call $C$ the admissible closure of $C$. We introduce a subalgebra $Z_C$ of $B(M)$ which centralizes $c_C$. 

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Definition 4.2.1. Denote by $Z_C$ the subalgebra generated by elements $\hat{w}_{ji}^g \hat{w}_{ij} e_C$ with $i \in C$ and satisfying one of the following properties,
(i) $i = j$ and $\alpha_k \in C^\perp \cup C$,
(ii) $k$, $j$ and $i$ are distinct nodes with $k \sim i \sim j$ where $k \neq j$ and $\alpha_j, \alpha_k \in (C \setminus \{\alpha_i\})^\perp$,
(iii) $k$, $j$ and $i$ are distinct nodes and there is a node $i'$ such that $k \sim i' \sim j$ and the shortest path $p_{ij}$ in $M$ from $i$ to $j$ contains $i'$ but all nodes $h \neq i$ with $\alpha_h \in C$ are not equal to or adjacent to $k$ or any node in $p_{ij}$.

When $C = \{i\}$, we write $Z_i$ instead of $Z_{\{i\}}$. Here $Z_i$ is the subalgebra (not necessarily containing the identity) of $B(M)$ generated by all elements of the form $\hat{w}_{ji}^g \hat{w}_{ij} e_i$ for $j$ and $k$ non-adjacent nodes of $M$. We allow for $j$ and $k$ to be equal, so that, for example in case $M = A_2$, the subalgebras $Z_i$ are one-dimensional (scalar multiples of $e_i$).

By Lemma 2.4.1(iv),(v), the generating elements of $Z_i$ can be written in various ways:
$$e_i \hat{w}_{ji}^g \hat{w}_{ij}^{-1} = \hat{w}_{ji}^g \hat{w}_{ij}^{-1} e_i = \hat{w}_{ji}^g \hat{w}_{ij} e_i.$$

We will need an integral version of $Z_C$ and $B(M)$. We shall work with the coefficient ring $E = \mathbb{Q}(\langle l^\perp \rangle)$ inside our field $\mathbb{Q}(l, \delta)$. Observe $m \in E$ by (1). Let $B(M)^{(0)}$ be the subalgebra of $B(M)$ over $E$ generated by all $g_i$ and $e_i$, and let $Z_C^{(0)}$ be the subalgebra of $Z_C$ over $E$ generated by the same elements as taken above for generating $Z_C$. Then $Z_C^{(0)}$ is a subalgebra of $B(M)^{(0)}$.

Proposition 4.2.2. The subalgebra $Z_C^{(0)}$ of $B(M)^{(0)}$ satisfies the following properties.

(i) It centralizes $e_C$ and has identity element $\delta^{-r} e_C$.
(ii) $Z_C^{(0)} = \hat{w}_{ji}^g \hat{w}_{ij}^{-1}$ when $C$ is the same set as $D$ with $j \in D$ replaced by $i$ and all other $k \in D$ are not in the shortest path $p_{ij}$ from $i$ to $j$ in $M$ or adjacent to any node in $p_{ij}$. In particular, $Z_i^{(0)} = \hat{w}_{ji}^g \hat{w}_{ij}^{-1}$ for all nodes $j \in M$.
(iii) The scaled versions $\delta^{-r} e_C \hat{w}_{ji}^g \hat{w}_{ij}^{-1}$ of the generators of $Z_C^{(0)}$ satisfy the quadratic relation $X^2 + mX - 1_C = 0 \mod I_{r+1}$, where $1_C$ stands for the identity element $\delta^{-r} e_C$ of $Z_C^{(0)}$.

Proof. (i). Since $\delta^{-r} e_C$ is an idempotent (cf. Proposition 2.3.1), it suffices to verify that the generators of $Z_C$ centralize $e_C$.

We do this first for $Z_i$. Here this follows from the following computation, in which Lemmas 2.4.1 and 2.4.6 are used.
$$\hat{w}_{ji}^g \hat{w}_{ij}^{-1} e_i = \hat{w}_{ji}^g e_j \hat{w}_{ij} = \hat{w}_{ji}^g \hat{w}_{ij}^{-1} = e_i \hat{w}_{ji}^g \hat{w}_{ij}^{-1}.$$

As every $\hat{w}_{ji}^g \hat{w}_{ij} e_i$ commutes with $e_i$, we only have to show all $\hat{w}_{ji}^g \hat{w}_{ij} e_C$ commute with every other $e_j$ with $j \in C$.

Consider the three possible shapes of $\hat{w}_{ji}^g \hat{w}_{ij} e_C$ separately in the same order as listed in Definition 4.2.1.

First, $\hat{w}_{ji}^g \hat{w}_{ij} e_C = g_k e_C$ where $k \neq j$ for all $j \in C$. Commutation follows immediately from the braid relation and the defining relation for $e_j$. 

We now prepare for considerations of \( \dim(\mathcal{C}) \). Whence recall from Definition 2.4.3 that \( \alpha_j, \alpha_k \in (C \setminus \{ \alpha_i \})^\perp \). As \( \alpha_j \in (C \setminus \{ \alpha_i \})^\perp \) we have \( g_i, g_j \) and \( g_k \) commuting with all \( e_h \) with \( \alpha_h \in C \setminus \{ \alpha_i \} \) and the statement holds.

Finally, \( \overline{w}_{ij} g_k \overline{w}_{ij} e_C = g_j g_k g_v g_i e_C \) and \( \alpha_j, \alpha_k \in (C \setminus \{ \alpha_i \})^\perp \) we have \( g_i, g_j \) and \( g_k \) commuting with all \( e_h \) with \( \alpha_h \in C \setminus \{ \alpha_i \} \) and the statement holds.

(ii). Consider \( e_D \overline{w}_{ij} \overline{e}_j \overline{e}_i e_C \), a generator of \( Z_D^{(0)} \), with \( h \in \mathcal{D} \) and where \( j \perp k \). Let \( \mathcal{C} \) be the same set as \( \mathcal{D} \) with \( h \in \mathcal{D} \) replaced by \( i \) and all other nodes of \( \mathcal{D} \) are not in the shortest path \( p_h \) from \( i \) to \( h \) in \( M \) or adjacent to any node in \( p_h \).

Denote by \( e_D e_C \) the product of the \( e_i \) contained in both \( \mathcal{C} \) and \( \mathcal{D} \). Hence \( e_D = e_h e_{DC} \) and \( e_C = e_i e_{DC} \). When \( |D| = 1 \), take \( e_{DC} = 1 \). We have

\[
\overline{w}_{hi} e_D \overline{w}_{ij} \overline{e}_j \overline{e}_i \overline{w}_{hi}^{-1} \mathcal{C} = e_D \overline{w}_{hi} e_h \overline{w}_{ij} \overline{e}_j \overline{e}_i \overline{w}_{hi}^{-1} \mathcal{C} = e_D e_C \overline{w}_{hi} \overline{w}_{ij} \overline{e}_j \overline{e}_i \overline{w}_{hi}^{-1} \mathcal{C} \]

whence \( \overline{w}_{hi} Z_D^{(0)} \overline{w}_{hi}^{-1} \subseteq Z_C^{(0)} \). The rest follows easily.

(iii). Substituting \( \delta^{-1} e_C \overline{w}_{ji} \overline{e}_i \overline{w}_{ji}^{-1} \) for \( X \), we find

\[
(\delta^{-1} e_C \overline{w}_{ji} \overline{e}_i \overline{w}_{ji}^{-1})^2 + m(\delta^{-1} e_C \overline{w}_{ji} \overline{e}_i \overline{w}_{ji}^{-1}) - \delta^{-1} e_C \overline{e}_i \overline{w}_{ji} \overline{e}_i \overline{w}_{ji}^{-1} = \delta^{-1} e_C \overline{w}_{ji} (\delta^{-1} e_C \overline{w}_{ji} \overline{e}_i \overline{w}_{ji}^{-1}) \in \mathcal{B} e_D e_k \mathcal{B} \subseteq I_{r+1},
\]

where \( e_D \) is the same product as \( e_C \) with \( e_i \) replaced by \( e_j \).

\[
4.3. \text{Structure of } I_1/I_2
\]

We now prepare for considerations of \( \mathcal{B}(M) \) modulo \( I_2 \). The aim is to find a linear spanning set for \( I_1/I_2 \) of size \( |\Phi^+|^2 |W_Y| \). In particular, we obtain an upper bound for \( \dim(I_1/I_2) \), which by Theorem 6.1.1 will be an equality. In the next section we will generalize these results to any quotient \( I_C/I_{r+1} \).

Recall from Definition 2.4.3 that \( w_{\beta,i} \in W \) is the element of minimal length with the property that \( w_{\beta,i} \alpha_i = \beta \) with \( \alpha_i, \beta \in \Phi^+ \).
Lemma 4.3.1. Suppose $i$, $j$, and $k$ are distinct nodes of $M$. Then

\[
e_i\widehat{je_k} = \begin{cases} 
e_i\widehat{e_kj} & \text{if } j \neq k \text{ and } i \neq k, \\
\widehat{w_{\alpha,i,k}e_k} & \text{if } j \neq k \text{ and } i \sim k, \\
\widehat{w_{\alpha,i,k}e_k(i+m)} - me_i e_k & \text{if } j \sim k \text{ and } i \sim j, \\
\widehat{e_i e_k w_{ik,j} w_{ki}} & \text{if } j \sim k, i \neq j, \text{ and } i \neq k.
\end{cases}
\]

In each case the result is in $\widehat{w_{\alpha,i,k}Z_k^{(0)}} + I_2$.

Proof. In the first two cases as $j \neq k$ we have $e_i\widehat{je_k} = e_i\widehat{e_kj}$. If $i \neq k$, $e_ie_k$ is in $I_2$. If $i \sim k$, $e_ie_k = w_{\alpha,i,k}e_k$. These are the only possibilities when $j \neq k$.

Suppose next that $j \sim k$. In the last case $e_i$ commutes with $j$ and $e_ie_k$ is in $I_2$. Suppose then $i \sim j$. Of course then $i \neq k$ since the type is spherical. Now by (R2)

\[
e_i\widehat{je_k} = (e_ie_j e_i)\widehat{je_k} = (e_i e_j \widehat{e_k})\widehat{je_k} = e_i e_j \widehat{e_k} - me_i e_j e_k + me_j e_k = e_i e_k \widehat{e_k (i+m)} - me_i e_k = \widehat{w_{\alpha,i,k}e_k(i+m)} - me_i e_k.
\]

As $e_ie_k \in I_2$ the result follows.

Finally, if $i \sim k$ then necessarily $i \neq j$, and

\[
e_i\widehat{je_k} = e_i e_k e_i \widehat{je_k} = e_i e_k \widehat{e_i e_k} = e_i e_k \widehat{\alpha_i k k i} = e_i e_k \widehat{\alpha_k j k i j} = e_i e_k \widehat{\alpha_k j k i j k}
\]

In each of the cases the elements are in $\widehat{w_{\alpha,i,k}Z_k^{(0)}} + I_2$ from the definition. \qed

If some of $i$, $j$, $k$ are equal, similar results follow from the defining relations and Propositions 2.3.3 and 2.3.5.

Lemma 4.3.2. Let $i, j, k \in \{1, \ldots, n\}$ and let $\gamma$ be the shortest path from $j$ to $k$. Then

\[
\widehat{iw_{\alpha,i,k}e_k} = \begin{cases} \widehat{w_{\alpha,i,k}e_k} & \text{if } i \neq \text{ any point of } \gamma, \\
l^{-1}\widehat{w_{\alpha,i,k}e_k} & \text{if } i = j, \\
\widehat{w_{\alpha,i,k}e_k h' \mod I_2} & \text{if } i \in \gamma, i \neq j, h' \text{ on } \\
\text{the path from } i \text{ to } j \text{ and at distance 2 to } i \text{ in } M, \\
\widehat{w_{\alpha,i,k}e_k \widehat{w_{h'k} \mod I_2}} & \text{if } i \notin \gamma, i \sim h, h \in \gamma, \\
\text{and } h \neq j, h' \sim h, \text{ and } h' \text{ on the path from } h \text{ to } j, \\
w_{\alpha,\alpha_i} e_k + mw_{\alpha,i,k} e_k - mw_{\alpha,\alpha_j} e_k & \text{if } i \in \gamma \text{ and } i \sim j, \\
w_{\alpha,\alpha_i} + w_{\alpha,\alpha_j} e_k & \text{if } i \notin \gamma \text{ and } i \sim j.
\end{cases}
\]

Also

\[
e_i w_{\alpha,i,k} e_k = \begin{cases} \widehat{\delta w_{\alpha,i,k} e_k} & \text{if } i = j, \\
0 \mod I_2 & \text{if } i \neq j, \\
\widehat{w_{\alpha,i,k} e_k} & \text{if } i \sim j.
\end{cases}
\]

In each case the result is in $\widehat{w_{\alpha,i,k}Z_k^{(0)} + mw_{\alpha,i,k}Z_k^{(0)}} + mw_{\alpha,i,k}Z_k^{(0)} + I_2$.

Proof. Consider the shortest path $\gamma = k, \ldots, j$ from $k$ to $j$ in $M$. If $i$ is non-adjacent to each element of this path, then the statement holds. Also if $i = j$ the statement follows immediately. This leaves two possibilities, $i$ is in $\gamma$, or $i$ is not in $\gamma$ but is adjacent to some $h$ in $\gamma$. 

Assume that \(i\) occurs in \(\gamma\). If \(i \sim j\), then by (9) of Proposition 2.3.3
\[
\hat{w}_{\alpha_i, k e_k} = \hat{e}_j e_i \cdots e_k = \hat{e}_j - 1 \hat{w}_{\alpha_i + \alpha_j, k e_k} + \hat{w}_{\alpha_i + \alpha_j, k e_k} - \hat{w}_{\alpha_i, k e_k}.
\]
Suppose, therefore, that \(i \not\sim j\). Then \(\hat{w}_{\alpha_i, k e_k} = e_j \cdots e_h \hat{e} \hat{e}_i e_r \cdots e_k\) with \(h' \sim h \sim i \sim i'\). Substitution of \(\hat{e} \hat{e}_i e_r = \hat{e} \hat{e}_i e_r \hat{e} \hat{e}_i e_r \cdots e_k\) begins with \(\hat{e} \hat{e}_i e_r \hat{e} \hat{e}_i e_r \cdots e_k\) except for \(\hat{e} \hat{e}_i e_r \hat{e} \hat{e}_i e_r \cdots e_k\) and use of Lemma 4.3.2 gives
\[
\hat{w}_{\alpha_i, k e_k} = e_j \cdots e_h \hat{e} \hat{e}_i e_r \cdots e_k = e_j \cdots e_h \hat{e} \hat{e}_i e_r \cdots e_k - \hat{w}_{\alpha_i, k e_k} + me_j \cdots e_h e_i \cdots e_k
\]
\[
\in e_j \cdots e_h \hat{e} \hat{e}_i e_r (\hat{h} + m) e_i \cdots e_k - \hat{w}_{\alpha_i, k e_k} + I_2
\]
\[
= e_j \cdots e_h \hat{e} \hat{e}_i e_r \hat{h}' e_i \cdots e_k + I_2 = e_j \cdots e_h e_i e_r e_k \hat{h}' + I_2
\]
\[
= \hat{w}_{\alpha_i, k e_k} \hat{h}' + I_2
\]

Next assume \(i\) is not in \(\gamma\) but is adjacent to some \(h\) in \(\gamma\). Suppose there exists \(h' \sim h\) but \(i\) is not in \(\gamma\). Then \(\hat{w}_{\alpha_i, k e_k} = \hat{w}_{\alpha_i + \alpha_j, k e_k}\). This ends the proof of the equalities involving \(\hat{w}_{\alpha_i, k e_k}\).

We now consider \(e_i \hat{w}_{\alpha_i, k e_k}\). If \(i = j\), we have trivially \(e_i \hat{w}_{\alpha_i, k e_k} = \hat{e} w_{\alpha_i + \alpha_j, k e_k}\). So let \(i \not\sim j\). If \(i \not\sim j\) we find \(e_i \hat{w}_{\alpha_i, k e_k} = e_i \hat{w}_{\alpha_i + \alpha_j, k e_k}\). So assume \(i \sim j\). If \(i\) occurs in \(\gamma\), the path \(\gamma\) begins with \(j \sim i\) and so
\[
e_i \hat{w}_{\alpha_i, k e_k} = e_i e_j e_i \cdots e_k = e_i \cdots e_k = \hat{w}_{\alpha_i, k e_k}
\]
and if \(i\) does not occur in \(\gamma\), we have \(e_i \hat{w}_{\alpha_i, k e_k} = e_i e_j \cdots e_k = \hat{w}_{\alpha_i, k e_k}\).

Let \(i\) be a node of \(M\) and \(\beta \in \Phi^+\). We shall use the following notation.

(i) \(\text{Geod}(i, \beta)\) is the set of nodes of the shortest path from \(i\) to a node in the support of \(\beta\) that are not in the support themselves. So \(\text{Geod}(i, \beta) = \emptyset\) if \(i \in \text{Supp}(\beta)\).

(ii) \(\text{Proj}(i, \beta)\) is the node in the support of \(\beta\) nearest \(i\). So \(\text{Proj}(i, \beta) = i\) if \(i \in \text{Supp}(\beta)\).

(iii) \(C_{\beta, i}\) is the coefficient of \(\alpha_i\) in the expression of \(\beta\) as a linear combination of the fundamental roots. So \(\beta = \sum_i C_{\beta, i} \alpha_i\).

(iv) \(J_{\beta, k}\) is the subset of \(M\) of all nodes \(j\) such that \((\alpha_j, \beta) = 1\) and \(\hat{w}_{\beta - \alpha_j, h} = \hat{w}_{\beta, h}\) where \(h = \text{Proj}(\beta, k)\). This set is empty only if \(\beta\) is a fundamental root.

For \(i\) a node of \(M\), denote by \(i^1\) the set of all nodes distinct and non-adjacent to \(i\).
**Lemma 4.3.3.** Let $\beta$ be a root and let $k$ be a node of $M$ such that $i = \text{Proj}(\beta, k)$ satisfies $(\alpha_i, \beta) = 0$ and $C_{\beta,i} = 1$. If $J_{\beta,k} \cap i^+ = \emptyset$ then

$$\hat{w}_{\beta,k}e_k = w_{\beta,k} - e_k w_{\beta,k} - \text{op}e_k w_{\beta,k} e_k \in w_{\beta,k} - \text{op}Z_k^{(0)}.$$

**Proof.** We only have to prove that $e_k \hat{w}_{\beta,k}^{-1} w_{\beta,k} \hat{w}_{\beta,k}$ belongs to $Z_k^{(0)}$. Moreover, $J_{\beta,k} = J_{\beta,1}$, so, by Proposition 4.2.2(ii), it suffices to consider the case where $k = i$.

We prove this by induction on the height of $\beta$. The smallest possible root that satisfies the conditions of the lemma is a root of the form $\alpha_j + \alpha_i + \alpha_k$ with $j \sim i \sim h$. In this case $w_{\beta,i} = h_i$. Straightforward computations give

$$e_i \hat{w}_{\beta,j}^{-1} w_{\beta,j} \hat{w}_{\beta,j} = e_i h_i h_j = e_i \hat{w}_{j} h_i \hat{w}_{j} = e_i \hat{w}_{ij} \hat{w}_{ij},$$

which belongs to $Z_k^{(0)}$ by definition.

Let $\beta$ be a positive root of height at least 4 and assume that the lemma holds for all positive roots of height less than $ht(\beta)$. Now $w_{\beta,k} e_k = \hat{w}_{\beta,k} e_i \cdots e_k$ with no $i$ in $w_{\beta,i}$. Let $j \in J_{\beta,k}$. Then, by the hypothesis $J_{\beta,k} \cap i^+ = \emptyset$, we have $i \sim j$. Clearly $w_{\beta,j} = j w_{\beta-j, i}$. As $(\alpha_i, \beta) = 0$ and $C_{\beta,i} = 1$, the sum of $C_{\beta,j}$ for $j$ running over the neighbors of $i$ in $M$, must be 2. Hence there are either two nodes $j, h$ say, in $M$ with $C_{\beta,j} = C_{\beta,h} = 1$ or there is a single node $j$ of $M$ adjacent to $i$ with $C_{\beta,j} = 2$.

In the former case, as $ht(\beta) \geq 4$, there is an end node $p$ of $\beta$ distinct from $j, i, h$ and non-adjacent to $i$ with $C_{\beta,p} = 1$, which implies $(\alpha_p, \beta) = 1$, whence $p \in J_{\beta,1} \cap i^+ = \emptyset$, a contradiction. Hence $i$ is an end node of $\beta$ and has a neighbor $j$ with $C_{\beta,j} = 2$ and $(\alpha_j, \beta) = 1$. This implies that $\hat{w}_{\beta-j, i} = \hat{w}_{\gamma, j}$, where $\gamma = \beta - \alpha_i - \alpha_j$.

As $(\alpha_j, \gamma) = 0$ and $J_{\gamma,j} \cap j^+ \subseteq J_{\beta,i} \cap i^+ = \emptyset$, we can apply induction to find $e_j \hat{w}_{\gamma,j}^{-1} j \hat{w}_{\gamma,j}$ belongs to $Z_j^{(0)}$. Consequently,

$$e_i \hat{w}_{\beta,i}^{-1} w_{\beta,i} \hat{w}_{\beta,i} = e_i \hat{w}_{\gamma,j}^{-1} j \hat{w}_{\gamma,j} \hat{w}_{\beta,j} \hat{w}_{\gamma,j} = e_i \hat{w}_{\gamma,j}^{-1} j \hat{w}_{\gamma,j} \hat{w}_{\beta,j} \hat{w}_{\gamma,j} = e_i \hat{w}_{\gamma,j}^{-1} j \hat{w}_{\gamma,j} \hat{w}_{\beta,j} \hat{w}_{\gamma,j}$$

$$\in \hat{w}_{\beta,j} Z_j^{(0)} / \hat{w}_{\beta,j} Z_k^{(0)} = Z_k^{(0)}.$$

**Lemma 4.3.4.** Let $\beta$ be a root and let $i$ be a node with $(\alpha_i, \beta) = 0$. Then the following hold.

(i) If $j$ is a node in $J_{\beta,k} \cap i^+$ then $\hat{w}_{\beta,k} e_k = \hat{j} \hat{w}_{\beta-j, i} e_k$.

(ii) If $i = \text{Proj}(\beta, k)$ and $C_{\beta,i} = 1$ and $J_{\beta,k} \cap i^+ = \emptyset$, then $\hat{w}_{\beta,k} \hat{w}_{\beta,k} e_k \in Z_k^{(0)}$ and

$$\hat{w}_{\beta,k} e_k = w_{\beta,k} - e_k w_{\beta,k} - \text{op}e_k w_{\beta,k} e_k.$$

(iii) If $i \neq \text{Proj}(\beta, k)$ or $C_{\beta,i} > 1$, then, for $j \in J_{\beta,k} \setminus i^+$,

$$\hat{w}_{\beta,k} e_k = \hat{j} \hat{w}_{\beta-j, i} e_k.$$

In each case, $\hat{w}_{\beta,k} e_k \in \hat{w}_{\beta,k} Z_k^{(0)}$.

**Proof.** (i). Straightforward from $\hat{i} \hat{j} = \hat{j} i$.

For the remainder of the proof, we can and will assume there is a node $j$ with $(\alpha_j, \beta) = 1$, $w_{\beta,k} = r_j w_{\beta-j, i}$ and $i \sim j$. Then $(\alpha_i, \beta - \alpha_j) = 1$.

(ii). This follows from Lemma 4.3.3.
(iii). Here $\hat{w}_{\beta,k} = \hat{j}w_{\beta-\alpha_i,\alpha_i,k}$ and the statement follows from the braid relation $ij = jij$. \hfill \Box

4.3. STRUCTURE OF $I_1/I_2$

Theorem 4.3.5. Let $B(M)$ be a BMW algebra of type $M \in ADE$, let $\beta \in \Phi^+$, and let $i$, $k$ be nodes of $M$.

If $(\alpha_i, \beta) = -1$, then

$$\hat{w}_{\beta,k} e_k = \begin{cases} w_{\beta-\alpha_i,k} e_k - mw_{\beta,\alpha_i,k} e_k + mw_{\alpha_i,\beta,k} w_{\beta,h} & \text{if } i \notin \text{Geod}(k, \beta), \\ w_{\beta+\alpha_i,k} e_k & \text{if } i \in \text{Geod}(k, \beta) \text{ and } h = \text{Proj}(k, \beta). \end{cases}$$

If $(\alpha_i, \beta) = 1$, then

$$\hat{w}_{\beta,k} e_k = \begin{cases} w_{\beta-\alpha_i,k} e_k - mw_{\beta,\alpha_i,k} e_k + ml^{-1}e_i w_{\beta-\alpha_i,k} e_k & \text{if } i \in J_{\beta,k}, \\ w_{\beta-\alpha_i,k} e_k & \text{if } i \notin J_{\beta,k}. \end{cases}$$

If $(\alpha_i, \beta) = 0$, then

$$\hat{w}_{\beta,k} e_k = \begin{cases} w_{\beta,k} e_k (w_{\beta,k}^{-1} \hat{w}_{\beta,k}) - jiw_{\beta-\alpha_i,k} e_k & \text{if } j \in J_{\beta,k} \cap i^+, \\ jiw_{\beta-\alpha_i,k} e_k & \text{if } j \in J_{\beta,k} \cap i^-= \emptyset, \\ C_{\beta,j} = 1, i = \text{Proj}(\beta, k), \text{ and } J_{\beta,k} \cap i^+ = \emptyset \text{ if } j \in J_{\beta,k} \cap i^+ \text{ and } i \in J_{\beta-\alpha_j,k}. \end{cases}$$

If $(\alpha_i, \beta) = 2$, then $\beta = \alpha_i$ and $\hat{w}_{\beta,k} e_k = l^{-1}w_{\beta,k} e_k$.

In each case, the result is in $w_{\gamma, k}Z_k^{(0)} + mw_{\beta, k}Z_k^{(0)} + mw_{\alpha_i, k}Z_k^{(0)} + I_2$, where $\gamma = \beta$ if $\beta = \alpha_i$ and $\gamma = r_1\beta$ otherwise.

**Proof.** By Lemma 4.3.2 the theorem holds for all fundamental roots $\beta$ in $\Phi^+$.

Suppose $\beta$ is a non-fundamental root in $\Phi^+$, and consider $\hat{w}_{\beta,k} e_k$. Now $(\alpha_i, \beta) < 2$, for otherwise $\beta = \alpha_i$. First let $(\alpha_i, \beta) = 1$. If $i \in J_{\beta,k}$, then

$$\hat{w}_{\beta,k} e_k = \hat{j}w_{\beta,k} e_k = w_{\beta-\alpha_i,k} e_k - mw_{\beta,\alpha_i,k} e_k + ml^{-1}e_i w_{\beta-\alpha_i,k} e_k.$$

Assume $i \notin J_{\beta,k}$ then $i = \text{Proj}(k, \beta)$ and $C_{\beta,i} = 1$. There must be a single node $j \in J_{\beta,k} \setminus i^+$ with $C_{\beta,j} = 1$, and the remaining nodes in the support of $\beta$ are on the side of $j$ in $M$ other than $i$. This means $w_{\beta,k} = \hat{j}w_{\alpha_i,k}$ where the elements in $u$ are on the side of $j$ other than $i$ and so $i$ commutes with $u$. Now

$$\hat{j}w_{\alpha_i,k} = \hat{j}w_{\alpha_i,k} = w_{\alpha_i,k} e_k$$

so $\hat{w}_{\beta,k} e_k = w_{\beta-\alpha_i,k} e_k$ as required.

Next let $(\alpha_i, \beta) = 0$ and assume $i$ is not in the support of $\beta$. Put $h = \text{Proj}(k, \beta)$ and $\rho = \text{Geod}(k, \beta)$. If $i$ is not in $\rho$ and not adjacent to an element of $\rho$, then $i$ commutes with $w_{\beta,k} = w_{\beta-\alpha_i,k} e_k$. Now

$$\hat{w}_{\beta,k} e_k = \hat{j}w_{\alpha_i,k} = \hat{j}w_{\alpha_i,k} = \hat{j}w_{\alpha_i,k}.$$}

We know that $\hat{j} \neq h$ so $\hat{w}_{\alpha_i,k}^{-1} \hat{w}_{\alpha_i,k} e_k \in Z_k^{(0)}$ by Lemma 4.3.2. We conclude

$$\hat{w}_{\beta,k} e_k = \hat{j}w_{\beta,k}^{-1} \hat{w}_{\beta,k} e_k = \hat{j}w_{\beta,k}^{-1} \hat{w}_{\alpha_i,k} e_k = w_{\beta,k}^{-1} \hat{w}_{\alpha_i,k} e_k \in w_{\beta,k}Z_k^{(0)}.$$
If $(\alpha_i, \beta) = 0$ with $i \in \text{Supp}(\beta)$, then the assertion follows from Lemma 4.3.4.

Finally let $(\alpha_i, \beta) = -1$. Here $\hat{w}_{\beta,k} e_k = w_{\beta+\alpha_i,k} e_k$ by definition if $i$ is not in Geod($k, \beta$). So suppose $i \in \text{Geod}(k, \beta)$. Write $h = \text{Proj}(k, \beta)$. Since $(\alpha_i, \beta) = -1$, we must have $i \sim h$. Therefore $\hat{w}_{\beta,k} = \hat{w}_{\beta,h} w_{k,h}$ and $\hat{w}_{\beta,k} e_k = w_{\beta,h} e_k \cdots \cdots e_k$. The set $\text{Supp}(\beta) \setminus \{k\}$ is a connected component of the Dynkin diagram connected to $h$ and disconnected from Geod($k, \beta$). Hence $h$ does not appear in $\hat{w}_{\beta,h}$. This means $\hat{i}$ commutes with $\hat{w}_{\beta,h}$. Moreover, by definition of $w_{\gamma,h}$, we have $\hat{w}_{\beta,h} = w_{\beta+\alpha_i,k}$ and so $\hat{w}_{\beta,h} \hat{w}_{k,i} = w_{\beta+\alpha_i,k}$. Consequently, by (9) of Proposition 2.3.3,

$$\hat{w}_{\beta,h} e_k e_i \cdots e_k = w_{\beta,h} (h + m(1 - e_h)) e_i \cdots e_k = w_{\beta,h} h w_{k,h} + m w_{k,h} w_{\beta,h} - m w_{\beta,h} kh = w_{\beta+\alpha_i,k} e_k + m w_{k,h} w_{\beta,h} - m w_{\beta,h} e_k.$$

□

**Corollary 4.3.6.** Let $\mathcal{B}(M)$ be a BMW algebra of type $M \in \text{ADE}$, let $\beta \in \Phi^+$, and let $i, k$ be nodes of $M$.

(i) $\hat{w}_{\beta,k}^{-op} e_k \in \hat{w}_{\beta,k} e_k + m \sum_{\gamma} \text{ht}(\gamma) < \text{ht}(\beta) \hat{w}_{\gamma,k} Z^{(0)}_k + I_2$,

(ii) $e_i \hat{w}_{\beta,k} e_k \in \hat{w}_{\alpha_i,k} Z^{(0)}_k + I_2$,

(iii) $\hat{w}_{\beta,k} e_k \in \sum_{\gamma \in \mathcal{H}_{\beta,i}} \hat{w}_{\gamma,k} Z^{(0)}_k + I_2$,

where $\mathcal{H}_{\beta,i} = \{\beta' \in \Phi^+ \mid \text{ht}(\beta') < \text{ht}(\beta)\} \cup \{\beta, \beta + \alpha_i\} \cap \Phi^+$.

**Proof.** We prove the statements simultaneously by induction on the height of $\beta$. If $\beta$ is a fundamental root then statement (i) holds by Lemma 2.4.1 and the statements (ii) and (iii) by Lemma 4.3.2.

Let $\beta \in \Phi^+$ with $\text{ht}(\beta) \geq 2$ and assume the lemma holds for all $\gamma \in \Phi^+$ with $\text{ht}(\gamma) < \text{ht}(\beta)$. Let $i, k$ be nodes and consider $\hat{w}_{\beta,k}^{-op} e_k, e_i \hat{w}_{\beta,k} e_k$ and $\hat{w}_{\beta,k} e_k$. There is (at least one) $j$ such that $\hat{w}_{\beta,k} = \hat{j} \hat{w}_{\beta+\alpha_i,k}$; then $\text{ht}(\beta - \alpha_j) = \text{ht}(\beta) - 1$.

Now

$$\hat{w}_{\beta,k}^{-op} e_k = \gamma^{-1} \hat{w}_{\beta+\alpha_j,k}^{-op} e_k$$

$$\leq \hat{J} + m - m e_j (w_{\beta+\alpha_j,k} e_k + m \sum_{\gamma} \hat{w}_{\gamma,k} Z^{(0)}_k + I_2)$$

$$= w_{\beta,k} e_k + m w_{\beta+\alpha_j,k} e_k - m e_j w_{\beta+\alpha_j,k} e_k$$

$$+ m \sum_{\gamma} \gamma \hat{w}_{\gamma,k} Z^{(0)}_k + m^2 \sum_{\gamma} \hat{w}_{\gamma,k} Z^{(0)}_k$$

$$- m^2 \sum_{\gamma} e_j \hat{w}_{\gamma,k} Z^{(0)}_k + I_2$$

$$\leq w_{\beta,k} e_k + m \sum_{\gamma} \hat{w}_{\gamma,k} Z^{(0)}_k + m \sum_{\gamma} e_j \hat{w}_{\gamma,k} Z^{(0)}_k + I_2$$

$$\leq w_{\beta,k} e_k + m \sum_{\gamma} \hat{w}_{\gamma,k} Z^{(0)}_k + I_2.$$

4. IDEALS OF THE BMW ALGEBRA
To see that \( \sum_{\text{ht}(\gamma')<\text{ht}(\beta-\alpha_j)} \hat{j} \hat{w}_{\gamma',k} Z_k^{(0)} \) is contained in \( \sum_{\text{ht}(\gamma)<\text{ht}(\beta)} \hat{w}_{\gamma,k} Z_k^{(0)} \), observe that by the induction hypothesis on (iii) we have \( \hat{j} \hat{w}_{\gamma,k} e_k \in \sum_{\gamma' \in H_{\gamma,i}} \hat{w}_{\gamma',i} Z_k^{(0)} + I_2 \). Here \( \text{ht}(\gamma') \leq \text{ht}(\gamma) + 1 < \text{ht}(\beta) \) while \( \text{ht}(\gamma) < \text{ht}(\beta-\alpha_j) \).

The sum \( \sum_{\text{ht}(\gamma)<\text{ht}(\beta)} \hat{e}_j w_{\gamma,k} Z_k^{(0)} \) is in \( \hat{w}_{\alpha,j,k} Z_k^{(0)} \) by our induction hypothesis on (ii) and this gives (i) for \( \beta \).

Now focus on \( e_i \hat{w}_{\beta,k} e_k = e_i \hat{j} \hat{w}_{\beta-\alpha_j,k} e_k \). If \( i = j \) then, by the induction hypothesis, \( e_i \hat{w}_{\beta,k} e_k = l^{-1} e_i \hat{w}_{\beta-\alpha_j,k} e_k \in \hat{w}_{\alpha,j,k} Z_k^{(0)} + I_2 \). If \( i \neq j \) then \( e_i \hat{j} \hat{w}_{\beta-\alpha_j,k} e_k = \hat{j} e_i \hat{w}_{\beta-\alpha_j,k} e_k = j e_i \hat{w}_{\alpha,j,k} Z_k^{(0)} + I_2 \) and by Lemma 4.3.2 this is contained in \( \hat{w}_{\alpha,j,k} Z_k^{(0)} + I_2 \).

So, for the remainder of the proof, we may (and shall) assume \( i \sim j \). By (9), we have \( e_i \hat{j} = e_i e_j + m e_i e_j - m e_i \), so

\[
e_i \hat{w}_{\beta,k} e_k = e_i \hat{j} \hat{w}_{\beta-\alpha_j,k} e_k = e_i \hat{w}_{\beta-\alpha_j,k} e_k + m e_i e_j \hat{w}_{\beta-\alpha_j,k} e_k - m e_i \hat{w}_{\beta-\alpha_j,k} e_k.
\]

By our induction hypothesis the last two terms are in \( \hat{w}_{\alpha,j,k} Z_k^{(0)} + I_2 \). This leaves the first term, \( e_i e_j \hat{w}_{\beta-\alpha_j,k} e_k \). Because \((\beta-\alpha_j, \alpha_i) = (\beta, \alpha_1) + 1\) the inner product of \( \alpha_i \) with \( \beta - \alpha_j \) can only take values 0, 1, and 2 and thus \( H_{\beta-\alpha_j} \) consists of roots with height at most \( \text{ht}(\beta-\alpha_j) \).

The induction hypothesis on (iii) now gives \( \hat{j} \hat{w}_{\beta-\alpha_j,k} e_k \in \sum_{\gamma \in H_{\beta-\alpha_j,i}} \hat{w}_{\gamma,k} Z_k^{(0)} \) where \( \text{ht}(\gamma) < \text{ht}(\beta) \) for all \( \gamma \). By applying the induction hypothesis twice we obtain

\[
e_i e_j \hat{w}_{\beta-\alpha_j,k} e_k \in \sum_{\text{ht}(\gamma)<\text{ht}(\beta-\alpha_j)} e_i e_j \hat{w}_{\gamma,k} Z_k^{(0)} + I_2 \subseteq e_i \hat{w}_{\alpha,j,k} Z_k^{(0)} \subseteq \hat{w}_{\alpha,j,k} Z_k^{(0)} + I_2.
\]

This establishes (ii). Finally consider \( \hat{w}_{\beta,k} e_k \). If \((\alpha_1, \beta) = -1 \) then \( \beta + \alpha_1 \notin \Phi^+ \) and the statement holds by Theorem 4.3.5. Also, if \((\alpha_1, \beta) = 1 \) then Theorem 4.3.5 applies. Here \( e_i \hat{w}_{\beta-\alpha_j,k} e_k \in \hat{w}_{\alpha,j,k} Z_k^{(0)} + I_2 \) by the induction hypothesis for (ii).

For the remainder of the proof we assume \((\alpha_1, \beta) = 0 \). Again Theorem 4.3.5 gives an expression for \( \hat{w}_{\beta,k} e_k \) in each of the four cases discerned. In the first cases, where \( i \notin \text{Supp}(\beta) \), the statement is immediate from this expression. By our induction hypothesis for (iii) the second case gives an expression contained in \( \sum_{\gamma \in H_{\beta-\alpha_j}} \hat{j} \hat{w}_{\gamma,k} Z_k + I_2 \) whence in \( \sum_{\gamma \in H_{\beta,i}} \hat{w}_{\gamma,k} Z_k + I_2 \). Now the fourth case goes by the same argument and only the third case remains to be verified. Above we have shown that \( \hat{w}_{\beta,k} e_k \in \hat{w}_{\beta,k} e_k + m \sum_{\text{ht}(\gamma)<\text{ht}(\beta)} \hat{w}_{\gamma,k} Z_k^{(0)} + I_2 \) and that completes the proof.

In the next section we generalize this result for every quotient ideal \( I_C/I_{r+1} \).

### 4.4. A spanning set for the quotient ideals

In the remaining sections of this chapter we discuss a strategy how it might be possible to construct a spanning set for every quotient ideal \( I_C/I_{r+1} \). Such a spanning set could be used to determine an upper bound for the dimension of the BMW algebras over \( \mathbb{Q}(l, \delta) \). Unfortunately we are not able to establish this objective here.

We give a general description of a spanning set and conjecture that it is indeed a spanning set for every quotient ideal \( I_C/I_{r+1} \). The final section is devoted to the
description of a partial proof that this spanning set is closed under multiplication and some open problems whose positive solutions would imply the conjecture.

For the remainder of this chapter we fix a coclique $C$ of $M$ of size $r$ and its admissible closure $C$, the corresponding set of mutually orthogonal positive roots in an admissible $W$-orbit $B$ of the positive root system $\Phi^+$. We write $e_C$ for the product of the $r$ commuting elements $e_i$ with $i \in C$.

We start with some definitions concerning the poset structure of an admissible orbit $W$-orbit $B$. Recall the notion of distance we introduced on page 50.

**Definition 4.4.1.** Let $B \in B$. Denote by $J_{B,C}$ the set of nodes $i \in M$ with the property

$$d(r_i B, C) < d(B, C).$$

We denote the subset of $J_{B,C}$ of nodes $i$ with $r_i B < B$ by $J^-_{B,C}$ and its complement in $J_{B,C}$ by $J^+_{B,C}$.

We generalize Definition 2.4.3 using properties of admissible orbits.

**Definition 4.4.2.** Let $B \in B$ and let $w \in W$ be a shortest word with $wC = B$. Denote by $w_{B,C}$ the image $\hat{w}$ of $w$ in $B(M)$.

We denote by $\hat{D}_C$ a set containing precisely one $w_{B,C}$ for every $B \in B$.

**Remark 4.4.3.** The elements $w_{B,C}$ are not uniquely determined by $B$ and $C$. Take for instance $C = \{\alpha_1, \alpha_4\}$ and consider $e_C = e_1 e_4$. Now both $w_1 = r_1 r_2 r_3 r_2$ and $w_2 = r_1 r_2 r_3 r_2$ are shortest words taking $C$ to $B = \{\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$. Later we will conjecture that although $w_{B,C}$ is not uniquely determined, the element $w_{B,C} e_C$ behaves well, that is, if the words $w_1$ and $w_2$ of shortest length take $C$ to $B$ then $\hat{w}_1 e_C$ is equal to $\hat{w}_2 e_C$ up to $m$ times elements $w_{B',C} e_C$, where $d(B', C) < d(B, C)$.

For $w_1 = r_1 r_2 r_3 r_2$ and $w_2 = r_4 r_2 r_3 r_2$ as above this gives

$$g_1 g_2 g_3 g_2 e_1 e_4 = g_4 g_2 g_3 g_2 e_1 e_4 + m (g_4 g_2 g_3 e_1 e_4 - g_1 g_2 g_3 e_1 e_4 + g_2 e_3 e_1 e_4 - g_4 e_2 e_1 e_4) + m^2 (e_3 e_1 e_4 - e_2 e_1 e_4).$$

We conjecture that the action of the generating elements of $B(M)$ on this spanning set is as follows.

**Conjecture 4.4.4.** Fix a coclique $C$ of $M$ and let $C$ be its admissible closure in the admissible $W$-orbit $B$. Then for all $B \in B$ and $i \in M$ the following holds.
Lemma 4.4.5. Suppose $F$ is a field, $E$ is a subring of $F$ which is a principal ideal domain. If $V$ is a vector space over $F$ and $V^{(0)}$ is an $E$-submodule of $V$ containing a spanning set of $V$, then $V^{(0)}$ is a free $E$-module on a basis of $V$. Moreover, if $a \in E$ generates a maximal ideal of $E$, then
\[ \dim_F(V) = \dim_{E/aE}(V^{(0)}/aV^{(0)}). \]

Proof. As $E$ is a principal ideal domain, it is well known, see [DF], Theorem 12.5, that each $E$-module of finite rank without torsion is free. Applying this observation to $V^{(0)}$, we let $X$ be a basis of the $E$-module $V^{(0)}$. By the hypothesis that $V^{(0)}$ spans $V$, it is also a basis of $V$, so $\dim_F(V) = |X|$. On the other hand, $X$ spans onto a basis of $V^{(0)}/aV^{(0)}$ over $E/aE$ (for, it clearly maps onto a spanning set and if $\sum_{x \in X} \lambda_x x = 0 \mod aV^{(0)}$ for $\lambda_x \in E$, then, as $V^{(0)} = EX$, with $X$ a basis, we have $\lambda_x = 0 \mod a$ for each $x \in X$, so the linear relation in $V^{(0)}/aV^{(0)}$ is the trivial one). This proves $\dim_F(V) = |X| = \dim_{E/aE}(V^{(0)}/aV^{(0)}).$ □

As we did our computations over the ring $\mathbb{Q}(\delta)[l^{\pm 1}]$ inside our field $\mathbb{Q}(l, \delta)$ this result, provided that the above conjecture holds, can be used to derive an upper bound for each quotient ideal $I_C/I_{r+1}$.

Corollary 4.4.6. Let $M \in ADE$ and let $C$ be a coclique of $M$. If Conjecture 4.4.4 holds, then $\overline{D_C}Z_C\overline{D_C}^{op}$ is a linear spanning set for $I_C/I_{r+1}$. Moreover, the dimension of $Z_C$ is at most $|W_C|$.

Proof. By Lemma 2.4.6 $I_C$ is spanned by a set of multiples of $e_C$ by generators $g_l$, so $I_C = B e_C B$. According to Conjecture 4.4.4, $B e_C = \overline{D_C}Z_C + I_{r+1}$. Applying Remark 2.3.2, we derive from this that $e_C B = Z_C \overline{D_C}^{op} + I_{r+1}$ (observe that $Z_C$ and $I_{r+1}$ are invariant under the anti-involution). Therefore,
\[ I_C = B e_C B = \overline{D_C}Z_C\overline{D_C}^{op} + I_{r+1}. \]

It remains to establish that the dimension of $Z_C$ mod $I_{r+1}$ is at most $|W_C|$. To this end we consider the integral versions $Z_C^{(0)}$ and $B^{(0)}$ of $Z_C$ and $B$ over $E = \mathbb{Q}(\delta)[l^{\pm}]$ defined at the beginning of § 4.2, and look at the quotients modulo $(l-1)$. Observe that, by (1), the scalar $m$ belongs to the ideal $(l-1)E$. 

\[
\begin{align*}
(24) & \quad w_{B,C}^{op} e_C \in w_{B,C} Z_C^{(0)} + \sum_{B' \in H_n^{-1}} w_{B',C} e_C Z_C^{(0)} + I_{r+1}, \\
(25) & \quad g_i w_{B,C} e_C \in w_{B,C} Z_C^{(0)} + \sum_{B' \in H_n^{-1}} w_{B',C} e_C Z_C^{(0)} + I_{r+1}, \quad \text{if } \alpha_i \in B^2, \\
(26) & \quad g_i w_{B,C} e_C \in \sum_{B' \in H_n^{-1}} w_{B',C} e_C Z_C^{(0)} + I_{r+1}, \quad \text{if } \alpha_i \notin B^2, \\
(27) & \quad e_i w_{B,C} e_C \in \sum_{B' \in H_n^{-1}} w_{B',C} Z_C^{(0)} + I_{r+1}.
\end{align*}
\]

where $H_B = \{ B' \in B \mid \text{ht}(B') \leq \text{ht}(B) + u \} \cap B$ and $H_i = \{ B \in B \mid \alpha_i \in B \}$.
By Lemma 4.4.5, applied with $\phi$ to \( W \), consequently, the dimension of $W$ of the form $Z^0 \in C$ is linearly spanned by the set $D_C(Z^0) \mod I_{r+1}$, it is linearly spanned by $B(C)$, and hence by $Z^0 \mod I_{r+1}$.

For brevity of notation, we set $m_1 = l - 1$. (The remainder of the proof would also work for $m_1 = l + 1$.) Since $\delta^{-r}e_C$ is a central idempotent belonging to $Z^0_C$, we have $m_1 B(0) \cap (Z^0_C + I_{r+1}) = m_1 e_C B(0) e_C \cap (Z^0_C + I_{r+1}) = m_1 Z^0_C \cap (Z^0_C + I_{r+1})$.

Therefore, the quotient $Z^0_C / m_1 Z^0_C$, viewed as a vector space over $\mathbb{Q}(\delta)$, is isomorphic to $(Z^0_C + m_1 B(0) + I_{r+1}) / (m_1 B(0) + I_{r+1})$. But this algebra is readily seen to be a quotient of a subalgebra of the group algebra over $\mathbb{Q}(\delta)$ by the stabilizer in $W(M)$ of the set $C \in B$, for the image of $\{ \tilde{w} \mid w \in W \}$ modulo $m_1 B(0)$ is the group $W$ and the image of the algebra $Z^0_C$ is generated by the products of the elements of the form $w_{j_1} v_{k_1} w_{j_1} \cdots w_{j_t} v_{k_t} w_{j_t}$ as described in Definition 4.2.1, all of which are contained in $W_C$, the stabilizer in $W$ of $C$.

Consequently, the dimension of $Z^0_C / m_1 Z^0_C$ over $\mathbb{Q}(\delta)$ is at most $|W_C|$, the order of the stabilizer in $W$ of $C \in B$ (a group conjugate to the stabilizer in $W$ of $C$).

By Lemma 4.4.5, applied with $F = \mathbb{Q}(\delta, l)$, $E = \mathbb{Q}(\delta)[l^\pm]$, $V = Z^0_C$, $V^{(0)} = Z^0_C$, and $n = m_1$, we see that $Z^0_C$ has dimension at most $|W_C|$, over $\mathbb{Q}(l, \delta)$. □

**Remarks 4.4.7.** (i). By Corollary 4.3.6 the statement above holds for $I_1 / I_2$ and the intended upper bound for the dimension of this quotient can therefore be established. In Chapter 6 we will show that the upper bound is met for $I_1 / I_2$.

(ii). Recall that we defined $n!!$ to be the product of the first $n$ odd integers. Consider Table 2 on page 38. When the above conjecture holds, we find the following upper bounds for the dimension of the ideal $I_1$ for $B(M)$ over $\mathbb{Q}(l, \delta)$.

\[
\begin{align*}
(n + 1)!! - (n + 1)! & \quad \text{if } M = A_n, \\
(2^n + 1)(n!! - n!) & \quad \text{if } M = D_n, \\
1, 388, 745 & \quad \text{if } M = E_6, \\
436, 766, 985 & \quad \text{if } M = E_7, \\
52, 631, 339, 625 & \quad \text{if } M = E_8.
\end{align*}
\]

The result for $M = D_n$ is derived by use of $\sum_t \frac{t^{(n+1)(n+1)}}{2^{2t} t! t! (n - 2t)!} = (n + 1)!! - (n + 1)!$, the result found for $B(A_n)$.

For $I_1$ of type $D$ observe that we have the summations

\[
\begin{align*}
\sum_t (2^{n-2t} (n - 2t)!) \left( \frac{n!}{t! (n - 2t)!} \right)^2 & = 2^n \sum_t \frac{n! n!}{2^{2t} t! t! (n - 2t)!}, \\
\sum_t (n - 2t)! \left( \frac{n!}{2^{2t} t! t! (n - 2t)!} \right)^2 & = \sum_t \frac{n! n!}{2^{2t} t! t! (n - 2t)!}.
\end{align*}
\]

Together these two summations are precisely $2^n + 1$ times the value we found for $I_1$ of type $A_{n-1}$.  


(iii). Finally, this would lead to the following upper bound for the dimension of the BMW algebra $B(M)$ over $\mathbb{Q}(l, \delta)$.

\[(n + 1)!! \text{ if } M = A_n,\]
\[(2^n + 1)(n - 1 + 1)!! \text{ if } M = D_n,\]
\[1,440,585 \text{ if } M = E_6,\]
\[439,670,025 \text{ if } M = E_7,\]
\[53,328,069,225 \text{ if } M = E_8.\]

4.5. About the conjecture

In this section we discuss the open problems which prevented us from proving the conjecture. We introduce three properties of the admissible orbit and the elements of the proposed spanning set as hypotheses. We sketch a step by step conceivable proof of Conjecture 4.4.4 under the assumption that the mentioned hypotheses hold.

**Definition 4.5.1.** Define the following three hypotheses.

(i) **Poset hypothesis**: $J_{B,C} = \emptyset \implies \exists \ i \in M \alpha_i \in B$.

(ii) **Well behaved hypothesis**: Let $B \in B$. Then $w_{B,C}e_C$ behaves well, that is, if the Weyl group elements $w_1$ and $w_2$ of shortest length taking $C$ to $B$ then $\hat{w}_1e_C$ is equal to $\hat{w}_2e_C$ up to $m$ times elements $w_{B',C}e_C$, with $d(B',C) < d(B,C)$.

(iii) **Fundamental root hypothesis**: Let $B \in B$ with $\alpha_i \in B$. Then

$$w_{B,C}e_C = \delta^{-1}e_iw_{B,C}e_C.$$ 

For every BMW algebra $B(M)$, the three hypotheses hold in $I_1/I_2$, the single root case. They also have been checked to hold for any quotient ideal $I_C/I_{r+1}$ in BMW algebras of simply laced type $M$ of rank $n \leq 5$. In the remainder of this section a sketch of a step by step proof of Conjecture 4.4.4 is discussed under the assumption that these three hypotheses hold.

**Lemma 4.5.2.** Assume the poset hypothesis holds. Let $B, C \in B$ with $C$ fundamental. Then the set $J_{B,C}$ has the following properties.

(i) $J_{B,C} = \emptyset \iff B = C$,

(ii) $B$ is minimal $\implies J_{B,C} = \emptyset$,

(iii) $\beta \in B$ of smallest height and $\beta$ is not fundamental $\implies J_{B,C} \neq \emptyset$.

**Proof.** (i). If $J_{B,C} = \emptyset$, then there is no reflection lowering the distance from $B$ to $C$. This is not possible unless $d(B,C) = 0$, hence $B = C$. It is also directly clear that $J_{C,C} = \emptyset$.

(ii). When $B$ is minimal, $r_iB > B$ for all reflections $r_i$ which move $B$, so clearly $J_{B,C} = \emptyset$.

(iii). Restatement of the poset hypothesis.

**Remark 4.5.3.** The fundamental root hypothesis leaves two possibilities when $\alpha_k \in B$. It is possible that $e_k$ occurs in the product $e_C$ and commutes with $w_{B,C}$. Hence, $w_{B,C}e_C = w_{B,D}e_De_k$ with $C = D \cup \{k\}$. Here $w_{B',D}e_D$ is part of the spanning set.
of an ideal \( I_D \) with \(|D| < |C|\). As most steps of the proof use induction on the size \( r \) of the coclique, this observation is sufficient to cover the proof for this particular case.

The other possibility is that \( e_k \) does not occur in the product \( e_C \). Hence, by the fundamental root hypotheses, there has to be a node \( j \in M \) with \( k \sim j \) and

\[
w_{B,C}e_C^j = \delta^{-1} e_ke_j w_{r_t r_j B, C} e_C = e_k w_{r_t r_j B, C} e_C.
\]

These observations lead to the following step in the proof.

**Lemma 4.5.4.** Assume the three hypotheses of Definition 4.5.1 hold. Let \( B \in \mathcal{B} \) with \( \text{ht}(B) = 0 \) and \( \alpha_i \not\in B^\perp \). Then the action of \( g_i \) and \( e_i \) on \( w_{B,C} e_C \) is as follows.

\[
g_i w_{B,C} e_C = \begin{cases} 
  l^{-1} w_{B,C} e_C & \text{if } \alpha_i \in B, \\
  w_{r_t B, C} e_C & \text{if } d(r_t B, C) > d(B, C), \\
  w_{r_t B, C} e_C - mw_{B,C} e_C & \text{if } d(r_t B, C) < d(B, C), \\
  +mw_{r_t r_j B, C} e_C & \text{if } d(r_t B, C) = d(B, C).
\end{cases}
\]

and

\[
e_i w_{B,C} e_C = \begin{cases} 
  \delta w_{B,C} e_C & \text{if } \alpha_i \in B, \\
  w_{r_t r_j B, C} e_C & \text{if } \alpha_j \in B \text{ with } i \sim j.
\end{cases}
\]

**Proof.** We prove the lemma by induction on the distance from \( B \) to \( C \).

Assume \( B = C \). When \( \alpha_i \in C \) we have \( e_C = \delta^{-1} e_i e_C \) by the fundamental root hypothesis and the action of \( g_i \) and \( e_i \) follows from the defining relation (R1) and the idempotent relation \( e_i^2 = \delta e_i \) respectively.

When \( r_t C > C \), there is a node \( j \sim i \) such that \( e_j \) occurs in the product \( e_C \). Now \( r_t C \) contains \( \alpha_j + \alpha_i \) and we find \( g_i e_C = w_{r_t \gamma C} e_C \). By the relation \( e_i e_j = g_j g_i e_j \) we find \( e_i e_C = w_{r_t r_j C} e_C \) with \( \alpha_i \in r_t r_j C \). So the lemma holds for \( B = C \).

Next assume the lemma holds for every fundamental set \( B' \) at distance less than \( t \) from \( C \), and let \( B \in \mathcal{B} \) with \( d(B, C) = t \). Again, when \( \alpha_i \in B \) we have \( w_{B, C} e_C = \delta^{-1} e_i w_{B,C} e_C \) and the action of \( g_i \) and \( e_i \) follows.

Suppose \( d(r_t B, C) > d(B, C) \). As \( B \) is minimal, we have \( r_t B > B \) and thus \( g_i w_{B,C} e_C = w_{r_t B, C} e_C \). Also, for some \( k \in M \), we have \( w_{B, C} e_C = \delta^{-1} e_k w_{B,C} e_C \). If \( e_k w_{B,C} = w_{B, C} e_k \), we must have \( i \sim k \) and we find, for the action of \( e_i \),

\[
e_i w_{B,C} e_C = \delta^{-1} e_i e_k w_{B,C} e_C = \delta^{-1} g_k g_i e_k w_{B,C} e_C = w_{r_t r_i B, C} e_C,
\]

where \( \text{ht}(r_t r_i B) = \text{ht}(B) \).

If \( e_k w_{B,C} \neq w_{B,C} e_k \), there is a node \( h \sim k \) such that \( w_{B,C} e_C = e_k w_{r_h r_k B, C} e_C \).

When \( i \sim k \) we find

\[
e_i w_{B,C} e_C = e_i e_k w_{r_h r_k B, C} e_C = g_k g_i e_k w_{r_h r_k B, C} e_C = w_{r_t r_i B, C} e_C.
\]

When \( k \sim i \) and \( h \sim i \) it gives

\[
e_i w_{B,C} e_C = e_k e_i w_{r_h r_k B, C} e_C
\]

and the induction step proves the statement.

When \( h \sim i \) there needs to be an \( \alpha_h' \in r_h r_k B \) with \( k \sim h' \) or \( i \sim h' \), for otherwise \( \alpha_i \in B^\perp \). Replace \( h \) by \( h' \) and we get one of the two previous cases.
Finally, assume \(d(r, B, C) < d(B, C)\). Again \(r, B > B\), so
\[
w_{B, C}e_C = \delta^{-1}e_e w_{r, r, B, C} e_C
\]
for some \(j \sim i\). Now, with \(g, i, j, e_i = g, j, e_i - me_j e_i + me_i\), we find
\[
g_i w_{B, C} e_C = g_i w_{r, r, B, C} e_C - mw_{B, C} e_C + mw_{r, r, B, C} e_C
\]
where \(g_i w_{r, r, B, C} e_C = w_{r, r, B, C} e_C\) with \(ht(r, B) = 1\). Here
\[
e_i w_{B, C} e_C = \delta^{-1}e_i e_j w_{r, r, B, C} e_C = w_{r, r, B, C} e_C,
\]
which completes the proof. \(\square\)

The next step of the proof involves the action of \(g_i\) and \(e_i\) on \(w_{B, C} e_C\) where \(B\) is fundamental and \(\alpha_i \in B^+\).

**Lemma 4.5.5.** Assume the three hypotheses of Definition 4.5.1 hold. Let \(B \in B\) with \(ht(B) = 0\) and \(\alpha_i \in B^+\). Then
\[
g_i w_{B, C} e_C, e_i w_{B, C} e_C \in w_{B, C} Z^{(0)} + I_{r+1}.
\]

**Proof.** We prove the lemma, first by induction on the size of the coclique \(C\), second by induction on the distance from \(B\) to \(C\).

Lemma 4.3.2 proves the lemma for \(|C| = 1\). Assume the lemma holds for any \(w_{B, C} e_D\) with \(|D| < r\) and assume we have a coclique \(C\) of size \(r\).

Assume \(B = C\). As \((\alpha_i, \alpha_j) = 0\) for all \(\alpha_j \in C\) we have both \(g_i\) and \(e_i\) commuting with \(e_C\). This gives \(g_i e_C = e_C g_i \in Z^{(0)}\) and \(e_i e_C \in I_{r+1}\) as \(e_i e_C\) is a product of \(r + 1\) commuting \(e_i\)’s.

Assume the lemma holds for any set \(B' \in B\) with \(ht(B') = 0\) and \(d(B, C) < t\). Let \(B \in B\) with \(ht(B) = 0\) and \(d(B, C) = t\). Notice \(e_i = l^{-1}(g_i^2 + mg_i - 1)\), so it suffices to prove the lemma for the action of \(g_i\) on \(w_{B, C} e_C\).

As \(B\) is minimal, there is a node \(k \in M\) with \(w_{B, C} e_C = \delta^{-1}e_k w_{B, C} e_C\). Obviously \(g_i\) commutes with \(e_k\). By Remark 4.5.3, there is a node \(j \in M\) with \(j \sim i\) and \(w_{B, C} e_C = e_k w_{r, r, r, B, C} e_C\). When \(\alpha_i \in r r_j B^+\) as well, induction gives immediately the desired result.

Suppose \(\alpha_i \notin r r_j B^+\). Then we have \(j \sim i\). Two options remain,
\[
\]
with \(i = h\) and \(j \sim k\). For \(i \neq h\) with \(i, h, k\) adjacent to \(j\). In the latter case, the node \(j\) has valency \(3\), so \(w_{r, r, r, B, C} = w_{r, h}\) with \(h'\) the node closest to \(h\) with \(\alpha_{h'} \in C\). Hence \(w_{B, C} e_C = w_{h' k} e_C\) and
\[
g_i w_{B, C} e_C = w_{h' k} w_{h} g_i w_{h' k} e_C \in w_{h' k} Z^{(0)}.
\]

For the other case we apply the rewrite rules \(g_i e_j e_i = g_j e_i - me_j e_i + me_i\) and \(e_k g_j = e_k e_j g_k + me_k e_j - me_k\) and find
\[
g_i w_{B, C} e_C = e_k e_j g_k w_{r, r, r, r, B, C} e_C
\]
In the next lemma a general description of the action of simple reflection and by induction we find, as where the root 

Next assume the lemma holds for any root \( r_j \) with these properties is \( \alpha_j + \alpha_i + \alpha_k \) with \( j \sim i \sim k \). Here \( \alpha_i + \alpha_k \in r_j B, \) so we can write \( g_j g_k e_i \) as the most left part of \( w_{B,C} e_C \). Now \( J_{B,C} = \{ j, k \} \), and therefore \( w_{B,C} e_C = g_j g_k \hat{w}_h e_C \) for \( h \in M \) the node closest to \( i \) on a path not containing \( j \) or \( k \). So \( \alpha_i \in C \) or \( i \) is the node in the Coxeter diagram of type \( D \) or \( E \) with valency \( 3 \). This means \( \hat{w}_h e_C = e_D \hat{w}_h \) where \( D \) is the set \( C \) with \( h \) replaced by \( i \). As

\[
g_k g_j g_i g_k e_i = e_i g_j g_k e_i
\]

we find,

\[
w_{B,C}^{op} g_i w_{B,C} e_C = \hat{w}_{hi} g_k g_j g_i g_k \hat{w}_{hi} e_C = e_C \hat{w}_{hi} g_k g_j g_i g_k \hat{w}_{hi} e_C \in Z_C^{(0)}.
\]

Next assume the lemma holds for any root \( \beta \) of height less than \( t \). Consider a \( B \) where the root \( \beta \in B \) is of smallest height such that \( \text{ht}(\beta) = t \) and assume the simple reflection \( r_j \), with \( j \sim i \), lowers this root. Clearly \( r_i (\beta - \alpha_j) = \beta - \alpha_j - \alpha_i \) and by induction we find, as \( \alpha_j \in r_i r_j B^\perp, \)

\[
w_{B,C}^{op} g_i w_{B,C} e_C = w_{r_i r_j B, C}^{op} g_j g_i g_j \hat{w}_{r_i r_j B, C} e_C = w_{r_i r_j B, C}^{op} g_j g_i g_j \hat{w}_{r_i r_j B, C} e_C = w_{r_i r_j B, C}^{op} g_j \hat{w}_{r_i r_j B, C} e_C = e_C w_{r_i r_j B, C}^{op} g_j \hat{w}_{r_i r_j B, C} e_C
\]

In the next lemma a general description of the action of \( g_i \) and \( e_i \) on any element \( w_{B, C} e_C \) is given.
Lemma 4.5.7. Assume the three hypotheses of Definition 4.5.1 hold. Let $B \in \mathcal{B}$ with $d(B, C) > 0$. Then

$$g_i w_{i, B, C} e_C = \begin{cases} 
    l^{-1} w_{B, C} e_C & \text{if } \alpha_i \in B, \\
    w_{r_i, B, C} e_C - mw_{B, C} e_C & \text{if } i \in J_{B, C}, \\
    +ml^{-1} e_i w_{r_i, B, C} e_C & \text{if } d(r_i B, C) > d(B, C), \\
    w_{r_i, B, C} e_C & \text{if } \alpha_i \in B^1 \text{ and } j \notin J_{B, C}, \\
    g_j g_i w_{r_i, r_j, B, C} e_C & \text{if } \alpha_i \in B^1 \text{ and } j \notin J_{B, C}, \\
    g_j g_i w_{r_i, r_j, B, C} e_C & \text{if } \alpha_i \in B^1 \text{ and } j \notin J_{B, C}.
\end{cases}$$

and

$$e_i w_{i, B, C} e_C = \begin{cases} 
    l^{-1} w_{r_i, B, C} e_C & \text{if } \alpha_i \in B, \\
    \delta w_{i, B, C} e_C & \text{if } i \in J_{B, C},
\end{cases}$$

and when $\alpha_i \notin B$ and $i \notin J_{B, C}$ then

$$e_i w_{i, B, C} e_C = \begin{cases} 
    g_j g_i w_{r_i, B, C} e_C & \text{if } \exists j \in J_{B, C} \ni j \neq i, \\
    e_i (e_j g_i - mg_i + m) w_{r_i, B, C} e_C & \text{if } \forall j \in J_{B, C} \ni j \neq i, \\
    \delta^{-1} e_i w_{i, B, C} e_C & \text{if } J_{B, C} = \emptyset \text{ and } j \sim k \neq i, \\
    \delta^{-1} e_i w_{r_i, B, C} e_C & \text{if } J_{B, C} = \emptyset \text{ and } j \sim k \sim i,
\end{cases}$$

Proof. We discuss the lemma case by case.

(i). $\alpha_i \in B$. Then, by the fundamental root hypothesis, $w_{B, C} e_C = \delta^{-1} e_i w_{B, C} e_C$ hence $g_i w_{B, C} e_C$ and $e_i w_{B, C} e_C$ are equal to $l^{-1} w_{B, C} e_C$ and $\delta w_{B, C} e_C$ respectively by the defining relation (R1) and the idempotent relation $e_i^2 = \delta e_i$.

(ii). $i \in J_{B, C}$. Now $w_{B, C} e_C = g_i w_{r_i, B, C} e_C$ and the results follow from $g_i^2 = 1 - mg_i + ml^{-1} e_i$ and $e_i g_i = l^{-1} e_i$.

(iii). $d(r_i B, C) > d(B, C)$. By definition now $g_i w_{B, C} e_C = w_{r_i, B, C} e_C$.

(iv). $\alpha_i \in B^1$. As $d(B, C) > 0$ we have a $j \in J_{B, C}$.

When $i \sim j$, we have $g_i g_j = g_j g_i$ and the result follows.

When $i \sim j$ and $i \in J_{r_j B, C}$ we have $w_{B, C} e_C = g_j g_i w_{r_i, r_j, B, C} e_C$ and the result follows from the braid relation (B1).

Finally when $i \sim j$ and $i \notin J_{r_j B, C}$ for all $j \in J_{B, C}$, we can write

$$g_i w_{B, C} e_C = w_{B, C} e_C g_i w_{B, C} e_C$$

where $w_{B, C} e_C g_i w_{B, C} e_C = e_C w_{B, C} e_C g_i w_{B, C} e_C$ by lemma 4.5.6.

This completes the proof of the lemma for the action of $g_i$. For the action of $e_i$ remains the case where $\alpha_i \notin B$ and $i \notin J_{B, C}$. We consider again several options.

(v). When there is a $j \in J_{B, C}$ with $i \sim j$ then obviously $g_j$ and $e_i$ commute, hence $e_i w_{B, C} e_C = g_j e_i w_{r_i, B, C} e_C$.

(vi). When there is a $j \in J_{B, C}$ with $i \sim j$ and $i \in J_{r_j B, C}$ then
We sketch a proof of the statements of Conjecture 4.4.4 by induction, first on the size of \( C \) and second on the height of a set \( B \) and third on the distance of \( B \) to \( C \).

We end this section with a sketch of the final step of the proof of Conjecture 4.4.4 under the assumption that the three hypotheses of Definition 4.5.1 hold. Along these lines and by use of the previous steps we obtain a conceivable proof for the conjecture. This implies that is sufficient, for the conjecture to hold, to prove the three hypotheses need to be proven.

We sketch a proof of the statements of Conjecture 4.4.4 by induction, first on the size \( r \) of \( C \), second on the height of a set \( B \) and third on the distance of \( B \) to our fundamental set \( C \).

For the ideal \( I_1/I_2 \) generated by \( e_C \) with \( |C| = 1 \) a spanning set was already constructed in § 4.3. It is readily verified that this spanning set and the spanning set of the conjecture for \( r = 1 \) are the same.

For \( \text{ht}(B) = 0 \) and for \( B \) at any distance of \( C \), statement (24) holds by Lemma 2.4.1 and the other three are proven in Lemma 4.5.4 and Lemma 4.5.5.

Assume the theorem holds for every \( w_{B',C}e_C \) where \( e_C \) is the product of less than \( r \) elements, for every \( w_{B',C}e_C \) where \( e_C \) is the product of \( r \) elements and \( \text{ht}(B') < k \) and for every \( w_{B',C}e_C \) with \( |C| < r \) and \( \text{ht}(B') = k \) but \( d(B',C) < t \).

Now consider \( w_{B,C}e_C \) with \( |C| = r \) and \( B \in B \) such that \( \text{ht}(B) = k \) and \( d(B,C) = t \). When \( e_k \) occurs in the product \( e_C \) such that \( w_{B,C}e_k = e_k w_{B,C} \), the theorem follows for such a \( w_{B,C}e_C \) by our induction step as mentioned in Remark 4.5.3. So for the remainder of the proof we can assume no such elements occur in the product \( e_C \). In particular, when \( J_{B,C} = \emptyset \) and \( B \not\sim C \), we have, by the poset hypothesis

\[
e_i w_{B,C}e_C = e_i e_j w_{r,r,B,C} e_C
\]

as \( e_i g_j g_i = e_i e_j \).

(vii). When there is a \( j \in J_{B,C} \) with \( i \sim j \) but \( i \notin J_{r,B,C} \) we substitute the relation \( e_i g_j = e_i e_j g_i + me_i e_j - me_i \).

(viii). Finally, when \( J_{B,C} = \emptyset \) we have, by the poset hypothesis and the fundamental root hypothesis,

\[
w_{B,C}e_C = \delta^{-1} e_k w_{B,C}e_C.
\]

When \( k \sim i \) the two elements commute and we get \( e_i w_{B,C}e_C = \delta^{-1} e_k e_i w_{B,C}e_C \).

When \( k \not\sim i \) we get

\[
e_i w_{B,C}e_C = \delta^{-1} g_k g_i e_k w_{B,C}e_C = w_{r,r,B,C}e_C
\]

with \( \alpha_i \in r_k r_i B \).

\[\square\]

**Lemma 4.5.8.** Assume the three hypotheses of Definition 4.5.1 hold. Let \( \alpha_j \in B \) and \( i \in M \) with \( i \neq j \). Then

\[
e_i w_{B,C}e_C \in \sum_{B' \in H \cap H_i} w_{B',C}e_C + I_{r+1}
\]

where \( H_k = \{ B \in B \mid \alpha_k \in B \} \).

**Proof.** Straightforward from the equations in Lemma 4.5.7. Proof by induction on first the size of \( C \) and second on the distance of \( B \) to \( C \). \[\square\]
and the fundamental root hypothesis, that there are nodes \( k, j \in M \) such that \( w_{B, C}eC = e_kw_{r_j, r}B, C eC \). Also notice, as \( t > 0 \) we always have \( J_{B, C} \neq \emptyset \).

We start with \( w_{B, C}^{-op}eC \). Assume \( j \in J_{B, C} \).

Write \( w_{B, C}^{-op}eC = g_j^{-1}w_{r_j, B, C}eC \). Substitution of \( g_j^{-1} = g_j + m - m \alpha_j \) gives

\[
g_jw_{r_j, B, C}eC + mw_{r_j, B, C}eC - me_jw_{r_j, B, C}eC.
\]

Here \( \text{ht}(r_jB) = k - 1 \), so our induction step gives

\[
w_{r_j, B, C}^{-op}eC \in w_{r_j, B, C}eC [0] + \sum_{B' \in H_{r_jB}^{-1}} w_{B'}eC [0] + I_{r+1},
\]

and the statement for the first two terms follows immediately. Applying induction for the action of \( e_j \) completes this part of the proof.

Assume \( J_{B, C} = \emptyset \). By the poset hypothesis and the fundamental root hypothesis, there is a node \( k \sim j \), such that \( w_{B, C}eC = e_kw_{r_k, B, C} \) and \( w_{B, C}^{-op}eC = e_kw_{r_k, B, C}^{-op} \) as \( g_j^{-1}g_k^{-1}e_j = e_k e_j \). Obviously \( \text{ht}(r_k r_jB) = k \) but \( d(r_k r_jB, C) < t \) and by our induction step,

\[
w_{B, C}^{-op}eC \in e_kw_{r_k, B, C}eC [0] + \sum_{B' \in H_{r_kB}^{-1}} e_kw_{B'}eC [0] + I_{r+1}.
\]

This is sufficient to prove the statement as the induction step on every \( e_kw_{B', C} \) with \( B' \in H_{r_kB}^{-1} = H_{B}^{-1} \) gives the desired result.

Next consider \( g_jw_{B, C}eC \) and \( e_iw_{B, C}eC \). We distinguish the following cases.

(i). \( \alpha_i \in B \). Straightforward from the rewrite rules in Lemma 4.5.7.

(ii). \( i \in J_{B, C} \). Here \( g_jw_{B, C}eC = (1 - mg_i + ml^{-1}e_i)w_{r_k, B, C}eC \). The first two terms are automatically contained in \( \sum_{B' \in H_{r_kB}^{-1}} w_{B'}eC [0] + I_{r+1} \). So the statement for the action of \( g_i \) holds when the action for \( e_i \) does.

For \( e_iw_{B, C}eC \) we get \( l^{-1}e_iw_{r_j, B, C}eC \) when \( i \in J_{B, C} \). Now \( \text{ht}(r_iB) < B \) and with induction this gives

\[
l^{-1}e_iw_{r_j, B, C}eC \in \sum_{B' \in H_{r_iB}^{+} \cap H_{i}} w_{B'}eC + I_{r+1}
\]

where every \( B' \in H_{r_iB}^{0} \cap H_{i} \) obviously is also contained in \( H_{B}^{0} \cap H_{i} \).

Suppose \( i \in J_{B, C} \). Now we have \( \alpha_j \in B \) with \( i \sim j \) or \( k \in J_{B, C} \) with \( k \neq i \).

In the first case we find

\[
e_iw_{B, C}eC = \delta^{-1}e_i e_j e_i w_{r_j, B, C}eC = w_{r_j, B, C}eC \in \sum_{B' \in H_{r_jB}^{0} \cap H_{i}} w_{B'}eC + I_{r+1}
\]

with \( \text{ht}(r_j r_iB) = \text{ht}(B) \).

In the second case, we have, by induction,
\[ e_i w_{B,C} e_C = g_k e_i w_{r_k B, C} e_C \]
\[ = \sum_{B' \in H_{r_k B}^0 \cap H_i} g_k w_{B', C} e_C + I_{r+1} \]
\[ \in \sum_{B' \in H_{r_k B}^0 \cap H_i} w_{B', C} e_C + I_{r+1}. \]

(iii). \( \alpha_i \in B^\perp \). Recall \( e_i = ml^{-1}(g_i g_1 + mg_i - 1) \), so when the statement holds for \( g_i w_{B,C} e_C \), we automatically find
\[ e_i w_{B,C} e_C \in w_{B,C} Z_C^{(0)} + \sum_{B' \in H_{r_k B}^0} w_{B', C} Z_C^{(0)} + I_{r+1}. \]
So we only have to check the action of \( g_i \) for \( \alpha_i \in B^\perp \).

First assume for all \( j \in J_{B,C} \) we have \( i \sim j \) and \( i \not\sim J_{r_j B, C} \). In this case we have
\[ g_i w_{B,C} e_C = w_{B,C} e_i w_{r_j B, C} g_i w_{r_j B,C} \]
which is, by induction, contained in
\[ w_{B,C} Z_C^{(0)} + \sum_{B' \in H_{r_k B}^0} w_{B', C} Z_C^{(0)} + I_{r+1} \]
as we have shown above.

Next assume \( J_{B,C} \neq \emptyset \). Recall \( \alpha_i \notin B^\perp \). The rewrite rules given in Lemma 4.5.7 for when \( \alpha_i \in B^\perp \)
show the statement follows immediately from applying the induction step in all cases except the single case we already handled first.

Assume \( J_{B,C} = \emptyset \), so again, by the poset hypothesis and the fundamental root hypothesis, we have \( w_{B,C} e_C = e_k w_{r_k r_j B, C} \) with \( k \sim j \). Clearly \( k \not\sim i \) as \( \alpha_i \in B^\perp \).

When also \( j \not\sim i \) we find
\[ g_i w_{B,C} e_C = e_k g_i w_{r_k B, C} e_C = e_k w_{r_k B, C} Z_C^{(0)} + \sum_{B' \in H_{r_k B}^0} e_k w_{B', C} e_C Z_C^{(0)} + I_{r+1} \]
which is, by induction, contained in
\[ w_{B,C} Z_C^{(0)} + \sum_{B' \in H_{r_k B}^0} w_{B', C} Z_C^{(0)} + I_{r+1}. \]

Finally, when \( i \sim j \), we have \( i \in J_{r_j B, C} \). Now \( r_j r_j B < r_j B \) and \( r_j r_j B < r_j B \). By the poset relations this means that \( \alpha_i \in B \), which contradicts \( \alpha_i \in B^\perp \).

(iv). \( d(r_k B, C) > d(B, C) \). For the action of \( g_i \) the statement follows immediately by Lemma 4.5.7. Remains to check this case for the action of \( e_i \).

Suppose \( J_{B,C} = \emptyset \). Now, by the poset hypothesis and the fundamental root hypothesis, \( w_{B,C} e_C = e_k w_{r_k r_j B, C} e_C \) for some \( k \sim j \). When \( i \sim k \) we have \( e_i e_k w_{r_k r_j B, C} e_C = w_{r_k r_j B, C} e_C \) with \( \alpha_i \in r_k r_j B \) and \( \text{ht}(r_k r_j B) = \text{ht}(B) \).

When \( i \not\sim k \), we have \( e_i w_{B,C} e_C = \delta^{-1} e_k e_i w_{r_k r_j B, C} e_C \). We can assume \( i \not\sim j \), as otherwise \( r_i \) would not move \( B \) or there would be a \( j' \) with \( e_i w_{B,C} e_C = \delta^{-1} e_k e_i e_j w_{r_k r_j B, C} e_C \) and \( i \not\sim j \). We find by induction and Lemma 4.5.8,
\[ e_i w_{r_k r_j B, C} e_C \in \sum_{B' \in H_{r_k r_j B, C} \cap H_i} w_{B', C} e_C + I_{r+1}. \]
As \( \alpha_j \in B' \) for every set \( B' \) in \( H^0_{r_jB,C} \cap H_1 \cap H_j \), the action by \( e_k \) now gives

\[
e_k w_{B,C} e_C = \sum_{B' \in H^0_{r_jB,C} \cap H_1} w_{B',C} e_C + I_{r+1}.
\]

This completes this part of the proof as clearly \( H^0_{B,C} \cap H_1 \cap H_k \subseteq H^0_{B,C} \cap H_1 \).

Finally suppose \( J_{B,C}^- \neq \emptyset \). When there is a \( j \in J_{B,C}^- \) with \( i \neq j \), we find after applying induction

\[
e_i w_{B,C} e_C = g_j e_i w_{r_jB,C} e_C
\]

and, as \( i \in J_{r_jB,C} \), we find

\[
e_i e_j w_{r_jB, C} e_C = e_i e_j w_{r_jB, C} e_C = e_i e_j w_{r_jB, C} e_C
\]

with \( \text{ht}(r_jB) \leq \text{ht}(B) \). Now by induction

\[
e_j w_{r_jB, C} e_C \in \sum_{B' \in H^0_{r_jB,C} \cap H_j} w_{B',C} Z_c^{(0)} + I_{r+1}
\]

and, as \( \alpha_j \) in every set,

\[
e_i e_j w_{r_jB, C} e_C \in \sum_{B' \in H^0_{r_jB,C} \cap H_j} w_{B',C} Z_c^{(0)} + I_{r+1}
\]

This leaves the possibility that for all \( j \in J_{B,C}^- \) we have \( i \neq j \) and \( i \not\in J_{r_jB,C}^- \). If \( \alpha_i \in r_jB^- \), we can substitute \( e_i g_j = e_i e_j g_i + me_i e_j - me_i \) and the statement follows straightforward by applying the induction step on \( g_i w_{r_jB, C} e_C \).

When \( r_jB < B \), we have \( \alpha_i \in r_jB \) as there are no \( k \in J_{B,C}^- \) with \( k \neq i \). Hence,

\[
e_i w_{B, C} e_C = e^{-1}_i g_j e_i w_{r_jB, C} e_C = l w_{r_jB, C} e_C.
\]

When \( r_jB < B \), substitute \( e_i g_j = e_i e_j g_i + me_i e_j - me_i \) to find

\[
e_i e_j g_i w_{r_jB, C} e_C + me_i e_j w_{r_jB, C} e_C - me_i w_{r_jB, C} e_C.
\]

By induction the last two terms give the desired result. For the first observe we have \( g_i w_{r_jB, C} e_C = w_{r_jB, C} e_C \) by Lemma 4.5.7 with \( \text{ht}(r_jB) = \text{ht}(B) - 2 \). Now induction gives also the desired result here.

This leaves one option, \( r_jB > r_jB \). We prove this last case by induction on the minimal height of the roots moved by \( r_j \) and \( r_i \). Let \( \beta, \gamma \in r_jB \) such that \( \beta \) is the
smallest root in $r_j B$ moved by $r_i$ and $\gamma$ is the smallest root in $r_j B$ moved by $r_j$.

Obviously $\beta \neq \gamma$ as then $(\alpha_i, \beta + \alpha_j) = -2$.

When $\beta$ or $\gamma$ is a fundamental root, say $\alpha_k$, we have

$$ w_{r_j B, C} e_C = \delta^{-1} e_k w_{r_j B, C} e_C. $$

As $j \neq k$ or $i \neq k$ we have $\alpha_k \in B$ or $\alpha_k \in r_i r_j B$ as well.

When $j \neq k$ we have straightforward

$$ e_i w_{r_j B, C} e_C = \delta^{-1} e_i e_k w_{r_j B, C} e_C = \delta^{-1} g_k g_i e_k w_{r_j B, C} e_C = w_{r_k B, r_j B} e_C $$

with $\alpha_i \in r_t r_j B$ and $\text{ht}(r_k r_j B) = \text{ht}(B)$.

When $i \neq k$ we can proceed in a similar way after substitution of $e_i g_j = e_i e_j g_i + me_i e_j - me_i$.

Now $g_i w_{r_i r_j B, C} e_C = w_{r_i r_j B, C} e_C$ and

$$ e_i e_j g_i w_{r_i r_j B, C} e_C = w_{r_i r_j B, C} e_C $$

with $\alpha_i \in r_i r_j r_k r_j B$ and $\text{ht}(r_i r_j r_k r_j B) = \text{ht}(B)$. (The two additional terms are in $\sum B \in B, j \in B$ as easily is shown by applying the induction step.)

Next assume the statement holds for any pair of roots $\beta$ and $\gamma$ with minimal height less than $q$ and assume we have a $B$ where $\beta$ and $\gamma$ have a minimal height of $q$.

We saw that by substitution of $e_i g_j = e_i e_j g_i + me_i e_j - me_i$ we can interchange the roles played by $i$ and $j$ so without loss of generality we can assume $\beta$ is of minimal height $q$. Notice that every reflection lowering $r_j B$ has to be adjacent to either $i$ or $j$ by our initial assumptions.

There is reflection, say $r_k$, with $k \in J_{r_j B, C}^-$. We can assume $i \sim k$ (otherwise interchange the $i$ and $j$) and $i \notin J_{r_j B, C}$. Namely, if $i \in J_{r_j B, C}$ we find, using the braid relation,

$$ e_i e_j g_i g_k w_{r_i r_j r_k r_j B, C} e_C = e_i g_k g_i e_j g_k w_{r_i r_j r_k r_j B, C} e_C, $$

and the statement follows by induction.

The reflection $r_k$ lowers $\beta$ or a root $\beta'$ of smaller height as there has to be a reflection which does this and we ruled out reflections not adjacent to $i$ or $j$. Also $r_k$ can not increase $\beta$ as this would give $(\alpha_i, \beta + \alpha_k) = -2$.

So either $\beta - \alpha_k < \alpha_k$ or $\beta' - \alpha_k$ are in $r_i r_j B$. Consequently, the set $r_k B$ contains $\beta - \alpha_k$ or at most the root $\beta' - \alpha_k + \alpha_j$. ($r_j$ does not move $\beta$ or $\beta - \alpha_k$ as otherwise $r_j B < r_k B$.) As we had $\text{ht}(\beta') < \text{ht}(\beta)$, we see both these roots have a height less than $\beta$.

Now we can apply induction on the set $r_k B$. In the worst case, we have for $w_{r_k B, C} e_C$ the same conditions as we had for $r_j B$, for instance $\beta - \alpha_k \in r_i r_k B$ as well, but for $r_k B$ there is a root of smaller height involved. In all other cases we can apply one of the arguments we used prior to this last induction proof to verify the statement.

The substitutions of the form $e_i g_j = e_i e_j g_i + me_i e_j - me_i$ give three terms of which the last two give straightforward terms satisfying the lemma using the induction step. So we only consider $e_i g_j = e_i e_j g_i$ to prove the statement. Here

$$ e_i g_j g_k w_{r_i r_j r_k} B, C e_C = e_i e_k g_i g_j w_{r_i r_j r_k} B, C e_C = e_i e_k w_{r_i r_k B, C} e_C + I_{r+1} $$

where $r_k$ moves a root of height less than $t$, so by induction this gives
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Figure 1. The partial poset for the case $d(r_i, B, C) > d(B, C)$
where $j \in J_{\overline{B}, C}$ and $k \sim i \sim j$.

$$e_i e_k w_{r_i r_k B, C} e_C \in \sum_{B' \in H_{r_i r_k B, C \cap H_k}} e_{i'} w_{B', C} e_C + I_{r+1} \in \sum_{B' \in H_{i' r_k B, C \cap H_i}} w_{B', C} e_C + I_{r+1}.$$  

Hence, the induction step holds also in this case. This completes the sketch of the proof which implies that the conjecture holds when the three hypotheses of Definition 4.5.1 are true.
CHAPTER 5

A tangle algebra of type D

5.1. Introduction

In [MW], Morton and Wasserman described an explicit isomorphism between \( B(A_n) \) and \( KT(A) \). In [GH], Goodman and Hauschild gave a similar construction for affine BMW algebras and affine Kauffman tangle algebras.

Here we proceed in a similar way and define a tangle algebra \( KT(D) \) of type D. Although our tangle algebra differs from the algebras introduced in [GH], this work is partially inspired by that paper and the article by Morton and Wasserman. Our first result connects this tangle algebra to the BMW algebra of type D.

**Theorem 5.1.1.** There exists a surjective homomorphism \( \varphi : B(D_n) \to KT(D) \) defined by \( \varphi(g_i) = G_i \) and \( \varphi(e_i) = E_i \) for all \( i \).

The proof of this theorem is given at the end of §5.5. In §5.6 we introduce a variation of the Brauer algebra. The dimension of this algebra \( C(D_n) \) over \( \mathbb{Q}(\delta) \) is determined and a surjective homomorphism from \( KT(D) \) to \( C(D) \) leads to the main result of this chapter.

**Theorem 5.1.2.** The dimension of \( B(D_n) \), the BMW algebra of type \( D_n \), over \( \mathbb{Q}(l, \delta) \) is at least \((2^n + 1)n! - (2^{n-1} + 1)n!\).

It is our firm belief that \( B(D_n) \) and \( KT(D) \) are in fact isomorphic. A proof of the conjecture discussed in the previous chapter would be sufficient to establish this as a direct consequence of those results.

5.2. Tangles with a pole

Recall the definition of a tangle as a piece of a link diagram in \( \mathbb{R} \times I \), given in Definition 1.3.1 on page 10. In this chapter a new set of tangles is introduced which we will call tangles of type D. These diagrams feature an additional strand, called a pole, with properties different from the other, ordinary strands in the tangle. We start with a general definition of a tangle with a pole. To this end we fix, besides \( b_1, \ldots, b_n \), another point \( p \) on the same horizontal line as the points \( b_i \), such that \( p < b_1 \).

**Definition 5.2.1.** An \((n,n)\)-tangle with a pole is an \((n,n)\)-tangle which includes the distinguished line \( \{p\} \times I \), connecting \((p,1)\) with \((p,0)\) called the pole.

The role of this pole is intuitively clear. When we regard normal strands of a diagram as pieces of rope or rubber bands, we can treat the pole as an iron pipe or bar. It is a fixed strand which does not allow any interaction with the other strands. In this sense it is clear that deformation of the pole by planar isotopy is not allowed. Also Reidemeister moves I and III are not allowed when the pole is
any of the strands involved. This leaves only Reidemeister move II as a valid action with the pole involved. Here two consecutive under or over crossings of one strand with the pole can be removed without deformation of the pole itself.

The (one time) encircling of the pole by a normal strand of the tangle is called a twist \textit{(around the pole)}. See Figure 1. The pole is depicted as a bold vertical strand.

It makes sense to talk about the number of twists of a tangle, which denotes the total number of times all the normal strands in the diagram twist around the pole.

The interaction of the pole with the other strands is determined by its order. This is a relation regarding consecutive twists around the pole of a single strand. Here consecutive means that the involved strand does not cross any other strand between these twists. Nor are there any closed loops around the pole between these twists. The order of the pole is the number of consecutive twists which can be removed. If no such number exists, we say the pole has infinite order. Here we will only work with poles of order two, which implies the relation shown in Figure 2.

We now define tangles of type D.

\textbf{Definition 5.2.2.} An \((n, n)\)-tangle of type D is an \((n, n)\)-tangle with a pole of order two and an even number of twists around the pole.
As only isotopy of the plane is allowed which does not affect the pole, the tangle will contain only crossings (of its strands) at the right side of the pole. Moreover, all the twists around the pole in a tangle diagram are not nested. So when traversing the pole from top to bottom, the twists of a tangle are met one by one.

Like the Kauffman tangle algebra for the ordinary tangles, we define an algebra for the tangles of type D. Let \( \hat{U}(n) \) be the monoid of \((n, n)\)-tangles of type D modulo regular isotopy. As the tangles contain a pole of order two, the tangles in the monoid satisfy the double twist relation, as shown in Figure 3. To see this, just compose both sides of Figure 2 with the left hand side of Figure 3.

We introduce a family of tangle algebras \( KT^*_n \) over the ring

\[
E = \mathbb{Q}(\delta)[t^{\pm 1}],
\]

inside the field \( \mathbb{Q}(l, \delta) \).

**Definition 5.2.3.** The tangle algebra \( KT^*_n \) over \( E \) is the algebra constructed from the monoid algebra \( E[\hat{U}(n)] \) by factoring out the following relations:

(i) The Kauffman skein relation

\[
\begin{align*}
\begin{array}{ccc}
\includegraphics[width=0.2\textwidth]{skein_relation.png}
\end{array}
\end{align*}
\]

(ii) The commuting relation

\[
\begin{align*}
\begin{array}{ccc}
\includegraphics[width=0.2\textwidth]{commuting_relation.png}
\end{array}
\end{align*}
\]

(iii) The self-intersection relations
(iv) The idempotent relation

\[ T \cup O = \delta T, \]

where \( T \cup O \) is the union of a tangle \( T \) and a closed loop \( O \) having no crossings with \( T \) nor self-intersections or twists around the pole. Here, the pictures indicate tangles which differ only in the region shown.

For the remainder of this chapter, a crossing will be called positive if the strand moving from top right to bottom left crosses over the other strand; the opposite crossing will be called negative.

**Proposition 5.2.4.** Composition of tangles of type D induces a bilinear multiplication on \( KT_n^* \), making \( KT_n^* \) an algebra over \( E \).

**Proof.** The relations (i)-(iv) hold in \( E[\hat{U}(n)] \). □

In the next section a number of additional relations will be derived from the defining relations. These relations will prove to be extremely useful in the full understanding of these algebras as they give a clear insight in the interaction between the pole and the other strands of the tangles.

**5.3. Relations**

The defining relations of the tangle algebras \( KT_n^* \) give rise to a number of new relations concerning small regions of the tangle diagrams containing (a part of) the pole. In this section we derive them and use them to introduce two new components arising in the involved tangle diagrams, namely the preserved self-intersection and the closed loop around the pole.

The self-intersection relation shows that in general a self-intersection of a strand can be replaced by \( l \pm 1 \). This, however, is not the case when the strand twists around the pole before intersecting itself. This leads to the following self-intersection relations.

**Lemma 5.3.1.** The tangles in \( KT_n^* \) satisfy the following properties.

(i) The second self-intersection relation

\[
= l^{-1} \quad \text{and} \quad = l
\]

(ii) The third self-intersection relation
Here, the pictures indicate tangles which differ only in the region shown.

**Proof.** (i). Consider the (partial) diagram

When we apply the Kauffman skein relation to the top crossing we find the equality

Similarly, applying the Kauffman skein relation to the bottom crossing gives
As the left hand side of both equations is the same tangle, equality has to hold between the right hand sides. Using the commuting relation and the double twist relation it is clear that the third terms on both right hand sides are equal. Also the first terms are equal as we get

Here on the first and last step the commuting relation is used. The two steps in between are done by using the double twist relation and Reidemeister move II. This obviously implies equality between the remaining two terms. We apply the commuting relation followed by removing a double twist. Now, by the first self-intersection relation, the two terms give the desired result as shown below.
(ii). Consider the (partial) diagram

When we apply the Kauffman skein relation to the top crossing we find

Similarly, applying the Kauffman skein relation to the bottom crossing gives

As the left hand side of both equations is the same, equality has to hold between the right hand sides. Using the commuting relation it is not only clear that both third terms are equal, but also the first terms on either side.
The first and last step are justified by the commuting relation. The steps in between are both done by applying Reidemeister move II. These equalities obviously also establish equality between the remaining two terms. The commuting relation followed by the self-intersection relation completes the proof.

To avoid confusion, from now on we will refer to the self-intersection relation from Definition 5.2.3 (iii) as the first self-intersection relation.

Using the Kauffman skein relation and the self-intersection relations it can be shown that the self-intersection relations and the commuting relation also hold when the crossings are changed. This way we get a second version of both the second and third self-intersection relations and three other versions of the commuting relation.

In contrast with the first self-intersection relation, the two new self-intersection relations preserve a self-twist of a strand. This strand twists around the pole and intersects itself. The relations show that such a preserved self-intersection can be moved freely to every strand in the tangle, even if the strand has no twist around
the pole. Such a strand can be given two twists around the pole using the double twist relation of Figure 2 backwards. Now the preserved self-intersection can be moved to this strand using one of the new self-intersection relations.

This leads to the following observation. If a tangle has more than one preserved self-intersection, they can all be moved to one single strand. The tangle obtained this way can be written as a linear combination of tangles with fewer preserved self-intersections using the Kauffman skein relation. Thus in general we only consider tangles with at most one preserved self-intersection.

A second phenomenon arising while working with tangles with a pole is the occurrence of closed loops with a twist around the pole. Some relations concerning closed loops with a twist around the pole can readily be derived from the defining relations of $KT^*_n$ and the self-intersection relations.

**Proposition 5.3.2.** The tangles of $KT^*_n$ satisfy the following additional relations.

(i) *The first pole relation,*

\[
\begin{array}{cc}
\includegraphics[width=0.2\textwidth]{first_pole_relation.png}
\end{array}
\]

(ii) *The second pole relation,*

\[
\begin{array}{cc}
\includegraphics[width=0.2\textwidth]{second_pole_relation.png}
\end{array}
\]

(iii) *The third pole relation,*

\[
\begin{array}{cc}
\includegraphics[width=0.2\textwidth]{third_pole_relation.png}
\end{array}
\]
Proof. All relations follow directly from the defining relations for $KT_n^*$ and Lemma 5.3.1. The first pole relation follows from the commuting relation, which can be applied, after the use of Reidemeister move II. The partial strand is deformed in such a way that it has, after its twist around the pole, two crossing with the closed loop on which the commuting relation can be applied. Again the use of Reidemeister move II provides the desired result.

The second pole relation follows from applying the Kauffman skein relation to both crossings of the first version of the second self-intersection relation. The third relation follows from the second pole relation, applied to one partial strand and one closed loop of the left hand side picture to obtain the first equality and applied to both partial strands to find the second equality.

Remark 5.3.3. In the proofs of Lemma 5.3.1 and in the proof of the second pole relation we assume invertibility of $m$. Later we will use a specialization taking $m \mapsto 0$ to study the algebra over $\mathbb{Q}(\delta)$. In order to do so we will extend Definition 5.2.3 of the algebra by adding the self-intersection relations from Lemma 5.3.1 and the second pole relation.

The second pole relation illustrates that when a tangle contains one closed loop around the pole, all other twists around the pole can be moved freely between all strands. In particular, all these twists can be moved to one single strand and, by use of the double twist relation, all but one twist can be removed.
So every tangle with closed loops with a twist around the pole can be deformed to a tangle containing one closed loop with a twist around the pole and precisely one other strand with a twist around the pole.

We now turn our attention to closed components with twists around the pole. Denote by $\Theta$ the (0,0)-tangle of type D consisting of only two separate loops with both a single twist around the pole, as shown in Figure 4. Denote by $\Xi^+$ and $\Xi^-$ the (0,0)-tangle of type D consisting of precisely one closed loop with two twists around the pole and a positive/negative self-intersection between these two twists. These (0,0)-tangles have some very useful properties.

**Lemma 5.3.4.** The (0,0)-tangles $\Theta$, $\Xi^+$ and $\Xi^-$ satisfy the following properties.

\[
\Xi^+ - \Xi^- = m(\Theta - \delta)
\]

\[
(\Xi^+)^2 = \delta^2 - m\delta\Xi^+ + ml^{-1}\delta\Theta,
\]

\[
\Xi^+\Theta = \Theta\Xi^+ = \delta l^{-1}\Theta,
\]

\[
\Theta^2 = \delta^2\Theta.
\]

**Proof.** Applying the Kauffman skein relation to the single crossing in $\Xi^+$ gives $\Xi^+ + m\delta = \Xi^- + m\Theta$ and (33) follows from this, showing that $\Xi^-$ can be expressed as a linear combination of $\Xi^+$ and $\Theta$.

Observe that $(\Xi^+)^2$ is equal to $\delta$ times the (0,0)-tangle containing one closed loop around the pole with two self-intersections, as shown in Figure 5. Property (34) is a direct result from applying the Kauffman skein relation to one of the two self-intersections.

![Figure 5](image-url)

**Figure 5.** $(\Xi^+)^2$ is equal to $\delta$ times a closed loop with two self-intersections.

(35) follows from applying Lemma 5.3.1 (ii) to $\Xi^+\Theta$, moving the self crossing to the other loop.

Finally, (36) is obtained by using of Lemma 5.3.2 (iii) to move two twists to the same loop, resulting in two loops without twists around the pole. □
Remarks 5.3.5. (i) The tangle $\Xi^-$ is not the inverse of $\Xi^+$. However, the second intersection relation can be used to verify that the tangle $\delta^{-2}\Xi^-$ is the inverse of $\Xi^+$.

(ii) Every $(0,0)$-tangle can be written as a linear combination of $\Theta$ and $\Xi^+$ over $E$. For example the $(0,0)$-tangle with one closed loop with one twist around the pole and one closed loop with three twists around the pole combined with two self-intersections is equal to $l^{-2}\Theta$, as shown in Figure 6.

![Figure 6. All $(0,0)$-tangles which are not equal to $\Theta$ or $\Xi^+$ can be rewritten to a linear combination of $\Theta$ and $\Xi^$.](image)

Lemma 5.3.6. The $(0,0)$-tangles $\Theta$, $\Xi^+$ commute with every tangle of type D.

Proof. Both tangles obviously commute with any tangle of type D which contains no twists around the pole. If we can show that both tangles commute with every twist around the pole we are done.

For a closed loop with a twist around the pole this holds by Lemma 5.3.2 (i), the first pole relation. Hence, $\Theta$ commutes with every tangle of type D.

It remains to prove that $\Xi^+$ commutes with every twist around the pole. We illustrate this in Figure 7. Using the second self-intersection relation we move the self intersection to the twist. The two twists around the pole of the closed loop can now be removed by the double twist relation. As a closed loop can move freely through the tangle, we can move it to the other side of the twist using the Reidemeister moves and apply the reverse of the procedure just described to bring the preserved self-intersection back in the closed loop.

5.4. Closed tangles

In the previous section we have seen that the definition of $\text{KT}_n^+$ and the additional relations show that besides the twists of strands around the pole, tangles of type D have two new features which are properties of the full tangle instead of properties of a single strand.

A closed loop around the pole allows the free movement of other twists around the pole and also a preserved self-intersection can move freely through the tangle. In
order to obtain a more generic description of a tangle we relate these elements to the (0,0)-tangles we introduced at the end of the previous section.

For the Kauffman tangle algebra $\text{KT}(\Lambda)_n$, the algebra of (0,0)-tangles is the ring $E = \mathbb{Q}(\delta)[l^\pm 1]$. This is due to the idempotent relation which replaces all closed components without twists around the pole by $\delta$. As the closed components around the pole, $\Theta$ and $\Xi^+$, commute with all other strands and twists around the pole, it makes sense to treat them as coefficients as well.

To do so, we introduce two coefficients $\theta$ and $\xi$ which satisfy the relations

$(\xi)^2 = \delta^2 - m\delta \xi + ml^{-1}\delta \theta,$
$\xi \theta = \theta \xi = \delta l^{-1} \theta,$
$\theta^2 = \delta^2 \theta.$

Now, replacing $\Theta$ by $\theta$ and $\Xi^+$ by $\xi$ we obtain that $\text{KT}^*_0$, the algebra of (0,0)-tangles of type D is the ring $E + E\xi + E\theta$. The algebra $\text{KT}^*_0$ is indeed contained in this ring. It remains to be shown that it is the full ring. Kauffman made a huge effort to show that $\text{KT}(\Lambda)_0$ is indeed the full ring $E$. From the relations it is clear that $\theta$ generates an ideal. So the main concern is the possible existence of some relation with the invertible element $\xi$, which is not likely to occur. When this is not the case, the algebra of (0,0)-tangles of type D is the full ring $E + E\xi + E\theta$.

Consideration of the tangles over the ring $E + E\xi + E\theta$, enables us to clearly describe a standard expression for tangles of type D.

**Proposition 5.4.1.** Let $T$ be a tangle in $\text{KT}_n^*$.

(i) Let $T$ contain a positive preserved self-intersection. Then $T = \delta^{-1} \xi T'$ where $T'$ is the tangle obtained from $T$ by removing the preserved self-intersection.

(ii) Let $T$ contain one closed loop around the pole without a self-intersection and $2r + 1$ other twists around the pole. Then $T = \delta^{-1} \theta T'$ where $T'$ is the tangle obtained from $T$ by removing the closed loop with a twist around the pole and all twists of other strands around the pole.
Proof. (i) If a tangle $T$ contains a positive preserved self-intersection, part of the tangle is similar to one of the partial diagrams shown in the second and third self-intersection relation as shown in Lemma 5.3.1.

Consider Figure 8. A closed loop with no twists around the pole can be brought into the tangle by applying the idempotent relation backwards. Now the self-intersection relations allow moving the preserved self-intersection inside this loop, obtaining the tangle $T'$ as described above.

(ii) Let $T$ contain one closed loop around the pole without a self-intersection and $2r + 1$ other twists around the pole. Consider the third pole relation of Proposition 5.3.2. The second equality of this relation shows that one of the $2r + 1$ twists can be replaced by $\delta^{-1}$ and a second loop around the pole.

Now the first equality of the third pole relation can be used to remove the $2r$ remaining twists around the pole and we obtain the desired tangle. $\square$

This standard expression can also be used for the tangles over the ring $E$ by replacing the coefficients again with $(0,0)$-tangles.

There exists a natural homomorphism $i: KT_{n-1}^* \to KT_n^*$ defined by adding a strand without crossings to the right side of the tangle.

We have a map $\epsilon_n: KT_n^* \to KT_{n-1}^*$ defined by

$$\epsilon_n(T) = \delta^{-1} cl_n(T),$$

where $cl_n: KT_n^* \to KT_{n-1}^*$ is the map defined by connecting the two most right end points $(b_n,0)$ and $(b_n,1)$ of the $(n,n)$-tangle by a strand with no crossings, self-intersections or twists around the pole, see Figure 9. This map obviously respects regular isotopy and the relations of $KT_n^*$.

![Figure 8](image)

**Figure 8.** A preserved self-intersection is moved to a closed loop around the pole.

As $\epsilon_n \circ i(T) = T$ for $T \in KT_{n-1}^*$, we can regard $KT_{n-1}^*$ as a subalgebra of $KT_n^*$. 

![Figure 9](image)

**Figure 9.** The closure of the most right strand of a tangle.
Our goal is to establish a surjective homomorphism between the BMW algebras of type $D$ and an algebra of tangles of type $D$. The tangle algebra introduced in Definition 5.2.3 is too large for this purpose. We therefore turn our attention to a subalgebra of $\mathbb{K} T_n^*$ and establish the mentioned homomorphism for this particular algebra.

**Definition 5.5.1.** Let $\mathbb{K} T_n^*$ be a tangle algebra. For $n \geq 1$ denote by $\mathbb{K} T(D)_n$ the linear span of all tangles in $\mathbb{K} T_n^*$ except tangles with a closed component around the pole but without horizontal strands. Let $\mathbb{K} T(D)_0 = \mathbb{K} T_0^*$.

Our first obligation is to show this is a subalgebra of $\mathbb{K} T_n^*$.

**Lemma 5.5.2.** $\mathbb{K} T(D)_n$ is a subalgebra of $\mathbb{K} T_n^*$ over $E$.

**Proof.** To prove $\mathbb{K} T(D)_n$ is indeed a subalgebra it suffice to show no multiplication of tangles in $\mathbb{K} T(D)_n$ gives a tangle with a closed component around the pole but without horizontal strands.

Multiplication of a random tangle in $\mathbb{K} T(D)_n$ by a tangle with a horizontal bar always results in a tangle with again a horizontal bar. This will never give a tangle with a closed component around the pole and no horizontal strands. So the multiplication of two tangles will only give a tangle with no horizontal strands when both tangles contain no horizontal strands themselves.

On the other hand, the only way multiplication of two tangles results in a tangle with a closed component around the pole is when both tangles contain horizontal strands themselves which leads to a contradiction with the previous argument.

Multiplication of tangles in $\mathbb{K} T(D)_n$ will never result in a tangle with a closed component around the pole but without horizontal strands, making $\mathbb{K} T(D)_n$ a subalgebra of $\mathbb{K} T_n^*$. □

We now set up a homomorphism from $\mathbb{B}(D_n)$ to $\mathbb{K} T(D)_n$. The generators of $\mathbb{B}(D_n)$ are to be mapped onto simple tangles, tangles which contain at most one crossing.

The intention here is to decompose every tangle as a product of simple tangles. We introduce $2n$ of those simple tangles, which we denote by $G_i$ and $E_i$ for $i = 1, \ldots, n$.

For $i \neq 1$ we define the $G_i$ to be just the simple tangles where the $(i - 1)$-th and $i$-th node are connected by two strands with a positive crossing. All other nodes

![Diagram of $G_i$ and $E_i$](image.png)
are connected by straight lines without crossings. These tangles do not have any twists around the pole.

The corresponding \( E_i \) for \( i \neq 1 \) connect the \((i - 1)\)-st and \( i \)-th node by horizontal strands. All other nodes are again connected by straight lines without crossings. These tangles have no twists around the pole.

The two tangles \( G_1 \) and \( E_1 \) are tangles with twists around the pole. The tangle \( G_1 \) is obtained from \( G_2 \) in a natural way by twisting the two strands connecting the first and second node around the pole. The tangle \( E_1 \) can be constructed from \( G_1 \) using the Kauffman skein relation.

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The relation (B1), $G_i G_i = G_i G_1$ for $i \notin \{1, 3\}$ is straightforward for $i \geq 4$ as the single crosses in the two tangles do not interfere. For $i = 2$, commutation is provided by the commuting relation.

The second braid relation (B2), $G_1 G_3 G_1 = G_3 G_1 G_3$ can be proven using only Reidemeister moves II and III.

The defining relation (D1) is covered by the Kauffman skein relation and the two relations (R1) and (R2) follow by applying the double twist relation followed by the first self-intersection relation.

Recall that twists of a tangle in $K T(D)^n$ are not nested. For the remainder of this section we ignore $K T^* n$, so when we refer to a tangle $T$, we always assume the tangle is in the subalgebra $K T(D)^n$ and in one of the forms described in Proposition 5.4.1.

In the remaining part of this section we prove surjectivity of the homomorphism. To do so we proceed in a similar way as Morton and Wasserman did in [MW] in order to prove surjectivity of the homomorphism from $B(A)^n$ to $K T(A)^{n+1}$.

**Definition 5.5.5.** Given a tangle $T$, choose a sequence of base points, consisting firstly of one end point of each strand, and then of one point on each closed loop. We say $T$ is totally descending (with this choice of base points) if, on traversing all the strands from $T$ in order of the base points, we meet each crossing for the first time as an over crossing. Such a crossing is called descending.

These tangles span the subalgebra $K T(D)^n$.

**Lemma 5.5.6.** $K T(D)^n$ is spanned by totally descending tangles.

**Proof.** The proof is similar to the proof of Theorem 2.6 in [MW] and is done by induction on first the number of crossings, then on the number of non-descending crossings.

Let $T$ be a tangle in $K T(D)^n$. Choose a sequence of base points for $T$. Follow the strands of $T$ in this determined order. At the first non-descending crossing apply the skein relation. This results in a tangle $T'$ with this particular crossing changed to a descending one and two tangles with the crossing removed. Notice $T'$ has fewer non-descending crossings. So $T$ is a linear combination of two tangles with one lesser crossing and one tangle with fewer non-descending crossings. Induction shows we can write every tangle in $K T(D)^n$ as a linear combination of totally descending crossings this way.
The strands of a totally descending tangle with no twists around the pole do not interfere with each other, that is, each strand lies completely above or completely below the other strands. When a totally descending tangle has twists, the argument is very much the same as the twists around the pole are not nested.

**Lemma 5.5.7.** Let $T$ be a totally descending tangle in $\mathbf{KT}(D)_n$. Then $T$ can be represented by a tangle where each strand without a self-intersection twists around the pole at most once.

**Proof.** Let $T$ be a totally descending tangle in $\mathbf{KT}(D)_n$. Assume $T$ has a strand $q$ which twists around the pole twice but has no self-intersection. As our pole has order 2, the strand has to cross other strands between its two twists around the pole otherwise we can remove the twists with the double twist relation. Notice that we have shown before that all closed loops with a twist around the pole commute with any other twist around the pole and therefore we can assume there is no such loop around the pole between the two twists of $q$ around the pole.

First assume that no other strand twists around the pole between the two twists of $q$ around $p$. So when traversing the pole from top to bottom, the second twist of $q$ is met directly after the first twist of $q$.

Consider the region $Q$ of the tangle bounded by the pole and the strand $q$ between its two crossings with the pole.

![Figure 14. The region $Q$ between two twists around the pole.](image)

Each strand crossing $q$ between its two twists enters and leaves $Q$ at least once. We first show there are no crossings in $Q$.

Assume there are crossings in $Q$. Take a crossing such that at least two of its four outgoing strands leave $Q$ (so cross $q$) without crossing any other strand in $Q$. This results in a triangle within $Q$ where the two strands cross each other in one corner and cross $q$ in one of the other corners.

As $T$ is totally descending, one of the three strands crosses over both other strands and one crosses under both other strands. Using Reidemeister move III the crossing can be moved outside of $Q$. By induction this shows the region $Q$ contains no crossings.

So every strand entering $Q$ leaves the region again without crossing any other strand. Again, as $T$ is totally descending, every crossing of such strand with $q$ will be of the same shape (always an over- or under-crossing) and by Reidemeister
move II the crossings with \( q \) can be removed. So there are no strands in \( Q \) and both twists of \( q \) vanish as the pole has order two.

Next assume there are strands which enter \( Q \) and twist around the pole before leaving. As all crossings of strands lie in \( Q \) and not on the right side of the pole, the same argument as above can be used to show that again the region \( Q \) does not contain any crossings. So all strands which enter \( Q \) leave \( Q \) again without crossing any other strand (besides the pole). A strand \( q' \) enters \( Q \), twists around the pole (precisely once) and leaves \( Q \) again. By the previous argument and the Reidemeister II move, there are no strands crossing \( q \) between its two crossings with \( q' \). Consider the first of such strands \( q' \) which is encountered while traversing \( q \) starting at its first crossing with the pole.

Locally the tangle looks like the picture above (or with the two crossings mirrored). Now, by the commuting relation of Definition 5.2.3, we can move \( q' \) completely outside \( Q \). The proof can now be completed by induction on the number of strands in \( Q \) which twist around the pole. \( \square \)

**Proposition 5.5.8.** \( \text{KT}(D)_{n} \) is spanned by totally descending tangles \( T \) satisfying one of the following three properties.

(i) The tangle \( T \) contains no closed components at all.

(ii) \( T = \theta T' \) where \( T' \) contains no closed components and no twists around the pole.

(iii) \( T = \xi T' \) where \( T' \) contains no closed components.

**Proof.** The three distinct shapes are due to the rewrite rules of Proposition 5.4.1. Without loss of generality we can assume that the base point on \( \Xi^{+} \) is chosen in such a way that the positive self-intersection is totally descending. Other configurations of closed loops around the pole are already shown to be rewritable to the ones listed here. So we only have to show that all these tangles have no closed components without twists around the pole.
Assume $T$ is a totally descending tangle with a closed component which is not twisted around the pole. Crossings of such a component with other strands can be removed using Reidemeister moves II and III as such crossings are either all over or all under crossings. So the component can be isolated from the rest of the tangle. Using the first self-intersection relation we can strip it from its self-intersections and finally by the idempotent relation we find $T = \delta T'$ where $T'$ is the same tangle as $T$ with the closed loop removed. We can perform this procedure for every closed component in $T$ which contains no twists around the pole. □

This leads to one of the main results of this chapter.

**Proof of Theorem 5.1.1.** We need to show that each totally descending tangle in $\text{KT}(D)_n$ is generated by $\{G_i^{\pm 1}, E_i\}$ for $i = 1, \ldots, n$.

A tangle with no twists around the pole is in $\text{KT}(A)_n$ and generated by $\{G_i^{\pm 1}, E_i\}$ with $i \neq 1$ as shown in [MW]. So we only need to check elements with twists around the pole. Let $T$ be such a totally descending tangle with $2q$ twists around the pole and assume the statement holds for all tangles with fewer twists around the pole.

Traverse the pole starting at the top and label the first two twists around the pole by $q_1$ and $q_2$.

Consider three options.

(i). Assume both $q_1$ and $q_2$ are part of the same strand. By Lemma 5.5.7, there is only one possibility, the strand is a closed loop around the pole with a preserved self-intersection. By regular isotopy we can now write $T = AG_1^{\pm 1} B$ with $A$ a tangle without twists around the pole and $B$ a tangle with $2q - 2$ twists around the pole, which by induction both can be written as a linear combination of $\{G_i^{\pm 1}, E_i\}$ for $i = 1, \ldots, n$.

From now on assume $q_1$ and $q_2$ are part of different strands we will also denote by $q_1$ and $q_2$.

(ii). Assume the part of the strand of $q_1$ after twisting around the pole and the part of $q_2$ before twisting around the pole do not cross anywhere in the tangle. Regular isotopy gives $T = AE_1 B$, again with $A$ a tangle without twists around the pole and $B$ a tangle with $2q - 2$ twists around the pole.

Notice this procedure also holds when $q_1$ and/or $q_2$ is a closed component around the pole.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure17.png}
\caption{The tangle contains a preserved self-intersection.}
\end{figure}
(iii). Finally assume the part of the strand of $q_1$ after twisting around the pole and the part of $q_2$ before twisting around the pole cross somewhere in the tangle. This means we can isolate a "region" in the plane bounded by the pole and the two strands $q_1$ and $q_2$.

As $T$ is totally descending and all strands traversing through $Q$ do not twist around the pole between $q_1$ and $q_2$ we can use Reidemeister moves II and III to move every crossing out of $Q$. So $T$ is regular isotopic to the tangle where $q_1$ and $q_2$ cross each other right after twisting around the pole.

Again, using regular isotopy, we can transform the tangle to get $T = A G_1^{2q+1} B$, again with $A$ a tangle without twists around the pole and $B$ a tangle with $2q - 2$ twists around the pole.
5.6. The Brauer algebra of type D

In this section we introduce a variation of the original Brauer algebra defined on page 13. The usual Brauer algebra is a monomial algebra defined by Brauer diagrams over the rational function field $\mathbb{Q}(\delta)$. This algebra can be obtained from the Kauffman tangle algebra $\mathbf{KT}(A)_n$ over $E = \mathbb{Q}(\delta)[t^{\pm1}]$. This ring has an ideal generated by $l - 1$ with quotient ring the rational functions field $\mathbb{Q}(\delta)$. Hence, it makes sense to consider the specialization $\delta \mapsto \delta$, $l \mapsto 1$ and $m \mapsto 0$. Notice that $m \mapsto 0$ is a direct consequence from $l \mapsto 1$ as $m = \frac{l - 1}{1 - \delta}$.

Using this specialization we obtain from $\mathbf{KT}(D)_n$ an algebra over $\mathbb{Q}(\delta)$ we denote by $\mathbf{CT}(D)_n$. Consider the homomorphism $e : E \to \mathbb{Q}(\delta)$, defined by $\delta \mapsto \delta$, $l \mapsto 1$ and $m \mapsto 0$. Via this specialization $\mathbf{CT}(D)_n$ is the algebra obtained from $\mathbb{Q}(\delta)[\hat{U}(n)]$ by factoring out the Kauffman skein relation, the commuting relation, all self-intersection relations, the idempotent relation and the second pole relation.

As mentioned in Remark 5.3.3, the second and third self-intersection relations and the second pole relation have to be factored out as well to retain the definition of the algebra after taking $m \mapsto 0$.

The Kauffman skein relation and the first self-intersection relation can be simplified using the specialization, to obtain the following reduced versions of these relations.

(i) The reduced skein relation

(ii) The reduced self-intersection relation

Here, the pictures indicate tangles which differ only in the region shown. Notice that by the reduced skein relation over and under crossings are no longer distinguished and the reduced self-intersection relation shows that non preserved self-intersections can be ignored as well. This leads to the observation that the obtained diagrams are only defined by the way its $2n$ end points are connected and by the twists of the strands around the pole.

To describe the resulting diagrams, we return to the notion of an $n$-connector. We introduce a variation here, the notion of a decorated $n$-connector.

**Definition 5.6.1.** A decorated $n$-connector is an $n$-connector in which an even number of pairs is labelled 1. All other pairs are labelled 0. A pair labelled 1 will be called decorated.

We shall now describe an algebra on the linear span of the decorated $n$-connectors which is isomorphic to $\mathbf{CT}(D_n)$. We introduce some collections of decorated $n$-connectors.
Definition 5.6.2. Let $F_n$ be the collection of all decorated $n$-connectors. Denote by $F^0_n$ the subset of $F_n$ of decorated $n$-connectors with no decorations and denote by $F^\leq_n$ the subset of $F_n$ of decorated $n$-connectors with at least one horizontal pairing.

Recall that $n!! = (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1$, the product of the first $n$ odd integers.

Lemma 5.6.3. $F_n$ contains $2^{n-1}n!!$ distinct decorated $n$-connectors. Moreover, $|F^0_n| = n!!$ and $|F^\leq_n| = 2^{n-1}(n!! - n!)$ and their intersection $F^\leq_n \cap F^0_n$ contains $n!! - n!$ distinct decorated $n$-connectors.

Proof. The $2n$ points can be paired in $n!!$ ways, so $|F^0_n| = n!!$. Each strand can randomly be labelled 0 or 1, except for the last one which is obliged to make the total number of decorations even. Hence, $|F_n| = 2^{n-1}n!!$.

To obtain a decorated $n$-connector without horizontal strands the $2n$ points can be paired in $n!$ ways. So there are $n!! - n!$ pairings which have at least one horizontal pair. These are all decorated $n$-connectors with at least one horizontal strand but without decorations, so $|F^\leq_n \cap F^0_n| = n!! - n!$. Again, these pairings can be decorated in $2^{n-1}$ ways.

A (Brauer) diagram is obtained from a tangle by removing all crossings using the specialization rules and by replacing each twist of a strand around the pole with a solid dot. Using the multiplication rules of the tangles, multiplication rules for the Brauer diagrams follow. Like for the tangles, we distinguish three types of closed loops, see Figure 21.

\[
\begin{align*}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) circle (0.5);
\end{tikzpicture} & = \delta \\
\begin{tikzpicture}[scale=0.5]
\draw (0,0) circle (0.5);
\draw (0,-1) circle (0.5);
\end{tikzpicture} & = \theta \\
\begin{tikzpicture}[scale=0.5]
\draw (0,0) circle (0.5);
\draw (0,-1) circle (0.5);
\draw (0,0) -- (0,-1);
\draw (0,0) -- (0,-1);
\end{tikzpicture} & = \xi
\end{align*}
\]

Figure 20. A (Brauer) diagram of a decorated 4-connector with two decorated strands.

Figure 21. Three closed loops introduced to define multiplication of Brauer diagrams.
We will refer to the three loops as the non-decorated loop, the decorated loop and the crossed loop. A crossed loop can not have a decoration. (In fact, from the tangle algebra we know that a crossed loop gets actually two decorations which will cancel each other.)

In Figure 22, some basic calculation rules concerning the decorations and loops are listed. They are derived from the relations between the tangles. The first relation is related to the double twist relation and the second one one the left is related to the idempotent relation. The top right relation comes from the pole relations and the three other relations come from the relations between \((0,0)\)-tangles as described in Lemma 5.3.4.

\[
\begin{align*}
\begin{array}{c}
\mid \quad \mid \\
\circ \quad \delta \\
\otimes \otimes = \delta^2 \\
\end{array}
\end{align*}
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure22.png}
\caption{The basic calculation rules for decorated \(n\)-connectors.}
\end{figure}

Multiplication of two diagrams of decorated \(n\)-connectors \(f_1\) and \(f_2\) is done in a few steps and is a direct consequence of the multiplication of two tangles.

(i) Like for ordinary \(n\)-connectors, draw the corresponding diagrams and stack them by associating the two lines of points in the middle of the new diagram. Next, pick a point on the top line of \(f_1\).

(ii) Look for its (new) pairing by following the strands until it ends in a point on the top line of \(f_1\) or the bottom line of \(f_2\). This results in a new pair for the resulting decorated \(n\)-connector, say \(f\). While following the path of the concatenated strands, use the rules in Figure 23 to determine whether crossed loops come in.

(iii) Determine whether the new pair is decorated by counting the number of solid dots on the new strand. If it has an odd number of dots, the strand gets a decoration. When the number of solid dots on the new strand is even, the strand gets label 0.

(iv) Repeat the three steps above by picking a point on the top line of \(f_1\) or the bottom line of \(f_2\) which has not been paired again until a new pairing, say \(f\), is found.

(v) When all \(2n\) points are paired again, only closed loops remain. Closed loops come from strands which have no end points in the new diagram or from the concatenation of the two pairings. Substitute the \(\delta\), \(\theta\) and \(\xi\) and apply the rules described in Figure 22.

This way we find \(f_1f_2 = xd\) with \(x \in \mathbb{Q}(\delta)\) and \(d \in \{f, f\xi, f\theta\}\).

As the multiplication of decorated \(n\)-connectors is directly related to the associative multiplication of tangles, it is straightforward that this multiplication is also associative.
Remarks 5.6.4. (i). The rules in Figure 23 where the two vertical strands do not intersect imply the other rules listed there. Take for instance the top left rule. Denote the vertical strand with the decoration by $j$ and the vertical strand without by $i$, see Figure 25.

We can write both strands as the concatenation of two vertical strands $j_1, j_2$ and $i_1, i_2$ respectively. The strands $i_1$ and $j_1$ go straight down without intersections. Here $j_1$ has a decoration. The strands $i_2$ and $j_2$ intersect and have no decoration.
Now, as the multiplication of the strands and the decorations is associative, we can first multiply the strands $i_2$ and $j_2$ with the horizontal strand. Both $i_2$ and $j_2$ vanish and no crossed loop comes in here.

Secondly, we consider the concatenation of $i_1$ and $j_1$ with the horizontal strand. This gives the expected crossed loop.

(ii). By the same argument used in the proof above, the diagrams given in Figure 24, where the common node of the two horizontal strands is in the middle, are sufficient to describe all computations with multiple horizontal strands involved. The strands can be considered locally as partial diagrams of Figure 23 and treated as such. An example is given in Figure 26.

This way it is straightforward to check that when two horizontal strands have their common node on the left, the crossed loop only comes in when precisely one of the two strands has a decoration. Similarly, when two horizontal strands have their common node on the right, the crossed loop never appears.

\[ t \quad t = t \quad t = \delta^{-1} \quad \circ \]  

Figure 25. The rules where diagrams contain intersections are a direct consequence of the other rules.

\[ t \quad t = t \quad t = \delta^{-1} \quad \circ \]  

\[ t \quad t = t \quad t = \delta^{-1} \quad \circ \]  

\[ t \quad t = t \quad t = \delta^{-1} \quad \circ \]  

Figure 26. Multiplication concerning multiple horizontal strands is a direct consequence of the given rules.

**Definition 5.6.5.** Denote by $C_n^*$ the algebra over $\mathbb{Q}(\delta)$ of all the decorated $n$-connectors with the defined multiplication.

From the discussed multiplication rules we derive that for all monomials $d \in C_n^*$ we have
Clearly, to the left side of the diagram of a monomial $d \in C_n^*$ by attaching a non-decorated strand to the left of the diagram of a monomial $d \in C_n$.

We have a homomorphism $i : C_{n-1} \to C_n^*$ by attaching a non-decorated strand to the left of the diagram of a monomial $d \in C_n^*$. We also have a mapping $cl_n : C_n^* \to C_{n-1}$ by joining the most left end points by an additional non-decorated strand. This new strand is part of a concatenation of strands, which is treated in the same way as done for strands in a usual multiplication described above. Now define a morphism $\varepsilon_n : C_n^* \to C_{n-1}$ by

$$\varepsilon_n(d) = \delta^{-1} cl_n(d).$$

Clearly, $\varepsilon_n(dd') = de(d')$ if $d \in C_{n-1}$, i.e., the most left strand of $d$ goes straight down and is not decorated. Also $\varepsilon_n \circ i(d) = d$, so we can consider $C_{n-1}$ as a subalgebra of $C_n^*$.

**Definition 5.6.6.** Denote by $C(D)_n$ the linear span of the subset of all decorated $n$-connectors $d \in C_n^*$ except those where $d = f \theta$ with $f \not\in F_n^0 \cap F_n^\infty$.

**Lemma 5.6.7.** $C(D)_n$ is a subalgebra of $C_n^*$ over $\mathbb{Q}(\delta)$.

**Proof.** We need to show $C(D)_n$ is closed under multiplication. The only decorated $n$-connectors in $C_n^*$ which are not in $C(D)_n$, are elements $f \theta$ with $f \not\in F_n^0 \cap F_n^\infty$. These are elements whose Brauer diagrams contain no horizontal strands but do contain two decorated closed loops.

All decorated $n$-connectors in $C(D)_n$ which have a decorated loop also have horizontal strands. Multiplication of two such elements where one of them has at least one horizontal strand will always result in a new element containing also a horizontal strand. The only way to obtain an element $f \theta$ with $f \not\in F_n^0 \cap F_n^\infty$ is by multiplying two elements without horizontal strands.

By definition $C(D)_n$ contains such monomials only without decorated or crossed loops. So the only possibility left is finding a $d, d' \in C(D)_n$ with $d = f$ and $d' = f'$ with $f, f' \not\in F_n^0$. But multiplication of two Brauer diagrams without horizontal strands will never give a decorated closed loop, proving multiplication is closed in $C(D)_n$. \qed

By definition of $C(D)_n$, a monomial $d \in C(D)_n$ can be written as

$$d = \begin{cases} f & \text{with } f \in F_n^0, \\ f \xi & \text{with } f \in F_n^\infty, \\ f \theta & \text{with } f \in F_n^0 \cap F_n^\infty. \end{cases}$$

**Lemma 5.6.8.** The algebra $C(D)_n$ has dimension $(2^n + 1)n!! - (2^{n-1} + 1)n!$ over $\mathbb{Q}(\delta)$.

**Proof.** From the three possible shapes of a monomial $d \in C(D)_n$ we obtain

$$\dim(C(D)_n) = |F_n^0| + |F_n^\infty| + |F_n^0 \cap F_n^\infty|$$

and the result follows from Lemma 5.6.3. \qed

Notice the full algebra $C_n^*$ has dimension $(2^n + 1)n!! - (2^{n-1})n!$ over $\mathbb{Q}(\delta)$. 


5.7. A lower bound for the dimension of $\text{B}(D_n)$

A basis of $\text{KT}(D)_n$ related to the monomials of $\text{C}(D)_n$ can be defined as follows. Construct a tangle out of any element $d$ of the monomial basis of $\text{C}(D)_n$ by drawing its (Brauer) diagram and change every intersection of the curves into a crossing. A decorated strand is drawn with a twist around the pole. The decorated loops $\theta$ are replaced by two loops around the pole represented by $\Theta$ and a crossed loop $\xi$ is replaced by the closed loop with a positive preserved self-intersection, the closed loop $\Xi^+$. Denote the tangle obtained this way $T_d$. Note that by Lemma 5.3.6 it does not matter where the closed and crossed loops are placed.

As we can randomly choose each crossing to be positive or negative, the tangle $T_d$ is clearly not unique.

Therefore assume for each $d \in \text{C}(D)_n$ a choice of base points for its pairs has been made, and the tangle $T_d$ inherits this order of base points. Now we can choose the crossings to be positive and negative in such a way that the resulting tangle is totally descending.

Next we define a map $\varphi : \text{KT}(D)_n \to \text{C}(D)_n$. Let $T$ be a tangle in $\text{KT}(D)_n$. Pair the $2n$ end points of $T$ such that each pair contains the two end points of one of the $n$ strands of the tangle. For every strand, count the number of twists around the pole. If it is odd, label the pair 1, otherwise label it 0. Replace every preserved self-intersection in the tangle by $\delta^{-1}\xi$. When the tangle has $r$ closed loops around the pole, replace them by $\delta^{-2}\theta$. Finally, replace every closed loop without a twist around the pole by $\delta$.

This maps every tangle in $\text{KT}(D)_n$ to a monomial in the $E$-algebra $E\text{C}(D)_n$. Denote this image of $T$ by $nc(T)$.

Secondly, by use of the specialization $e : E \to \mathbb{Q}(\delta)$ determined by $l \mapsto 1$, $m \mapsto 0$ and $\delta \mapsto \delta$, the images $nc(T)$ can be mapped to the elements of the algebra $\text{C}(D)_n$ over $\mathbb{Q}(\delta)$ as we defined in the previous section.

**Lemma 5.7.1.** There is a homomorphism of algebras $\varphi : \text{KT}(D)_n \to \text{C}(D)_n$, determined by

$$\varphi(\lambda T) = e(\lambda)nc(T).$$

**Proof.** The relations defining $\text{KT}(D)_n$ also hold in $\text{C}(D)_n$.

The Kauffman skein relation holds as by $m \mapsto 0$ the relation reduces to the equality of over and under crossings. The first self-intersection relation obviously holds in $\text{C}(D)_n$ when $l \mapsto 1$.

The idempotent relation is carried over to the same relation in $\text{C}(D)_n$. The double twist relation is equivalent with the rule in the Brauer algebra that two decorations on one strand vanish.

The partial diagrams which satisfy the self-intersection relations concerning a preserved self-intersection are mapped to equivalent diagrams with a crossed loop.

Finally, the two partial diagrams satisfying the commuting relation are both mapped to the same partial diagram containing two decorated vertical strands. \(\square\)

Proposition 5.5.8 describes the totally descending tangles which span the tangle algebra $\text{KT}(D)_n$. 
Lemma 5.7.2. Let $T_1$ and $T_2$ be totally descending tangles as described in Proposition 5.5.8. When $\varphi(T_1) = \varphi(T_2)$, the tangles $T_1$ and $T_2$ are ambient isotopic, so $T_1 = l^k T_2$ in $\text{KT}(\mathbb{D}_n)$ for some $k \in \mathbb{Z}$.

Proof. As $\varphi(T_1) = \varphi(T_2)$, the strands of the two tangles connect the same points. Obviously both tangles satisfy the same property of Proposition 5.5.8. We saw before, on page 94, that the strands of a totally descending tangle do not interfere with each other. Using planar isotopy we can even establish that they twist around the pole in an isolated region, also without interference of other strands. Each individual strand is unknotted as the tangle is totally descending. So using Reidemeister move I all possible self-intersections can be removed to obtain strands without them. This way both tangles can be shaped in the same crossing pattern proving they are ambient isotopic.

Proposition 5.7.3. $\text{KT}(\mathbb{D}_n)$ is spanned, for every fixed choice of base points, by the finite set $T_d$, with $d$ an element of the monomial basis of $\mathbb{C}(\mathbb{D}_n)$.

Proof. Proposition 5.5.6, Proposition 5.5.8 and Lemma 5.7.2 show $\text{KT}(\mathbb{D}_n)$ is spanned by tangles which are ambient isotopic to the tangles $T_d$, with $d$ an element of the monomial basis of $\mathbb{C}(\mathbb{D}_n)$. Hence the set $T_d$ where $d$ runs over all elements of the monomial basis of $\mathbb{C}(\mathbb{D}_n)$, spans $\text{KT}(\mathbb{D}_n)$.

Corollary 5.7.4. The homomorphism $\varphi : \text{KT}(\mathbb{D}_n) \to \mathbb{C}(\mathbb{D}_n)$ is surjective.

Proof. As the finite set $T_d$, with $d$ an element of the monomial basis of $\mathbb{C}(\mathbb{D}_n)$, spans $\text{KT}(\mathbb{D}_n)$ and clearly $\varphi(T_d) = d$, the homomorphism is surjective.

This gives the desired lower bound for the dimension of $\mathbb{B}(\mathbb{D}_n)$.

Proof of Theorem 5.1.2. A direct consequence of Theorem 5.1.1 and Corollary 5.7.4.

In the previous chapter we discussed a strategy to construct a spanning set for the quotient ideals $I_C/I_{r+1}$ in order to find an upper bound for the dimension of $\mathbb{B}(\mathbb{D}_n)$ over $\mathbb{Q}(l, \delta)$. It is our firm belief that this leads to $(2^n + 1)(n)! - (2^{n-1} + 1)n!$ as an upper bound for this dimension. Hence, proving the conjecture of Chapter 4 would determine the dimension of $\mathbb{B}(\mathbb{D}_n)$ over $\mathbb{Q}(l, \delta)$. Manual computations provided us with the knowledge that this upperbound is valid for $n \leq 5$. Hence, we have that the dimension of $\mathbb{B}(\mathbb{D}_4)$ and $\mathbb{B}(\mathbb{D}_5)$ is 1569 and 29145 respectively.
CHAPTER 6

Generalized Lawrence-Krammer representations

6.1. Introduction

One of the motivations for the study of the BMW algebras of simply laced type was a result of Zinno that a faithful representation of the Braid group factors through the BMW algebra of type A. (cf. \[Z\]) In this chapter we show that the generalized version of this so called Lawrence-Krammer representation factors through the BMW algebra for all simply laced Coxeter graphs \(M\).

In § 6.2 we construct the analog of this representation and show it factors through \(\mathcal{B}(M)/I_2\). By doing so we find an irreducible representation of \(\mathcal{B}(M)/I_2\). This will lead to the following result.

**Theorem 6.1.1.** Let \(\mathcal{B}(M)\) be the BMW algebra of type \(A_n\) (\(n \geq 1\)), \(D_n\) (\(n \geq 4\), or \(E_n\) (\(n = 6, 7, 8\)) over \(\mathbb{Q}(l, \delta)\). Then \(\mathcal{B}(M)/I_2\) is semi-simple over \(\mathbb{Q}(l, \delta)\).

Let \(Z_0\) be the Hecke algebra of type \(Y\), as defined in Corollary 3.4.5. For each irreducible representation \(\theta\) of \(Z_0\), there is a corresponding representation \(\Gamma_\theta\) of \(\mathcal{B}(M)\) of dimension \(|\Phi^+| \dim(\theta)|\) and, up to equivalence, these are the irreducible representations of \(\mathcal{B}(M)\) occurring in \(I_1/I_2\). In particular, the dimension of \(I_1/I_2\) as a vector space over \(\mathbb{Q}(l, \delta)\) equals \(|\Phi^+|^2 |W_Y|\).

In analogy to this construction we define a right free \(Z_0\)-module with basis \(x_B\) \((B \in \mathcal{B})\) which is a left module for the positive monoid \(A^+\) of the Artin group \(A\) of type \(M\). For each node \(i\) of \(M\), the \(i\)-th fundamental generator \(s_i\) of \(A^+\) maps onto the linear transformation \(\tau_i\) on \(V\) given by the following case division.

\[
\tau_i x_B = \begin{cases} 
0 & \text{if } \alpha_i \in B, \\
x_B h_{B,i} & \text{if } \alpha_i \in B^+, \\
x_{r_iB} & \text{if } r_iB < B, \\
x_{r_iB} - mx_B & \text{if } r_iB > B.
\end{cases}
\]

This leads to the second result of this chapter, established in § 6.3.

**Theorem 6.1.2.** Let \(W\) be a Weyl group of simply laced type. For \(B\) an admissible \(W\)-orbit of sets of mutually orthogonal positive roots, there is a partial order \(<\) on \(B\) such that the above defined map \(s_i \mapsto \tau_i\) determines a homomorphism of monoids from \(A^+\) to \(\text{End}(V)\).

Here the poset \((\mathcal{B}, <)\), introduced in Chapter 3, is used. In the final section of this chapter, we give an extension of this action in an attempt to find a representation which factors through any quotient \(I_C/I_{r+1}\). This results in a recursive method to construct such a representation. We conjecture that this method indeed leads to a representation.

In the final section, we discuss how the results could be extended to \(I_r\) with \(r \geq 2\).
6.2. Generalized Lawrence-Krammer representations

In this section we construct the analog of the Lawrence-Krammer representation of \( A \) with coefficients in \( Z_0 \), the Hecke algebra of type \( Y \), where \( Y \) is the parabolic of the highest root centralizer. We show the representation factors through \( B(M)/I_2 \). By taking an irreducible representation of \( Z_0 \), we find an irreducible representation of \( B(M)/I_2 \). Finally, by counting dimensions of irreducible representations, we are able to conclude that all representations of \( B(M)/I_2 \) that do not vanish on \( I_1 \) are of this generalized Lawrence-Krammer type, and we can finish the proof of Theorem 6.1.1.

Let \((W, R)\) be the Coxeter system of type \( M \). We write \( \Phi^+ \) for the set of positive roots of the Coxeter system of type \( M \). By \( \alpha_0 \) we denote its highest root, and by \( Y \) the set of nodes \( j \) in \( M \) with \((\alpha_j, \alpha_0) = 0\). In case \( A_n \), the type of \( Y \) is \( A_{n-2} \); in case \( D_n \), it is \( A_1 \times D_{n-2} \), in case \( E_6 \), it is \( A_5 \), \( D_6 \), and \( E_7 \) for \( n = 6, 7, 8 \), respectively. (So \( Y \) has corank 2 if \( M \) is of type \( A \) and 1 otherwise.) If \( X \) is a set of nodes of \( M \), we denote by \( W_X \) the parabolic subgroup of \( W \) corresponding to \( X \). This means that \( W_X \) is the subgroup of \( W \) generated by all \( r_j \) for \( j \in X \).

Since the construction for disconnected \( M \) is a direct sum of the representations of \( B(M) \) for the distinct connected components, we simply take \( M \) to be connected, so \( M \in \text{ADE} \). We let \( \Phi \) be the root system in \( \mathbb{R}^n \) of type \( M \), and denote by \( \alpha_1, \ldots, \alpha_n \) the fundamental roots corresponding to the reflections \( r_1, \ldots, r_n \), respectively. As usual, by \( \Phi^+ \) we denote the set of positive roots in \( \Phi \).

For a root \( \beta \), the set of roots \( \{ \gamma \in \Phi \mid (\beta, \gamma) = 0 \} \) is also a root system. Its type can be read off from \( M \) as follows: the extended Dynkin diagram \( \tilde{K} \) of the connected component \( K \) of \( M \) involving \( \beta \) (i.e., having nodes in the support of \( \beta \)) has a single node \( \alpha_0 \) in addition to those of \( K \); now take \( Y \) to consist of all nodes of \( M \) that are not connected to \( \alpha_0 \). Then the type of the roots orthogonal to \( \beta \) is \( M|_Y \). In fact, if \( \beta = \alpha_0 \), then \( \{ \alpha_i \mid i \in Y \} \) is a set of fundamental roots of the root system \( \{ \gamma \in \Phi \mid (\beta, \gamma) = 0 \} \). For \( A_n \) with \( \beta = \alpha_0 \) this is the diagram of type \( A_{n-2} \) on \{2, \ldots, n-1\}, for \( D_n \), it is the diagram of type \( A_1D_{n-2} \) on \{0\} \cup \{1, \ldots, n-2\}, for \( E_6 \) it is the diagram of type \( A_5 \) on \{1, 3, 4, 5, 6\}, for \( E_7 \) it is the diagram of type \( D_6 \) on \{2, 3, 4, 5, 6, 7\}, and for \( E_8 \) it is the diagram of type \( E_7 \) on \{1, 2, 3, 4, 5, 6, 7\}. Here we have used the labelling as given in Figure 12 on page 15.

Recall the coefficients of \( Z_0 \) are in \( Q(l, \delta) \). We take the coefficients of our representation in the Hecke algebra \( Z_0^{(0)} \) of type \( M|_Y \) over the subdomain \( Q[l^{\pm 1}, m] \) of \( Q(l, \delta) \), where \( m \) is defined in (1). Observe that the fraction field of \( Q(l, m) \) coincides with \( Q(l, \delta) \). The generators \( z_i \) \((i \in Y)\) of \( Z_0^{(0)} \) satisfy the quadratic relations \( z_i^2 + mz_i - 1 = 0 \). For the proof of irreducibility at the end of this section, we need however a smaller version of this Hecke algebra, namely the subalgebra \( Z_0^{(1)} \) with same generators \( z_i \), but over \( Q[m] \). Thus, \( Z_0^{(0)} = Z_0^{(1)}Q[l^{\pm 1}] \).

By Lemma 2.4.8, the element \( h_{\beta,i} \) of \( A \) defined in (15), where \( \beta \in \Phi^+ \) and \( i \) is a node with \((\alpha_i, \beta) = 0\), maps onto an element of \( Z_0^{(1)} \) upon substitution of \( s_j \) by \( z_j \) and \( s_j^{-1} \) by \( z_j + m \). We shall also write \( h_{\beta,i} \) for the image of this element in \( Z_0^{(1)} \). We write \( V^{(0)} \) for the free right \( Z_0^{(0)} \) module with basis \( x_\beta \) indexed by \( \beta \in \Phi^+ \).
Theorem 6.2.1. Let $M \in \text{ADE}$ and let $A$ be the Artin group of type $M$. Then, for each $i \in \{1, \ldots, n\}$ and each $\beta \in \Phi^+$, there are elements $T_{i,\beta}$ in $Z_0^{(1)}$ such that the following map on the generators of $A$ determines a representation of $A$ on $V^{(0)}$:

$$s_i \mapsto \sigma_i = \tau_i + l^{-1}T_i,$$

where $\tau_i$ is determined by

$$\tau_i(x_\beta) = \begin{cases} 0 & \text{if } (\alpha_i, \beta) = 2 \\ x_\beta - \alpha_i & \text{if } (\alpha_i, \beta) = 1 \\ x_\beta \alpha_i & \text{if } (\alpha_i, \beta) = 0 \\ x_\beta + \alpha_i - mx_\beta & \text{if } (\alpha_i, \beta) = -1, \end{cases}$$

and where $T_i$ is the $Z_0^{(0)}$-linear map on $V^{(0)}$ determined by $T_ix_\beta = x_\alpha T_{i,\beta}$ on the generators of $V^{(0)}$ and by $T_{i,\alpha} = 1$.

When tensored with $\mathbb{Q}(\delta, l)$, the representation of $A$ on $V^{(0)}$ becomes a representation on the vector space $V$ which factors through the quotient $B(M)/I_2$ of the BMW algebra $B(M)$ of type $M$ over $\mathbb{Q}(\delta, l)$.

Remark 6.2.2. The connection with Definition 1.4.4, used in [CW], is given by $m = r - r^{-1}$, $l = 1/(tr^3)$.

Recall that $A^+$ is the positive monoid of $A$.

Throughout this section we use several properties of the elements $h_{\beta,i}$ listed in Lemma 2.4.7. In addition, we shall use the Hecke algebra relation for the image of $h_{\beta,i}$ in $Z_0^{(0)}$:

$$h_{\beta,i}^{-1} = h_{\beta,i} + m.$$ 

The proof of the theorem follows the lines of the proof in [CW]. We shall first describe the part modulo $l^{-1}$ of the representation of the Artin monoid $A^+$ on $V^{(0)}$.

Lemma 6.2.3. There is a monoid homomorphism $A^+ \rightarrow \text{End}(V^{(0)})$ determined by $s_i \mapsto \tau_i$ ($i = 1, \ldots, n$).

**Proof.** We must show that, if $i$ and $j$ are not adjacent, then $\tau_i\tau_j = \tau_j\tau_i$ and, if they are adjacent, then $\tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j$. We evaluate the expressions on each $x_\beta$ and show they are equal. We begin with the case where $\beta = \alpha_i$. Suppose first that $i$ and $j$ are not adjacent. Then $\tau_i x_{\alpha_i} = 0$ and $\tau_j x_{\alpha_i} = x_{\alpha_i} h_{\beta,j}$. Now $\tau_i \tau_j x_{\alpha_i} = 0$ and $\tau_i \tau_j x_{\alpha_i} = \tau_j x_{\alpha_i} h_{\beta,j} = 0$, so the result holds. Suppose next that $i$ and $j$ are adjacent. Then $\tau_i x_{\alpha_i} = \tau_j x_{\alpha_i} = 0$ and $\tau_j x_{\alpha_i} = -m x_{\alpha_i} + x_{\alpha_i + \alpha_j}$. Now

$$\tau_i \tau_j x_{\alpha_i} = \tau_j(0) = 0, \quad \text{and} \quad \tau_i \tau_j x_{\alpha_i} = \tau_j(\tau_i(-m x_{\alpha_i} + x_{\alpha_i + \alpha_j})) = \tau_j(\tau_i x_{\alpha_i + \alpha_j}) = 0.$$ 

This ends the verification for the case where $\beta = \alpha_i$. We now divide the verifications into the various cases depending on the inner products $\langle \alpha_i, \beta \rangle$ and $\langle \alpha_j, \beta \rangle$. By the above, we may assume $(\alpha_i, \beta), (\alpha_j, \beta) \neq 2$.

First assume that $(\alpha_i, \alpha_j) = 0$. The computations verifying $\tau_i \tau_j = \tau_j \tau_i$ are summarized in the following table. The last column indicates the formulas that are used.
We demonstrate how to derive these expressions by checking the third line.

\[ \tau_i \tau_j x_\beta = \tau_j (x_\beta h_{\beta,j}) = x_{\beta - \alpha_j} h_{\beta,j}. \]

In the other order,

\[ \tau_j \tau_i x_\beta = \tau_j (x_\beta - \alpha_i) = x_{\beta - \alpha_i} h_{\beta - \alpha,j}. \]

Equality between \( h_{\beta,j} \) and \( h_{\beta - \alpha_i,j} \) follows from (18).

Suppose next that \( i \sim j \). The same situation occurs except the computations are sometimes longer and one case does not occur. This is the case where \((\alpha_i, \beta) = (\alpha_j, \beta) = -1\). For then \( \beta + \alpha_i \) is also a root, and \((\beta + \alpha_i, \alpha_j) = -1 - 1 = -2\). This means \( \beta + \alpha_i = -\alpha_j \) and \( \beta \) is not a positive root. The table is as follows.

<table>
<thead>
<tr>
<th>((\alpha_i, \beta))</th>
<th>((\alpha_j, \beta))</th>
<th>(\tau_i \tau_j x_\beta = \tau_j \tau_i x_\beta)</th>
<th>ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(x_{\beta - \alpha_i} - \alpha_j)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>(x_{\beta + \alpha_j} - \alpha_i - mx_{\beta - \alpha_i})</td>
<td>(18)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>(x_{\beta - \alpha_i} h_{\beta,j})</td>
<td>(16)</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td>(x_{\beta + \alpha_j} h_{\beta,i} - mx_{\beta - \alpha_i})</td>
<td>(18)</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>(m^2 x_{\beta} - m(x_{\beta + \alpha_i} + x_{\beta - \alpha_i}) + x_{\beta + \alpha_i + \alpha_j})</td>
<td></td>
</tr>
</tbody>
</table>

We next study the possibilities for the parameters \( T_{k,\beta} \) occurring in Theorem 6.2.1. Recall that there we defined \( \sigma_k = \tau_k + l^{-1} T_k \), where \( T_k x_\beta = x_{\alpha_k} T_{k,\beta} \). We shall introduce \( T_{k,\beta} \) as elements of the Hecke algebra \( Z_0 \) of type \( M/|Y| \).

**Proposition 6.2.4.** Set \( T_{i,\alpha_i} = 1 \) for all \( i \in \{1, \ldots, n\} \). For \( \sigma_i \mapsto \tau_i + l^{-1} T_i \) to define a linear representation of the group \( A \) on \( V \), it is necessary and sufficient that the equations in Table 1 are satisfied for each \( k, j = 1, \ldots, n \) and each \( \beta \in \Phi^+ \).

**Proof.** The \( \sigma_k \) should satisfy the relations (B1), (B2). Substituting \( \tau_k + l^{-1} T_k \) for \( \sigma_k \), we find relations for the coefficients of \( l^{-i} \) with \( i = 0, 1, 2, 3 \). The constant part involves only the \( \tau_k \). It follows from Lemma 6.2.3 that these equations are satisfied. We shall derive all of the equations of Table 1 except for (53) from the \( l^{-1} \)-linear part and the remaining one from the \( l^{-1} \)-quadratic part of the relations. The coefficients of \( l^{-1} \) lead to

\[
(39) \quad T_i \tau_j = \tau_j T_i \quad \text{and} \quad T_j \tau_i = \tau_i T_j \quad \text{if} \quad i \neq j,
\]

\[
(40) \quad \tau_i T_j + T_j \tau_i T_j + \tau_j T_i T_j = \tau_j T_i + T_i \tau_j + \tau_i \tau_j T_i \quad \text{if} \quad i \sim j.
\]

We focus on the consequences of these equations for the \( T_{i,\beta} \). First consider the case where \( i \neq j \). Then \( \tau_i x_{\alpha_j} = x_{\alpha_j} h_{\alpha_j,i} \) and so, for the various values of \((\alpha_i, \beta)\) we find the following equations
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$\{\alpha_i, \beta\}$ & $T_j \tau_i x_{\beta} = \tau_j T_i x_{\beta}$ & equation \\
\hline
0 & $x_{\alpha_j} T_j \beta h_{\beta,i} = x_{\alpha_i} h_{\alpha_j,i} T_j \beta$ & $T_j \beta h_{\beta,i} = h_{\alpha_j,i} T_j \beta$ \\
1 & $x_{\alpha_j} T_j \beta - \alpha_i = x_{\alpha_i} h_{\alpha_j,i} T_j \beta$ & $T_j \beta - \alpha_i = h_{\alpha_j,i} T_j \beta$ \\
-1 & $x_{\alpha_j} T_j \beta + \alpha_i - m x_{\alpha_i} T_j \beta = x_{\alpha_i} h_{\alpha_j,i} T_j \beta$ & $T_j \beta + \alpha_i = h_{\alpha_j,i} T_j \beta$ \\
2 & $0 = x_{\alpha_i} h_{\alpha_i,i} T_j \beta$ & $0 = T_j \beta$ \\
\hline
\end{tabular}
\end{table}

The first equation gives

\[(41) \quad T_j \beta h_{\beta,i} = h_{\alpha_j,i} T_j \beta\]

and the second

\[(42) \quad T_j \beta = h_{\alpha_j,i}^{-1} T_j \beta - \alpha_i.\]

The third case gives an equation that is equivalent to (42). The fourth equation is part of (53) in Table 1 (namely the part where $j \neq i$).

Next, we assume $i \sim j$. A practical rule is

$$\tau_i T_j x_{\alpha_i} = \tau_i (-m x_{\alpha_i} + x_{\alpha_i + \alpha_j}) = x_{\alpha_j}.$$ 

We distinguish cases according to the values of $(\alpha_i, \beta)$ and $(\alpha_j, \beta)$. Since each inner product, for distinct roots is one of 1, 0, −1, there are six cases to consider up to interchanges of $i$ and $j$. However, as in the proof of Lemma 6.2.3 for $i \sim j$, the case $(\alpha_i, \beta) = (\alpha_j, \beta) = -1$ does not occur.

For the sake of brevity, let us denote the images of the left hand side and the right hand side of (40) on $x_{\beta}$ by LHS and RHS, respectively.

Case $(\alpha_i, \beta) = (\alpha_j, \beta) = 1$. Then $(r_i, \beta, \alpha_j) = (\beta - \alpha_i, \alpha_j) = 2$, so $\beta = \alpha_i + \alpha_j$. Now

$$\text{RHS} = x_{\alpha_j} (T_i \beta - m T_i \alpha) + x_{\beta} T_i \alpha_i.$$ 

Comparison with the same expression but then $j$ and $i$ interchanged yields LHS.

This leads to the equations $T_i \beta = m T_i \alpha_j$ and $T_j \alpha_j = T_i \alpha_i$. In view of the latter, and connectedness of the diagram there is an element $z$ in $Z_{0}^{(0)}$ such that

\[(43) \quad T_i \alpha_i = z \quad \text{for all} \quad i.\]

Consequently, the former equation reads

\[(44) \quad T_i \beta = mz.\]

By the requirement $T_i \alpha_i = 1$ in the hypotheses, we must have $z = 1$.

Case $(\alpha_i, \beta) = (\alpha_j, \beta) = 0$. This gives

$$\text{RHS} = x_{\alpha_j} (T_i \beta - m T_j \beta h_{\beta,i}) + x_{\alpha_j - \alpha_j} T_j \beta h_{\beta,i} + x_{\alpha_i} T_i \beta h_{\beta,j} h_{\beta,i}$$

and LHS can be obtained from the above by interchanging the indices $i$ and $j$.

Comparison of each of the coefficients of $x_{\alpha_i}, x_{\alpha_j - \alpha_j}, x_{\alpha_j}$ gives

\[(45) \quad T_i \beta h_{\beta,j} = T_j \beta h_{\beta,i} \quad \text{if} \quad (\alpha_i, \beta) = (\alpha_j, \beta) = 0 \quad \text{and} \quad (\alpha_i, \alpha_j) = -1.\]

Since the other cases come down to similar computations, we only list the results.

Case $(\alpha_i, \beta) = 0, (\alpha_j, \beta) = -1$. Here we have

$$\text{RHS} = x_{\alpha_j} (-m T_i \beta h_{\beta,i} + T_i \beta + \alpha_j h_{\beta,i}) + x_{\alpha_j} (-m T_j \beta h_{\beta,i} + T_i \beta)$$

$$+ x_{\alpha_j + \alpha_j} (T_j \beta h_{\beta,i}).$$
and
\[
LHS = x_{\alpha_i}(m^2T_{i,\beta} + T_{j,\beta} - mT_{i,\beta + \alpha_j}) + x_{\alpha_j}(-mT_{j,\beta\bar{h}}_{\beta,i} - mT_{j,\beta + \alpha_j} + T_{j,\beta + \alpha_j}) + x_{\alpha_i + \alpha_j}(-mT_{i,\beta} + T_{i,\beta + \alpha_j})
\]

which gives
\[
T_{i,\beta + \alpha_j} = T_{j,\beta\bar{h}}_{\beta,i} + mT_{i,\beta},
\]
\[
T_{j,\beta + \alpha_j + \alpha_i} = T_{i,\beta + mT_{j,\beta + \alpha_j}}.
\]

Case \((\alpha_i, \beta) = 0, (\alpha_j, \beta) = 1\).
\[
RHS = x_{\alpha_i}T_{i,\beta - \alpha_j}h_{\beta,i} + x_{\alpha_j}(-mT_{j,\beta\bar{h}}_{\beta,i} + T_{i,\beta}) + x_{\alpha_i + \alpha_j}T_{j,\beta\bar{h}}_{\beta,i}
\]

and
\[
LHS = x_{\alpha_i}(T_{j,\beta} - mT_{i,\beta - \alpha_j}) + x_{\alpha_j}T_{j,\beta - \alpha_j - \alpha_i} + x_{\alpha_i + \alpha_j}T_{i,\beta - \alpha_j}
\]

whence
\[
T_{i,\beta} = T_{j,\beta - \alpha_j - \alpha_i} + mT_{i,\beta - \alpha_j},
\]
\[
T_{j,\beta} = T_{j,\beta - \alpha_j - \alpha_i - \alpha_j}.
\]

Case \((\alpha_i, \beta) = 1, (\alpha_j, \beta) = -1\). Now
\[
RHS = x_{\alpha_i}(T_{i,\beta - \alpha_j}h_{\beta,i}) + x_{\alpha_j}(T_{i,\beta} - mT_{j,\beta - \alpha_i}) + x_{\alpha_i + \alpha_j}(T_{j,\beta - \alpha_i})
\]

and
\[
LHS = x_{\alpha_i}(m^2T_{i,\beta} - mT_{i,\beta + \alpha_j} + T_{j,\beta}) + x_{\alpha_j}(T_{j,\beta + \alpha_j}h_{\beta,i} + mT_{j,\beta - \alpha_i}) + x_{\alpha_i + \alpha_j}(T_{i,\beta + \alpha_j} - mT_{i,\beta})
\]

whence
\[
T_{j,\beta} = T_{i,\beta - \alpha_j}h_{\beta,i} + mT_{j,\beta - \alpha_i},
\]
\[
T_{j,\beta + \alpha_j} = T_{i,\beta}h_{\beta,i}^{-1},
\]
\[
T_{i,\beta + \alpha_j} = T_{j,\beta}h_{\beta,i}^{-1} + mT_{i,\beta}.
\]

We now consider the coefficients of \(l^{-2}\) and of \(l^{-3}\) in the equations (B1), (B2) for \(\sigma_i\). We claim that, given (42)–(52), a necessary condition for the corresponding equations to hold is
\[
T_{k,\alpha_j} = 0 \quad \text{if} \quad k \neq j.
\]
To see this, note that, if \(k \neq j\), the coefficient of \(l^{-2}\) gives \(T_kT_j = T_jT_k\) which, applied to \(x_{\alpha_j}\), yields (53). If \(k \sim j\), note
\[
T_kT_jx_{\alpha_k} = T_k(-mT_{\alpha_k} + x_{\alpha_k + \alpha_j}) = 0
\]
as \(T_{k,\alpha_k + \alpha_j} = mT_{k,\alpha_k}\) by (44). Now use the action of
\[
T_jT_kT_j + T_jT_kT_j + T_jT_kT_j = T_kT_j + \tau_kT_k + \tau_kT_kT_k + T_kT_j
\]
on \( x_{\alpha_j} \). We see only the middle terms do not vanish because of the relation above and so

\[
\tau_j x_{\alpha_k} T_{k, \alpha_j} z = \tau_k x_{\alpha_j} T_{j, \alpha_k} T_{k, \alpha_j}.
\]

By considering the coefficient of \( x_{\alpha_k} \), which occurs only on the left hand side, we see that (53) holds.

A consequence of this is that \( T_i T_j = 0 \) if \( i \neq j \). Now all the equations for the \( l^{-2} \) and \( l^{-3} \) coefficients are easily satisfied. In the noncommuting case of \( l^{-2} \), the first terms on either side are 0 by the relation above and the other terms are 0 as \( T_j T_k = 0 \).

We have seen that, in order for \( s_i \mapsto \sigma_i \) to determine a representation, the \( T_{i, \beta} \) have to satisfy the equations (41)–(53). This system of equations, however, is redundant. Indeed, when the root in the index of the left hand side of (46) is set to \( \gamma \), we obtain (50) for \( \gamma \) instead of \( \beta \). Similarly, (47) is equivalent to (48), while (51) is equivalent to (49), and (52) is equivalent to (48). Consequently, in order to finish the proof that Table 1 contains a sufficient set of relations, we must show that (45) and (41) follow from those of the table. These proofs are given in Lemmas 6.2.6 and 6.2.8 below.

It remains to establish that the matrices \( \sigma_k \) are invertible. To prove this, we observe that the linear transformation \( \sigma_k^2 + m \sigma_k - 1 \) maps \( V \) onto the submodule spanned by \( x_{\alpha_k} \) and that the image of \( x_{\alpha_k} \) under \( \sigma_k \) is \( x_{\alpha_k} l^{-1} \). This is easy to establish and will be shown in Lemma 6.2.11 below.

**Corollary 6.2.5.** If the \( T_{i, \beta} \in Z_0^{(0)} \) satisfy the equations in Table 1, then these obey the following rules.

- (i) \( T_{i, \beta} = 0 \) whenever \( i \notin \text{Supp}(\beta) \).
- (ii) If \( (\alpha_i, \beta) = 1 \), then \( T_{i, \beta} = m d_{\alpha_i}^{-1} s_{\beta}^{-1} s_i s_{\beta} d_{\beta} \).

**Proof.** (i) follows from (53) by use of (42) and (50). Observe that, if \( i \notin \text{Supp}(\beta) \) and \( (\alpha_j, \beta) = 1 \) for some \( j \sim i \), then \( j \notin \text{Supp}(\beta - \alpha_j) \).

(ii). By induction on \( \text{ht}(\beta) \). The assertion is vacuous when \( \text{ht}(\beta) = 1 \). Suppose \( \text{ht}(\beta) = 2 \). Then \( s_{\beta} = s_j s_i s_j \) for some node \( j \) adjacent to \( i \) in \( M \). Therefore,
md_{\alpha}^{-1}s_{j}^{-1}s_{i}s_{j}d_{\beta} = md_{\alpha}^{-1}s_{j}^{-1}s_{i}s_{j}d_{\beta} = md_{\beta}^{-1}s_{j}^{-1}s_{i}s_{j}d_{\beta} = m \text{ and, by (44) } T_{i,\beta} = m, \text{ as required.}

Now suppose $ht(\beta) > 2$.

If $j$ is a node distinct from $i$ such that $(\alpha_{j}, \beta) = 1$, then, necessarily, $i \not\sim j$ (for otherwise $(\alpha_{i} - \alpha_{j}) = 2$, so $\beta = \alpha_{i} + \alpha_{j}$, contradicting $ht(\beta) > 2$). Now (42) applies, giving

\[
T_{i,\beta} = h_{\alpha_{i},j}^{-1} T_{i,\beta - \alpha_{i}} \quad \text{by (42)},
\]

\[
= md_{\alpha_{i}}^{-1}s_{j}^{-1}s_{\beta - \alpha_{i}}s_{\beta - \alpha_{i}}s_{j}d_{\beta} \quad \text{by induction},
\]

\[
= md_{\alpha_{i}}^{-1}s_{\beta - \alpha_{i}}^{-1}s_{j}s_{\beta - \alpha_{i}}s_{j}d_{\beta} \quad \text{by definition of } s_{\beta}
\]

\[
= md_{\alpha_{i}}^{-1}s_{j}^{-1}s_{\beta}s_{j}d_{\beta}, \quad \text{as } s_{i}s_{j} = s_{j}s_{i}
\]
as required.

Suppose $l$ is a node distinct from $i$ such that $(\alpha_{l}, \beta) = 0$ and $i \sim l$. Then (49) applies, giving

\[
T_{i,\beta} = T_{i,\beta - \alpha_{l}}h_{\beta,\beta_{l}}^{-1} \quad \text{by (49)}
\]

\[
= md_{\alpha_{l}}^{-1}(s_{\beta - \alpha_{l}}^{-1}s_{l})s_{\beta - \alpha_{l}}(d_{\beta - \alpha_{l}}d_{\beta_{l}}^{-1})s_{l}^{-1}d_{\beta} \quad \text{by induction},
\]

\[
= md_{\alpha_{l}}^{-1}(s_{l}^{-1}s_{\beta}^{-1}d_{\beta}^{-1})(s_{l}^{-1}s_{\beta}^{-1}d_{\beta}^{-1})s_{l}^{-1}d_{\beta} \quad \text{by definition of } d_{\beta} \text{ and } s_{\beta},
\]

\[
= md_{\alpha_{l}}^{-1}s_{\beta}^{-1}s_{l}^{-1}s_{\beta}^{-1}s_{l}^{-1}s_{\gamma}^{-1}s_{l}^{-1}s_{\gamma}^{-1}s_{l}d_{\beta} \quad \text{by Lemma 2.4.9}
\]

\[
= md_{\alpha_{l}}^{-1}(s_{l}^{-1}s_{\beta}^{-1}d_{\beta}^{-1})(s_{l}^{-1}s_{\beta}^{-1}d_{\beta}^{-1})s_{l}^{-1}d_{\beta} \quad \text{by the braid relation}
\]

\[
= md_{\alpha_{l}}^{-1}s_{\beta}^{-1}s_{l}^{-1}s_{\beta}^{-1}s_{l}d_{\beta} \quad \text{by definition of } s_{\beta}
\]
as required.

\[\square\]

**Lemma 6.2.6.** The equations for $\beta$ in (45) are consequences of the relations of Table 1 and those of (45) and (41) for positive roots of height less than $ht(\beta)$.

**Proof.** The equation says that $T_{k,\beta}h_{\beta,j} = T_{j,\beta}h_{\beta,k}$ whenever $(\alpha_{k}, \beta) = 0$, $(\alpha_{j}, \beta) = 0$ and $k \sim j$. The initial case of $\beta$ having height 1 is direct from (55). Suppose therefore, $ht(\beta) > 1$. There exists $m \in \{1, \ldots, n\}$ such that $(\alpha_{m}, \beta) = 1$. If $(\alpha_{m}, \alpha_{k}) = \alpha_{m}, \alpha_{j}) = 0$, then, by the induction hypothesis and (18), $T_{k,\beta - \alpha_{m}}h_{\beta,j} = T_{k,\beta - \alpha_{m}}h_{\beta,m} = T_{j,\beta - \alpha_{m}}h_{\beta,m} = T_{j,\beta - \alpha_{m}}h_{\beta,k}, \text{ so, applying (42) twice, we find}

\[
T_{k,\beta}h_{\beta,j} = h_{\alpha_{k},m}T_{k,\beta - \alpha_{m}}h_{\beta,j} = h_{\alpha_{k},m}T_{j,\beta - \alpha_{m}}h_{\beta,k} = T_{j,\beta}h_{\beta,k},
\]
as required.

Therefore, interchanging $k$ and $j$ if necessary, we may assume that $j \sim m$, whence $k \not\sim m$ (as the Dynkin diagram contains no triangles). Now $\delta = \beta - \alpha_{m} - \alpha_{j}$ and $\gamma = \delta - \alpha_{k}$ are positive roots and $(\alpha_{k}, \delta) = 1$, so (42) gives $T_{m,\gamma} = h_{\alpha_{m},k}T_{m,\delta}$, which, by induction on height, and (22), leads to $h_{\alpha_{k},m}T_{j,\gamma} = h_{\alpha_{m},k}T_{m,\gamma}h_{\gamma,m} = T_{j,\beta}h_{\gamma,j}h_{\gamma,m}^{-1}$. Observing that, by straightforward application of the braid relations and the definition of $h_{\beta,k}$, we also have

\[
h_{\gamma,j}h_{\gamma,m}^{-1}h_{\beta,j} = h_{\beta,k}
\]

\[
h_{\gamma,m}^{-1}h_{\beta,j} = h_{\beta,m}^{-1}h_{\beta,k}
\]
Lemma 6.2.7. Let \( h, k \) be generators (or conjugates thereof) in the Hecke algebra \( \mathbb{Z}_q^{(0)} \). Then, for any \( t \in \mathbb{Z}_q^{(0)} \),

(1) \( h^{-1}t - tk^{-1} = ht - tk \),
(2) \( h^{-1}(t + h^{-1}tk^{-1})k = t + h^{-1}tk^{-1} \).

**Proof.** (i). Expand the left hand side and use that \( z^{-1} = z + m \) for every conjugate of a generator.

(ii). By (i), \( tk + h^{-1}t = ht + tk^{-1} \). Multiplying both sides from the left by \( h^{-1} \) and pulling out a factor \( k \) at the right of the left hand side, we find the required relation.

Lemma 6.2.8. The equations for \( \beta \) in (41) are consequences of the relations of Table 1 and those of (45) and (41) for positive roots of height less than \( ht(\beta) \).

**Proof.** Suppose that the positive root \( \beta \) and the distinct nodes \( l \), \( i \) satisfy \( (\alpha_i, \beta) = 0 \) and \( i \neq l \). By Corollary 6.2.5(i), we know that \( T_{i,\beta} = 0 \) if \( i \notin \text{Supp}(\beta) \), so we need only consider cases where \( i \in \text{Supp}(\beta) \).

If \( ht(\beta) = 1 \), then, by (43) and (53), \( T_{i,\beta} = 0 \) and there is nothing to prove unless \( \beta = \alpha_i \). In the latter case \( T_{i,\beta} = 1 \) and \( h^{-1}_{\alpha_i,l}T_{i,\beta}h_{\beta,l} = h^{-1}_{\alpha_i,l}h_{\alpha_i,l} = 1 \), so (41) is satisfied.

If \( ht(\beta) = 2 \), then \( \beta = \alpha_i + \alpha_j \) for some \( j \) and \( T_{i,\beta} = m \) by (44). As \( \alpha_i \) is orthogonal to both \( \beta \) and \( \alpha_j \), it must be orthogonal to \( \alpha_j \) as well. Now \( h^{-1}_{\alpha_i,j}T_{i,\beta}h_{\beta,l} = mh^{-1}_{\alpha_i,j}h_{\alpha_i + \alpha_j,l} = md_{\alpha_i}a^{-1}_{\alpha_i}d_{\alpha_j}a^{-1}_{\alpha_j}a_{\beta}d_{\beta}a_{\alpha_i} = m \), as required.

Case (42): there is a node \( j \) with \( (\alpha_j, \beta) = 1 \) and \( (\alpha_i, \alpha_j) = 0 \). Then \( T_{i,\beta} = h^{-1}_{\alpha_i,j}T_{i,\beta - \alpha_i} \). If \( j \neq l \), we find

\[
T_{i,\beta} = h^{-1}_{\alpha_i,j}T_{i,\beta - \alpha_i},
\]

as required. 

The relation (41) is new compared to [CW]. But it is superfluous. In order to see this, we first prove some auxiliary claims.
If \( j \sim l \), we find
\[
    h^{-1}_{\alpha,i} T_{i,\beta} h_{\beta,l} = h^{-1}_{\alpha,i} h^{-1}_{\alpha,j} T_{i,\beta-\alpha_j} h_{\beta,l}
\]
by (42)
\[
    = h^{-1}_{\alpha,i} h^{-1}_{\alpha,j} T_{i,\beta-\alpha_j-\alpha_i} h_{\beta,l}
\]
by (42)
\[
    = h^{-1}_{\alpha,i} h^{-1}_{\alpha,j} T_{i,\beta-\alpha_j-\alpha_i} h_{\beta-\alpha_j-\alpha_i,j}
\]
by (17) and (20)
\[
    = h^{-1}_{\alpha,i} h^{-1}_{\alpha,j} T_{i,\beta-\alpha_j-\alpha_i}
\]
by induction
\[
    = T_{i,\beta}.
\]
by (42) applied twice
This ends Case (42).

Case (48): \( (\alpha_i, \beta) = 0 \) and there is a node \( j \sim i \) with \( (\alpha_j, \beta) = 1 \). Then \( T_{i,\beta} = T_{j,\beta-\alpha_j-\alpha_i} + m T_{i,\beta-\alpha_i} \). Now \( h^{-1}_{\alpha,i} T_{i,\beta} h_{\beta,l} = h^{-1}_{\alpha,i}(T_{j,\beta-\alpha_j-\alpha_i} + m T_{i,\beta-\alpha_i}) h_{\beta,l} \). If \( j \not\sim l \), we find
\[
    h^{-1}_{\alpha,i} T_{i,\beta} h_{\beta,l} = h^{-1}_{\alpha,i}(T_{j,\beta-\alpha_j-\alpha_i} + m T_{i,\beta-\alpha_i}) h_{\beta,l}
\]
by (48)
\[
    = h^{-1}_{\alpha,i} T_{j,\beta-\alpha_j-\alpha_i} h_{\beta-\alpha_j-\alpha_i,j} + m h^{-1}_{\alpha,i} T_{i,\beta-\alpha_j} h_{\beta-\alpha_j-\alpha_i,j}
\]
by (18)
\[
    = T_{j,\beta-\alpha_j-\alpha_i} + m T_{i,\beta-\alpha_i}
\]
by induction
\[
    = T_{i,\beta}.
\]
by (48)
If \( j \sim l \), we claim
\[
    T_{i,\beta} = T_{i,\delta} + m(T_{j,\gamma} + h^{-1}_{\alpha,i} T_{j,\gamma} h^{-1}_{\beta,l}),
\]
where \( \gamma = \beta - \alpha_i - \alpha_j \) and where \( \delta = \gamma - \alpha_j \) are positive roots. For
\[
    T_{i,\beta} = T_{j,\beta-\alpha_j-\alpha_i} + m T_{i,\beta-\alpha_j}
\]
by (48)
\[
    = (T_{i,\delta} + m T_{j,\gamma}) + m h^{-1}_{\alpha,i} T_{i,\beta-\alpha_j-\alpha_i}
\]
by (48) and (42)
\[
    = T_{i,\delta} + m T_{j,\gamma} + m h^{-1}_{\alpha,i} T_{j,\gamma} h^{-1}_{\beta,l}
\]
by (49)
\[
    = T_{i,\delta} + m T_{j,\gamma} + m h^{-1}_{\alpha,i} T_{\gamma} h^{-1}_{\beta,l}.
\]
by (20)
By (20), we have \( h\beta,l = h_{\beta-\alpha_j-\alpha_i,j} = h\delta,i \), so, by induction we find \( h^{-1}_{\alpha,i} T_{i,\delta} h_{\beta,l} = T_{i,\delta} h^{-1}_{\beta,i} h_{\beta,l} = T_{i,\delta} \). So the first summand of (54) is invariant under simultaneous left multiplication by \( h^{-1}_{\beta,i} \) and right multiplication by \( h_{\beta,l} \). The same holds for the second summand, \( m(T_{j,\gamma} + h^{-1}_{\alpha,i} T_{j,\gamma} h^{-1}_{\beta,l}) \) by Lemma 6.2.7 applied with \( h = h_{\alpha,i}, k = h_{\beta,l}, \) and \( t = T_{j,\gamma} \). Consequently (41) holds for \( T_{i,\beta} \) in Case (48).

Case (50): \( (\alpha_i, \beta) = -1 \) and there is a node \( j \sim i \) with \( (\alpha_j, \beta) = 1 \). Then \( T_{i,\beta} = T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,i} + m T_{i,\beta-\alpha_i} \). Now \( h^{-1}_{\alpha,i} T_{i,\beta} h_{\beta,l} = h^{-1}_{\alpha,i}(T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,i} + m T_{i,\beta-\alpha_i}) h_{\beta,l} \). If \( j \not\sim l \), we find
\[
    h^{-1}_{\alpha,i} T_{i,\beta} h_{\beta,l} = h^{-1}_{\alpha,i}(T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,i} + m T_{i,\beta-\alpha_i}) h_{\beta,l}
\]
by (48)
\[
    = h^{-1}_{\alpha,i} T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,i} + m h^{-1}_{\alpha,i} T_{i,\beta-\alpha_j} h_{\beta-\alpha_j,i}
\]
by (18), (16)
\[
    = T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,i} + m T_{i,\beta-\alpha_i}
\]
by induction
\[
    = T_{i,\beta}.
\]
by (48)
If \( j \sim l \), we claim
\[
    T_{i,\beta} = T_{i,\gamma} h_{\beta-\alpha_j,i} + m(T_{j,\gamma} h_{\beta-\alpha_j,i} + h^{-1}_{\alpha,i} T_{i,\gamma}),
\]
\]
where $\gamma = \beta - \alpha_j - \alpha_i$ is a positive root. For

$$T_{i,\beta} = T_{j,\beta-\alpha_j,\alpha_i} + mT_{i,\beta-\alpha_j} \quad \text{by (50)}$$

$$= T_{i,\gamma}h_{\gamma,j}h_{\beta-\alpha_j,i} + mT_{j,\gamma}h_{\beta-\alpha_j,i} + m^{-1}h_{\alpha_i,\beta}T_{i,\gamma} \quad \text{by (50), (42)}$$

By Lemma 38, we have

$$h_{\gamma,j}^{-1}h_{\beta-\alpha_j,i}h_{\beta,j} = d_\beta^{-1}(s_{j}^{-1}s_i^{-1}s_is_j)(s_{j}^{-1}s_i^{-1}s_js_i)(s_i^{-1}s_is_j)(s_i^{-1}s_is_j)(s_i^{-1}s_is_j)(s_i^{-1}s_i^{-1}s_is_j)(s_i^{-1}s_i^{-1}s_is_j)s_i^{-1}s_is_j d_\beta$$

$$= d_\beta^{-1}(s_j^{-1}s_i^{-1}s_is_j)(s_j^{-1}s_i^{-1}s_js_i)(s_i^{-1}s_is_j)(s_i^{-1}s_is_j)(s_i^{-1}s_is_j)(s_i^{-1}s_i^{-1}s_is_j)(s_i^{-1}s_i^{-1}s_is_j)s_i^{-1}s_is_j d_\beta$$

$$= d_\beta^{-1}s_j^{-1}s_is_j s_is_j s_is_j d_\beta = d_\beta^{-1}s_j^{-1}s_i^{-1}s_is_j s_is_j d_\beta$$

$$= d_\beta^{-1}s_j^{-1}s_i^{-1}s_is_j s_is_j d_\beta = d_\beta^{-1}s_j^{-1}s_i^{-1}s_is_j d_\beta$$

$$= h_{\gamma,j}h_{\beta-\alpha_j}.$$ 

Hence, using induction, we find for the first summand of (55)

$$h_{\alpha_i,\beta}^{-1}(T_{i,\gamma}h_{\gamma,j}h_{\beta-\alpha_j,i}h_{\beta,j}) = T_{i,\gamma}h_{\gamma,j}^{-1}h_{\beta-\alpha_j,i}h_{\beta,j} = T_{i,\gamma}h_{\gamma,j}h_{\beta-\alpha_j},$$

proving that it is invariant under simultaneous left multiplication by $h_{\alpha_i,\beta}$ and right multiplication by $h_{\beta,j}$. The same holds for the second summand, $m(T_{j,\gamma}h_{\beta-\alpha_j,i} + h_{\alpha_i,\beta}T_{i,\gamma})$ as we shall establish next. First of all, note that $h_{\gamma,j} = h_{\beta,j}$ by (20) and that $h_{\gamma,i} = h_{\beta-\alpha_j,i}$ by (18). Moreover, by (45) for $\gamma$, we have $T_{i,\gamma}h_{\gamma,j} = T_{j,\gamma}h_{\gamma,i}$. Substituting all this in the second summand, we obtain

$$m(T_{j,\gamma}h_{\beta-\alpha_j,i} + h_{\alpha_i,\beta}T_{i,\gamma}) = m(T_{j,\gamma}h_{\gamma,i} + h_{\alpha_i,\beta}T_{i,\gamma}) = m(T_{i,\gamma}h_{\gamma,j} + h_{\alpha_i,\beta}T_{i,\gamma})$$

Again, using Lemma 6.2.7 applied with $h = h_{\gamma,i}$, $k = h_{\gamma,j}$, and $t = T_{i,\gamma}$, we find the required invariance. Consequently (41) holds for $T_{i,\beta}$ in Case (48). Case (49): $\langle \alpha_i, \beta \rangle = 1$ and there is a node $j \sim i$ with $\langle \alpha_j, \beta \rangle = 0$. Then $T_{i,\beta} = T_{j,\beta-\alpha_j}h_{\beta,j}^{-1}$. Now $h_{\alpha_i,\beta}^{-1}h_{i,\beta}h_{\beta,j} = h_{\alpha_i,\beta}^{-1}T_{i,\beta-\alpha_j}h_{\beta,j}^{-1}h_{\beta,j}$. If $j \neq l$, we find

$$h_{\alpha_i,\beta}^{-1}T_{i,\beta}h_{\beta,j} = h_{\alpha_i,\beta}^{-1}T_{j,\beta-\alpha_j}h_{\beta,j}^{-1}h_{\beta,j} \quad \text{by (49)}$$

$$= h_{\alpha_i,\beta}^{-1}T_{j,\beta-\alpha_j}h_{\beta,j}^{-1}h_{\beta,j} \quad \text{by (16) and (18)}$$

$$= T_{j,\beta-\alpha_j}h_{\beta,j}^{-1} \quad \text{by induction}$$

$$= T_{i,\beta}. \quad \text{by (49)}$$

If $j \sim l$, observe that $h_{\beta-\alpha_j}^{-1}h_{\beta,j} = h_{\beta-\alpha_j}^{-1}T_{j,\beta-\alpha_j}h_{\beta,j}^{-1}h_{\beta,j}$ in view of (18), (20), and (17). Also, $h_{\alpha_i,\beta} = h_{\alpha_i,\beta}$ by a double application of (21). Therefore,

$$h_{\alpha_i,\beta}^{-1}T_{i,\beta}h_{\beta,j} = h_{\alpha_i,\beta}^{-1}T_{i,\beta-\alpha_j}h_{\beta,j}^{-1}h_{\beta,j} \quad \text{by (49) twice}$$

$$= h_{\alpha_i,\beta}^{-1}T_{i,\beta-\alpha_j}h_{\beta,j}^{-1}h_{\beta,j} \quad \text{by the above}$$

$$= T_{i,\beta-\alpha_j}h_{\beta,j}^{-1}h_{\beta,j} \quad \text{by induction}$$

$$= T_{i,\beta}. \quad \text{by (49) twice}$$

□

The proposition enables us to describe an algorithm computing the $T_{i,\beta}$. 
Algorithm 6.2.9. The Hecke algebra elements $T_{i,\beta}$ of Theorem 6.2.1 can be computed as follows by using Table 1.

(i) If $i \notin \text{Supp}(\beta)$, then, in accordance with (53), set $T_{i,\beta} = 0$.

From now on, assume $i \in \text{Supp}(\beta)$.

(ii) If $\text{ht}(\beta) \leq 2$, Equations (43) and (44), that is, the second and third lines of Table 1, determine $T_{i,\beta}$.

From now on, assume $\text{ht}(\beta) > 2$. We proceed by recursion, expressing $T_{i,\beta}$ as a $Z_0^{(0)}$-bilinear combination of $T_{k,\gamma}$’s with $\text{ht}(\gamma) < \text{ht}(\beta)$.

(iii) If $(\alpha_i, \beta) = 1$, in accordance with Corollary 6.2.5(ii), set

$$T_{i,\beta} = m\alpha_i^{-1}s_{\beta}^{-1}s_i^*s_{\beta}d_{\beta}. $$

From now on, assume $(\alpha_i, \beta) \notin \{0, -1\}$.

(iv) Search for a $j \in \{1, \ldots, n\}$ such that $(\alpha_i, \alpha_j) = 0$ and $(\alpha_j, \beta) = 1$. If such a $j$ exists, then $\beta - \alpha_j \in \Phi$ and (42) expresses $T_{i,\beta}$ as a multiple of $T_{i,\beta-\alpha_j}$.

(v) So, suppose there is no such $j$. There is a $j$ for which $\beta - \alpha_j$ is a root, so $(\alpha_j, \beta) = 1$. As $(\alpha_i, \beta) \neq 1$, we must have $i \sim j$. According as $(\alpha_i, \beta) = 0$ or $-1$, the identities (48) or (50) express $T_{i,\beta}$ as a $Z_0^{(0)}$-bilinear combination of $T_{i,\beta-\alpha_j}$ and some $T_{j,\gamma}$ with $\text{ht}(\gamma) < \text{ht}(\beta)$.

This ends the algorithm. Observe that all lines of Table 1 have been used, with (49) implicitly in (iii).

The algorithm computes a Hecke algebra element for each $i, \beta$ based on Table 1, showing that there is at most one solution to the set of equations. The next result shows that the computed Hecke algebra elements do indeed give a solution.

Proposition 6.2.10. The equations of Table 1 have a unique solution.

Proof. We will first show that the Hecke algebra elements $T_{i,\beta}$ defined by Algorithm 6.2.9 are well defined by the algorithm and then that they satisfy the equations of Table 1. Both assertions are proved by induction on $\text{ht}(\beta)$, the height of $\beta$.

If $\beta$ has height 1 or 2, $T_{i,\beta}$ is chosen in Step (i) if $\beta = \alpha_j$ with $j \neq i$ and in Step (ii) otherwise. Indeed there is a unique solution.

Now assume $\text{ht}(\beta) \geq 3$. Suppose first that $T_{i,\beta}$ is determined in Step (iii). This means that $(\alpha_i, \beta) = 1$. This is unique as it is a closed form.

We now suppose that $T_{i,\beta}$ is chosen in Step (iv). This means there is a $j$ for which $(\alpha_i, \alpha_j) = 0$ and $(\alpha_j, \beta) = 1$. We must show that if there are two such $j$ the result is the same. Suppose there are distinct $j$ and $j'$ for which $(\alpha_i, \beta) = (\alpha_j', \beta) = 1$ and $(\alpha_j, \alpha_i) = (\alpha_j', \alpha_i) = 0$. Then by our definition $T_{i,\beta} = h_{\alpha_i,\beta}^{-1}T_{i,\beta-\alpha_j}$, and we must show that $T_{i,\beta} = h_{\alpha_i,\beta}^{-1}T_{i,\beta-\alpha_j}$. If $j \sim j'$, then $(\beta - \alpha_j, \alpha_j') = 2$ and $\beta = \alpha_j + \alpha_j'$ has height 2. This means we can assume $j \neq j'$. Then $(\beta - \alpha_j, \alpha_j') = 1$ and $(\beta - \alpha_j, \alpha_j') = 1$. In particular, $\beta - \alpha_j - \alpha_j'$ is also a root. Now apply (42) and the induction hypothesis to see $T_{i,\beta-\alpha_j} = h_{\alpha_i,\beta}^{-1}T_{i,\beta-\alpha_j}$ and $T_{i,\beta-\alpha_j'} = h_{\alpha_i,\beta}^{-1}T_{i,\beta-\alpha_j'}$, and so by (16), we find $h_{\alpha_i,\beta}^{-1}T_{i,\beta-\alpha_j} = h_{\alpha_i,\beta}^{-1}T_{i,\beta-\alpha_j'}$. This shows the definitions are the same with either choice.
We may now assume that $T_{\alpha, \beta}$ was chosen in Step (v). If $j$ is the one chosen in Step (v), then $T_{\alpha, \beta}$ was chosen to satisfy (48) or (50). Suppose now that there is another index $j'$ which was used in Step (v) to define $T_{\alpha, \beta}$. For these the conditions are $(\alpha_j, \beta) = (\alpha_j, \beta) = 1$ and $(\alpha_j, \alpha_j) = (\alpha_j, \alpha_j) = 1$. Clearly $j \neq j'$ for otherwise there would be a triangle in the Dynkin diagram $M$. Therefore, $(\alpha_j, \beta - \alpha_j) = 1$, and so $\beta - \alpha_j - \alpha_{j'}$ is a root. We distinguish according to the two possibilities for $(\alpha_i, \beta)$.

Assume first $(\alpha_i, \beta) = 0$. Then, $(\alpha_i, \beta - \alpha_j - \alpha_{j'}) = 2$, and so $\beta = \alpha_i + \alpha_j + \alpha_{j'}$. By using (48), with either $j$ or with $j'$, we find $T_{\alpha, \beta} = m^2$, independent of the choice of $j$ or $j'$.

Next assume $(\alpha_i, \beta) = -1$. Then $(\alpha_i, \beta - \alpha_j - \alpha_{j'}) = 1$, so $\gamma = \beta - \alpha_j - \alpha_{j'} - \alpha_i$ is a root. We need to establish that the result of application of (50) to $T_{\alpha, \beta}$ does not depend on the choice $j$ or $j'$. We do so by showing that the result can be expressed in an expression symmetric in $j$ and $j'$. Observe that $\gamma$ is an expression symmetric in $j$ and $j'$. The expression of $T_{\alpha, \beta}$ obtained by applying (50) to $j$ is

$$T_{j, \beta - \alpha_j, h_{\beta - \alpha_j, i} + mT_{j, \beta - \alpha_j}}.$$ 

By (48), the second summand of the right hand side equals

$$mT_{j, \beta - \alpha_j} = mT_{j', \gamma} + m^2T_{j, \beta - \alpha_j - \alpha_{j'}}.$$ 

For the first summand of (56) we find

$$T_{j, \beta - \alpha_j, h_{\beta - \alpha_j, i}} = h_{\alpha_j, j}^{-1}T_{j, \beta - \alpha_j, h_{\beta - \alpha_j, i}} \quad \text{by (42)}$$

$$= h_{\alpha_j, j}^{-1}(T_{\gamma, \gamma} + T_{\gamma, \gamma'}) h_{\beta - \alpha_j, i}.$$ 

Expanding (56) with these expressions, we find by use of $h_{\alpha_j, j'} = h_{\alpha_j, j}$ (see (21), $h_{\gamma, j'} = h_{\beta - \alpha_j, i}$ (see (22)), and (41),

$$h_{\alpha_j, j}^{-1}T_{\gamma, \gamma} h_{\beta - \alpha_j, i} + m \left(h_{\alpha_j, j}^{-1}T_{j, \gamma} h_{\beta - \alpha_j, i} + T_{j', \gamma} \right) + m^2T_{j, \beta - \alpha_j - \alpha_{j'}} =$$

$$= h_{\alpha_j, j}^{-1}T_{\gamma, \gamma} h_{\beta - \alpha_j, i} + m \left(T_{j, \gamma} + T_{j', \gamma} \right) + m^2T_{j, \beta - \alpha_j - \alpha_{j'}}.$$ 

Since $h_{\gamma, j}$ and $h_{\gamma, j'}$ commute, cf. (16), the result is indeed symmetric in $j$ and $j'$. This shows that the algorithm gives unique Hecke algebra elements $T_{\alpha, \beta}$. We now show that the relations of Table 1 all hold for $T_{\alpha, \beta}$ as computed by the algorithm. If the height of $\beta$ is one or two the values are given by (53) and (43) of the table and none of the other relations holds as there are no applicable $j$.

We consider each of the remaining relations, one at a time, and show that each holds by assuming the relations all hold for roots of lower height. If $(\alpha_i, \beta) = 1$ the value of $T_{\alpha, \beta}$ is given in Step (iii). The relevant equations are (42) and (49). The proof of Corollary 6.2.5(ii) shows that both equations are satisfied by the closed formula which is the outcome of our algorithm.

We have yet to check (48) and (50) in which case $(\beta, \alpha_j)$ is 0 or $-1$. Notice (42) and (49) require $(\alpha_i, \beta) = 1$ and do not apply here. In these cases $T_{\alpha, \beta}$ is chosen in Step (iv) or Step (v).

Suppose first $T_{\alpha, \beta}$ was chosen by Step (iv). In this case there is a $j'$ with $(\alpha_{j'}, \beta) = 1$, $(\alpha_i, \alpha_j) = -1$. As $T_{\alpha, \beta}$ is determined by Step (iv) of the algorithm, $T_{\alpha, \beta} = h_{\alpha_{j'}, \beta}^{-1}T_{\alpha, \beta - \alpha_{j'}}$. We have already seen that this is independent of the choice of $j'$ and so if there is another $j$ for which $(\alpha_j, \beta) = 1$ with $(\alpha_i, \alpha_j) = 1$, (42) holds. To check (48) we suppose there is a $j$ for which $(\alpha_i, \beta) = 1$ with $(\alpha_i, \alpha_j) = -1$. We
must have \( j \not\sim j' \), for otherwise we would again be in the height 2 case. In order to obtain (48) we must show that
\[
h_{\alpha, j'} T_{i, \beta - \alpha - j'} = T_{j, \beta - \alpha - j} + mT_{i, \beta - \alpha - j}.
\]
As for the left hand side, \((\beta - \alpha - j, \alpha_j) = 1\) and \((\alpha_i, \alpha_j) = -1\), so by (48), we have
\[
h_{\alpha, j'} T_{i, \beta - \alpha - j'} = h_{\alpha, j'} T_{j, \beta - \alpha - j} - \alpha_i + m h_{\alpha, j'} T_{i, \beta - \alpha - j}.
\]
As for the right hand side, as \((\alpha_j, \alpha_j') = 0\), we can use (42) to obtain
\[
T_{j, \beta - \alpha - \alpha} = h_{\alpha, j'} T_{j, \beta - \alpha - \alpha - \alpha} + T_{i, \beta - \alpha} = h_{\alpha, j'} T_{i, \beta - \alpha - \alpha} = h_{\alpha, j'} T_{i, \beta - \alpha - \alpha} - \alpha_i + m h_{\alpha, j'} T_{i, \beta - \alpha - \alpha - \alpha}.
\]
We have yet to consider the case \((\alpha_i, \beta) = 0\) and so this equation is satisfied also. We have now shown all the equations. In particular we need to show
\[
h_{\alpha, j'} T_{i, \beta - \alpha - j'} = T_{j, \beta - \alpha - j} + mT_{i, \beta - \alpha - j}.
\]
Use (50) on the left hand side to get
\[
h_{\alpha, j'} T_{i, \beta - \alpha - j'} = T_{j, \beta - \alpha - j} + mT_{i, \beta - \alpha - j}.
\]
On the right hand side use (42) to get
\[
h_{\alpha, j'} T_{j, \beta - \alpha - j} + m h_{\alpha, j'} T_{i, \beta - \alpha - j}.
\]
The needed equation will hold provided \( h_{\alpha, j'} = h_{\alpha, j'} \) and \( h_{\beta - \alpha - j, j'} = h_{\beta - \alpha - j, i} \).
The first is (23) and the second is (18). The second is (23) and the second is (18). This shows that all the projections are satisfied if \( T_{\alpha, \beta} \) is chosen in Step (iv). If \( T_{\alpha, \beta} \) was chosen in Step (v) we have already checked any two choices of \( j \) give the same answer for (50) and so this equation is satisfied also. We have now shown all the relations in Table 1 hold.

At this point we have established the existence of a linear representation \( \sigma \) of A on \( V^{(0)} \). We need some properties of projections which have already arisen in [CW]. In particular let \( f_i = ml^{-1} e_i \). The following lemma shows these elements are multiples of projections.

**Lemma 6.2.11.** The endomorphisms \( \sigma(f_i) \) of \( V^{(0)} \) satisfy
\[
\sigma(f_i) x_{\alpha} = \begin{cases} 
(l^2 + ml^{-1} - 1) x_{\alpha} & \text{if } (\alpha_i, \beta) = 2, \\
-1 x_{\alpha} T_{i, \beta}(h_{\beta, i} + m + l^{-1}) & \text{if } (\alpha_i, \beta) = 0, \\
-1 x_{\alpha} (T_{\beta - \alpha} + l^{-1}) & \text{if } (\alpha_i, \beta) = -1, \\
-1 x_{\alpha} (T_{i, \beta - \alpha} + (m + l^{-1}) T_{\beta, \beta}) & \text{if } (\alpha_i, \beta) = 1.
\end{cases}
\]
In particular, \( \sigma(f_i) x_{\alpha} \in x_{\alpha} l^{-1} Z_0^{(1)} [l^{-1}] \) if \( \beta \neq \alpha_i \) and \( \sigma(f_i) x_{\alpha} \in x_{\alpha} (-1 + l^{-1} Z_0^{(1)} [l^{-1}]) \).

**Proof.** Suppose first \((\alpha_i, \beta) = 2\) in which case \( \beta = \alpha_i \). Using the definition of \( \sigma \) and (43) gives \( \sigma(x_{\alpha}) = l^{-1} x_{\alpha} \). Now \( \sigma(f_i) x_{\alpha} = (l^{-2} + ml^{-1} - 1) x_{\alpha} \).

Suppose \((\alpha_i, \beta) = 0\). Then \( \sigma(x_{\alpha}) = x_{\alpha} h_{\beta, i} + l^{-1} x_{\alpha} T_{i, \beta} \). Now \( \sigma^2 x_{\alpha} = x_{\alpha} h_{\beta, i}^2 + l^{-1} x_{\alpha} T_{i, \beta} h_{\beta, i} + l^{-2} x_{\alpha} T_{i, \beta} \). Evaluating \( \sigma(f_i) \) on \( x_{\alpha} \), and using the Hecke algebra quadratic relation for \( h_{\beta, i} \), gives that the coefficient of \( x_{\alpha} \) is 0. Adding the other terms gives \( l^{-1} x_{\alpha} T_{i, \beta} (h_{\beta, i} + m + l^{-1}) \) as stated.
Suppose \((\alpha_i, \beta) = -1\). Now \(\sigma_i \alpha_i = x_{\beta+\alpha_i} - mx_{\beta} + l^{-1}x_{\alpha_i} T_{i,\beta}\). Applying \(\sigma_i\) again gives \(\sigma_i^2 x_{\beta} = x_{\beta} + l^{-1}x_{\alpha_i} T_{i,\beta} + m(x_{\beta} - mx_{\beta} + l^{-1}x_{\alpha_i} T_{i,\beta}) + l^{-2}x_{\alpha_i} T_{i,\beta}\). Again adding gives the result.

If \((\alpha_i, \beta) = 1, \sigma_i x_{\beta} = x_{\beta - \alpha_i} + l^{-1}x_{\alpha_i} T_{i,\beta}\). Now

\[
\sigma_i^2 x_{\beta} = x_{\beta} - mx_{\beta - \alpha_i} + l^{-1}x_{\alpha_i} T_{i,\beta - \alpha_i} + l^{-2}x_{\alpha_i} T_{i,\beta}.
\]

Adding and again using the quadratic relation gives the result.

The final statement follows from the fact that the \(T_{i,\gamma}\) and \(h_{\beta,\delta}\) belong to \(Z_0^{(1)}[l^{-1}]\) (that is, there is no \(l\) involved).

**Proof of Theorem 6.2.1.** In view of Proposition 6.2.4 we need only check (D1), (R1), (R2), and that \(\sigma(e_i e_j) = 0\) for \(i \neq j\). But (D1) is just the definition. By Lemma 6.2.11 we know \(\sigma(e_i) x_{\beta} \in \text{span} \{x_{\alpha_i}\}\). Now (R1) follows as \(\sigma_i x_{\alpha_i} = l^{-1}x_{\alpha_i}\). For \(i \neq j\) we know \(\sigma(e_i e_j) = \sigma(e_j e_i)\). By Lemma 6.2.11 this is in \(x_{\alpha_i} Z_{0}^{(0)}\) and also in \(x_{\alpha_i} Z_{0}^{(0)}\), and so it is 0. As for (R2) again \(\sigma(e_i) x_{\beta}\) is a multiple of \(x_{\alpha_i}\). Now \(\sigma_j x_{\alpha_i} = x_{\alpha_i + \alpha_j} - mx_{\alpha_i}\). Lemma 6.2.11 gives \(\sigma(j)(x_{\alpha_i + \alpha_j} - mx_{\alpha_i}) = x_{\alpha_i}(l^{-1}(m + l^{-1})m - (l^2 + ml^{-1} - 1)m = mx_{\alpha_i}\). Now scaling to get \(\sigma(e_i)\) gives the result. We have shown that Theorem 6.2.1 holds.

We now show how to construct irreducible representations of \(B(M)\) which have \(I_2\) in the kernel.

**Lemma 6.2.12.** For each node \(i\) of \(M\), we have \(\sigma(Z_{i}^{(0)}) x_{\alpha_i} = x_{\alpha_i} Z_{0}^{(0)}\).

**Proof.** For \(j\) and \(i\) adjacent nodes, the following computation shows \(\sigma_i \sigma_j x_{\alpha_i} = x_{\alpha_j}\).

\[
\sigma_i \sigma_j x_{\alpha_i} = \sigma_i(x_{\alpha_i + \alpha_j} - mx_{\alpha_i}) = x_{\alpha_j} + l^{-1}T_{i,\alpha_i + \alpha_j} x_{\alpha_i} - ml^{-1}x_{\alpha_i} = x_{\alpha_j} + l^{-1}x_{\alpha_i} m - ml^{-1}x_{\alpha_i} = x_{\alpha_j}.
\]

By induction on the length of a path from \(i\) to \(k\) in \(M\), this gives

\[
\sigma(w_{ik}) x_{\alpha_i} = x_{\alpha_k}.
\]

Therefore, for \(j\) and \(k\) distinct nonadjacent nodes of \(M\),

\[
\delta^{-1} \sigma(w_{ik} j \hat{w}_{ik} e_i) x_{\alpha_i} = \sigma(w_{ik} j \hat{w}_{ik} e_i) x_{\alpha_k} = \sigma(w_{ik}) x_{\alpha_k} = \sigma(w_{ik}) x_{\alpha_k} h_{\alpha_k, j} = x_{\alpha_k} h_{\alpha_k, j}.
\]

As \(\sigma(Z_{i}^{(0)})\) is generated by elements of the form \(\sigma(w_{ik} j \hat{w}_{ik} e_i)\), it follows that \(\sigma(Z_{i}^{(0)}) x_{\alpha_i} \subseteq x_{\alpha_i} Z_{0}^{(0)}\). Note it follows from Lemma 6.2.11 that \(\delta^{-1} \sigma(e_i) x_{\alpha_i} = x_{\alpha_i}\).

As for the converse, this follows from Lemma 2.4.8(ii), which implies that \(Z_{0}^{(i)}\) is generated by \(h_{\alpha_k, i}\), for \(i \neq k\), \(i \neq k\). (For, by definition, \(Z_{0}^{(0)}\) is generated by \(\hat{Y}\) mod \(I_2\)).

Suppose \(\theta\) is any representation of \(Z_0\), acting on a vector space \(U\) over \(K\), where \(K = \mathbb{Q}(r)\), or an algebraic extension thereof. Then we can form a representation of \(B(M)\) on the vector space \(V \otimes Z_0 U\) over \(K(l)\) which is the direct sum of vector spaces \(x_{\beta} U\) where each is a vector space isomorphic to \(U\). Let \(V\) be the representation space of Theorem 6.2.1. For each \(i\) define an action of \(\sigma_i\) on \(V \otimes Z_0 U\) by letting elements of \(Z_0\) act directly on \(U\). In particular, \(\sigma_i x_{\alpha_i} u = l^{-1}x_{\alpha_i} u; \) if \((\alpha_i, \beta) = 0\), then \(\sigma_i x_{\beta} u = x_{\beta}(h_{\beta, i}) u + l^{-1}x_{\alpha_i} \theta(T_{i, \beta}) u; \) for \((\alpha_i, \beta) = 1\) we have \(\sigma_i x_{\beta} u = x_{\beta - \alpha_i} u + l^{-1}x_{\alpha_i} \theta(T_{i, \beta}) u\). This is a representation by Theorem 6.2.1. Denote it \(\Gamma_\theta\).
Lemma 6.2.13. If $\theta$ is an irreducible representation of $Z_0^{(0)}$, then the representation $\Gamma_\theta$ is also irreducible. For inequivalent representations $\theta$, $\theta'$, the resulting representations $\Gamma_\theta$ and $\Gamma_{\theta'}$ are also inequivalent.

Proof. Suppose $V_1$ is a proper nontrivial invariant subspace of $V \otimes Z_0 U$. We show first that $\sigma(f_i)V_1 = 0$ for all nodes $i$ of $M$. By Lemma 6.2.11, $\sigma(f_i)V \otimes Z_0 U$ is in $x_\alpha, \theta(Z_0^{(0)}) U$ which is in $x_\alpha U$. This means that $\sigma(f_i)V_1 = 0$. Suppose there is a node $i$ with $\sigma(f_i)V_1$ nonzero. This means there is a nonzero element of $u \in U$ such that $x_\alpha u \in V_1$. In Lemma 6.2.12, we have seen that $Z_0^{(0)} x_\alpha = x_\alpha Z_0^{(0)}$. Hence $x_\alpha, \theta(Z_0^{(0)}) u = Z_0^{(0)} x_\alpha \subseteq V_1$. But $\theta$ is irreducible and so all of $x_\alpha U$ is contained in $V_1$.

By Lemma 6.2.12, $x_\alpha U$ is in $V_1$ for all $k$. We show by induction on the height of a positive root ht$(\beta)$ that $x_\beta U$ is in $V_1$. Assume ht$(\beta) \geq 2$. Choose a node $j$ with $\beta = r_j(\beta - \alpha_j)$. By induction, $x_{\beta - \alpha_j} U$ is in $V_1$. But for each $u \in U$, the vector $\sigma_j x_{\beta - \alpha_j} u$ is a sum of $x_\beta u$ and vectors already known to be in $V_1$ and so $x_\beta U$ is in $V_1$. But this means all of $V \otimes Z_0 U$ is in $V_1$, contradicting that $V_1$ is proper. This shows $\sigma(f_i)V_1 = 0$ for each node $i$.

As $V_1$ is invariant, its image $\sigma(\overline{w_{\beta,j} f_j w_{\beta,j}^{-1}})V_1$ under a conjugate of $\sigma(f_i)$ is also trivial. We will derive from this that $V_1 = 0$. To this end, choose an order on $\Phi^+$ that is consistent with height. For each $\beta$ choose a node $j(\beta)$ in the support of $\beta$. Notice that Lemma 6.2.11 shows that the image of $\sigma(f_i)$ is in $x_\alpha Z_0^{(0)}$. Let $L$ be the matrix whose rows and columns are indexed by $\Phi^+$ in the fixed order and whose $\beta, \gamma$ entry is the coefficient of $x_\beta$ in $\sigma(\overline{w_{\beta,j(\beta)} f_{j(\beta)} w_{\beta,j(\beta)}^{-1}}) x_\gamma$. This means the entries are elements of $\theta(Z_0^{(0)})$. As each $\sigma(\overline{w_{\beta,j(\beta)} f_{j(\beta)} w_{\beta,j(\beta)}^{-1}}) V_1 = 0$, we have $LV_1 = 0$.

Observe that $L$ can be viewed as a matrix with entries in $K[l^{-1}]$ by interpreting the entries from $\theta(Z_0^{(0)})$ as submatrices over $K[l^{-1}]$. We claim that $L$ is nonsingular. By the Lawrence-Krammer action rules, the $\beta, \gamma$ entry of $L$ mod $l^{-1}$ is readily seen to be the coefficient of $x_{\alpha_{j(\beta)}}$ in $\sigma(\overline{f_{j(\beta)} w_{\beta,j(\beta)}}^{-1}) x_\gamma$. If $\beta = \gamma$, then this coefficient is equal to $-1$ modulo $l^{-1}$, and if $\beta$ is less than $\gamma$ in the given order, then there is no summand $x_{\alpha_{j(\beta)}}$ present in the expansion of $\sigma(\overline{w_{\beta,j(\beta)}}^{-1}) x_\gamma$ and so the $\beta, \gamma$ coefficient of $L$ is $0$. This means $L$ modulo $l^{-1}$ is lower-triangular with $-1$ on the diagonal, whence non-singular. Therefore, the equality $LV_1 = 0$ implies $V_1 = 0$. We conclude that there is no invariant subspace and the representation is irreducible.

Finally, we argue that inequivalent $\theta$ lead to inequivalent $\Gamma_\theta$. To this end we consider the trace of each element $\overline{w_z z w_k e_i}$ of $Z_0$ in $\Gamma_\theta$, where $z$ is in $W_{k \perp}$. By Lemma 6.2.11, the only contributions to the trace occur for vectors in $x_\alpha \theta(Z_0)$, and, in view of Lemma 6.2.12, this contribution is $m^{-1}(l^{-1} + m - l^{-1}) \text{tr}(\theta(d_{z \alpha}^{\perp} z d_{z \alpha}))$. Since $d_{z \alpha}^{\perp} z d_{z \alpha}$, for $k$ a node of $M$ and $z \in W_{k \perp}$, span $Z_0$ over $K[l]$, these values uniquely determine $\theta$. \qed

With these results in hand we are now ready to show that the dimension of $I_1/I_2$ is at least the dimension we need for Theorem 6.1.1.
Proof of Theorem 6.1.1. In Theorem 6.2.13 we have constructed irreducible representations \( \Gamma_\theta \) of \( B(M)/I_2 \) of dimension \( |\Phi^+| \dim \theta \) for any irreducible representation \( \theta \) of \( Z_0 \). Since \( I_1 \) is not in the kernel of these representations, they are irreducible representations of \( I_1/I_2 \). Moreover, \( Z_0 \), being a Hecke algebra over \( \mathbb{Q}(l,m) \) of spherical type, is semi-simple, so summing the squares of the dimensions of the irreducibles of \( Z_0 \) gives \( \dim(Z_0) \). Hence the dimension of \( I_1/I_2 \) is at least \( |\Phi^+|^2 \dim(Z_0) \). By Theorem 4.3.5, this is also an upper bound for the dimension, whence equality. The semisimplicity follows as \( B(M)/I_1 \), being the Hecke algebra of type \( M \), is semisimple, and the sum of the squares of the irreducible representations of \( I_1/I_2 \) is the dimension of \( I_1/I_2 \).

To end this section, we observe that the usual Lawrence-Krammer representation is the representation \( \Gamma_\theta \), where \( \theta \) is the linear character of \( Z_0 \) determined by \( \theta(h_{\beta,i}) = r^{-1} \) for all pairs \( (\beta,i) \in \Phi^+ \times M \) with \( (\alpha_i, \beta) = 0 \).

6.3. The monoid action

In the Lawrence-Krammer representation we discussed in the previous section, the coefficients were obtained from the Hecke algebra whose type is the subdiagram of \( M \) induced on the set of nodes \( i \) of \( M \) whose corresponding fundamental root \( \alpha_i \) is orthogonal to the highest root. Here, the coefficients are obtained from the Hecke algebra \( Z \) whose type is the subdiagram of \( M \) induced on the nodes \( i \) whose corresponding fundamental root \( \alpha_i \) is orthogonal to each element of \( B_0 \). Moreover, in the Lawrence-Krammer representation, to each pair of a positive root \( \beta \) and a node \( i \) with corresponding fundamental root \( \alpha_i \) such that \( (\alpha_i, \beta) = 0 \), we assigned an element \( h_{\beta,i} \) of the coefficient algebra. It occurs in the definition of the action of a fundamental generator of the Artin group \( A \) in the Lawrence-Krammer representation, on the basis element \( \beta \).

For the analogous purpose, we use the elements \( h_{B,i} \) (Definition 3.4.1) in the corresponding coefficient algebra \( Z \). These elements are parameterized by pairs consisting of an element \( B \) of \( \mathcal{B} \) and a node \( i \) of \( M \) such that the corresponding fundamental root \( \alpha_i \) is orthogonal to all of \( B \).

Let \( B \) be an admissible \( W \)-orbit of sets of mutually orthogonal positive roots, let \( (B, <) \) be the corresponding monoidal poset (cf. Proposition 3.3.1), let \( B_0 \) be the maximal element of \( (B, <) \) (cf. Corollary 3.3.6), and let \( Y \) be the set of nodes \( i \) of \( M \) with \( \alpha_i \in B_0^+ \). As before (Proposition 3.4.3), \( Z \) is the Hecke algebra over \( \mathbb{Q}(m) \) of type \( Y \). These are listed in Table 1 under column \( Y \). In analogy to § 6.2 we define a free right \( Z \)-module \( V \) with basis \( x_B \) indexed by the elements \( B \) of \( \mathcal{B} \). By Lemma 3.4.2 the linear transformations \( \tau_i \) of (37) are completely determined. We are ready to prove the main theorem.

Proof of Theorem 6.1.2. Let \( M \) be connected (see a remark following the theorem). We need to show that the braid relations hold for \( \tau_i \) and \( \tau_j \), that is, they commute if \( i \not\sim j \) and \( \tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j \) if \( i \sim j \).

Take \( B \in \mathcal{B} \). By linearity, it suffices to check the actions on \( x_B \). We first dispense with the case in which either \( \tau_i x_B \) or \( \tau_j x_B \) is 0. This happens if \( B \) contains \( \alpha_i \) or \( \alpha_j \). If both roots are in \( B \) both images are 0 and the relations hold.

Suppose then that \( \alpha_i \) is in \( B \) but \( \alpha_j \) is not in \( B \). Consider first the case in which \( i \not\sim j \). Then \( \tau_i x_B = 0 \) and so \( \tau_j \tau_i x_B = 0 \). Now \( \tau_j x_B \) is in the span of \( x_B \) and
Notice as \((\alpha_i, \alpha_j) = 0\) that \(\alpha_i\) is in \(r_j B\) as well as \(B\) and so \(\tau_i \tau_j x_B = 0\) also. Suppose \(i \sim j\). Clearly \(\tau_i \tau_j x_B = 0\) as \(\tau_i x_B = 0\). As \(\alpha_i \in B\), the root \(\alpha_i + \alpha_j\) belongs to \(r_j B\). Also \(r_j\) raises \(B\) as a height one element, \(\alpha_i\), becomes height 2. This means \(\tau_j x_B = x_r r_j B - m x_B\). If \(r_i\) lowers \(r_j B\), \(\tau_i x_{r_j B} = x_{r_i r_j B}\). But \(r_i r_j B\) contains \(r_i (\alpha_i + \alpha_j) = \alpha_i\), so \(\tau_i x_{r_i r_j B} = 0\). Also \(\tau_i x_B = 0\) as \(\alpha_i \in B\). This proves the result unless \(\tau_i\) raises \(\tau_j B\). We know \(\tau_i\) takes the root \(\alpha_i + \alpha_j\) to \(\alpha_j\) and so lowers a root of height 2. The only way \(r_i\) could raise \(r_j B\) is if \(r_j B\) contained an \(\alpha_k\) with \(k \sim i\). This would be \(r_j \beta\) for \(\beta \in B\). If \(r_j \beta = \beta\) we would have \(\alpha_k \in B\) but all elements of \(B\) except \(\alpha_i\) are orthogonal to \(\alpha_i\). This means \(\alpha_k\) is not orthogonal to \(\alpha_j\) and we have \(j \sim k\), \(j \sim i\), and \(i \sim k\) a contraction as there are no triangles in the Dynkin diagram. We conclude that the braid relations hold if either \(\tau_i\) or \(\tau_j\) annihilates \(x_B\).

We now consider the cases in which \(i \not\sim j\) with neither \(\alpha_i\) nor \(\alpha_j\) being in \(B\). We wish to show \(\tau_i \tau_j = \tau_j \tau_i\).

We suppose first that both \(\alpha_i\) and \(\alpha_j\) are in \(B^\perp\). This means that \(\tau_i x_B = x_B h_B i\) and that \(\tau_j x_B = x_B h_B j\). We need only ensure that \(h_B i\) and \(h_B j\) commute, which is Proposition 3.4.3(ii). Suppose now that \(\alpha_i\) is in \(B^\perp\) and \(\alpha_j\) is not in \(B^\perp\). In this case \(\tau_j \tau_i x_B = \tau_j x_B h_B i\). Also \(\tau_j x_B = x_{r_j B} - \delta m x_B\) where \(\delta\) is 0 or 1. This gives

\[
\tau_j \tau_i x_B = (x_{r_j B} - \delta m x_B) h_B i.
\]

We also get \(\tau_i \tau_j x_B = \tau_i x_{r_j B} - \delta m \tau_i x_B\). Notice \(\alpha_i \in B^\perp\) and \(i \not\sim j\) imply \(\alpha_i \in (r_j B)^\perp\). In particular

\[
\tau_i \tau_j x_B = x_{r_j B} h_{r_j B} i - \delta m x_B h_{r_j B} i.
\]

In order for this to be \(\tau_j \tau_i x_B\) we need \(h_{r_j B} i = h_{r_j B} i\), which is satisfied by Proposition 3.4.3(iv).

We are left with the case in which neither \(\alpha_i\) nor \(\alpha_j\) is in \(B\) or in \(B^\perp\). In this case the relevant actions are \(\tau_i\) on \(x_B\) and \(\tau_j\) on \(x_B\). If \(r_i B = r_j B\) it is clear \(\tau_i\) and \(\tau_j\) commute. This gives the table

<table>
<thead>
<tr>
<th>(\tau_i) on (x_B)</th>
<th>(\tau_j) on (x_B)</th>
<th>(\tau_i \tau_j x_B = \tau_j \tau_i x_B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>lower (x_{r_i B})</td>
<td>lower (x_{r_j B})</td>
<td>(x_{r_i r_j B})</td>
</tr>
<tr>
<td>lower (x_{r_i r_j B} - m x_{r_i B} - m^2 x_B)</td>
<td>raise (x_{r_i r_j B} - m x_{r_i B} - m^2 x_B)</td>
<td></td>
</tr>
</tbody>
</table>

Notice that \(\alpha_i \not\in (r_j B)^\perp\) as \(\alpha_i \not\in B^\perp\). Similarly \(\alpha_j \not\in (r_j B)^\perp\).

Suppose first that \(\tau_i\) and \(\tau_j\) both lower \(B\). By Proposition 3.3.1(iii) this means \(\tau_i\) also lowers \(r_j B\) and \(\tau_j\) lowers \(r_j B\). Now

\[
\tau_i \tau_j x_B = \tau_i x_{r_j B} = x_{r_i r_j B}.
\]

The same result occurs in the reverse order as \(r_i\) and \(r_j\) commute.

Suppose next that \(\tau_i\) and \(\tau_j\) both raise \(B\). Then by Lemma 3.3.4(ii), \(\tau_i\) raises \(r_j B\) and \(\tau_j\) raises \(r_i B\). In particular we have

\[
\tau_j \tau_i x_B = \tau_j(x_{r_i B} - m x_B) = x_{r_i r_j B} - m x_{r_i B} - m^2 x_B + m^2 x_B.
\]

The same is true for the reverse order.
Suppose then \( \tau_i \) lowers \( B \) and \( \tau_j \) raises \( B \). By Lemma 3.3.4(i), applied to \( \{ \tau_i B < B < \tau_j B \} \), the reflection \( \tau_i \) also lowers \( r_j B \) and \( r_j \) raises \( r_i B \). This means
\[
\tau_i \tau_j x_B = \tau_i(x_{r_i B} - mx_B) = x_{r_i r_j B} - mx_{r_i} B.
\]

In the other order
\[
\tau_j \tau_i x_B = \tau_j x_{r_i B} = x_{r_i r_j B} - mx_{r_i} B.
\]
These are the same. Notice here the assumptions imply \( r_i B \neq r_j B \) and \( r_i r_j B \neq B \). We conclude that \( \tau_i \) and \( \tau_j \) commute whenever \( i \neq j \).

We now suppose \( i \sim j \) and wish to show \( \pi_i \tau_j \tau_i = \tau_j \tau_i \tau_j \). Suppose first \( \alpha_i \) and \( \alpha_j \) are in \( B^\perp \). Then \( \tau_i x_B = x_B h_{B,i} \) and \( \tau_j x_B = x_B h_{B,j} \). The condition needed is
\[
h_{B,i} h_{B,j} h_{B,i} = h_{B,j} h_{B,i} h_{B,j},
\]
which is Proposition 3.4.3(iii).

Suppose now \( i \sim j \) and \( \alpha_i \in B^\perp \) but \( \alpha_j \notin B^\perp \). We are still assuming neither \( \alpha_i \) nor \( \alpha_j \) is in \( B \). The relevant data here are the actions of \( r_j \) on \( B \) and \( r_i \) on \( B \).

The table below handles the cases where \( r_j \) lowers \( B \) and \( r_i \) lowers \( B \) as well as those where \( r_j \) raises \( B \) and \( r_i \) raises \( B \).

The other cases, of \( r_i \) raising \( r_j B \) when \( r_j \) lowers \( B \) and of \( r_i \) lowering \( r_j B \) when \( r_j \) raise \( B \), are ruled out by Condition (ii) of Proposition 3.3.1.

<table>
<thead>
<tr>
<th>( r_i ) on ( r_j B )</th>
<th>( r_j ) on ( B )</th>
<th>( \tau_i \tau_j \tau_i x_B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>lower raise</td>
<td>lower raise</td>
<td>( x_{r_i r_j B} h_{r_i r_j B,j} = x_{r_i r_j B} h_{B,i} + m^2 x_B h_{B,i} )</td>
</tr>
</tbody>
</table>

Notice that \( \alpha_i \notin (r_j B)^\perp \) as if \( (\alpha_j, \beta) \neq 0 \), then \( (\alpha_i, \beta - (\alpha_j, \beta) \alpha_j) = (\alpha_j, \beta) \neq 0 \).

Suppose first \( r_j \) lowers \( B \) and \( r_i \) lowers \( r_j B \) as in the first row. Then
\[
\tau_j \tau_i \tau_j x_B = \tau_j \tau_i x_{r_i B} = \tau_j x_{r_i r_j B} = x_{r_i r_j B} h_{r_i r_j B,j}.
\]

Note here \( \alpha_j \in (r_j B)^\perp \) by application of \( r_j r_i \) to \( \alpha_i \in B^\perp \). Also
\[
\tau_i \tau_j \tau_i x_B = \tau_i \tau_j x_{B,j} = \tau_i x_{r_j B} h_{B,j} = x_{r_j B} h_{B,j} = x_{r_j B} h_{B,j} + m^2 x_B h_{B,i}.
\]

Now the braid relation is satisfied according to Proposition 3.4.3(v).

Suppose \( r_j \) raises \( B \) and \( r_i \) raises \( B \).
\[
\tau_i \tau_j \tau_i x_B = \tau_i \tau_j x_{B} = \tau_i(x_{r_j B} - mx_B) = x_{r_j r_i B} - mx_{r_j B} h_{B,i} + m^2 x_B h_{B,i}.
\]

Here we used \( h_{B,i}^2 = 1 - mh_{B,i} \). In the other order we have
\[
\tau_j \tau_i \tau_j x_B = \tau_j \tau_i(x_{r_j B} - mx_B) = \tau_j(x_{r_j r_i B} - mx_{r_j B} h_{B,i} + m^2 x_B h_{B,i})
\]
\[
= x_{r_j r_i B} h_{r_j r_i B,j} = x_{r_j r_i B} h_{B,i} = x_{r_j r_i B} h_{B,i} + m^2 x_B h_{B,i}.
\]

Once again we need \( h_{r_j r_i B,j} = h_{B,i} \) which is Proposition 3.4.3(v).
We can finally consider the case in which \( i \sim j \) and neither \( \alpha_i \) nor \( \alpha_j \) is in \( B^+ \cup B \). Here relevant data are the actions of \( r_i \) and \( r_j \) on \( B \), where for the first row we assume \( \alpha_i + \alpha_j \notin B \) (for otherwise, each side equals zero).

<table>
<thead>
<tr>
<th>( r_i ) on ( B )</th>
<th>( r_j ) on ( B )</th>
<th>( \tau_i \tau_j \tau_i x_B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>lower</td>
<td>lower</td>
<td>( x_{r_ir_j},B )</td>
</tr>
<tr>
<td>lower</td>
<td>raise</td>
<td>done below</td>
</tr>
<tr>
<td>raise</td>
<td>raise</td>
<td>( x_{r_ir_j},B )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( + m^2(x_{r_i}B + x_{r_j}B - (m^3 + m)x_B) )</td>
</tr>
</tbody>
</table>

We start with the first row in which both \( r_i \) and \( r_j \) lower \( B \). We may assume \( r_i B \neq r_j B \) or \( \tau_i \) and \( \tau_j \) act on \( x_B \) and \( x_{r_i}B \) in the same way. By Proposition 3.3.1(iv) and Lemma 3.3.2 all the actions we encounter are lowering actions. Therefore,

\[
\tau_i \tau_j \tau_i x_B = \tau_i \tau_j x_{r_i}B = \tau_i x_{r_i}x_{r_j}B = x_{r_j}x_{r_i}B.
\]

This gives the same result with the other product.

Next take the bottom row in which both \( r_i \) and \( r_j \) raise \( B \). By Lemma 3.3.4 (iii), the actions we encounter are all raising actions.

\[
\tau_i \tau_j x_B = \tau_i \tau_j (x_{r_i}B - mx_B) = \tau_i (x_{r_i}r_jB - mx_{r_j}B - m(x_{r_i}B - mx_B)) = x_{r_i}r_jx_{r_j}B - mx_{r_j}B - m^2x_{r_i}B - (m^3 + m)x_B.
\]

This also gives the same result with the other product.

We now tackle the remaining cases. Here \( r_i \) lowers \( B \) and \( r_j \) raises \( B \). There are two cases depending on how \( r_j \) acts on \( r_i B \).

<table>
<thead>
<tr>
<th>( r_i ) on ( B )</th>
<th>( r_j ) on ( B )</th>
<th>( \tau_i \tau_j \tau_i x_B = \tau_j \tau_i \tau_i x_B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>lower</td>
<td>lower</td>
<td>( x_{r_i}r_jx_{r_j}B - m x_{r_j}r_iB )</td>
</tr>
<tr>
<td>lower</td>
<td>raise</td>
<td>( x_{r_i}r_jx_{r_j}B - m x_{r_j}r_iB )</td>
</tr>
</tbody>
</table>

Consider first the second row, where \( r_j \) lowers \( r_i B \). By the Lemma 3.3.4(iv) applied to \( r_i r_j B \), this means \( r_i \) raises \( r_j r_i B \) and the remaining raising and lowering actions can be determined by this. Notice \( r_j r_i B \neq B \), for otherwise \( r_i B = r_j B \) which is not consistent with the assumption.

\[
\tau_i \tau_j x_B = \tau_i \tau_j x_{r_i}B = \tau_i x_{r_i}r_jB = x_{r_i}r_jx_{r_j}B
\]

For the other product

\[
\tau_j \tau_i x_B = \tau_j \tau_i (x_{r_j}B - mx_B) = \tau_j (x_{r_j}r_iB - mx_{r_i}B) = x_{r_j}r_ix_{r_i}B - mx_{r_i}r_jB
\]

These are the same as indicated in the table.
For the first row suppose next that \( r_j \) raises \( r_i B \). By Lemma 3.3.4(iii) applied to \( r_i B \), \( r_j \) raises \( r_i B \), \( r_i \) raises \( r_j r_i B \), \( r_j \) lowers \( r_i r_j B \) and \( r_i \) raises \( r_j B \). Again we use \( B \neq r_j r_i B \).

\[
\tau_i \tau_j \tau_i x_B = \tau_i \tau_j x_{r_i B} \\
= \tau_i (x_{r_j r_i B} - m x_{r_i B}) \\
= x_{r_j r_i r_i B} - m x_{r_j r_i B} - m (x_B - m x_{r_i B}).
\]

For the other product

\[
\tau_j \tau_i \tau_j x_B = \tau_j \tau_i (x_{r_j B} - m x_B) \\
= \tau_j (x_{r_i r_j B} - m x_{r_j B} - m x_{r_i B}) \\
= x_{r_j r_i r_j B} - m x_B - m (x_{r_j r_i B} - m x_{r_i B}).
\]

This gives the same for either product.

These are also the same as indicated in the table finishing the last case. In particular Theorem 6.1.2 has been proven.

\[\square\]

### 6.4. More LK-representations

In Chapter 5, it was shown that the BMW algebra of type \( D \) does not collapse and a lower bound was established for its dimension over \( \mathbb{Q}(l, \delta) \). For \( B(A_n) \) the dimension over \( \mathbb{Q}(l, \delta) \) was already established by [BW], so only for the BMW algebras of simply laced type \( E_6, E_7 \) and \( E_8 \) we have not been able to establish such a lower bound for their dimension. Attempts to construct a tangle algebra variation were not successful.

In this chapter we sketch an alternative approach which could lead to the proof that these algebras also really exist in a non trivial way.

For this purpose we follow the construction of the Lawrence-Krammer representation of § 6.2 in an attempt to find a general description of an irreducible representation which factors through \( B(M)/I_r \).

It remains an open problem whether the algorithm described here indeed gives a unique representation. Using the rules described in Proposition 6.4.3 and listed in Table 2 we might be able to prove that the algorithm always gives an unique solution. Unfortunately the rewrite rules do not imply this in a straightforward way. Calculations for small cases like \( M = D_4 \) and \( M = D_5 \) show that in these cases the algorithm works fine. This might indicate that the following conjecture indeed holds and can be proven in the near future.

**Conjecture 6.4.1.** Set \( T_{i,B} = 1 \) when \( \alpha_i \in B \). For \( \sigma_i \mapsto \tau_i + l^{-1} T_i \) to define a linear representation of the group \( A \) on \( V \), it is necessary and sufficient that the equations in Table 2 are satisfied for each \( k, j = 1, \ldots, n \) and each \( \beta \in \Phi^+ \).

As in § 6.2, we take the coefficients of our representation in the Hecke algebra \( Z_0^{(0)} \) of type \( M|_\gamma \) over the subdomain \( \mathbb{Q}[l^{\pm 1}, m] \) of \( \mathbb{Q}(l, x) \). We write \( V^{(0)} \) for the free right \( Z_0^{(0)} \) module with basis \( x_B \) indexed by \( B \in B \). Throughout this section we use the properties of the elements \( h_{B,i} \) listed in Lemma 2.4.7.

The representation we want to construct is defined \( \sigma_k = \tau_k + l^{-1} T_k \), where \( T_k x_B = \sum_{\alpha_i \in C} T_{k,B,C} x_{C} \). Here \( \tau_k \) is the in § 6.3 described monoid action. We study the possibilities for the parameters \( T_{k,B,C} \).
Remark 6.4.2. In Chapter 3 we used \( C \) to denote a fundamental set in an orbit \( \mathcal{B} \). Here the sets \( C \) and \( D \) are not necessarily fundamental. The only property the sets carry here is that they contain at least one fundamental root.

We show that the \( T_{k,B,C} \) satisfy the equations as listed in Table 2.

\[
\begin{array}{|c|c|c|}
\hline
T_{i,B,C} & \text{condition} & \text{reference} \\
\hline
1 & \alpha_i \in B \text{ and } C = B & (60) \\
0 & \alpha_i \in B \text{ and } C \neq B & (61) \\
0 & \alpha_j \in B \text{ with } i \sim j & (62) \\
m & \alpha_i + \alpha_j \in B \text{ and } C = r_j B & (63) \\
0 & \alpha_i + \alpha_j \in B \text{ and } C \neq r_j B & (64) \\
\hline
\end{array}
\]

\[
T_{j,r_j,B,D} - mT_{j,r_j,B,D} + mT_{i,r_j,B,C} \\
T_{j,r_j,B,D} + mT_{i,r_j,B,C} \\
T_{j,r_j,B,D} + mT_{i,r_j,B,C} \\
\hline
\exists_{j \neq i} \text{ with } r_j B < B & & (66) \\
\exists_{j \neq i} \text{ with } r_j B < B \text{ and } \alpha_i \in C & & (65) \\
mT_{j,r_j,B,C} + h_{j,B,B}T_{j,r_j,B,C} & & \exists_{j \neq i} \text{ with } r_j B < B \text{ and } \alpha_i \notin C & & (68) \\
mT_{i,B,C} + T_{j,r_j,B,C} & & \alpha_i \in r_j B^+ & & (69) \\
f_{j,B} = r_j B & & \alpha_i \in r_j B & & (70) \\
f_{j,B} = r_j B & & r_j B > r_i B & & (71) \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\hline
T_{i,B,C} & \text{condition} \\
\hline
1 & \alpha_j \in C^+ & (72) \\
0 & r_j C < C & (73) \\
0 & r_j C > C & (74) \\
0 & \exists_{i \neq j} \text{ with } r_j B \notin B & & (75) \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
h_{j,B}T_{i,B} & \text{condition} \\
\hline
r_j B < B \text{ and } \alpha_j \in B^+ \text{ with } j \sim i & (76) \\
r_i B < B \text{ and } j \sim i & (77) \\
\text{with } r_j B > B \text{ and } r_j r_i B < r_i B & & (81) \\
\alpha_j \in B \text{ and } i \neq k \sim j & (81) \\
\alpha_j \in B \text{ and } i \sim k \sim j & (84) \\
\hline
\end{array}
\]

Table 2. Equations for \( T_{i,B} \).

Proposition 6.4.3. Set \( T_i x_B = x_B \) if \( \alpha_i \in B \) for all \( i \in \{1, \ldots, n\} \).

For \( \sigma_i \mapsto \tau_i + l^{-1}T_i \) to satisfy the braid relations (B1), (B2) on \( V \), it is necessary and sufficient that the equations in Table 2 are satisfied for each \( k, j = 1, \ldots, n \) and each \( B \in \mathcal{B} \).

Proof. To check the \( \sigma_k \) satisfy the relations (B1), (B2) we substitute \( \tau_k + l^{-1}T_k \) for \( s_k \), we find relations for the coefficients of \( l^{-1} \) with \( i = 0, 1, 2, 3 \). The constant part involves only the \( \tau_k \). It follows from Theorem 6.1.2 that these equations are satisfied. We shall derive all the equations of Table 2 from the \( l^{-1} \)-linear part and show they hold for the other linear parts.

The coefficients of \( l^{-1} \) lead to

\[
T_i \tau_j + \tau_i T_j = \tau_j T_i + T_j \tau_i \quad \text{if} \quad i \neq j, \tag{58}
\]

\[
T_i \tau_j + T_j \tau_i = \tau_i T_j + T_j \tau_i + \tau_j T_i \quad \text{if} \quad i \sim j. \tag{59}
\]

When \( \alpha_i \notin B \) we have set \( T_i x_B = x_B \), so

\[
T_{i,C,C} = 1 \tag{60}
\]
and

\[ T_{i,B,C} = 0 \]  

when \( C \neq B \).

When \( \alpha_j \in B \) with \( i \sim j \) we use (59) to find

\[
x_{r_i r_j B} = \sum_{\alpha_i \in C} T_{i,B,C} x_{r_i r_j C} + \sum_{\alpha_j \in D} T_{j,r_i B,D} (x_{r_i D} - m x_D) + m^2 x_B - m x_{r_i B} + x_{r_j r_i B}.
\]

Comparison of the coefficients of \( x_K \), with \( K \) containing \( \alpha_i \), \( \alpha_j \) or \( \alpha_i + \alpha_j \) respectively, gives

\[ T_{i,B,C} = 0 \]  

and

\[ T_{j,r_i B,D} = m \text{ if } D = B \]  

\[ T_{j,r_i B,D} = 0 \text{ if } D \neq B \]  

For the remainder of the proof assume that \( \alpha_i \), \( \alpha_j \) nor \( \alpha_i + \alpha_j \) with \( j \sim i \) are elements of \( B \).

Assume there is a \( j \sim i \) with \( r_j B < B \). From the poset relations we know that the following cases arise:

(i) \( \alpha_i \in B^+ \) with \( r_i r_j B < r_j B \),
(ii) \( r_i B < B \) with \( r_i r_j B < r_j B \) and \( r_i r_j r_i B < r_j r_i B < r_i B \),
(iii) \( r_i B > B \) and \( \alpha_i \in r_j B^+ \),
(iv) \( r_i B > B \) with \( r_i r_j B > r_j B \) and \( r_i r_j B > r_i B \) or
(v) \( r_i B > B \) with \( r_i r_j B < r_j B \) and \( r_i r_j B < r_i B \).

In general the left hand side (LHS) of (59) for \( \sigma_i \) and \( \sigma_j \) on \( x_B \) is

\[ LHS = T_j \tau_i x_{r_i B} + \sum_{\alpha_j \in D} T_{j,B,D} x_{r_j r_i D} + \sum_{\alpha_i \in C} T_{i,r_i B,C} (x_{r_i C} - m x_C) \]

and the right hand side (RHS) is

\[ RHS = \tau_i T_j \tau_i x_B + T_i \tau_j \tau_i x_B + \sum_{\alpha_i \in C} T_{i,B,C} x_{r_i r_j C}. \]

When we substitute the five possible cases, we obtain, from the comparison of the coefficients of \( x_D \) with \( \alpha_j \in D \) or \( x_{r_i D} = x_{r_j C} \) with \( \alpha_j + \alpha_i \in r_i D \),

\[ T_{i,B,C} = T_{j,r_i r_j B,D} + m T_{i,r_j B,C} \]

for the cases (i), (ii) and (v). For case (iii) we find
(66) \[ T_{i,B,C} = T_{j,r_j,B,D}h_{r_j,B,i} + mT_{i,r_j,B,C} \]

and finally for case (iv) we get

(67) \[ T_{i,B,C} = T_{j,r_j,B,D} - mT_{j,r_j,B,D} + mT_{i,r_j,B,C}. \]

For the remainder of the proof assume for all \( j \sim i \) we have \( r_j B \neq B \). Suppose there exists a \( j \not\sim i \) with \( r_j B < B \). Again, from the poset relations we know the following cases arise:

(i) \( \alpha_i \in B^\perp \),

(ii) \( r_i B > B \) and \( r_j B < r_j B < r_i B \),

(iii) \( r_i B < B \) and \( r_j B = r_i B \) or

(iv) \( r_i B < B \) and \( r_j B > r_j r_j B = r_j r_i B < r_i B \).

We use (58) for \( \sigma_i \) and \( \sigma_j \) on \( x_{r_j B} \) to find the right equalities here.

In general we have

\[ T_j x_{r_j B} = \sum_{\alpha_j \in D} T_{j,r_j,B,D} x_D + \sum_{\alpha_i \in C} (T_{i,B,C} - mT_{i,r_j,B,C}) x_C - T_{i,r_j,B,C} \tau_j x_C \]

The coefficient for the \( x_C \) with both \( \alpha_i, \alpha_j \in C \) now comes from the equality

\[ \sum_{\alpha_i, \alpha_j \in C} T_{i,B,C} x_C = T_j x_{r_j B} + m \sum_{\alpha_i, \alpha_j \in C} T_{i,r_j,B,C} x_C \]

and the coefficient for the \( x_C \) with \( \alpha_i \in C \) but \( \alpha_j \notin C \) from the equality

\[ \sum_{\alpha_i \in C} T_{i,B,C} x_C = \sum_{\alpha_i \in C} (T_{i,r_j,B,C} \tau_j + mT_{i,r_j,B,C}) x_C. \]

In the four cases this results in the following relations if \( \alpha_i, \alpha_j \in C \)

(68) \[ T_{i,B,C} = mT_{i,r_j,B,C} + T_{j,r_j,B,C} h_{r_j,B,i}, \]

(69) \[ T_{i,B,C} = mT_{i,r_j,B,C} + T_{j,r_j,B,C}, \]

(70) \[ T_{i,B,C} = mT_{i,r_j,B,C} + T_{j,B,C} - mT_{i,r_j,B,C}, \]

(71) \[ T_{i,B,C} = mT_{i,r_j,B,C} + T_{j,r_j,B,C} - mT_{j,r_j,B,C}. \]

If \( \alpha_i \in C \) and \( \alpha_j \notin C \) we find

(72) \[ T_{i,B,C} = T_{j,r_j,B,C} h_{r_j,B,i}^{-1} \quad \text{if} \quad \alpha_j \in C^\perp, \]

(73) \[ T_{i,B,C} = T_{i,r_j,B,C} + mT_{i,r_j,B,C} \quad \text{if} \quad r_j C < C, \]

(74) \[ T_{i,B,C} = T_{i,r_j,B,C} \quad \text{if} \quad r_j C > C. \]

For case (ii) we check also (58) for \( \sigma_i \) and \( \sigma_j \) on \( x_B \). Here we find

\[ T_{i,r_j,B,C} = T_{j,r_i,B,C} \]
when \( \alpha_i, \alpha_j \in C \). Together with (70) this gives

\[ T_{i,B,C} = T_{j,B,C}. \]

Next assume \( r_j B \not< B \) for all \( j \neq i \). Hence, \( r_i B < B \) or \( B \) is minimal.

Let \( r_i B < B \) and assume there is a \( j \sim i \) with \( \alpha_j \in B^\perp \). We find for the left hand side (LHS) of (59) for \( \sigma_i \) and \( \sigma_j \) on \( x_B \)

\[ LHS = \sum_{\alpha_j \in D} (T_{j,B,D}x_{r_j r_i,D} + T_{j,r_i B,D}x_Dh_{B,j}) + \sum_{\alpha_i \in C} T_{i,B,C}(x_{r_j C} - mx_C)h_{B,j}. \]

The right hand side (RHS) becomes

\[ RHS = \sum_{\alpha_j \in D} T_{j,r_i r_j B,D}(x_{r_j D} - mx_D) + \sum_{\alpha_i \in C} (T_{i,B,C}x_{r_j r_i C} + T_{i,r_j r_i B,D}x_C). \]

Comparison of the coefficients for \( x_{r_i D} = x_{r_j C} \) with \( \alpha_i + \alpha_j \in r_i D \) gives

\[ T_{i,B,C} = h_{B,j}^{-1}T_{j,r_i B,r_j r_i C}. \]

Next assume there is no \( j \sim i \) with \( \alpha_j \in B^\perp \). Suppose there is a \( j \sim i \) with \( r_j B > B \) and \( r_j r_i B > r_i B \). Then (59) for \( \sigma_i \) and \( \sigma_j \) on \( x_B \) gives

\[ T_{i,B,C} = T_{j,r_i r_j B,D}. \]

again after comparison of the coefficients for \( x_{r_i D} = x_{r_j C} \) with \( \alpha_i + \alpha_j \in r_i D \).

Notice that \( r_i B \) is minimal when \( r_j r_i B > r_i B \) for all nodes adjacent to \( i \) as no other reflection can lower \( r_i B \). A minimal set always contains at least one fundamental root. Here \( B \) contains the same fundamental roots as \( r_i B \). If this was not the case, then the reflection \( r_i \) increases a fundamental root \( \alpha_k \) which implies \( r_k B < B \).

So in the remaining case we have a set \( B \) containing a fundamental root \( \alpha_j \) with \( j \neq i \). We introduce a third node \( k \) adjacent to \( j \), so \( r_k B > B \) and \( r_j r_k B < r_k B \). We distinguish two options, \( k \neq i \) and \( k \sim i \). First consider the case where \( i \not\sim k \).

We check (58) for \( \sigma_j \) and \( \sigma_i \) on \( x_B \) and find

\[ \tau_i x_B = T_{j} \tau_i x_B + \sum_{\alpha_i \in C} T_{i,B,C} \tau_j x_C. \]

As \( \tau_i x_B \) and \( T_j \tau_i x_B \) only give non-zero coefficients for \( x_K \) with \( \alpha_j \in K \) and \( \alpha_i \notin K \), this results in

\[ \sum_{\alpha_i \in C} T_{i,B,C} \tau_j x_C = 0. \]

A case by case check gives

\[ T_{i,B,C} = 0 \quad \text{if} \quad \alpha_j \not\in C. \]

When we check (58) for \( \sigma_k \) and \( \sigma_i \) on \( x_{r_j B} \) we find,
6. GENERALIZED LAWRENCE-KRAMMER REPRESENTATIONS

Figure 1. The partial poset when $\alpha_j \in B$ and $i \sim j \sim k$.

\[
m r_i x_{r_j r_k B} + \sum_{\alpha_i \in C} T_{i,B,C} x_C = T_k \tau_i x_{r_k B} + \sum_{\alpha_i \in C} T_{i,r_k B,C} x_C.
\]

Hence, by (78),

(79) \[ T_{i,B,C} = T_{i,r_j r_k r_x B,C} \text{ if } \alpha_j \in C. \]

The same computations for $\sigma_j$ and $\sigma_i$ on $r_j r_k B$, with $\alpha_k \in r_j r_k B$, give

(80) \[ T_{i,r_j r_k r_x B,r_j r_k C} = T_{i,r_j r_k B,r_j r_k C} = T_{i,r_k B,r_k C}. \]

Combining (79) and (80) gives the desired equality,

(81) \[ T_{i,B,C} = T_{i,r_j r_k B,r_j r_k C} \text{ if } \alpha_j \in C. \]

This leaves the final case where $i \sim k \sim j$. Besides $r_k B > B$ and $r_j r_k B < r_k B$ we have $r_i r_k B > r_k B$, $r_i r_j r_k B > r_j r_k B$ and $r_k r_j r_i r_k B < r_j r_i r_k B < r_i r_k B$.

We have the partial poset as depicted in Figure 1. Notice that $\alpha_j \in r_j r_k B$ and $\alpha_i \in r_k r_j r_k B$.

By (78), $T_{i,B,C} = 0$ when $\alpha_j \notin C$ and $T_{j,r_i r_j r_k B,D} = 0$ when $\alpha_i \notin D$. We check (59) for $\sigma_i$ and $\sigma_k$ on $x_{r_k B}$. The left hand side (LHS) of the equality becomes

\[ LHS = T_k \tau_i x_B + m x_{r_k B} + \sum_{\alpha_i, \alpha_j \in C} T_{i,B,C}(x_{r_k C} - m x_C). \]

The right hand side (RHS) equals
\[
RHS = T_{i,r_k,r_k,B} - m^2 x_{r_i,r_j,r_k,B} + m^3 x_{r_j,r_k,B} - m \sum_{\alpha_i, \alpha_j \in C} T_{i,B,C} \alpha C \\
+ \sum_{\alpha_i \in C} T_{i,r_k,B,C} \alpha r_i r_k C + \sum_{\alpha_k \in D} T_{k,r_i,r_k,B,D} (x_{r_i,D} - m x_D).
\]

Comparison of the coefficients for \(x_K\) with \(\alpha_i + \alpha_k, \alpha_j + \alpha_k \in K\) gives

\[(82) \quad T_{i,B,C} = T_{k,r_i,r_k,B,r_i r_k C}.
\]

The same calculations for \(\sigma_j\) and \(\sigma_k\) on \(x_{r_j,r_i,r_k,B}\) result in a similar way in

\[(83) \quad T_{j,r_i,r_k,B,C} = T_{k,r_i,r_k,B,r_j r_k C}.
\]

As both \(\alpha_i, \alpha_j \in C\) we have \(r_i r_k C = r_j r_k C\) and combining (82) and (83) gives the desired result

\[(84) \quad T_{i,B,C} = T_{j,r_i,r_j,r_k,B,C}
\]

for all \(C\) with \(\alpha_i, \alpha_j \in C\) and \(i \sim k \sim j\). \qed
Epilogue

The research in this thesis gives rise to new questions and topics of further research. It seems likely that it will not take long before the open problems which arose in Chapter 4 and 6 are solved, the conjectures are proven and the dimension of the BMW algebras of simply laced type is determined. Also the connection between the tangles and the admissible orbit with the poset structure seem to have more connections than the ones discussed here.

The BMW algebras of simply laced type and the tangle algebras $KT(D)_n$ described in this thesis are not only topics which require more study themselves, but also might inspire further research in some other, related mathematical areas.

A good illustration of such a relation are the Temperley-Lieb algebras. They arise not only as a quotient of the Hecke algebra, but also as a subalgebra of the ideal $I_1$ of the BMW algebra. The Temperley-Lieb algebras were originally defined in the context of Potts models in statistical mechanics. But they also have connections with many other areas of mathematics and physics. One of them is knot theory, as the algebras are related to link invariants like the Jones polynomial and the Kauffman polynomial.

The Temperley-Lieb algebras also occur as a natural quotient of the Hecke algebra. Like the Hecke algebra, the Temperley-Lieb algebras were originally defined as one infinite family of algebras. Later these algebras were related to the Coxeter graph of type $A_n$ and the definition as a Hecke algebra quotient was generalized to obtain Temperley-Lieb algebras $TL(M)$ for $M$ an arbitrary Coxeter graph. Graham, in [Gra], studied these generalized Temperley-Lieb algebras and classified those with irreducible $M$ as belonging to seven finite dimensional families, $M = A, B, D, E, F, H$ or $I$.

In [Gre], Green gives presentations by diagrams of the Temperley-Lieb algebras of type B and D. From these diagrams it is straightforward that $TL(D_n)$ arises as a subalgebra, spanned by diagrams without crossings, of the Brauer algebra variation $C(D)_n$ introduced in § 5.6. As the Hecke algebra $H(M)$ is isomorphic to $B(M)/I_1$, the Temperley-Lieb algebra is a natural quotient of $B(M)/I_1$. But the algebra also occurs as a subalgebra of $I_1$.

We finish by mentioning some ideas and topics of further research which arose during our study of the algebras. For Coxeter diagrams that are not simply laced, we expect a natural BMW algebra to exist as well. For type $B_n$, an approach is given in [Har]. More generally, by means of a folding $\varphi : M \rightarrow M'$ of Coxeter diagrams, a BMW algebra of spherical type $M'$ could be constructed as the subalgebra of $B(M)$ generated by suitable products of $g_i$ for $g_i \in \varphi^{-1}(a)$, one for each $a \in M'$, in much the same way the Artin group of type $M'$ in embedded into the one of
type $M$, see [Cri]. However, further research is needed to see if this definition is independent (up to isomorphism) of the choice of $\varphi$ for fixed $M'$, as well as to find an intrinsic definition of this algebra.

The BMW algebras of type $A_n$ play a role in algebraic topology, in particular, in the theory of knots. The versions of spherical type ADE are related to the topology of the quotient space by $W$ of the complement of the union of all reflection hyperplanes in the complexified space of the reflection representation of $(W, R)$.

After all, by [Bri], the Artin group is the fundamental group of this space. A direct relationship, for instance, a definition of the BMW algebra in terms of this topology, would be of interest.

Brauer algebras play a role in tensor categories for the representations of classical Lie groups, and the corresponding BMW algebras seem to play a similar role for the related quantum groups. It is conceivable that the new BMW algebras constructed in this thesis play a similar role for the tensor categories of representations of quantum groups for the other types.

Indications for the validity of our theorems were first found by experimental computations in GBNP, [CG]. However, the sheer size of the algebras involved makes the computations difficult. For instance, the dimension of $I_1/I_2$ in $B(E_8)$ is equal to 4180377600.

All of the above seem promising paths which not only can lead to the better understanding of the BMW algebras of simply laced type and their role in different parts of mathematics, but can also eventually lead to the description and understanding of BMW algebras of arbitrary Coxeter type.
Appendix. An admissible orbit in $\Phi_{D_5}^+$

This appendix is an example which illustrates most of the research of this thesis. Let $M = D_5$ and fix the coclique $C = \{3, 5\}$ of the Coxeter graph $D_5$. The coclique $C$ is matched to the admissible closure $C = \{\alpha_3, \alpha_5\}$, a set of two orthogonal positive roots contained in the admissible orbit $B$ with maximal element $\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4\}$ of size 60.

In Table 3 all 60 pairs of positive roots in the orbit are listed. Each root is described as a linear combination of the five fundamental roots of the root system of type $D_5$. To simplify the table and the pictures, only the indices of the fundamental roots are displayed. So $23, 123^24$ represents the orthogonal pair $\alpha_2 + \alpha_3$ and $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$. For each pair its height and distance to the fundamental set $C$ is listed. Also the simple reflections which move each set are determined.

Next, we illustrate the poset structure of this admissible $W$-orbit. On three pages the poset structure of $B$ is depicted. The three pages should be treated as separate layers lying on top of each other, making it a three dimensional construction. The first layer depicted should be regarded as the top layer, the second as the middle layer and the two pieces on the third page are together the bottom layer. The three layers should be positioned in such a way that the four squares of four sets they all have in common are directly above each other. So, $34, 1345$ should be on top of $3, 1345$ which should be directly positioned above $3, 134$.

The edges connecting two sets are labelled by the simple reflections of the Weyl group which action takes these sets to each other. The direction of the edges is related to the height of the sets. Movement in the displayed direction means the height of the set increases. See also Table 3 for the complete list of elements of $B$ and the reflections moving each set.

There are three different kinds of edges which are connected to only one set. The ones which start at filled dot and are directed towards the set indicate that the related simple reflections centralize the set it is pointing to. The other two types are edges which connect two sets on different layers, so two sets on different pages. An edge with an arrow pointing outwards indicates that the set is connected to a set directly above itself on the higher layer. The edges with an arrow at the end point where the edge is attached to a set indicate a connection with a set directly below itself on the lower layer. These last two kinds of edges obviously occur in pairs. The greyscale colouring is used to classify the pairs by their height.

Notice that on the second page the set $3, 123^24^25$ of height 2 is marked. This is the only minimal set in this poset which is not fundamental. It can only be reached via sets of height 3.
### Table 3

The 60 pairs in the $W$-orbit of $\Phi_{D_5}^+$ containing \{\(\alpha_3, \alpha_5\)\}.
Appendix. An admissible orbit in $\Phi_{D_5}$.
Next, 60 tangles are listed which correspond to 60 elements $w_{B,C}c_B$, introduced in Definition 4.4.2. Each tangle can also be associated with one of the 60 pairs of orthogonal roots in the admissible orbit $B$. The tangle represents a path from the fundamental set $C = \{\alpha_3, \alpha_5\}$ to the corresponding set. Below each tangle, this corresponding pair of roots is given. Also the element $w_{B,C}c_B$ is listed. The last line displays the distance from the set $B$ to the fundamental set $C = \{\alpha_3, \alpha_5\}$.

**Figure 2.** The fundamental set $C$.

**Figure 3.** All $B \in B$ with $ht(B) = 0$ and $B \neq C$. 
Figure 4. $B \in \mathcal{B}$ with $ht(B) = 1$. 
Figure 5. $B \in \mathcal{B}$ with $\text{ht}(B) = 2$. 
Figure 6. All $B \in \mathcal{B}$ with $ht(B) = 3$. 
Figure 7. All $B \in \mathcal{B}$ with $\text{ht}(B) = 4$.

Figure 8. All $B \in \mathcal{B}$ with $\text{ht}(B) > 4$. 
Notation

\(K\) a knot
\(M\) the type of the simply laced Coxeter diagram
\(n\) the rank of the simply laced Coxeter diagram of type \(M\)
\(\Sigma_n\) the symmetric group of rank \(n\)
\(B_n\) the braid group on \(n\) strands
\(W(M)\) the Weyl group of type \(M\)
\(A(M)\) the Artin group of type \(M\)
\(B(M)\) the BMW algebra of type \(M\)
\(C(M)_n\) the Brauer algebra of type \(M\) and rank \(n\)
\(H(M)\) the Hecke algebra of type \(M\)
\(\text{KT}(M)_n\) the (Kauffman) tangle algebra of type \(M\) and rank \(n\)
gi, ei the generators of \(B(M)\) and \(H(M)\)
Gi, Ei the generators of \(\text{KT}(M)_n\)
si the generators of \(B_n\) and \(A(M)\)
r\(i\) the generators of \(W(M)\)
\(\Phi^+_M\) the positive root system of type \(M\)
\(B\) a \(W(M)\)-orbit in the root system \(\Phi^+_M\)
\(E\) the ring \(\mathbb{Q}(\delta)[t^{\pm 1}]\)
\(T_{n,n}\) an \((n, n)\)-tangle
\(U[n]\) the collection of all \((n, n)\)-tangles
\(B\) a set in an orbit \(B\)
\(B_0\) the maximal element of an orbit \(B\)
\(C\) a fundamental set in an orbit \(B\)
\(J_{B, C}\) the nodes \(i \in M\) with \(d(r_i B, C) < d(B, C)\)
\(C\) a coclique of \(M\)
r the size of a coclique \(C\) of \(M\)
p the pole in a tangle of type \(D\)
\(R\) the set of reflections \(\{r_1, \ldots, r_n\}\) generating \(W(M)\).
\(S\) the set of generators \(\{s_1, \ldots, s_n\}\) of \(A(M)\).
\(Y\) the set of all nodes orthogonal to a set \(B_0\).
Bibliography


Samenvatting

Dit proefschrift beschrijft een onderzoek naar BMW-algebra’s van de drie enkelvoudig verbonden types $A_n$, voor $n \geq 1$, $D_n$ voor $n \geq 4$ en $E_6$ voor $n = 6, 7, 8$.

De algebra’s van type $A$ werden in 1989 geïntroduceerd door Birman en Wenzl en, onafhankelijk, door Murakami. De algebra’s zijn abstract gedefinieerd door voortbrengers en hun onderlinge relaties.

Deze algebra’s komen van origine uit de knopentheorie. Een manier om knopen te beschrijven is door middel van tangles. Voor niet negatieve $n$ vormen de $(n, n)$-tangles een algebra $KT(A)_n$ over de ring $Q(\delta)[l^{\pm 1}].$ De BMW-algebra $B(A_n)$ van type $A_n$ is isomorf met de tangle algebra $KT(A)_{n+1}$. De dimensie van $B(A_n)$ over het lichaam $Q(l, \delta)$ is gelijk aan

$$(n + 1)! = (2n + 1) \cdot (2n - 1) \cdot 5 \cdot 3 \cdot 1,$$

het product van de eerste $n + 1$ oneven getallen.

In 2000 werd bewezen dat Artin-groepen lineair zijn. In het bijzonder werd dit bewezen voor alle Artin-groepen van het enkelvoudig verbonden type. Voor dit bewijs werd gebruik gemaakt van een representatie die zijn oorsprong vindt in de natuurkunde en die bekend is geworden als de Lawrence-Krammer representatie. Een van de eigenschappen van deze representatie is dat voor type $A_n$ de representatie factoriseert via de BMW-algebra van type $A_n$. Dit leidde tot de uitbreiding van de definitie van BMW-algebra’s van type $A_n$ naar BMW-algebra’s van elk enkelvoudig verbonden type.

Hoofdstuk 2 behandelt de definitie van de algebra’s en beschrijft diverse eigenschappen van de voortbrengers en hun onderlinge relaties. Als belangrijk resultaat wordt bewezen dat alle BMW-algebra’s van het enkelvoudig verbonden type eindig dimensionaal zijn over $Q(l, \delta)$.

Het volgende hoofdstuk beschrijft banen in positieve wortelsystemen van een enkelvoudig verbonden type. Het begrip toelaatbare baan wordt geïntroduceerd en op iedere toelaatbare baan wordt een partiele ordening gedefinieerd.

In hoofdstuk 4 wordt een methode beschreven om een opspannende verzameling te vinden voor elk van de idealen die in de BMW-algebra’s voorkomen teneinde een bovengrens te vinden voor de dimensie van de BMW-algebra’s van type $D_n$ en $E_n$.

Deze verzameling wordt gedefinieerd met behulp van de in hoofdstuk 3 beschreven toelaatbare banen. Voor het quotiënt ideaal $I_1/I_2$ wordt bewezen dat de op deze manier geconstrueerde verzameling inderdaad het quotiënt ideaal opspruit. Voor het algemene geval worden de voorwaarden geformuleerd waar de partiele ordening van de baan en de bijbehorende verzameling aan dienen te voldoen opdat ook hier de verzameling het quotiënt ideaal $I_r/I_{r+1}$ opspruit. Als bewezen wordt dat aan deze
voorwaarden wordt voldaan, dan leidt dit, voor BMW-algebra’s van type $D_n$, tot een bovengrens $(2^n + 1)n!! - (2^{n-1} + 1)n!$ voor de dimensie van de BMW-algebra’s van type $D_n$ over het lichaam $\mathbb{Q}(l, \delta)$.

In hoofdstuk 5 wordt een nieuwe tangle algebra $\mathbf{KT}(D)_n$ geïntroduceerd, bestaande uit tangles with a pole. Met behulp van een surjectief homomorfisme van $\mathbf{B}(D_n)$, de BMW-algebra’s van type $D_n$, naar $\mathbf{KT}(D)_n$ wordt een ondervenreg voor de dimensie van $\mathbf{B}(D_n)$ over $\mathbb{Q}(l, \delta)$ vastgesteld op $(2^n + 1)n!! - (2^{n-1} + 1)n!$.

In hoofdstuk 6 wordt beschreven hoe de Lawrence-Krammer representatie factoriseert via het quotiënt ideaal $I_1/I_2$ van iedere BMW-algebra van het enkelvoudige verbonden type. Ook wordt een algoritme beschreven om een soortgelijke representatie te construeren die via een willekeurig quotiënt ideaal $I_r/I_{r+1}$ factoriseert.

In de appendix tenslotte is het quotiënt ideaal $I_2/I_3$ van type $D_5$ uitgewerkt. De in hoofdstuk 3 gegeven partiële ordening van de bijbehorende toelaatbare baan van 60 paren van orthogonale wortels wordt beschreven. Daarnaast is voor alle 60 paren het bijbehorende element van de opspannende verzameling uit hoofdstuk 4 gegeven alsmede de tangle van $\mathbf{KT}(D)_5$ uit hoofdstuk 5.
Terugkijkend zijn de ruim vier jaar die ik als promovendus bij de vakgroep Discrete Algebra en Meetkunde van de faculteit Wiskunde en Informatica aan de Technische Universiteit Eindhoven heb doorgebracht gevoelsmatig snel voorbijgegaan. Zou de tijd dan toch vliegen als je het naar je zin hebt? De werkomstandigheden waren er in elk geval prima voor.

De voornaamste reden hiervoor is het nimmer aflatende enthousiasme van Arjeh Cohen; hij is mijn eerste promotor, was mijn afstudeerbegeleider en is iemand voor wie ik grote bewondering heb. Tot het laatste moment wist zijn positieve houding mij te verbazen als ik weer eens vastgelopen was en met mijn probleem bij hem aanklopte. Ook het feit dat zijn belangstelling veel verder reikt dan alleen mijn onderzoek maakt hem in mijn ogen tot de meest ideale begeleider die ik me kon wensen.

Een belangrijke rol in de totstandkoming van dit proefschrift is daarnaast weggelegd voor David Wales, mijn tweede promotor. Zijn bezoeken aan Eindhoven gaven het onderzoek iedere keer weer een nieuwe impuls, zeker ook door zijn praktische manier om problemen te benaderen. Ook het bezoek van Vincent Florens heeft bijgedragen tot het juiste inzicht in de wirwar van tangles. Verder ben ik blij dat naast beide promotoren Andries Brouwer, Eric Opdam en Eduard Looijenga de tijd hebben genomen om mijn proefschrift te lezen en als kerncommissie te beoordelen.

De vakgroep Discrete Algebra en Meetkunde heb ik altijd als erg positief ervaren. Ik wil vooral mijn kamergenoten in de afgelopen jaren, Ernesto Reinaldo Barreiro en Erik Postma hiervoor bedanken.

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Eindhoven, september 2005,

Die Gijsbers
Curriculum Vitae

Dié Gijsbers werd op 5 juni 1975 geboren in het Radboudziekenhuis te Nijmegen en is opgegroeid in Maashees. Na het succesvol doorlopen van het gymnasium van de katholieke scholengemeenschap Jerusalem in Venray is hij in 1993 een gecombineerde studie Technische Wiskunde en Technische Informatica begonnen aan de Technische Universiteit Eindhoven.


Aansluitend is hij bij dezelfde vakgroep zijn promotieonderzoek gestart op het NWO-project "Braid group representations and tensor categories". Onder begeleiding van prof.dr. A.M. Cohen deed hij onderzoek naar BMW algebras of simply laced type. De resultaten van dit onderzoek zijn beschreven in dit proefschrift.