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Distribution theories based on representations of locally compact Abelian topological groups

by

A.F.M. ter Elst

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Introduction

In the monograph [EG2] the topological spaces $SH,A$ en $TH,A'$, which are both inductive and projective limit of Hilbert spaces, have been studied extensively. Here the label $H$ denotes a separable Hilbert space, the label $A$ denotes a positive self-adjoint operator in $H$. If $H = L^2(\mathbb{R})$ and $A = \sqrt{-\frac{d^2}{dx^2}}$ it turns out that $SH,A = C * L^2(\mathbb{R})$ as sets, with $C = \{x \mapsto \frac{1}{\pi} \frac{t}{t^2 + x^2} : t > 0\}$ a subset of $L^1(\mathbb{R})$ and $*$ the convolution product between $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. If we put a suitable topology on $C * L^2(\mathbb{R})$ we can prove that $SH,A = C * L^2(\mathbb{R})$ as locally convex topological vector spaces. The map $f \mapsto f * \delta$ from $L^1(\mathbb{R})$ into $L(L^2(\mathbb{R}))$ is a $\sigma$-representation of the algebra $L^1(\mathbb{R})$ by operators on the Hilbert space $L^2(\mathbb{R})$ and this map corresponds to the regular representation of $\mathbb{R}$ in $L^2(\mathbb{R})$. (See [HR1], Theorem 22.10.) So in fact the space $S_{L^2(\mathbb{R})}$, $\sqrt{-\frac{d^2}{dx^2}}$ is determined by the group $\mathbb{R}$, a subset $C$ of $L^1(\mathbb{R})$ and a representation of this group in a Hilbert space.

The aim of this report is to study the results in [EG2] from the viewpoint of commutative harmonic analysis and also to explore the possibilities for generalizations which naturally arise in that new context. We start with a locally compact Abelian group $G$, a subset $C$ of $L^1(G)$, a representation $U$ of $G$ in a (not necessarily separable) Hilbert space $H$ and we suppose that the pair $(C, U)$ satisfies some technical conditions (P1, P2).

In chapter I we define the locally convex topological vector space of smoothed elements $S_{U,C}$, the trajectory space $T_{U,C}$ and a duality between them. The space $S_{U,C}$ is an inductive limit of Hilbert space and $T_{U,C}$ is a projective limit of Hilbert spaces. Assuming a weak condition (P2') we prove that $S_{U,C} = H$ as sets if and only if $S_{U,C} = H$ as topological vector spaces.

In chapter II we define a projective limit topology on the vector space $S_{U,C}$ and we consider a special type of bounded subsets of $T_{U,C}$. It turns out that the projective limit topology for $S_{U,C}$ is equal to the inductive limit topology for $S_{U,C}$ if and only if all bounded subsets of $T_{U,C}$ are of this special type. Further, we consider an additional condition on the set $C$ (P3) and then we prove that $S_{U,C}$ is a projective limit of Hilbert spaces.

Assuming the conditions P1, P2, P3, in chapter III we show that the following conditions are equivalent: the pair $(C, U)$ has property P4; $T_{U,C}$ is homological; $T_{U,C}$ is reflexive; $S_{U,C}$ is complete; $S_{U,C}$ is sequentially complete; every bounded sequence in $S_{U,C}$ has a weakly convergent subsequence; all bounded subsets of $S_{U,C}$ look as if the inductive limit were strict; and several other conditions. Finally we present necessary and sufficient conditions for $S_{U,C}$ and $T_{U,C}$ to be nuclear or Montel.

In chapter IV we include several (old and new) examples. We show that all $SH,A$, $TH,A$, $\pi(H,A)$ and $\sigma(H,A)$ spaces, where $H$ is a Hilbert space and $A$ a positive self-adjoint operator in $H$, fit in the theory.

In chapter V we consider continuous linear maps between spaces of type $S_{U,C}$ and $T_{U,C}$. The theorems are nearly similar to the theorems in [EG2], Section 1.4, but the proofs are (of course) different.

In chapter VI we investigate conjugate linear maps. We prove that $S_{U,C}$ is conjugate linear
homeomorphic with the strong dual of $T_{U,C}$. Assuming property P3, it is shown that $T_{U,C}$ is conjugate linear homeomorphic with the strong dual of $S_{U,C}$ if and only if a symmetry property (P4) holds.

In chapter VII it is investigated whether Cartesian products of two $S_{U,C}$ spaces (resp. two $T_{U,C}$ spaces) are again of type $S_{U,C}$ ($T_{U,C}$). Here property P5 plays an essential role.

In chapter VIII we introduce explicit completions of $\theta$-topological tensor product of $S_{U_1,C_1}$ and $T_{U_2,C_2}$. (Four possibilities.) It is shown that the completed $\theta$-topological tensor product of two $T_{U,C}$-spaces is again of type $T_{U,C}$. We formulate and prove four kernel theorems. The latter is a generalisation of the corresponding chapter (III) in [EG2].

Appendix A contains a summary of the concepts of commutative harmonic analysis as used in this report. Appendix B summarizes the prerequisites from topological vector space theory. In Appendix C we introduce a so-called $\theta$-topology on the tensor product of two locally convex topological vector spaces which have enough seminorms which correspond to semi inner products. This $\theta$-topology corresponds to the Hilbert space tensor product of two Hilbert spaces.

In the final part of this introduction we sketch the relation between the results of the underlying report and the papers [EGK] and [EK]. These papers also expand the results of [EG2], be it in distinct directions.

The set $\Phi$ of nonnegative functions on $\mathbb{R}^n$ in [EGK] can be compared more or less with the sets of complex valued bounded functions $\hat{f}$ on $\hat{G}$ in this report. An important difference is that $\Phi$ may contain unbounded functions. On the other hand, $\hat{G}$ need not necessarily be equal to $\mathbb{R}^n$.

The condition A.I in [EGK] is superfluous and the (strong) condition A.III in [EGK] (cf. property P3 in this report) is always assumed. By condition A.II in [EGK], for every $x \in \mathbb{R}^n$ there exists $\phi \in \Phi$ such that $\phi(x) \neq 0$, whereas in this report all functions $\hat{f}$, with $f \in C$, may be 0 on a large subset of $\hat{G}$.

In [EK] the operators $a \in R$ are all bounded as is the case with the operators $U(f)$ in this report. Further, the operators $a \in R$ are required to be positive Hermitian and bounded by 1. Also a subsemigroup property is required which is not assumed in [EGK] nor in this report. A condition like property P3 is considered at the end of the paper [EK].

In both papers [EGK] and [EK] no theorems are proved which are similar to the theorems in chapters 5-8 in this report.
Some notations

Let $A$ be a set and let $V$ be a subset of $A$. Then $V^c$ is the complement of $V$ in $A$, so $V^c = \{ a \in A : a \notin V \}$. $1_V$ is the characteristic function of $V$, thus

$$1_V : A \rightarrow \mathbb{R}$$

$$1_V(a) = \begin{cases} 
1 & \text{if } a \in V, \\
(a \in A) & \\
0 & \text{if } a \notin V.
\end{cases}$$

Let $f$ be a complex valued bounded function on $A$. Then $\|f\|_\infty := \sup \{ |f(a)| : a \in A \}$. $\mathbb{N}$ is the set of positive integers, $\mathbb{N} = \{1, 2, 3, \cdots \}$.

Let $X$ be a topological space and let $V$ be a subset of $X$. Then $\text{clo} \ V = \overline{V}$ denotes the closure of $V$. By $C_c(X)$ we denote the set of all complex valued continuous functions $f$ on $X$ with compact support, i.e. there exists a compact subset $K$ of $X$ such that $f(x) = 0$ for all $x \in K^c$. Further, $C_0(X)$ denotes the Banach space of all complex valued continuous function $f$ on $X$ which vanish at infinity, i.e. for all $\varepsilon > 0$ there exists a compact subset $K$ of $X$ such that $|f(x)| < \varepsilon$ for all $x \in K^c$. The norm on $C_0(X)$ is $\| \|_\infty$. By $C(X)$ we denote the set of all complex valued continuous functions on $X$.

The abbreviation a.e. means almost everywhere.

Let $G$ be a locally compact Abelian topological group and let $f, g \in L^1(G)$. Then $\|f\|_1$ denotes the norm of $f$ and $f * g$ denotes the convolution product of $f$ and $g$. The adjoint $\hat{f}$ of $f$ is the function (equivalence class) with $\hat{f}(x) = \overline{f(x^{-1})}$, a.e. $x \in G$. For every subset $C$ of $L^1(G)$ let $\hat{C}$ denote the set $\{ \hat{f} : f \in C \}$. $\hat{f}$ is the Fourier transform of $f$. The dual group of $G$ is $\hat{G}$.

Let $H$ be a Hilbert space. The inner product in $H$ is denoted by $(,)$ and the norm by $\| \|$. Let $T$ be a densely defined operator from $H$ into a Hilbert space. Then $D(T)$ denotes the domain of $T$ and $T^*$ denotes the adjoint operator. In case $T$ is a bounded operator, we denote by $\| T \|$ the norm of $T$. 

Chapter I. The spaces \( S_{U, C} \) and \( T_{U, C} \)

1.1. The space \( S_{U, C} \)

Let \( G \) be a locally compact Abelian topological group with a Haar measure \( \mu \) and let \( U \) be a (continuous) representation of \( G \) in a Hilbert space \( H \). For definitions concerning topological groups the reader is referred to Appendix A. We suppose that every representation is continuous. For every \( f \in L^1(G) \) define a continuous operator on \( H \) by

\[
(U(f)u, v) = \int g f(x)(U_x u, v) \, d\mu(x) \quad (u, v \in H).
\]

(See [HRI], Theorem 22.3.) The operators \( U(f), f \in L^1(G) \) have the following properties:

\[
\| U(f)u \| \leq \| f \| \| u \| ,
\]

\[
U(f^*) = U(\hat{f}) ,
\]

\[
U(f * g) = U(f) U(g) = U(g) U(f) .
\]

Let \( C \) be a subset of \( L^1(G) \). Suppose the pair \((C, U)\) possesses the following properties.

PROPERTIES 1.1.

P1. For all \( f, g \in C \) there exists \( h \in C \) such that 1 and 2 hold:

1) \( f = h \) or there exists \( f_1 \in L^1(G) \) such that \( f = h * f_1 \),

2) \( g = h \) or there exists \( g_1 \in L^1(G) \) such that \( g = h * g_1 \).

P2. There exist a net \((f_\lambda)_{\lambda \in I}\) in \( C \) such that for all \( x \in H \) holds \( \lim \lambda U(f_\lambda) x = x \).

Throughout this report we suppose that the conditions P1 and P2 are satisfied by the pair \((C, U)\).

REMARK. These conditions are weak, but sufficient for the purpose of this chapter. In practice one meets sets \( C \) which satisfy much stronger properties such as:

P1'. For all \( f, g \in C \) there exist \( h \in C \) and \( f_1, g_1 \in L^1(G) \) such that \( f = h * f_1 \) and \( g = h * g_1 \).

P2'. There exists a \( L^1(G) \)-bounded net \((f_\lambda)_{\lambda \in I}\) in \( C \) such that for all \( x \in H \) holds \( \lim \lambda U(f_\lambda) x = x \).

P2''. There exists a \( L^1(G) \)-bounded net \((f_\lambda)_{\lambda \in I}\) in \( C \) such that for all \( g \in L^1(G) \) holds \( \lim \lambda f_\lambda * g = g \) in \( L^1(G) \).

Of course: P1' implies P1 and P2' implies P2. Also P2'' implies P2'. Indeed, let \( E := \{ x \in H : \lim \lambda U(f_\lambda) x = x \} \) and \( F := \{ U(f)x : f \in L^1(G), x \in H \} \) Then \( E \) is closed in \( H \) and \( F \subset E \). Since \( U \) is continuous, \( F \) is dense in \( H \). So \( E = H \).

REMARK. For all \( x \in H \) we have \( \lim \lambda U(f_\lambda)^* x = x \). Indeed, \( U(f_\lambda) - I \) is a normal operator, so

\[
\| (U(f_\lambda) - I)^* x \| = \| (U(f_\lambda) - I)x \| \quad \text{for all } \lambda \in I .
\]

Hence, \( \lim \lambda U(f_\lambda)^* x - x \| = \)
\[
\lim_{\lambda} \| (U(f_\lambda) - I)^n x \| = \lim_{\lambda} \| (U(f_\lambda) - I)x \| = 0.
\]

REMARK. Let \( \tilde{C} := \{ \tilde{f} : f \in C \} \). Then the pair \((\tilde{C}, U)\) satisfies properties P1 and P2. The pair \((C, U)\) satisfies properties P1', P2' and P2'' if and only if the pair \((C, U)\) satisfies properties P1', P2' resp. P2''.

**DEFINITION 1.2.**

Let \( f \in L^1(G) \). Define

\[
N_f := \{ x \in H : U(f)x = 0 \}, \quad \text{the kernel of } U(f),
\]

\[
R_f := U(f)(H), \quad \text{the range of } U(f),
\]

\[
\Omega_f := U(f) : N_f^1 \to R_f.
\]

Note that \( N_f = N_f^1 \) for all \( f \in L^1(G) \).

**LEMMA 1.3.**

Let \( f \in L^1(G) \). Then \( \Omega_f \) is a bijection. So there exists a unique norm \( \| \cdot \|_f \) on \( R_f \) such that \( R_f \) becomes a Hilbert space and \( \Omega_f \) is a unitary map. The identity map from \( R_f \) into \( H \) is continuous.

**Proof.**

\( \Omega_f \) is injective. Let \( x \in H \). Since \( U(f) \) is continuous, the set \( N_f \) is closed, hence there exist \( v \in N_f \) and \( w \in N_f^1 \) such that \( x = v + w \). Then \( U(f)x = U(f)v = \Omega_f w \). So \( \Omega_f \) is surjective. Let \( \phi \in R_f \). Then \( \| \phi \| \leq \| U(f) \| \| \Omega_f^{-1}(\phi) \| \leq \| f \| \| \phi \|_f \), so the identity map from \( R_f \) into \( H \) is continuous.

In the next lemma the spaces \( R_f \) and \( R_g \) are compared if \( f = g \ast h \) for some \( h \in L^1(G) \).

**LEMMA 1.4.**

Let \( f, g, h \in L^1(G) \). Suppose \( f = g \ast h \). Then \( N_g \subset N_f \) and \( R_f \subset R_g \). For all \( \phi \in R_f \) holds \( \Omega_g^{-1}(\phi) = U(h)(\Omega_f^{-1}(\phi)) \). The identity map from \( R_f \) into \( R_g \) is continuous.

**Proof.**

Let \( x \in N_g \). Then \( U(f)x = U(h)U(g)x = 0 \), so \( x \in N_f \). Similarly: \( R_f \subset R_g \). Let \( \phi \in R_f \) and let \( x := \Omega_f^{-1}(\phi) \). Assertion: \( U(h)x = N_f^1 \). Let \( v \in N_g \). Then \( U(g)U(h)v = U(h)U(g)v = 0 \), so \( U(h)v \in N_g \subset N_f \) and \( 0 = (x, U(h)v) = (U(h)x, v) \). This proves the assertion. Since \( \Omega_g(U(h)x) = U(g)U(h)x = U(f)x = \phi \) we have proved that \( \Omega_g^{-1}(\phi) = U(h)\Omega_f^{-1}(\phi) \). Hence, for all \( \phi \in R_f \) we obtain \( \| \phi \|_g = \| \Omega_g^{-1}(\phi) \| \leq \| U(h) \| \| \Omega_f^{-1}(\phi) \| \leq \| h \| \| \phi \|_f \). \( \square \)
DEFINITION 1.5.
Let $S_U, C := \bigcup_{f \in C} R_f$.

By Lemma 1.4 and property P1, $S_U, C$ is a linear vector space. The topology $\sigma_{\text{ind}}$ for $S_U, C$ is the inductive limit topology generated by the Hilbert spaces $R_f$, i.e. the finest locally convex topology for $S_U, C$ for which all natural maps from $R_f$ into $S_U, C$ are continuous ($f \in C$).

For the terminology of locally convex topological vector spaces we refer to Appendix B.

LEMMA 1.6.
The identity map from $S_U, C$ into $H$ is continuous. Hence the topology $\sigma_{\text{ind}}$ for $S_U, C$ is Hausdorff. The set $S_U, C$ is dense in $H$. Let $f \in C$. Then the map $U(f) : H \to S_U, C$ is continuous.

Proof.
The first assertion follows from Lemma 1.3 and the definition of $\sigma_{\text{ind}}$. Since $H$ is Hausdorff, also $(S_U, C, \sigma_{\text{ind}})$ is Hausdorff. By property P2, $\lim_{\lambda} U(f_\lambda)x = x$ for all $x \in H$, so $S_U, C$ is dense in $H$. Let $f \in C$ and let $P$ be the projection of $H$ onto $N_f^\perp$. Then $U(f) = \Omega_f \circ P : H \to S_U, C$ is continuous.

THEOREM 1.7.
The space $(S_U, C, \sigma_{\text{ind}})$ is bomological, barrelled and a Mackey space.

Proof. $\sigma_{\text{ind}}$ is the inductive limit topology of bomological and barrelled spaces, hence $(S_U, C, \sigma_{\text{ind}})$ is bomological and barrelled ([Wil], Theorem 13-1-13). Every bomological space is a Mackey space. (See [Wil], Theorem 8-4-9.)

1.2. The space $T_U, C$

Besides the space $S_U, C$ we introduce the space $T_U, C$ in this section.

DEFINITION 1.8.
A $C$-trajectory is a map $\Phi$ from $C$ into $H$ with the property that

$$\Phi(f \ast g) = (U(g))^* \Phi(f)$$

for all $f \in C$, $g \in L^1(G)$ such that $f \ast g \in C$.

Let $T_U, C$ be the vector space of all $C$-trajectories. The Hausdorff topology $\tau_{\text{proj}}$ for $T_U, C$ is the locally convex topology for $T_U, C$ generated by the seminorms $t_f$, defined by $t_f(\Phi) := \|\Phi(f)\|$, $\Phi \in T_U, C, f \in C$.

It follows that for all $f \in C$, the map $\Phi \mapsto \Phi(f)$ from $T_U, C$ into $H$ is continuous.

LEMMA 1.9.
Let $\Phi \in T_U, C$ and let $f, g \in C$. Then $(U(f))^* \Phi(g) = (U(g))^* \Phi(f)$.
Proof.
First, suppose there exists \( l \in L^1(G) \) such that \( f = g * l \). Then \( U(g)^* \Phi(f) = U(g)^* U(l)^* \Phi(g) = U(f)^* \Phi(g) \). In general, by property PI, there exists \( h \in C \) such that 1) and 2) holds:
1) \( f = h \) or there exists \( f_1 \in L^1(G) \) such that \( f = h * f_1 \).
2) \( g = h \) or there exists \( g_1 \in L^1(G) \) such that \( g = h * g_1 \).
If \( f = h \) or \( g = h \), then the lemma is trivial or already proved. Suppose \( f \neq h \) and \( g \neq h \). Let \( f_1 \) and \( g_1 \) as in 1) and 2). Then \( U(f)^* \Phi(g) = U(h)^* U(f_1)^* U(g_1)^* \Phi(h) = U(h)^* U(g_1)^* U(f_1)^* \Phi(h) = U(g)^* \Phi(f) \).

The Hilbert space \( H \) can be embedded in \( T_{U,C} \) in a natural manner.

DEFINITION 1.10.
Define
\[
emb : H \to T_{U,C} \\
[emb(x)](f) = U(f)^* x \quad (x \in H, f \in C).
\]

THEOREM 1.11.
The map \( emb \) is injective and continuous. For all \( \Phi \in T_{U,C} \) holds \( \Phi = \lim_{\lambda} \text{emb}(\Phi(\lambda)) \) in \( T_{U,C} \).

So the range \( \text{emb}(H) \) of the map \( \text{emb} \) is dense in \( T_{U,C} \).

Proof.
Let \( x \in H \) and \( \text{emb} x = 0 \). By property P2, then also \( x = \lim_{\lambda} U(f_\lambda)^* x = \lim_{\lambda} [emb(x)](f_\lambda) = 0 \). So \( \text{emb} \) is injective. It is trivial that \( \text{emb} \) is continuous. Let \( \Phi \in T_{U,C} \) and let \( f \in C \). Then by Lemma 1.9, \( \lim_{\lambda} \| \tau_{\text{emb}}(\Phi(f_\lambda) - \Phi) = \lim_{\lambda} \| U(f)^* \Phi(f_\lambda) - \Phi(f) \| = \lim_{\lambda} \| U(f)^* \Phi(f) - \Phi(f) \| = 0 \). So \( \Phi = \lim_{\lambda} \text{emb}(\Phi(\lambda)) \) and \( \text{emb}(H) \) is dense in \( T_{U,C} \).

COROLLARY 1.12.
I. For all \( \Phi \in T_{U,C} \), \( f \in C \) and \( g \in L^1(G) \) such that \( f * g = 0 \) holds \( U(g)^* \Phi(f) = 0 \).
II. For all \( \Phi \in T_{U,C} \) and \( f \in C \) holds \( \Phi(f) \in N_f^1 \).

Proof.
I. Let \( f \in C \), \( g \in L^1(G) \) and suppose \( f * g = 0 \). Let \( E := \{ \Phi \in T_{U,C} : U(g)^* \Phi(f) = 0 \} \). By definition of \( \tau_{\text{proj}} \), the set \( E \) is closed in \( T_{U,C} \). It is trivial that \( E \) contains \( \text{emb}(H) \). So \( E = T_{U,C} \).
II. Let \( x \in H \) and \( l \in E \). For all \( v \in N_f \), \([\text{emb}(x)](f), v) = (U(f)^* x, v) = (x, U(f)v) = 0\). So \([\text{emb}(x)](f) \in N_f^\perp\). Since \( N_f^\perp \) is closed and \( \text{emb}(H) \) is dense in \( T_{U,C} \), the corollary follows. 

**THEOREM 1.13.**

The space \((T_{U,C}, \tau_{\text{proj}})\) is complete.

**Proof.**

Let \( (\Phi_\alpha)_{\alpha \in A} \) be a Cauchy net in \( T_{U,C} \). For all \( f \in C \), \((\Phi_\alpha(f))_{\alpha \in A} \) is a Cauchy net in \( H \), so there exists \( \Phi_f \in H \) such that \( \lim_{\alpha} \Phi_\alpha(f) = \Phi_f \). Define \( \Phi : C \to H \) by \( \Phi(f) := \Phi_f \), \( f \in C \). Let \( f \in C \), \( g \in L^1(G) \) and suppose \( f \ast g \in C \). Then \( \Phi(f \ast g) = \lim_{\alpha} \Phi_\alpha(f \ast g) = \lim_{\alpha} U(g)^* \Phi_\alpha(f) = U(g)^* (\lim_{\alpha} \Phi_\alpha(f)) = U(g)^* \Phi(f) \). So \( \Phi \in T_{U,C} \). Since \( \lim_{\alpha} t_f(\Phi - \Phi_\alpha) = 0 \) for all \( f \in C \), \( \Phi = \lim_{\alpha} \Phi_\alpha \) in \( T_{U,C} \).

The next lemma turns out to be useful. (Cf. Theorem 1.18.)

**LEMMA 1.14.**

Let \( f, g \in C, h \in L^1(G) \) and suppose that \( f = g \ast h \). Then \( t_f \leq \| h \|_1 \). Hence, let \( B \subset T_{U,C} \) be a \( \tau_{\text{proj}} \)-neighbourhood of 0. Then there exist \( f \in C \) and \( c > 0 \) such that \((\Phi \in T_{U,C} : t_f(\Phi) \leq c) \subset B\).

**Proof.**

For all \( \Phi \in T_{U,C} \) we get \( t_f(\Phi) = \| U(h)^* \Phi(g) \| \leq \| U(h) \| \| \Phi(g) \| \leq \| h \|_1 \). From this and property P1, the second assertion follows.

I.3. The pairing between \( S_{U,C} \) and \( T_{U,C} \)

In this section we define a duality \( < , > \) between \( S_{U,C} \) and \( T_{U,C} \) with the properties that \( \sigma_{\text{ind}} \) and \( \tau_{\text{proj}} \) are compatible with \( < , > \) and \( < \phi, \text{emb} x > = (\phi, x) \) for all \( \phi \in S_{U,C} \) and \( x \in H \). It turns out that \( < U(f)x, \Phi > = (x, \Phi(f)) \) for all \( f \in C, x \in H \) and \( \Phi \in T_{U,C} \). (See Theorem 1.16 II.)

Let \( \phi \in S_{U,C} \) and \( \Phi \in T_{U,C} \). There exists \( f \in C \) such that \( \phi \in R_f \). We show that it makes sense to define \( < \phi, \Phi > := (\Omega_f^{-1}(\phi), \Phi(f)) \). Let \( g \in C \) and suppose also \( \phi \in R_g \). Then we have to prove that \( (\Omega_g^{-1}(\phi), \Phi(f)) = (\Omega_g^{-1}(\phi), \Phi(g)) \). First suppose there exists \( l \in L^1(G) \) such that \( f = g \ast l \). By Lemma 1.4:

\[
(\Omega_g^{-1}(\phi), \Phi(g)) = (U(l) \Omega_f^{-1}(\phi), \Phi(g)) = (\Omega_f^{-1}(\phi), U(l)^* \Phi(g)) = (\Omega_f^{-1}(\phi), \Phi(f)).
\]

In the general case there exists \( h \in C \) such that 1) and 2) hold:

1) \( f = h \) or there exists \( f_1 \in L^1(G) \) such that \( f = h \ast f_1 \),
2) \( g = h \) or there exists \( g_1 \in L^1(G) \) such that \( g = h \ast g_1 \).

In any case: \((\Omega_f^{-1}(\phi), \Phi(f)) = (\Omega^{-1}_h(\phi), \Phi(h)) = (\Omega^{-1}_g(\phi), \Phi(g))\).

**DEFINITION 1.15.**

Define

\[
<\cdot, \cdot>: \mathcal{S}_U, C \times T_U, C \to \mathcal{C}
\]

\[
<\phi, \Phi> := (\Omega_f^{-1}(\phi), \Phi(f)) \quad (f \in \mathcal{C}, \phi \in R_f, \Phi \in T_U, C).
\]

The pairing \(<\cdot, \cdot>\) is sesquilinear.

**THEOREM 1.16.**

I. Let \( \ell \in \mathcal{S}_U, C \) and \( x \in H \). Then \( <\phi, \text{emb} x> = (\phi, x) \).

II. Let \( f \in \mathcal{C}, x \in H \) and \( \Phi \in T_U, C \). Then \( <U(f)x, \Phi> = (x, \Phi(f)) \).

**Proof.**

I. Let \( f \in \mathcal{C} \) be such that \( \phi \in R_f \). Then \( <\phi, \text{emb} x> = (\Omega_f^{-1}(\phi), (\text{emb} x)(f)) = (\Omega_f^{-1}(\phi), U(f)^* x) = (U(f) \Omega_f^{-1}(\phi), x) = (\phi, x) \).

II. Let \( P \) be the projection of \( H \) onto \( N_f^\perp \). Then \( \Omega_f^{-1}(U(f)x) = Px \). Hence by Corollary 1.12 II:

\[
<U(f)x, \Phi> = (\Omega_f^{-1}(U(f)x), \Phi(f)) = (Px, \Phi(f)) = (x, P \Phi(f)) = (x, \Phi(f)).
\]

**LEMMA 1.17.**

The pairing \(<\cdot, \cdot>\) is nondegenerate.

**Proof.**

Let \( \phi \in \mathcal{S}_U, C \) and suppose \( <\phi, \Phi> = 0 \) for all \( \Phi \in T_U, C \). Then \( \|\phi\|^2 = (\phi, \phi) = <\phi, \text{emb} \phi> = 0 \). So \( \phi = 0 \).

Let \( \Phi \in T_U, C \) and suppose \( <\phi, \Phi> = 0 \) for all \( \phi \in \mathcal{S}_U, C \). Let \( f \in \mathcal{C} \). For all \( x \in H \) we obtain by Theorem 1.16 II, \( 0 = <U(f)x, \Phi> = (x, \Phi(f)) \). So \( \Phi(f) = 0 \).

**THEOREM 1.18.**

The topology \( \sigma_{\text{ind}} \) for \( \mathcal{S}_U, C \) is compatible with the dual pair \( <\mathcal{S}_U, C, T_U, C> \) and the topology \( \tau_{\text{proj}} \) for \( T_U, C \) is compatible with the dual pair \( <T_U, C, \mathcal{S}_U, C> \).

**Proof.**

For \( f \in \mathcal{C} \) let \( i_f : R_f \to \mathcal{S}_U, C \) be the natural map.

Let \( \Phi \in T_U, C \). Define

\[
l : \mathcal{S}_U, C \to \mathcal{C}
\]
\[ l(\phi) := \langle \phi, \Phi \rangle \quad (\phi \in S_{U,C}). \]

Let \( f \in C. \) For all \( \phi \in R_f : \| l \circ i_f(\phi) \| = \| \langle \Phi(f),\Phi(f) \rangle \| \leq \| \Phi(f) \| \| \Omega_f^{-1}(\phi) \| = \| \Phi(f) \| \| \phi \| \), so \( l \circ i_f : R_f \to C \) is continuous. Hence \( l : S_{U,C} \to C \) is continuous. (See [Wil], Theorem 13-1-8.) Conversely, let \( l : S_{U,C} \to C \) be continuous and linear. Let \( f \in C. \) The map \( l \circ U(f) : H \to C \) is continuous and linear by Lemma 1.6. So (Riesz) there exists a unique \( \Phi(f) \in H \) such that \( l \circ U(f) x = (x, \Phi(f)) \) for all \( x \in H. \) We prove that \( \Phi \) is a \( C \)-trajectory. Let \( g \in C, h \in L^1(G) \) and suppose \( f := g \ast h \in C. \) Then for all \( x \in H \) we have \( (x, U(h)^* \Phi(g)) = (U(h)x, \Phi(g)) = l \circ U(g) U(h)x = l \circ U(f)x = (x, \Phi(f)). \) We conclude that \( \Phi(f) = U(h)^* \Phi(g) \) and \( \Phi \in T_{U,C}. \) Finally, let \( \phi \in S_{U,C}. \) Let \( f \in C \) be such that \( \phi \in R_f. \) Then \( l(\phi) = (\Omega_f^{-1}(\phi),\Phi(f)) = \langle \phi, \Phi \rangle >. \) So the topology \( \sigma_{ind} \) for \( S_{U,C} \) is compatible with the dual pair \( < S_{U,C}, T_{U,C} >. \)

Let \( \phi \in S_{U,C}. \) Define
\[
\begin{align*}
l : T_{U,C} &\to C \\
l(\Phi) := \langle \phi, \Phi \rangle &\quad (\Phi \in T_{U,C}).
\end{align*}
\]

The map \( l \) is linear. There exists \( f \in C \) such that \( \phi \in R_f. \) Then \( \| l(\Phi) \| = \| \langle \phi, \Phi \rangle \| = \| (\Omega_f^{-1}(\phi),\Phi(f)) \| \leq \| \Omega_f^{-1}(\phi) \| \| \Phi(f) \| = \| \Phi(f) \| \| \phi \|. \) So \( l \) is continuous.

Conversely, let \( l : T_{U,C} \to C \) be continuous and linear. Since \( l \) is continuous, by Lemma 1.14, there exist \( f \in C \) and \( c > 0 \) such that \( \| l(\Phi) \| \leq c \| \Phi(\Phi) \| \) for all \( \Phi \in T_{U,C}. \) Define \( \alpha : U(\hat{f})(H) \to C \) by \( \alpha := l \circ \text{emb} \circ \Omega_f^{-1}. \) The map \( \alpha \) is linear. Let the topology for \( U(\hat{f})(H) \) be the induced topology of \( H. \) Let \( x \in U(\hat{f})(H). \) Then \( \| \alpha(x) \| \leq c \| \alpha(\text{emb}(\Omega_f^{-1}(x))) \| (f) \| = c \| x \|. \) So \( \alpha \) is continuous.

By Riesz' theorem there exists a unique \( v \in U(\hat{f})(H) = N_f^T \) such that \( \alpha(x) = (x, v) \) for all \( x \in U(\hat{f})(H). \)

We show that \( l(\Phi) = \langle U(f)v, \Phi \rangle \) for all \( \Phi \in T_{U,C}. \) Let \( x \in N_f. \) Then \( l(\text{emb} x) = \alpha(U(\hat{f})x) = (U(f)^* x, v) = \langle U(f)^* v, \Phi \rangle. \) Let \( x \in N_f. \) Then \( l(\text{emb} x) \leq c \| \Phi(\text{emb} x) \| = 0, \) so \( l(\text{emb} x) = 0 = (U(\hat{f})x, v) = \langle U(f)v, \Phi \|. \) So for all \( x \in H \) we obtain \( l(\text{emb} x) = \langle U(f)v, \Phi \|. \) Because \( l \) and \( \Phi \mapsto \langle U(f)v, \Phi \| > \) are continuous functions on \( T_{U,C} \) and \( \text{emb}(H) \) is dense in \( T_{U,C}, \) for all \( \Phi \in T_{U,C} \) holds \( l(\Phi) = \langle U(f)v, \Phi \|. \)

In Lemma 1.6 it is proved that the set \( S_{U,C} \) is dense in the Hilbert space \( H. \) It may happen that \( S_{U,C} \) equals \( H \) as a set. In a particular case, it can be proved that \( (S_{U,C}, \sigma_{ind}) \), as a topological vector space, is equal to the Hilbert space \( H. \)

**Theorem 1.19.**

Suppose \( S_{U,C} = H \) as sets and suppose property P2' holds. Then \( (S_{U,C}, \sigma_{ind}) = H \) as topological vector spaces.

**Proof.**

The identity map from \( S_{U,C} \) into \( H \) is continuous by Lemma 1.6, so we have to prove that the identity map \( i \) from \( H \) into \( S_{U,C} \) is continuous. For all \( \Phi \in T_{U,C} \) define
Let \((f_\lambda)_{\lambda \in J}\) be the net as in property \(P2'\) and let \(M > 0\) be such that \(\|f_\lambda\|_1 < M\) for all \(\lambda \in J\).

Let \(x \in H\). Then \(l_{\text{emb}(\lambda)}(y) = \langle y, \text{emb} x \rangle = (y, x)\) for all \(y \in H\). So \(l_{\text{emb}(\lambda)}\) is continuous \((x \in H)\).

Let \(\Phi \in T_{U, C}\). Assertion: the set \(\{l_{\text{emb}(\Phi(\lambda))} : \lambda \in J\}\) is pointwise bounded. Indeed, let \(x \in H\).

Since \(H = S_{U, C}\) as sets, there exist \(f \in C\) and \(y \in N_f^1\) such that \(x = U(f)y\). Then for all \(\lambda \in J\):
\[
\|l_{\text{emb}(\Phi(\lambda))}(x)\| = \|\langle x, \text{emb} \Phi(f_\lambda) \rangle\| = \|\langle y, \text{emb} \Phi(f_\lambda) \rangle\| = \|\langle y, U(f)^* \Phi(f_\lambda) \rangle\| \leq \|y\| \|U(f_\lambda)\| \|\Phi(f)\| \leq M \|y\| \|\Phi(f)\|. \]

This proves the assertion. By the uniform boundedness theorem ([Wil], Theorem 3-3-6), the set \(\{l_{\text{emb}(\Phi(\lambda))} : \lambda \in J\}\) is uniformly bounded. Let \(M_0\) be an upper bound for this set. Then for all \(x \in H\):
\[
\|l_\Phi(x)\| = \|\langle x, \Phi \rangle\| = \lim_\lambda \|\langle x, \text{emb} \Phi(f_\lambda) \rangle\| \leq \lim_\lambda \|l_{\text{emb}(\Phi(\lambda))}(x)\| \leq M_0 \|x\|. \]

So \(l_\Phi\) is continuous for all \(\Phi \in T_{U, C}\).

Let \(\alpha \in S_{U, C}\). There exists \(\Phi \in T_{U, C}\) such that \(\alpha(\phi) = \langle \phi, \Phi \rangle\) for all \(\phi \in S_{U, C}\). Then \(\alpha \circ i = l_\Phi\) is continuous. Since a Hilbert space is bornological, the map \(i : H \to S_{U, C}\) is continuous. \(\square\)

We finish this chapter with a lemma which will be used in Chapter 8, but belongs here.

**Lemma 1.20.**

Let \(f \in C\), \(g \in L^1(G)\) and suppose \(f \ast g = 0\). Then \(U(g)^* x = 0\) for all \(x \in N_f^1\).

**Proof.**

Let \(E := \{x \in H : U(g)^* x = 0\}\). Then \(E\) is closed in \(H\). Let \(y \in H\). Then \(U(g)^* U(f)^* y = U(f^* g)^* y = 0\), so \(U(f)^* y \in E\). Hence \(U(f)^* (H) \subseteq E\). By [Wei], Theorem 4.13(b), \(N_f^1 = U(f)^* (H) \subseteq E\). \(\square\)
II.1. The Stone-representant for a unitary representation

Let $U$ be a representation of a locally compact Abelian group $G$ in a Hilbert space $H$. The group $G$ and the representation $U$ are taken fixed throughout this chapter.

DEFINITION 2.1.

Let $A$ be a locally compact topological Hausdorff space, $m$ a measure on $A$, $I$ an index set, for all $i \in I$ let $A_i$ be an open subset of $A$ with induced topology and let $\tau_i : \hat{G} \to A_i$ be a topological homeomorphism. Let $W$ be a unitary operator from $H$ onto $L^2(m)$.

The tuple $(A, m, I, A_i, \tau_i, W)$ will be called a Stone-representant for $U$ if and only if:

- $A_i \cap A_j = \emptyset$ if $i \neq j$ ($i, j \in I$),
- $A = \bigcup_{i \in I} A_i$,
- The map $Y \mapsto m(\tau_i(Y))$, $Y$ a Borel measurable subset of $\hat{G}$, is a finite regular measure on $\hat{G}$ ($i \in I$),
- $0 < m(A_i) < \infty$ ($i \in I$),
- $m(Z) = \sum_{i \in I} m(Z \cap A_i)$ ($Z \subset A$ Borel measurable),
- For all $f \in L^1(G)$ let $\hat{f}$ be the continuous function on $A$ such that $\hat{f}(\tau_i(\gamma)) = \hat{f}(\gamma)$ for all $i \in I$ and all $\gamma \in \hat{G}$. Then $WU(f)W^{-1} \xi = \hat{f} \cdot \xi$ for all $f \in L^1(G)$ and $\xi \in L^2(m)$.

This definition is useless without the next theorem.

THEOREM 2.2.

There exists a Stone-representant for $U$.

Proof.

See [HR II], Remark 33.6 and [HR I], Theorem C.37. \qed

REMARK. The Stone-representant is not unique.

REMARK. If, in addition, the Hilbert space $H$ is separable, then the measure $m$ is regular.

Proof. For $i \in I$ let $e_i := (m(A_i))^{-\frac{1}{2}} 1_{A_i}$. Then $(e_i)_{i \in I}$ is an orthonormal set in $L^2(m)$, so $I$ is countable. We may suppose that the set $I$ is infinite, and hence suppose that $I = \mathbb{N}$.

Every compact subset $K$ of $A$ is contained in only finite many open subsets $A_i$, hence $m(K) < \infty$.

Let $Z$ be a Borel measurable set, $m(Z) < \infty$ and let $\epsilon > 0$. For all $n \in \mathbb{N}$ there exist open $U_n \subset \hat{G}$ such that $m(\tau_n(U_n)) < m(Z \cap A_n) + \epsilon 2^{-n}$ and $U_n = \tau_n^{-1}(Z \cap A_n)$, by the regularity of $Y \mapsto m(\tau_n(Y))$. Let $U := \bigcup_{n=1}^{\infty} \tau_n(U_n)$. Then $U$ is open in $A$, $Z \subset U$ and $m(U) = \sum_{n=1}^{\infty} m(\tau_n(U_n)) \leq$
\[ \sum_{n=1}^{\infty} m(Z \cap A_n) + \varepsilon 2^{-n} = m(Z) + \varepsilon. \]

So \( m(Z) \) is the infimum of \( \{ m(U) : U \text{ open}, Z \subset U \} \).

Let \( U \) be an open subset of \( A \). Suppose \( m(U) < \infty \). Let \( \varepsilon > 0 \). There exists \( N \in \mathbb{N} \) such that \( m(U) \leq \sum_{n=1}^{N} m(U \cap A_n) + \frac{\varepsilon}{2} \). For all \( n \in \mathbb{N} \), \( n \leq N \) there exists compact \( K_n \subset \tau_n^{-1}(U \cap A_n) \) such that \( m(U \cap A_n) \leq m(\tau_n(K_n)) + \frac{\varepsilon}{2n} \). Let \( K := \bigcup_{n=1}^{N} \tau_n(K_n) \). Since each \( A_n \) is open in \( A \), the set \( K \) is compact in \( A, K \subset U \) and

\[ m(U) \leq \sum_{n=1}^{N} m(\tau_n(K_n)) + \varepsilon = m(K) + \varepsilon. \]

Hence,

\[ m(U) = \sup \{ m(K) : K \text{ compact}, K \subset U \}. \]

Similar, if \( m(U) = \infty \),

\[ m(U) = \sup \{ m(K) : K \text{ compact}, K \subset U \}. \]

So \( m \) is a regular measure on \( A \).

**REMARK.** The measure \( m \) is locally finite, i.e. for every measurable set \( Z \subset A \) with \( m(Z) > 0 \), there exists a measurable \( Z_1 \subset Z \) such that \( 0 < m(Z_1) < \infty \).

**DEFINITION 2.3.**

Let \( \text{Bor}(\hat{G}, \mathcal{C}) \) be the set of all complex valued Borel measurable functions on \( \hat{G} \) and let \( \text{Bor}_b(\hat{G}, \mathcal{C}) \) be the subset of all bounded elements of \( \text{Bor}(\hat{G}, \mathcal{C}) \). Define similarly \( \text{Bor}(\hat{G}, \mathcal{R}) \) and \( \text{Bor}_b(\hat{G}, \mathcal{R}) \).

With the aid of the Stone-representant for \( U \) we can extend the set operators \( U(f) \) with \( f \in L^1(G) \) to a set operators \( U[F] \) with \( F \in \text{Bor}_b(\hat{G}, \mathcal{C}) \).

**DEFINITION 2.4.**

Let \((A, m, I, A_i, \tau_i, W)\) be a Stone-representant for \( U \) and let \( F \) be a Borel measurable function on \( \hat{G} \). Define the Borel measurable function \( F \) on \( A \) by

\[ F(\tau_i(y)) := F(y) \quad (i \in I, \gamma \in \hat{G}) \]

For every \( F \in \text{Bor}_b(\hat{G}, \mathcal{C}) \) define a continuous operator \( U[F, A] \) on \( H \) by \( U[F, A] := W^{-1} M_F W \), with \( M_F \) the multiplication operator by \( F \) on \( L^2(m) \).

The definition of \( F \) depends on the choice of \( A \), but from the context it will follow on which set \( A \) the function \( F \) is defined. We prove that the operator \( U[F, A] \) does not depend on \( A \).
REMARK. It is clear that for all $F, K \in \text{Bor}_0(\hat{G}, \mathcal{C})$, all $\lambda \in \mathcal{C}$ and all $f \in L^1(G)$


$$U [\lambda F, A] = \lambda U [F, A],$$

$$U [FK, A] = U [F, A] \cdot U [K, A],$$

$$U [\bar{F}, A] = U [F, A]^*,$$

$$U [\hat{f}, A] = U(f),$$

for every Stone-representant $(A, M, I, A_i, \tau_i, W)$ for $U$.

**LEMMA 2.5.**

Let $(A, m, I, A_i, \tau_i, W)$ be a Stone-representant for $U$.

**I.** Let $V$ be an open subset of $\hat{G}$. Let

$$X := \text{clo} \{ U [F, A](H) : F \in C_0(\hat{G}), 0 \leq F \leq 1_v \}.$$

Then $U [1_v, A]$ is the projection of $H$ onto $X$.

**II.** Let $F_1, F_2, \cdots$ be a uniformly bounded sequence in $\text{Bor}_0(\hat{G}, \mathcal{C})$ and suppose

$$F(\gamma) := \lim_{n \to \infty} F_n(\gamma)$$

exists for every $\gamma \in \hat{G}$. Then $U [F, A] = s - \lim_{n \to \infty} U [F_n, A]$.

**Proof.**

**I.** Let $X_0 := U [1_v, A](H)$. Then $X_0$ is closed in $H$ and $X \subset X_0$. Let $f \in X_0$ and let $\xi := Wf$.

Then $1_v \cdot \xi \in L^2(m)$. We have to prove that $\xi \in \text{clo} \{ F \eta : F \in C_0(\hat{G}), 0 \leq F \leq 1_v, \eta \in L^2(m) \} = W^{-1}(X)$. We may as well assume that $\xi$ is real valued and $\xi \geq 0$. Moreover, by Lebesgue's dominated convergence theorem, we may also assume that there exist $\alpha, \beta > 0$ such that $\xi = \xi_1 \cdot 1_2$ with $Z = \{ a \in A : \alpha \leq \xi(a) \leq \beta \}$ and $Z \cap A_i \subset \tau_i(V)$ for all $i \in I$. Then $m(Z) < \infty$. Let $\varepsilon > 0$. There exist $n \in \mathbb{N}$ and $i_1, \cdots, i_n \in I$ such that

$$m(Z) < \sum_{j=1}^n m(Z \cap A_{i_j}) + \frac{\varepsilon}{2\beta^2}.$$ Since $m \circ \tau_i$ is a regular measure on $\hat{G}$, there exists compact $K \subset \hat{G}$ such that $K \subset \tau_i^{-1}(Z \cap A_{i_j})$ and $m(Z \cap A_{i_j}) - m(\tau_i(K)) < \frac{\varepsilon}{2n\beta^2}$ for all $j \in \{1, \cdots, n\}$. Then $K \subset V$. By Urysohn's lemma there exists $F \in C_c(\hat{G})$ such that $1_k \leq F \leq 1_v$.

Let $K_1 := \bigcup_{j=1}^n \tau_i(K)$. Then $K_1 \subset Z$ and $m(Z) - m(K_1) < \varepsilon\beta^{-2}$. Let $\eta := 1_{K_1} \cdot \xi$. Then

$$\| \xi - F \eta \|^2 = \| \xi - F 1_{K_1} \xi \|^2 = \| 1_z \xi - 1_{K_1} \xi \|^2 \leq \beta^2 \| 1_z - 1_{K_1} \| \| 1_z \xi \|^2 \leq \beta^2 \cdot \varepsilon\beta^{-2} = \varepsilon.$$ So $X_0 \subset X$.

**II.** This follows from Lebesgue's dominated convergence theorem.

**THEOREM 2.6.**

Let $(A_1, m_1, I_1, A_{1_1}, \tau_{1_1}, W_1)$ and $(A_2, m_2, I_2, A_{2_1}, \tau_{2_1}, W_2)$ be two Stone-representants for $U$. 


Then $U[F, A_1] = U[F, A_2]$ for all $F \in \text{Borb}(\hat{G}, \mathcal{C})$.

**Proof.** Let $f \in L^1(G)$. Then $U[\hat{f}, A_1] = U(f) = U[\hat{f}, A_2]$. Since $\{\hat{f} : f \in L^1(G)\}$ is dense in $C_0(\hat{G})$, we obtain $U[F, A_1] = U[F, A_2]$ for all $F \in C_0(\hat{G})$ by Lemma 2.5 II. Let $V \subset \hat{G}$ be open. Then we have seen that $\{U[F, A_1](H) : F \in C_0(\hat{G}), 0 \leq F \leq 1_V\} = \{U[F, A_2](H) : F \in C_0(\hat{G}), 0 \leq F \leq 1_V\}$. Hence by Lemma 2.5 I, $U[1_V, A_1] = U[1_V, A_2]$.

Let $B := \{V : V \subset \hat{G}$ Borel measurable and $U[1_V, A_1] = U[1_V, A_2]\}$. Then $B$ contains all open subsets of $\hat{G}$ and $B$ is a $\sigma$-algebra by Lemma 2.5 II, so $B$ contains all Borel measurable subsets of $\hat{G}$. Using Lemma 2.5 II again, we obtain $U[F, A_1] = U[F, A_2]$ for all $F \in \text{Borb}(\hat{G}, \mathcal{C})$. 

We naturally arrive at the following definition.

**DEFINITION 2.7.**

Let $F \in \text{Borb}(\hat{G}, \mathcal{C})$. Define $U[F] := U[F, A]$ where $(A, m, I, A_i, \tau_i, W)$ is any Stone-representant for $U$.

We summarize some properties of $U[\cdot]$.

**PROPOSITION 2.8.**

Let $F, K \in \text{Borb}(\hat{G}, \mathcal{C})$, $\lambda \in \mathcal{C}$ and $f \in L^1(G)$. Then

- $\|U[F]\| \leq \|F\|_\infty$,
- $U[F + K] = U[F] + U[K]$,
- $U[\lambda F] = \lambda U[F]$, 
- $U[FK] = U[F] \circ U[K]$,
- $U[\hat{f}] = U[F]^*$,
- $U[1_{\hat{G}}] = I$. 

**LEMMA 2.9.**

Let $F \in \text{Bor}(\hat{G}, \mathcal{C})$ and let $(A, m, I, A_i, \tau_i, W)$ be a Stone-representant for $U$. For all $n \in \mathbb{N}$ let $V_n := \{\gamma \in \hat{G} : n - 1 \leq |F(\gamma)| < n\}$. Then the orthogonal sum $\sum_{n \in \mathbb{N}} U[F \cdot 1_{V_n}] = W^{-1} M_F W$, where $M_F$ denotes the (unbounded) multiplication operator by $F$ on $L^2(m)$.

**Proof.**

Trivial. 

[]
In the following definition we extend $U\ [\cdot\ ]$ to $\text{Bor}(\hat{G}, \mathcal{C})$. This definition makes sense by Lemma 2.9.

**DEFINITION 2.10.**
Let $F \in \text{Bor}(\hat{G}, \mathcal{C})$. Define a closed, densely defined operator $U\ [F\ ]$ on $H$ by $U\ [F\ ] := W^{-1} M_F W$ where $(A, m, I, A_t, \tau, W)$ is any Stone-representant for $U$ and $M_F$ denotes the multiplication operator by $F$ on $L^2(m)$.

**REMARK.** $U\ [F\ ]$ is a self-adjoint operator on $H$ for every $F \in \text{Bor}(\hat{G}, \mathcal{R})$.

**II.2. Seminorms on $S_{U,C}$ and bounded sets in $T_{U,C}$**

Let $C$ be a fixed subset of $L^1(G)$ which satisfies properties P1 and P2. Corresponding to the set $C$ we define three subsets of $\text{Bor}(\hat{G}, \mathcal{C})$ and seminorms on $S_{U,C}$.

**DEFINITIONS 2.11.**
Let

\[ C^\# := \{ F \in \text{Bor}(\hat{G}, \mathcal{C}) : \text{for all } f \in C \text{ is } F \cdot \hat{f} \text{ bounded} \}, \]

\[ C_u^\# := \{ F \in C^\# : \text{for all } \gamma \in \hat{G} \text{ is } F(\gamma) \neq 0 \}, \]

\[ C_p^\# := \{ F \in C^\# : \text{there exists } \varepsilon > 0 \text{ such that for all } \gamma \in \hat{G} \text{ holds } |F(\gamma)| > \varepsilon \}. \]

For all $F \in C^\#$ define

\[ s_F : S_{U,C} \to \mathcal{R} \]

\[ s_F(\phi) := \| U\ [F\ ] \phi \| \quad (\phi \in S_{U,C}). \]

**REMARK.** Since $\hat{\hat{f}} = \hat{f}$ for all $f \in C$, it is obvious that $(\hat{C})^\# = C^\#$, $(\hat{C})_u^\# = C_u^\#$ and $(\hat{C})_p^\# = C_p^\#$. We write $\hat{C}^\#$, $\hat{C}_u^\#$ and $\hat{C}_p^\#$ for $(\hat{C})^\#$, $(\hat{C})_u^\#$, resp. $(\hat{C})_p^\#$.

**REMARK.** $C^\#$ is a vector space. $\text{Bor}_b(\hat{G}, \mathcal{C}) \subset C^\#$. For all $F \in C^\#$ and $K \in \text{Bor}_b(\hat{G}, \mathcal{C})$, also $FK \in C^\#$ and $\overline{F} \in C^\#$.

**LEMMA 2.12.**
I. Let $F \in C^\#$. Then $s_F$ is $\sigma_{\text{ind}}$-continuous on $S_{U,C}$.

II. The set of seminorms $\{ s_F : F \in C_p^\# \}$ separates the points of $S_{U,C}$.

**Proof.**
I. Let \( f \in C \). For all \( \phi \in R_f \) holds \( s_F(\phi) = \|U[F] U(f) \Omega_f^{-1}(\phi)\| = \|U[F] \Omega_f^{-1}(\phi)\| \leq \|U[F]\| \Omega_f^{-1}(\phi)\| \leq \|F\| \|\phi\|_F \). By [Wil], Theorem 13-1-8, \( s_F \) is \( \sigma_{\text{ind}} \)-continuous.

II. Trivial because \( I_C \in C_p^* \).

DEFINITION 2.13.
Let \( \sigma_{\text{proj}} \) be the locally convex Hausdorff topology for \( S_{U,C} \) generated by the seminorms \( s_F \) with \( F \in C_p^* \).

COROLLARY 2.14.
The following inclusion holds for the topologies for \( S_{U,C} \):

\[ \sigma_{\text{proj}} \subset \sigma_{\text{ind}}. \]

Proof.
Lemma 2.12 1.

We introduce a special type of elements of \( T_{U,C} \).

DEFINITION 2.15.
Let \( F \in C_p^* \) and \( x \in H \). Define \( U[F] \star x : C \to H \)

\[ (U[F] \star x)(f) := U[F] U(f)^* x \quad (f \in C). \]

LEMMA 2.16.
Let \( F \in C_p^* \) and \( x \in H \). Then:

I. \( U[F] \star x \) is a \( C \)-trajectory.

II. If \( x \in D(U[F]) \), then \( U[F] \star x = \text{emb}(U[F]x) \).

III. Let \( \phi \in S_{U,C} \). Then \( \langle \phi, U[F] \star x \rangle = (U[\tilde{F}]\phi, x) \).

IV. Let \( K \in \text{Borel}(\tilde{G}, C) \). Then \( U[F] \star U[K] x = U[FK] \star x \).

V. Suppose \( F \in C_p^* \). Then \( U[F] \star x = 0 \) if and only if \( x = 0 \).

VI. The map \( U[F] \star \) from \( H \) into \( T_{U,C} \) is continuous.

Proof.
I. Let \( f \in C \), \( g \in L^1(G) \) and suppose \( f \star g \in C \). Then

\[ (U[F] \star x)(f \star g) = U[F] U(f \star g)^* x = U[F] U(f) \tilde{g} U(f) \tilde{g} x = U[F] U(f) \tilde{g} x = (U[F] \star x)(f). \]

So \( U[F] \star x \in T_{U,C} \).
II. Suppose $x \in D(U[F]),$ For all $f \in C$ holds

$$(U[F]*x)(f) = U[F] U(f)^* x = U(f)^* U[F] x = (\text{emb}(U[F]x))(f).$$

III. Let $f \in C$ and $\phi \in R_f.$ Then

$$<\phi, U[F]*x> = (\Omega_f^{-1}(\phi), U[F] U(f)^* x) = (\Omega_f^{-1}(\phi), U[F \tilde{f}] x) = (U[\tilde{F}] \Omega_f^{-1}(\phi), x) = (U[\tilde{F}] \phi, x).$$

IV. For all $f \in C$ we have


V. Let $(f_\lambda)_{\lambda \in J}$ be the net in property $P_2.$ Suppose $U[F]*x = 0.$ Then $U[F] (U(f_\lambda)*x) = (U[F]*x)(f_\lambda) = 0,$ so $U(f_\lambda)*x = 0$ for all $\lambda \in J.$ Hence $x = \lim U(f_\lambda)*x = 0.$

VI. Let $f \in C.$ For all $x \in H$ we obtain

$$t_f(U[F]*x) = \| U[F \tilde{f}] x \| \leq \| F \tilde{f} \| \| x \|.$$ So $U[F]*$ is continuous.

**DEFINITION 2.17.**

Let $F \in C_w.$ By Lemma 2.16 V there can be defined a unique norm $\| \|_F$ on $U[F]*H$ such that $U[F]*H$ becomes a Hilbert space and $U[F]*$ is a unitary map from $H$ onto $U[F]*H.$

**LEMMA 2.18.**

I. The identity map from $U[F]*H$ into $(T_{U,C}, \tau_{\text{proj}})$ is continuous for all $F \in C_w.$

II. Let $F, K \in C_w.$ Then $\| F + K \| \in C_w, U[F]*H \subset U[| F + K \|]*H$ and the identity map from $U[F]*H$ into $U[| F + K \|]*H$ is continuous.

**Proof.**

I. This follows from Lemma 2.16 VI and Definition 2.17.

II. It is clear that $\| F + K \| \in C_w.$ Define $L \in \text{Bor}_w(\hat{G}, C)$ by $L := \frac{F}{\| F + K \|}.$ Then $(\| F + K \|)*L = F.$ Let $x \in H.$ Then by Lemma 2.16. IV,

$$U[F]*x = U[| F + K \|]*U[L] x \in U[| F + K \|]*H$$

and

$$\| U[F]*x \|_{| F + K \|} = \| U[L] x \| \leq \| x \| = \| U[F]*x \|_F.$$

**DEFINITION 2.19.**

Let $T := \bigcup_{F \in C_w} U[F]*H.$ By Lemma 2.18 II, $T$ is a linear vector space. The topology $\tau_{\text{ind}}$ for $T$ is the inductive limit topology generated by the Hilbert spaces $U[F]*H, F \in C_w.$

We only use the space $T$ in this section and the following section. We give conditions such that
every \(\tau_{\text{proj}}\)-bounded subset of \(T_{U,C}\) is contained in a set of the form \(U[F] \ast B\) with \(F \in C_p^\#\) and \(B\) a bounded subset of \(H\). Then, in particular, \(T = T_{U,C}\) as sets and \(\tau_{\text{ind}}\) is a locally convex topology for the vector space \(T_{U,C}\).

**LEMMA 2.20.**

\(T\) is a subset of \(T_{U,C}\). The identity map from \((T, \tau_{\text{ind}})\) into \((T_{U,C}, \tau_{\text{proj}})\) is continuous. Hence the topology \(\tau_{\text{ind}}\) for \(T\) is Hausdorff.

**Proof.**

From Lemma 2.16 I it follows that \(T \subset T_{U,C}\) and by Lemma 2.18 I and [Wil] Theorem 13-1-8, the identity map from \((T, \tau_{\text{ind}})\) into \((T_{U,C}, \tau_{\text{proj}})\) is continuous. Since \(\tau_{\text{proj}}\) is Hausdorff, also \(\tau_{\text{ind}}\) is a Hausdorff topology. \(\square\)

The main result of this section is the following theorem.

**THEOREM 2.21.**

The following conditions are equivalent.

I. \((S_{U,C}, \sigma_{\text{ind}}) = (S_{U,C}, \sigma_{\text{proj}})\) as topological vector spaces.

II. For every \(\tau_{\text{proj}}\)-bounded subset \(B\) of \(T_{U,C}\) there exist \(F \in C_p^\#\) and a bounded subset \(B_0\) of \(H\) such that \(B = U[F] \ast B_0\).

**Proof.**

I \(\Rightarrow\) II. Let \(B\) be a bounded subset of \(T_{U,C}\). Let \(B^*\) be the polar of \(B\). By [Wil] Theorem 8-4-12, \(B^*\) is an absorbing subset of \(S_{U,C}\), which is also closed and absolutely convex. So \(B^*\) is a barrel in \(S_{U,C}\) and since \((S_{U,C}, \sigma_{\text{ind}})\) is barrelled, \(B^*\) is a \(\sigma_{\text{ind}}\)-neighbourhood of 0 in \(S_{U,C}\). Because \(\sigma_{\text{ind}} = \sigma_{\text{proj}}\), there exist \(F \in C_p^\#\) and \(\varepsilon > 0\) such that \(\{\phi \in S_{U,C} : s_F(\phi) \leq \varepsilon\} \subset B^*\).

Let \(\Phi \in B\). For all \(\phi \in S_{U,C}\) with \(\| U[F] \phi \| \leq \varepsilon\) holds \(\langle \phi, \Phi \rangle \leq 1\), so \(\langle \phi, \Phi \rangle \leq \frac{1}{\varepsilon} \| U[F] \| \) for all \(\phi \in S_{U,C}\). Define \(l : U[F] (S_{U,C}) \to C\) by \(l(U[F]\phi) := \langle \phi, \Phi \rangle\) (\(\phi \in S_{U,C}\)). Let the topology for \(U[F] (S_{U,C})\) be the induced topology of \(H\). Then the map \(l\) is continuous and linear. By Riesz' theorem there exists \(x_\Phi \in \overline{U[F]}(S_{U,C}) \subset H\) such that \(l(x) = (x, x_\Phi)\) for all \(x \in U[F] (S_{U,C})\). Then \(\| x_\Phi \| = l \| l \| \leq \varepsilon^{-1}\). For all \(\phi \in S_{U,C}\) we obtain \(\langle \phi, U[F] \ast x_\Phi \rangle = (U[F]\phi, x_\Phi) = l(U[F]\phi) = \langle \phi, \Phi \rangle\), so \(U[F] \ast x_\Phi = \Phi\) by Lemma 1.17.

Let \(B_0 := \{x_\Phi : \Phi \in B\}\). Then \(B_0\) is bounded in \(H\) and \(B = U[F] \ast B_0\).
II ⇒ I. Let V be a $\sigma_{\text{ind}}$-neighbourhood of 0 in $S_{U,C}$. Since $\sigma_{\text{ind}}$ is a regular locally convex topology for $S_{U,C}$ there exists a closed absolutely convex $\sigma_{\text{ind}}$-neighbourhood $W$ of 0 such that $W \subset V$. Then $W = W^{**}$ by [Wil] Theorem 8-3-8, hence $(W^{**})^\circ$ is absorbing. So ([Wil] Theorem 8-4-12) $W^*$ is $\tau_{\text{proj}}$-bounded in $T_{U,C}$. By assumption there exist $F \in C^K$ and a bounded subset $B_0$ of $H$ such that $W^* = U[F] \ast B_0$. Then $(U[F] \ast B_0)^\circ = W^{**} = W \subset V$. Let $M > 0$ be such that $\|x\| \leq M$ for all $x \in B_0$. Let $\phi \in S_{U,C}$ and suppose $\|U[\tilde{F}]\phi\| \leq M^{-1}$. Then for all $x \in B_0$:

$$\|\phi, U[F] \ast x \| > 1 = \|U[\tilde{F}]\phi, x\| \leq \|U[\tilde{F}]\phi\| \|x\| \leq 1,$$

so $\phi \in (U[F] \ast B_0)^\circ \subset V$. Then $\{\phi \in S_{U,C} : s_F(\phi) \leq M^{-1}\} \subset V$ and $V$ is a $\sigma_{\text{proj}}$-neighbourhood.

II.3. A growth condition

Let $C$ be a subset of $L^1(G)$ and suppose the pair $(C, U)$ satisfies properties P1 and P2. In this section we consider an additional condition on the set $C$. We then prove that every $\tau_{\text{proj}}$-bounded subset $B$ of $T_{U,C}$ is equal to $U[F] \ast B_0$ for some $F \in C^K$ and some bounded subset $B_0$ of $H$.

Suppose the set $C$ has the following property.

PROPERTY 2.22.

P3. There exist a sequence of Borel measurable disjoint sets $Q_1, Q_2, \cdots$ in $\hat{G}$ and a sequence of positive real numbers $b_1, b_2, \cdots$ such that $\hat{G} = \bigcup_{n=1}^{\infty} Q_n$ and $\sum_{n=1}^{\infty} b_n^{-1} < \infty$ and for all $f \in C$ there exist $g \in C$ and $\delta > 0$ such that for all $n \in \mathbb{N}$ holds:

$$b_n \sup \{\|\hat{f}(\gamma)\| : \gamma \in Q_n\} \leq \delta \inf \{\|\hat{g}(\gamma)\| : \gamma \in Q_n\}.$$ 

This property is inspired by [EGK] Definition 1.1, A III and improved by F.J.L. Martens.

REMARK. Property P3 is equivalent with the following property.

P3'. There exist a sequence of Borel measurable disjoint sets $Q_1, Q_2, \cdots$ in $\hat{G}$, a sequence of positive real numbers $b_1, b_2, \cdots$ and $\nu > 0$ such that $\hat{G} = \bigcup_{n=1}^{\infty} Q_n$ and $\sum_{n=1}^{\infty} b_n^{-\nu} < \infty$ and for all $f \in C$ there exist $g \in C$ and $\delta > 0$ such that for all $n \in \mathbb{N}$ holds:

$$b_n \sup \{\|\hat{f}(\gamma)\| : \gamma \in Q_n\} \leq \delta \inf \{\|\hat{g}(\gamma)\| : \gamma \in Q_n\}.$$ 

Of course, P3 implies P3'. Suppose P3' holds. Then, inductively, for all $k \in \mathbb{N}$ we obtain: for all $f \in C$ there exist $g \in C$ and $\delta > 0$ such that for all $n \in \mathbb{N}$ holds:

$$b_n^k \sup \{\|\hat{f}(\gamma)\| : \gamma \in Q_n\} \leq \delta \inf \{\|\hat{g}(\gamma)\| : \gamma \in Q_n\}.$$ 

Take $k > \frac{1}{\nu}$. 
REMARK. In case the set $C$ is supposed to satisfy property P3 then we write $Q_1, Q_2, \ldots$ and $b_1, b_2, \ldots$ for the sequences as in property P3 without explicitly defining them again, if no confusion can arise.

REMARK. The set $C$ satisfies property P3 if and only if the set $\tilde{C}$ satisfies property P3.

We start with some useful lemmas. Let $(A, m, I, A_i, \tau_i, W)$ be a Stone-representant for $U$, which is taken fixed throughout the remaining part of this chapter.

**Definition 2.23.**

Let $n \in \mathbb{N}$. Define the projection operator $P_n := U[1_{Q_n}]$, define the layer $L_n$ of $Q_n$ in $A$ by $L_n := \{ \tau_i(\gamma) : i \in I, \gamma \in Q_n \}$.

Note that $P_n = 0$ if and only if $m(L_n) = 0$ ($n \in \mathbb{N}$).

**Lemma 2.24.**

Let $f \in C$, $n \in \mathbb{N}$, $\varepsilon > 0$ and suppose $|\hat{f}(a)| \geq \varepsilon$ for a.e. $a \in L_n$. Then:

I. $P_n(H)$ is a subset of $R_f$.

II. The map $\Omega_f^{-1} P_n$ from $H$ into $H$ is bounded and $\| \Omega_f^{-1} P_n \| \leq \frac{1}{\varepsilon}$.

III. $\Omega_f^{-1} (\phi) \in P_n(H)$ for all $\phi \in P_n(H)$.

IV. $U(g) \Omega_f^{-1} P_n = \Omega_f^{-1} P_n U(g)$ for all $g \in L^1(G)$.

**Proof.**

Let $V := \{ \gamma \in \hat{G} : \hat{f}(\gamma) = 0 \}$ and let $P := U[1_V]$. Then $P$ is the projection of $H$ onto $N_f$. Define

$$\Lambda : A \rightarrow \mathcal{C}$$

$$\Lambda(a) := \begin{cases} (\hat{f}(a))^{-1} & \text{if } |\hat{f}(a)| \geq \varepsilon \text{ and } a \in L_n, \\ 0 & \text{else.} \end{cases} \quad (a \in A)$$

Then $\Lambda$ is Borel measurable and $\| \Lambda \|_m \leq \varepsilon^{-1}$. Let $M_{\Lambda}$ be the multiplication operator by $\Lambda$ on $L^2(m)$.

I. Let $x \in H$. Then $U(f) W^{-1} M_{\Lambda} Wx = W^{-1} (\hat{f} \Lambda \cdot Wx) = W^{-1} (1_{L_n} \cdot Wx) = P_n x$. So $P_n(H) \subset R_f$.

II. Let $x \in H$. Then $W^{-1} M_{\Lambda} Wx \in N_f^\perp$ since $1_V \cdot (\Lambda \cdot Wx) = 0$ a.e. So $\Omega_f^{-1} P_n x = W^{-1} M_{\Lambda} Wx$ and $\Omega_f^{-1} P_n = W^{-1} M_{\Lambda} W$. Then $\| \Omega_f^{-1} P_n \| \leq \| \Lambda \|_m \leq \varepsilon^{-1}$. 

III. Let \( \phi \in P_n(H) \). Then
\[
\Omega_f^{-1}(\phi) = P_n \Omega_f^{-1} \left( P_n \phi = P_n \omega^{-1} M \omega \phi = \omega^{-1} \left( \omega \Lambda \omega \phi \right) = \Omega_f^{-1} \phi \right) \in P_n(H).
\]

IV. Let \( x \in H \). Then \( U(g) \Omega_f^{-1} P_n x = W^{-1}(\hat{g} \Lambda \omega) x = \Omega_f^{-1} P_n U(g)x. \]

[Lemma 2.25]

For all \( n \in \mathbb{N} \) with \( P_n \neq 0 \) there exists \( f \in C \) and \( \varepsilon > 0 \) such that \( |\hat{f}(\gamma)| > \varepsilon \) for all \( \gamma \in Q_n \).

**Proof.**

Let \( n \in \mathbb{N} \) and suppose \( m(L_n) \neq 0 \). There exists \( i \in I \) such that \( m(\tau_i(Q_n)) \neq 0 \). Let \( \xi := 1_{\tau_i(Q_n)} \). Then \( \xi \in L^2(m) \) and \( \xi \neq 0 \). Let \( (f_\lambda)_{\lambda \in J} \) be the net in property P2. Suppose \( \hat{f}_\lambda(\gamma) = 0 \) for all \( \lambda \in J \) and \( \gamma \in Q_n \). Then \( 0 \neq W^{-1} \xi = \lim_{\lambda} F(\lambda) x = \lim_{\lambda} W^{-1}(\hat{f}_\lambda) \xi = 0 \), contradiction. So there exists \( \lambda \in J \) and \( \gamma \in Q_n \) such that \( \hat{f}_\lambda(\gamma) \neq 0 \). By property P3 there exist \( g \in C \) and \( \delta > 0 \) such that

\[
b_n \sup \{ |\hat{f}_\lambda(\omega)| : \omega \in Q_n \} \leq \delta \inf \{ |\hat{g}(\omega)| : \omega \in Q_n \}.
\]

Then \( |\hat{g}(\omega)| \geq \delta^{-1} |\hat{f}_\lambda(\gamma)| > b_n \) for all \( \omega \in Q_n \).

**Corollary 2.26.**

Let \( n \in \mathbb{N} \). Then \( P_n(H) \subset S_{U,C} \).

**Proof.**

If \( P_n = 0 \), then \( P_n(H) = \{0\} \subset S_{U,C} \). If \( P_n \neq 0 \), the corollary is proved by Lemmas 2.25 and 2.24.

**Lemma 2.27.**

Let \( \Phi \in T_{U,C} \) and \( n \in \mathbb{N} \). Define \( \Psi : C \rightarrow H \)

\[
\Psi(f) := P_n \Phi(f) \quad (f \in C).
\]

Then \( \Psi \in \text{emb}(H) \). More explicitly, suppose there exist \( g \in C \) and \( \varepsilon > 0 \) such that \( |\hat{g}(\gamma)| \geq \varepsilon \) for all \( \gamma \in Q_n \). Then \( \Psi = \text{emb}(\Omega_f^{-1} P_n \Phi(g)) \).

**Proof.**

We may assume that \( P_n \neq 0 \). By Lemma 2.25 there exist \( g \in C \) and \( \varepsilon > 0 \) such that \( |\hat{g}(\gamma)| > \varepsilon \) for all \( \gamma \in Q_n \). Then \( |\hat{g}(\gamma)| \geq \varepsilon \) for all \( \gamma \in Q_n \). Since the pair \((C, U)\) satisfies properties P1-P3, we obtain by Lemma 2.24 I that \( P_n \Phi(g) \in R_\hat{g} \). So \( \Omega_f^{-1} P_n \Phi(g) \) is well defined.

Let \( f \in C \). Then by the Lemmas 2.24 IV, 1.9 and 2.24 I,

\[
[\text{emb} \Omega_f^{-1} P_n \Phi(g)](f) = U(f)^* \Omega_f^{-1} P_n \Phi(g) = \Omega_f^{-1} P_n U(f)^* \Phi(g) = \Omega_f^{-1} P_n U(g)^* \Phi(f) = \Omega_f^{-1} U(\hat{g}) P_n \Phi(f) = \Omega_f^{-1} \Omega_f P_n \Phi(f) = P_n \Phi(f) = \Psi(f).
\]

So \( \Psi = \text{emb}(\Omega_f^{-1} P_n \Phi(g)) \).

The following definition makes sense by Lemma 2.27.
DEFINITION 2.28.
Let $\Phi \in T_{U,C}$ and $n \in \mathbb{N}$. Define $P_n \cdot \Phi \in H$ by the relation $P_n \Phi(f) = [\text{emb} (P_n \cdot \Phi)](f)$ for all $f \in C$.

LEMMA 2.29.
I. Let $\Phi \in T_{U,C}$. Then $\Phi = \lim_{N \to \infty} \sum_{n=1}^{N} \text{emb}(P_n \cdot \Phi)$ in $(T_{U,C}, \tau_{\text{proj}})$.

II. $P_n \cdot \text{emb} \cdot x = P_n x$ for all $x \in H$ and $n \in \mathbb{N}$.

III. $P_n \cdot \Phi \in P_n(H)$ for all $\Phi \in T_{U,C}$ and $n \in \mathbb{N}$.

IV. The map $\Phi \mapsto P_n \cdot \Phi$ from $(T_{U,C}, \tau_{\text{proj}})$ into $H$ is continuous for all $n \in \mathbb{N}$.

Proof.
I. Let $\Phi \in T_{U,C}$. For every $f \in C$ we obtain in the Hilbert space $H$:

$$\Phi(f) = \sum_{n=1}^{\infty} P_n \Phi(f) = \lim_{N \to \infty} \left( \sum_{n=1}^{N} \text{emb}(P_n \cdot \Phi) \right)(f).$$

So $\Phi = \lim_{N \to \infty} \sum_{n=1}^{N} \text{emb}(P_n \cdot \Phi)$ in $(T_{U,C}, \tau_{\text{proj}})$.

II. Let $n \in \mathbb{N}$ and $x \in H$. Then $P_n (\text{emb} x) (f) = P_n U(f) x = U(f)^* P_n x = (\text{emb} P_n x)(f)$ for all $f \in C$. By Definition 2.28, $P_n \cdot \text{emb} \cdot x = P_n x$.

Let $n \in \mathbb{N}$. For the proof of III and IV we may suppose that $P_n \neq 0$. By Lemma 2.25 there exist $g \in C$ and $\varepsilon > 0$ such that $|g(\gamma)| \geq \varepsilon$ for all $\gamma \in Q_n$.

III. Let $\Phi \in T_{U,C}$. By Lemmas 2.27 and 2.24 III, $P_n \cdot \Phi = \Omega_x^{-1} P_n \Phi(g) \in P_n(H)$.

IV. By Lemmas 2.27 and 2.24 II we obtain for all $\Phi \in T_{U,C}$:

$$\| P_n \cdot \Phi \| \leq \| \Omega_x^{-1} P_n \Phi(g) \| \leq \| \Omega_x^{-1} P_n \| \cdot \| \Phi(g) \| \leq \varepsilon^{-1} \cdot t_g(\Phi).$$

So the map $\Phi \mapsto P_n \cdot \Phi$ from $(T_{U,C}, \tau_{\text{proj}})$ into $H$ is continuous.

The main result of this section is the following theorem.

THEOREM 2.30.
Let $B \subset T_{U,C}$ be a $\tau_{\text{proj}}$-bounded set, $B \neq \emptyset$. Then there exist $F \in C_f^g$ and a bounded set $B_0 \subset H$ such that $U[F] \cdot B_0 = B$ and $U[F] \cdot \| B_0 \| \cdot \| \Phi \| \mapsto (B, \tau_{\text{proj}} | B)$ is a homeomorphism.

Proof.
Let $n \in \mathbb{N}$. By Lemma 2.29 IV, the map $\Phi \mapsto P_n \cdot \Phi$ from $(T_{U,C}, \tau_{\text{proj}})$ into $H$ is continuous, so $r_n := \sup \{ \| P_n \cdot \Phi \| : \Phi \in B \} < \infty$.

Define $F : \hat{G} \to \mathbb{R}$

$$F := \sum_{n=1}^{\infty} (1 + b_n r_n) 1_{Q_n}.$$ 

Assertion: $F \in C_f^g$. Let $f \in C$. By property P3, there exist $g \in C$ and $\delta > 0$ such that
Then \( \tilde{F} \tilde{f} \leq \| \tilde{f} \|_{\infty} + \delta \sup \{ t_g(\Phi) : \Phi \in B \} < \infty \). Indeed, for all \( n \in \mathbb{N} \), \( \gamma \in Q_n \) such that \( \tilde{f}(\gamma) \neq 0 \) and for all \( \Phi \in B \) we obtain by Lemma 2.24 II:

\[
(1 + b_n \| P_n \cdot \Phi \|) \| \tilde{f} \|_{\infty} + b_n \Omega \tilde{\Phi}^{-1} \| \Phi(g) \| \| \tilde{f}(\gamma) \| \leq \| \tilde{f} \|_{\infty} + b_n (\delta^{-1} b_n \| \tilde{f}(\gamma) \|)^{-1} \cdot \| \Phi(g) \| \| \tilde{f}(\gamma) \| \leq \| \tilde{f} \|_{\infty} + \delta \sup \{ t_g(\Psi) : \Psi \in B \}.
\]

So the assertion is proved, since \( \| \tilde{f}(\gamma) \| = \sup \{ (1 + b_n \| P_n \cdot \Phi \|) \| \tilde{f}(\gamma) \| : \Phi \in B \} \).

For all \( \Phi \in B \) define \( x_\Phi := \sum_{n=1}^{\infty} \frac{1}{1+b_n r_n} P_n \cdot \Phi \in H \). Let \( B_0 := \{ x_\Phi : \Phi \in B \} \). Then

\[
\| x_\Phi \|^2 = \sum_{n=1}^{\infty} \frac{1}{(1+b_n r_n)^2} \| P_n \cdot \Phi \|^2 \leq \sum_{n=1}^{\infty} \frac{1}{b_n^2} < \infty \text{ for all } \Phi \in B, \text{ so } B_0 \text{ is bounded in } H.
\]

Let \( \Phi \in B \). For all \( f \in C \) we obtain by Definition 2.28:

\[
P_n \Phi(f) = P_n U(F)^{\circ} (P_n \cdot \Phi) = P_n U(F) U(f)^{\circ} x_\Phi = P_n((U(F)^{\circ} x_\Phi)(f)) \text{ for all } n \in \mathbb{N}.
\]

So \( \Phi = U(F)^{\circ} x_\Phi \). By Lemma 2.16 VI, \( U(F)^{\circ} \mid B : B_0 \to B \) is a bijection. Obviously \( U(F)^{\circ} \mid B : B_0 \to B \) is continuous. (See Lemma 2.16 VI.) The inverse map is also continuous. Indeed, let \( (\Phi_\alpha)_{\alpha \in M} \) be a net in \( B \), let \( \Phi \in B \) and suppose \( \lim_{\alpha} \Phi_\alpha = \Phi \) in \( \tau_{\text{proj}} \mid B \).

Assertion: \( \lim_{\alpha} x_{\Phi_\alpha} = x_\Phi \) in \( H \). Let \( \epsilon > 0 \). Let \( N \in N \) be so large that \( \sum_{n=N+1}^{\infty} \frac{4}{b_n^2} \leq \frac{\epsilon^2}{2} \). For all \( n \in \mathbb{N} \), \( n \leq N \), and \( P_n \neq 0 \), there exist \( f_n \in C \) and \( \epsilon_n > 0 \) such that \( \| f_n \|_{Q_n} \geq \epsilon_n \) by Lemma 2.25. There exist \( \alpha_0 \in M \) such that for all \( \alpha \in M \), \( \alpha \geq \alpha_0 \), and all \( n \in N \), \( n \leq N \), \( P_n \neq 0 \) holds:

\[
\| f_n (\Phi_\alpha - \Phi) \| \leq (1+b_n r_n) \epsilon_n (2N)^{-\frac{3}{2}}.
\]

Then for all \( \alpha \in M \), \( \alpha \geq \alpha_0 \) we obtain by Lemma 2.24 II:

\[
\| x_{\Phi_\alpha} - x_\Phi \|^2 = \sum_{n=1}^{N} \| P_n (x_{\Phi_\alpha} - x_\Phi) \|^2 + \sum_{n=N+1}^{\infty} \| P_n (x_{\Phi_\alpha} - x_\Phi) \|^2
\]

\[
= \sum_{n=1}^{N} \| \frac{1}{1+b_n r_n} \Omega_{\alpha}^{-1} P_n ((\Phi_\alpha - \Phi)(f_n)) \|^2 + \sum_{n=N+1}^{\infty} \| \frac{1}{1+b_n r_n} (P_n \cdot \Phi_\alpha - P_n \cdot \Phi) \|^2
\]

\[
\leq \sum_{n=1}^{N} \left( \frac{1}{1+b_n r_n} \frac{1}{\epsilon_n} \| (\Phi_\alpha - \Phi)(f_n) \|^2 + \sum_{n=N+1}^{\infty} \frac{2}{b_n} \right)
\]

\[
\leq \sum_{n=1}^{N} \frac{\epsilon_n^2}{2N} + \frac{\epsilon_n^2}{2} \leq \epsilon^2.
\]

So the assertion and the theorem is proved.

**COROLLARY 2.31.**

\( (S_U, c \cdot \sigma_{\text{ind}}) = (S_U, c \cdot \sigma_{\text{proj}}) \) as topological vector spaces.

**Proof.**

Theorems 2.30 and 2.21.
Let $T$ be the subspace of $T_{U,C}$ as in Definition 2.19.

**COROLLARY 2.32.**

$T = T_{U,C}$ as sets. We have even the stronger equality: $T_{U,C} = \bigcup_{F \in C^*_u} U[F] \ast H$ as sets.

The identity map from $(T_{U,C}, \tau_{\text{ind}})$ into $(T_{U,C}, \tau_{\text{proj}})$ is continuous.

In the next chapter we give conditions such that $(T_{U,C}, \tau_{\text{ind}})$ is equal to $(T_{U,C}, \tau_{\text{proj}})$ as a topological vector space.

**COROLLARY 2.33.**

Let $B$ be a subset of $T_{U,C}$. Then:

I. $B$ is bounded in $(T_{U,C}, \tau_{\text{proj}})$ if and only if there exist $F \in C^*_u$ and bounded $B_0 \subset H$ such that $B = U[F] \ast B_0$.

II. $B$ is compact in $(T_{U,C}, \tau_{\text{proj}})$ if and only if there exist $F \in C^*_u$ and a compact $B_0 \subset H$ such that $B = U[F] \ast B_0$.

III. $B$ is sequentially compact in $(T_{U,C}, \tau_{\text{proj}})$ if and only if there exist $F \in C^*_u$ and a sequentially compact $B_0 \subset H$ such that $B = U[F] \ast B_0$.

IV. $B$ is sequentially compact in $(T_{U,C}, \tau_{\text{proj}})$ if and only if $B$ is compact in $(T_{U,C}, \tau_{\text{proj}})$.

V. Let $(\Phi_n)_{n \in \mathbb{N}}$ be a sequence in $(T_{U,C}, \tau_{\text{proj}})$. Then $(\Phi_n)_{n \in \mathbb{N}}$ is a Cauchy sequence if and only if there exist $F \in C^*_u$ and a Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in $H$ such that $\Phi_n = U[F] \ast x_n$ for all $n \in \mathbb{N}$.

VI. Every $\tau_{\text{proj}}$-bounded sequence in $T_{U,C}$ has a weakly convergent subsequence.

**Proof.**

Every compact and every sequentially compact set is bounded. Every Cauchy sequence is bounded. For all $F \in C^*_u$, the map $U[F] \ast$ from $H$ into $(T_{U,C}, \tau_{\text{proj}})$ is continuous. So I, II, III, V and VI follow. Assertion IV follows from II and III.
Chapter III. Nice properties of the spaces \( S_{U,C} \) and \( T_{U,C} \)

Let \( G \) be a locally compact Abelian group and let \( U \) be a representation of \( G \) in a Hilbert space \( H \). Let \( C \) be a subset of \( L^1(G) \) which satisfies properties P1-P3. We fix the group \( G \), the representation \( U \) and the set \( C \) throughout this chapter.

As we have seen \( T_{U,C} \) carries the topology \( \tau_{\text{proj}} \) in a natural way. In the previous chapter, a second topology, \( \tau_{\text{ind}} \), has been introduced on \( T_{U,C} \). (Note that we used property P3 to define the topology \( \tau_{\text{ind}} \).) It turns out that the assertion \( (T_{U,C}, \tau_{\text{ind}}) = (T_{U,C}, \tau_{\text{proj}}) \) as topological vector spaces is equivalent with the assertion that \( S_{U,C} \) is complete. As we shall see it is also equivalent with a lot more assertions such as: \( (T_{U,C}, \tau_{\text{proj}}) \) is bornological; \( (T_{U,C}, \tau_{\text{proj}}) \) is reflexive; \( S_{U,C} = \bigcap_{F \in C^*_r} D(U[F]) \). In particular it turns out to be equivalent with the following property of the pair \( (C, U) \). (Cf. [IE property A IV'.]

**PROPERTY 3.1. P4.**

\[
\forall F \in \text{Bor}(\mathcal{G}, C) \left[ \left( \forall K \in C^* \left[ F \cdot K \text{ is bounded} \right] \right) \Rightarrow \exists \varepsilon > 0 \left[ \| U(\mathcal{G}) : |F(y)| > \varepsilon \right] = 0 \right].
\]

**REMARK.** It is trivial that the following property implies property P4.

**P4'.** \( \forall F \in \text{Bor}(\mathcal{G}, C) \left[ \left( \forall K \in C^* \left[ F \cdot K \text{ is bounded} \right] \right) \Rightarrow \exists \varepsilon > 0 \left[ \| U(\mathcal{G}) : |F(x)| \leq \varepsilon \right] \right].
\]

**REMARK.** The pair \( (C, U) \) satisfies property P4 resp. P4' if and only if the pair \( (\mathcal{C}, U) \) satisfies property P4 resp. P4'.

**LEMMA 3.2.** The following conditions are equivalent.

I. Property P4 holds.

II. \( \forall F \in \text{Bor}(\mathcal{G}, C) \left[ \left( \forall K \in C^* \left[ F \cdot K \text{ is bounded} \right] \right) \Rightarrow \exists \varepsilon > 0 \left[ \| U(\mathcal{G}) : |F(y)| > \varepsilon \right] = 0 \right].
\]

III. \( \forall F \in \text{Bor}(\mathcal{G}, C) \left[ \left( \forall K \in C^* \left[ F \cdot K \text{ is bounded} \right] \right) \Rightarrow \exists \varepsilon > 0 \forall y \in H \left[ \| U(\mathcal{G}) : |F(x)| \leq \varepsilon \right] \right].
\]

IV. \( \forall F \in \text{Bor}(\mathcal{G}, C) \left[ \left( \forall K \in C^* \left[ F \cdot K \text{ is bounded} \right] \right) \Rightarrow \exists \varepsilon > 0 \forall x \in H \left[ \| U(\mathcal{G}) : |F(y)| \leq \varepsilon \right] \right].
\]

**Proof.** Note that a function \( F \in \text{Bor} \left( \mathcal{G}, C \right) \) is bounded if \( F \cdot K \) is bounded for all \( K \in C^* \), since \( 1_{\mathcal{G}} \in C^* \). So I, II. and IV. make sense.

I \( \Rightarrow \) II. For every \( K \in C^* \), \( 1 + 1_{K} \in C^*_p \). The implication follows.
II => I. Trivial.

I => III. Let $F \in \text{Bor}(\hat{G}, \mathcal{C})$ and suppose $F * K$ is bounded for all $K \in C^\delta$. By property P4 there exist $f \in C$ and $c > 0$ such that $U[1_Y] = 0$ with $Y := \{ \gamma \in \hat{G} : |F(\gamma)| > c |\hat{f}(\gamma)| \}$.

Define $L : \hat{G} \rightarrow \mathbb{R}$

$$L(\gamma) = \begin{cases} 
0 & \text{if } \hat{f}(\gamma) = 0 \text{ or } \gamma \in Y, \\
\frac{F(\gamma)}{\hat{f}(\gamma)} & \text{if } \hat{f}(\gamma) \neq 0 \text{ and } \gamma \in Y.
\end{cases}$$

Then $L \in \text{Bor}_0(\hat{G}, \mathcal{C})$ and $\|L\|_\infty \leq c$. Let $x \in H$. By Proposition 2.8:

$$\|U[F]x\| = \|U[1_Y]U[F]x\| = \|U[1_Y]F\| \leq \|U[L]F\| \leq \|U[\hat{f}]\| \leq c \|U[f]\| \leq \|x\|.$$  

III => I. Let $F \in \text{Bor}(\hat{G}, \mathcal{C})$ and suppose $F * K$ is bounded for all $K \in C^\delta$. By assumption, there exist $f \in C$ and $c > 0$ such that $\|U[F]x\| \leq c \|U[f]\| \leq \|x\|$ for all $x \in H$. Let $(A, m, I, A_i, \tau_i, W)$ be a Stone-representant for $U$.

Let $e > 0$ and define $Z_e := \{ a \in A : |F(a)| > 2c |\hat{f}(a)| > e \}$. Suppose $m(Z_e) \neq 0$. There exist $i \in I$ and $n \in \mathbb{N}$ such that $0 < m(Z_e \cap \tau_i(Q_n)) < \infty$. Let $A_1 := Z_e \cap \tau_i(Q_n)$.

Define $\xi : A \rightarrow \mathcal{C}$

$$\xi(a) := \begin{cases} 
\frac{1}{\hat{f}(a)} & \text{if } a \in A_1, \\
0 & \text{if } a \in A \setminus A_1.
\end{cases}$$

Then $\xi$ is bounded, so $\xi \in L^2(m)$. But $0 \neq 2c \|1_{A_1}\| \leq \|F\| \leq \|U[F]W^{-1}\| \leq c \|U(f)W^{-1}\| \leq c \|U(f)\|. \text{Contradiction. Hence } m(Z_e) = 0.$

Similar, define $Z_0 := \{ a \in A : |F(a)| > 0 \text{ and } |\hat{f}(a)| = 0 \}$. Then $m(Z_0) = 0$. Define $Y := \{ \gamma \in \hat{G} : |F(\gamma)| > 2c |\hat{f}(\gamma)| \}$. Then $Z_0 \cup \bigcup_{n=1}^{\infty} Z_{n^{-1}} = \bigcup_{i \in I} \tau_i(Y)$, hence $U[1_Y] = 0$.

So III implies I.

III => IV. Let $F \in \text{Bor}(\hat{G}, \mathcal{C}), f \in C$ and $c > 0$ and suppose that $\|U[F]x\| \leq c \|U(f)\| x \| \text{ for all } x \in H$. Let $x, y \in H$ and suppose that $\|U(f)x\| \leq 1$ and $\|y\| \leq 1$. Then $\|U[F]y, x\| = \|y \cdot U[F]x\| \leq \|y\| \|U[F]x\| = \|y\| \|U[F]x\| \leq c \|y\| \|U(f)x\| \leq c.$
IV $\Rightarrow$ III. Let $F \in \text{Bor}_0(\hat{G}, \mathbb{R}), f \in C$ and $c > 0$. Suppose there exists $x_0 \in H$ such that $\|U[F]x_0\| > c \|U(f)x_0\|$. We prove that there exist $x, y \in H$ with $\|U(f)x\| \leq 1$, $\|y\| \leq 1$ and $\| (U[F]y, x) \| > c$. Since $x_0 \neq 0$, there exists $x \in H$ such that $c^{-1} \|U[F]x\| > 1 > \|U(f)x\|$. Let $y := \|U[F]x\|^{-1} U[F]^* x$. Then $\|U(f)x\| \leq 1$, $\|y\| = 1$ and $\| (y, U[F]^* x) \| = \|U[F]x\| > c$.

REMARK. In lemma 3.2 III and IV we may take $C_p^b$ instead of $C^b$. The proof of this is similar to the equivalence of I and II in Lemma 3.2.

III.1. Nice properties of the space $T_{U, C}$

In this section we only prove one theorem. We start with a lemma.

**LEMMA 3.3.**

I. Let $F \in C^b$ and suppose for all $n \in \mathbb{N}$: $\sup \{ |F(\gamma)| : \gamma \in Q_n \} < \infty$.

Define $K : \hat{G} \to \mathbb{R}$

$$K(\gamma) := b_n \sup \{ |F(\omega)| : \omega \in Q_n \} \quad (n \in \mathbb{N}, \gamma \in Q_n).$$

Then $K \in C^b$.

Similar results hold if we take $F \in C_u^b$ or $F \in C_p^b$.

II. Let $F \in C^b$.

Define $K : \hat{G} \to [0, \infty]$.

$$K(\gamma) :=
\begin{cases}
  b_n \sup \{ |F(\omega)| : \omega \in Q_n \} & \text{if } P_n \neq 0, \\
  1 & \text{if } P_n = 0.
\end{cases}
\quad (n \in \mathbb{N}, \gamma \in Q_n)$$

Then $K \in C^b$. Here, $P_n := U[1_{Q_n}]$, as usual.

Similar results hold if we take $F \in C_u^b$ or $F \in C_p^b$.

**Proof.**

I. Clearly $K \in \text{Bor}(\hat{G}, C)$. Let $f \in C$. By property P3 there exist $g \in C$ and $\delta > 0$ such that

$$b_n \sup \{ |\tilde{f}(\gamma)| : \gamma \in Q_n \} \leq \delta \inf \{ |\hat{g}(\gamma)| : \gamma \in Q_n \}$$

for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $\gamma \in Q_n$. For all $\omega \in Q_n$, $|\tilde{f}(\gamma) b_n F(\omega)| \leq \delta |\hat{g}(\omega) F(\omega)| \leq \delta \|\hat{g} F\|_{\infty}$, so $|K(\gamma) \tilde{f}(\gamma)| \leq \delta \|\hat{g} F\|_{\infty}$.

II. Let $n \in \mathbb{N}$. If $P_n \neq 0$, then by Lemma 2.25 there exist $f \in C$ and $\varepsilon > 0$ such that $|\tilde{f}(\gamma)| > \varepsilon$ for all $\gamma \in Q_n$. Since $F \in C^b$, the function $\tilde{f} * F$ is bounded on $Q_n$, hence $F$ is bounded on $Q_n$. So $K \in \text{Bor}(\hat{G}, C)$.

Similar to the proof of I it follows that $K \in C^b$.

$\square$
THEOREM 3.4. The following conditions are equivalent.

I. Property P4 holds.

II. \((T_u, c, \tau_{proj}) = (T_u, c, \tau_{ind})\) as topological vector spaces.

III. \((T_u, c, \tau_{proj})\) is bornological.

IV. \((T_u, c, \tau_{proj})\) is barrelled.

V. \((T_u, c, \tau_{proj})\) is quasibarrelled.

VI. \((T_u, c, \tau_{proj})\) is reflexive.

Proof. Let \((A, m, I, A_i, \tau_i, W)\) be a Stone-representant for \(U\). For \(n \in \mathbb{N}\) define \(P_n := U[I_{Q_n}]\) and \(L_n := \{\tau_i[\gamma] : i \in I, \gamma \in \{Q_n\}\}\).

I \Rightarrow II. The technicalities in this proof are inspired by the proof of [EGK] Theorem 1.7 II. Because of Corollary 2.32 we only have to prove that \(\tau_{ind} \subset \tau_{proj}\). Let \(V\) be an absolutely convex \(\tau_{ind}\)-neighbourhood of 0 in \(T_u, c\). Then for every \(F \in C_p^b\) there exists \(\varepsilon_F > 0\) such that \(\{\Phi \in U[F] \ast H : \|\Phi\|_F < \varepsilon_F\} \subset V \cap U[F] \ast H\). For all \(F \in C_p^b\) and all \(n \in \mathbb{N}\) define:

\[\delta_{nF} := \sup\{d > 0 : \text{for a.e. } a \in L_n \text{ holds } |F(a)| \geq d\}.
\]

It turns out that \(\delta_{nF} > 0, F^{-1} \in \text{Bor}_b(\hat{G}, \mathcal{C})\) and \(\|U[F^{-1}]\|_F \leq \delta_{nF}^{-1}\) if \(\delta_{nF} \neq \infty\) \((F \in C_p^b, n \in \mathbb{N})\). Note, \(\delta_{nF} = \infty\) iff \(P_n = 0\). (See Lemma 3.3 II.)

Assertion: for all \(F \in C_p^b\) and all \(n \in \mathbb{N}\) holds:

\[\{\text{emb}x : x \in P_n(H), \|x\| < \delta_{nF} \varepsilon_F\} \subset V.
\]

Proof of the assertion. Let \(F \in C_p^b, n \in \mathbb{N}\) and let \(x \in P_n(H), \|x\| < \delta_{nF} \varepsilon_F\). By Lemma 2.16 II, \(\text{emb}(x) = U[F] \ast U[F^{-1}]P_n x\), hence \(\text{emb}x \|_F = \|U[F^{-1}]P_n x\| \leq \|U[F^{-1}]P_n\| \|x\|\). So if \(\delta_{nF} \neq \infty\), then \(\|\text{emb}x\|_F < \delta_{nF}^{-1} \delta_{nF} \varepsilon_F = \varepsilon_F\) and if \(\delta_{nF} = \infty\), then \(\|\text{emb}x\|_F \leq \|U[F^{-1}]P_n\| \|x\| = 0 \cdot \|x\| < \varepsilon_F\). In any case we obtain by definition of \(\varepsilon_F\) that \(\text{emb}x \in V\). This proves the assertion.

For \(n \in \mathbb{N}\) define \(r_n\) by

\[r_n := \sup\{\rho > 0 : \{\text{emb}x : x \in P_n(H), \|x\| < \rho\} \subset V\}.
\]

By the assertion: \(r_n \geq \delta_{nF} \varepsilon_F\) for all \(F \in C_p^b\) and all \(n \in \mathbb{N}\). Because \(C_p^b \neq \emptyset\), we obtain \(r_n > 0\) for all \(n \in \mathbb{N}\). Note that \(r_n = \infty\) for all \(n \in \mathbb{N}\) such that \(P_n = 0\).

Define \(F : \hat{G} \to \mathbb{R}\)

\[F := \sum_{n=1}^{\infty} \frac{2b_n}{r_n} I_{Q_n}, \text{ with } b := \sum_{n=1}^{\infty} b_n^{-1}.
\]

Then \(F \in \text{Bor}(\hat{G}, \mathcal{C})\). Let \(K \in C^b\). We prove that \(F \ast K\) is bounded. Without loss of generality, \(K \geq 1\). (Otherwise, take \(|K| + 1\).)
Define $L: \hat{G} \to \mathbb{R}$

$$L(\gamma) :=
\begin{cases}
  b_\gamma \sup \{ |K(\omega)| : \omega \in Q_n \} & \text{if } P_n \neq 0, \\
  1 & \text{if } P_n = 0.
\end{cases}
$$

By Lemma 3.3 II, $L \in C^0_\gamma$. Then for all $n \in \mathbb{N}$ such that $r_n \neq \infty$ and for all $\gamma \in Q_n : |F(\gamma)K(\gamma)| \leq \frac{2b_\gamma b_n}{\delta_n \varepsilon_L} \cdot \frac{L(\gamma)}{b_n} \leq \frac{2b}{\varepsilon_L}$, since $P_n \neq 0$. So $F \cdot K$ is bounded. By property P4 there exist $f \in G$ and $c > 0$ such that $\| U[F]x \| \leq c \| U(f) \| = c \| U(f) \| \| x \|$ for all $x \in H$. (See Lemma 3.2 I $\Rightarrow$ III.)

The implication $I \Rightarrow II$ is proved if we can show that $\{ \Phi \in T_{U,C} : t_f(\Phi) < \frac{1}{c} \} \subset V$.

Let $\Phi \in T_{U,C}, t_f(\Phi) < \frac{1}{c}$. By Corollary 2.32 there exist $K \in C^0_\gamma$ and $x \in H$ such that $\Phi = U[K] \cdot x$. Let $n \in \mathbb{N}$. Then $\text{emb}(2b_\gamma U[K]P_n x) \in V$. This is trivial in case $r_n = \infty$. If $r_n < \infty$, we have $\| U[K]P_n x \| = \| U[F]U[K]P_n x \| \leq c \| U(f) \| \| x \| 2b_\gamma$.

$\| \sum_{n=N+1}^\infty P_n x \| < \frac{\varepsilon_K}{2}$. By definition of $\varepsilon_K$, then $\| U[K] \cdot x \| = \| \sum_{n=N+1}^\infty P_n x \| < \frac{\varepsilon_K}{2}$. Since $V$ is absolutely convex, we obtain by Lemma 2.16 II:

$$\Phi = U[K] \cdot x = \sum_{n=1}^N \frac{1}{2b_\gamma} \cdot 2b_\gamma \text{emb}(U[K]P_n x) + \frac{1}{2} U[K] \cdot \sum_{n=N+1}^\infty P_n x \in V.$$ This completes the proof that I implies II.

**II $\Rightarrow I$**

Suppose $\tau_{ind} = \tau_{proj}$. Let $F \in \text{Bor}(\hat{G}, C)$ and suppose for all $K \in C^0_\gamma$ the function $F \cdot K$ is bounded on $\hat{G}$. We will define a map $E : T_{U,C} \to H$ by $E(U[K] \cdot x) = U[FK]x \in (K \in C^0_\gamma, x \in H)$. This defines indeed a map: let $K_1, K_2 \in C^0_\gamma$, $x_1, x_2 \in H$ and suppose $U[K_1] \cdot x_1 = U[K_2] \cdot x_2$. Define $K \in C^0_\gamma$ by $K := I K_1 I + I K_2 I$ and define $L_i \in \text{Bor}(\hat{G}, C)$ by $L_i := K_i I$ $(i \in \{1, 2\})$. Then $U[K] \cdot U[L_1]x_1 = U[K_1] \cdot x_1 = U[K_2] \cdot x_2 = U[K] \cdot U[L_2]x_2$, so $U[L_1]x_1 = U[L_2]x_2$. (See Lemmas 2.16 IV and 2.16 V.) Hence $U[FK_1]x_1 = U[FK_1]x_1 = U[FK_2]U[L_1]x_1 = U[FK_2]U[L_2]x_2 = U[FK_2]x_2$. So the definition of $E$ makes sense.

Let $K \in C^0_\gamma$. For all $x \in H$ holds $\| E(U[K] \cdot x) \| \leq \| FK \| \| x \| \leq \| FK \| \| U[K] \cdot x \| K$. By [Wil] Theorem 13-1-8 it follows that $E$ is a continuous map from $(T_{U,C}, \tau_{ind})$ into $H$. By assumption the map $E$ is continuous from $(T_{U,C}, \tau_{proj})$ into $H$. From Lemma 1.14 we obtain that there exist $f \in G$ and $c > 0$ such that $\| E(\Phi) \| \leq c \| t_f(\Phi) \|$ for all $\Phi \in T_{U,C}$.

Let $x \in H$. Then $\| U[F]x \| = \| E(U[1\hat{G}] \cdot x) \| \leq c \| t_f(U[1\hat{G}] \cdot x) \| = c \| U(f) \| = \| U(f) \|$. By Lemma 3.2 III $\Rightarrow I$, property P4 holds.
II \Rightarrow III and II 

III \Rightarrow IV are trivial since every inductive limit of Hilbert spaces is bornological and barrelled. (See [Wil] Theorem 13-1-13.)

III \Rightarrow V and IV \Rightarrow V are trivial.

V \Rightarrow II Again we have to prove that \( \tau_{ind} \subset \tau_{proj} \). Let \( V \subset T_{U,C} \) be a \( \tau_{ind} \)-neighbourhood of \( 0 \). Because \( \tau_{ind} \) is regular, there exists a absolutely convex \( \tau_{ind} \)-open subset \( V_1 \), of \( T_{U,C} \) such that \( 0 \in V_1 \subset \bar{V}_1 \subset V \). Assertion: \( V_1 \) is a bornivore in \((T_{U,C}, \tau_{proj})\). Let \( B \) be a \( \tau_{proj} \)-bounded subset of \( T_{U,C} \). By Corollary 2.33 there exist \( F \in C_u^\sigma \) and a bounded subset \( B_0 \) of \( H \) such that \( B = U[F] \cdot B_0 \). Let \( M > 0 \) be such that \( \|x\| \leq M \) for all \( x \in B_0 \). Since \( V_1 \) is \( \tau_{ind} \)-open, there exists \( \varepsilon > 0 \) such that for all \( x \in H \), \( \|x\| < \varepsilon \) holds \( U[F] \cdot x \in V_1 \). Then for all \( t \in C \), \( \|t\| < \varepsilon M^{-1} \) we get \( tB \subset V_1 \). This proves the assertion. Hence \( V_1 \) is a bornivore barrel and by assumption a \( \tau_{proj} \)-neighbourhood of \( 0 \). So \( \tau_{ind} \subset \tau_{proj} \).

V \Rightarrow VI Since \( S_{U,C} \) is barrelled, the dual pair \( <S_{U,C}, T_{U,C}> \) is barrelled. By [Wil] Theorem 10-2-4 the dual pair \( <T_{U,C}, S_{U,C}> \) is semireflexive, hence \((T_{U,C}, \tau_{proj})\) is semireflexive. Because \((T_{U,C}, \tau_{proj})\) is also quasibarrelled, \((T_{U,C}, \tau_{proj})\) is reflexive.

(See [Wil] Theorem 10-1-8.)

VI \Rightarrow V Any reflexive locally convex topological vector space is quasibarrelled. (See [Wil] Theorem 10-1-8.)

\[
\]

COROLLARY 3.5.

Suppose property P4 holds. Then \((T_{U,C}, \tau_{proj})\) is a Mackey space.

**Proof.** See [Wil] Theorem 10-1-9.

\[
\]

COROLLARY 3.6.

Suppose \((T_{U,C}, \tau_{proj})\) is of second category in itself or \((T_{U,C}, \tau_{proj})\) is metrizable or the set \( C \) is countable. Then property P4 holds.

**Proof.** Every locally convex space which is of second category in itself is barrelled ([Wil] Example 9-3-2). Every complete metric space is of second category in itself ([Wil] Theorem 1-6-1). If the set \( C \) is countable then \((T_{U,C}, \tau_{proj})\) is metrizable.

\[
\]

III.2. **Nice properties of the space** \( S_{U,C} \)

This section is a continuation of the previous section. We start with two lemmas concerning bounded subsets of \( S_{U,C} \).

**DEFINITION 3.7.**

Let \( S := \bigcap_{F \in G^\sigma_u} D(U[F]) \).
For all $F \in C_p^\#$ the seminorm $s_F$ defined on $S_{U,C}$ extends to a seminorm $s_F$ on $S$ by $s_F(\phi) := \| U[F] \phi \|$ ($\phi \in S$). Let $\sigma_{\text{proj}}$ be the locally convex Hausdorff topology for $S$ generated by the seminorms $s_F$ with $F \in C_p^\#$.

**REMARK.** The topology $\sigma_{\text{ind}} = \sigma_{\text{proj}}$ for the topological vector space $S_{U,C}$ is equal to the induced topology of the topological vector space $(S, \sigma_{\text{proj}})$. So no confusion can arise. It will be proved in Theorem 3.12 that $S = S_{U,C}$ as sets if and only if property P4 holds.

**LEMMA 3.8.**
Suppose property P4 holds. Let $B$ be a bounded subset of $(S, \sigma_{\text{proj}})'$. Then there exist $! \in C$ and a bounded subset $B_0$ of $H$ such that $B = \bigcup (f)(B_0)$.

**Proof.** Let $(A, m, I, A_i, \tau_i, W)$ be a Stone-representant for $U$. Let $P_n := U[1_{Q_n}]$ and let $L_n := \{ \tau_i(\gamma) : i \in I, \gamma \in Q_n \}$ ($n \in \mathbb{N}$). We may assume that $B \neq \emptyset$. For $n \in \mathbb{N}$ define $r_n := \sup \{ \| P_n \phi \| : \phi \in B \}$. Then $r_n \leq \sup \{ \| \phi \| : \phi \in B \} = \sup \{ s_{1d}(\phi) : \phi \in B \} < \infty$ for all $n \in \mathbb{N}$. Note that $r_n = 0$ for all $n \in \mathbb{N}$ such that $P_n = 0$.

Define $F : \hat{G} \to \mathbb{R}$

$$F := \sum_{n=1}^{\infty} b_n r_n 1_{Q_n}.$$ 

Let $K \in C_p^\#$. Define $L : \hat{G} \to \mathbb{R}$

$$L(\gamma) := \begin{cases} b_n \sup \{ |K(\omega)| : \omega \in Q_n \} & \text{if } P_n \neq 0, \\ 1 & \text{if } P_n = 0. \end{cases}$$

$(n \in \mathbb{N}, \gamma \in Q_n)$

By Lemma 3.3 II, $L \in C_p^\#$. Then $|FK| \leq \sup \{ s_{L}(\phi) : \phi \in B \}$. Indeed, for all $n \in \mathbb{N}$ with $P_n \neq 0$ and all $\gamma \in Q_n$ we have $|F(\gamma)K(\gamma)| \leq b_n r_n b_n^{-1} |L(\gamma)| = \sup \{ \| U[L] P_n \phi \| : \phi \in B \} \leq \sup \{ s_{L}(\phi) : \phi \in B \}$. By Lemma 3.2 I $\Rightarrow$ II, there exist $f \in C$ and $c > 0$ such $U[1_{0\leq |F(\gamma)| \leq c} |\hat{f}(\gamma)|] = 0$. Then $m(\{ a \in A : |F(a)| > c \land |\hat{f}(a)| \}) = 0$ since the measure $m$ is locally finite.

Define $K : \hat{G} \to \mathcal{C}$

$$K(\gamma) := \begin{cases} \frac{1}{|\hat{f}(\gamma)|} & \text{if } \hat{f}(\gamma) \neq 0, \\ 0 & \text{if } \hat{f}(\gamma) = 0. \end{cases}$$

$(\gamma \in \hat{G})$

Then $K \in \text{Bor}(\hat{G}, \mathcal{C})$ and $K \hat{f} = 1_{\{ m \in \hat{G} : |\hat{f}(m)| \neq 0 \}}$. For all $n \in \mathbb{N}$ such that $r_n \neq 0$ we obtain that for a.e. $a \in L_n$ holds $|K(a)| \leq \frac{c}{b_n r_n}$. Let $\phi \in B$. For all $n \in \mathbb{N}$, $P_n \phi \in D(U[K])$ and $\| U[K] P_n \phi \| = 0$ if $r_n = 0$ and $\| U[K] P_n \phi \| \leq \frac{c}{b_n r_n} \| P_n \phi \| \leq \frac{c}{b_n}$ if $r_n > 0$. So
\[ \sum_{n=1}^{\infty} \| U[K] P_n \phi \|^2 \leq \sum_{n=1}^{\infty} \left( \frac{c}{b_n} \right)^2 \cdot d < \infty. \] Define \( x_\phi := \sum_{n=1}^{\infty} U[K] P_n \phi \). Then \( \| x_\phi \| \leq \sqrt{d} \). Since \( b_n \| P_n \phi \| L_n(a) \leq \| F(a) \| \leq c \widehat{f}(a) \) for a.e. \( a \in A \) and all \( n \in \mathbb{N} \), it follows that
\[ U[1_{\{ \mathbb{R} \setminus \hat{f}(y) \neq 0 \}}] P_n \phi = P_n \phi \] for all \( n \in \mathbb{N} \). Hence
\[ U(f) x_\phi = \sum_{n=1}^{\infty} U[\hat{f} K] P_n \phi = \sum_{n=1}^{\infty} U[1_{\{ \hat{\mathbb{R} \setminus \hat{f}(y) \neq 0 \}}}] P_n \phi = \sum_{n=1}^{\infty} P_n \phi = \phi. \]

Let \( B_0 := \{ x_\phi : \phi \in B \} \). Then \( B_0 \) is a bounded subset of \( H \) and \( B = U(f) B_0 \).

**LEMMA 3.9.**

Let \( B \) be a bounded subset of \( S_{U,C} \), let \( f \in C \) and let \( B_0 \) be a bounded subset of \( H \) such that \( B = U(f) (B_0) \). Then there exist \( g \in C \) and a bounded subset \( B_1 \) of \( N_1^g \) such that \( U(g) \mid B_1 : (B_1, \| \cdot \|) \rightarrow (B, \sigma_{\text{ind}} \mid B) \) is a homeomorphism.

**Proof.** For every \( \phi \in B \) there exists \( x_\phi \in B_0 \) such that \( \phi = U(f) x_\phi \). Let \( c > 0 \) be such that \( \| x_\phi \| \leq c \) for all \( \phi \in B \). By property P3 there exist \( g \in C \) and \( \delta > 0 \) such that \( b_n \sup \{ | \hat{f}(\gamma) | : \gamma \in Q_n \} \leq \delta \inf \{ | \hat{g}(\gamma) | : \gamma \in Q_n \} \).

Define \( F : \hat{G} \rightarrow C \)
\[
F(\gamma) = \begin{cases} \frac{\hat{f}(\gamma)}{\hat{g}(\gamma)} & \text{if } \hat{g}(\gamma) \neq 0, \\ 0 & \text{if } \hat{g}(\gamma) = 0. \end{cases} (\gamma \in \hat{G})
\]

Then \( F \in \text{Bor}_b(\hat{G}, C) \) and \( \| F \|_{\infty} \leq \delta \sum_{n=1}^{\infty} b_n^{-1} \). Define \( y_\phi := U[F] x_\phi \) for all \( \phi \in B \). Then
\[ \| y_\phi \| \leq \delta c \sum_{n=1}^{\infty} b_n^{-1}, y_\phi \in N_1^g \quad \text{and} \quad U(g) y_\phi = U[\hat{g} F] x_\phi = U[\hat{f}] x_\phi = \phi \] for all \( \phi \in B \). Define \( B_1 := \{ y_\phi : \phi \in B \} \). By Lemma 1.3 the map \( U(g) \mid B_1 : B_1 \rightarrow B \) is a bijection.

Obviously the map \( U(g) \mid B_1 : B_1 \rightarrow (B, \sigma_{\text{ind}} \mid B) \) is continuous. (See Lemma 1.6.) The inverse map is also continuous. Indeed, let \( (\phi_\alpha)_{\alpha \in M} \) be a net in \( B \), let \( \phi \in B \) and suppose that \( \lim_{\alpha} \phi_\alpha = \phi \) in \( (B, \sigma_{\text{ind}} \mid B) \). We are ready if we prove the assertion: \( \lim_{\alpha} y_{\phi_\alpha} = y_\phi \) in \( H \). Let \( \varepsilon > 0 \). Let \( n \in \mathbb{N} \) be so large that
\[ \sum_{n=N+1}^{\infty} \frac{1}{b_n^2} < \varepsilon^2. \]

Define \( K : \hat{G} \rightarrow C \)
\[
K(\gamma) = \begin{cases} 
0 & \text{if } n > N, \\
0 & \text{if } n \leq N \quad \text{and } \sup \{ |\hat{f}(\omega)| : \omega \in Q_n \} = 0, \ (n \in \mathbb{N}, \gamma \in Q_n) \\
\frac{1}{\delta(\gamma)} & \text{if } n \leq N \quad \text{and } \sup \{ |\hat{f}(\omega)| : \omega \in Q_n \} \neq 0.
\end{cases}
\]

Since \( b_n \sup \{ |\hat{f}(\omega)| : \omega \in Q_n \} \leq \delta \inf \{ |\hat{g}(\gamma)| : \gamma \in Q_n \} \) for all \( n \in \mathbb{N} \), the function \( K \) is bounded, so \( K \in C^\theta \). For all \( n \in \mathbb{N} \) let \( P_n := U[1_Q_n] \). Let \( \psi \in B \). Then \( U[K] \psi = \sum_{n=1}^{N} P_n \psi \), because for every \( n \in \mathbb{N}, n \leq N \) we obtain if \( \sup \{ |\hat{f}(\omega)| : \omega \in Q_n \} = 0 \), then \( P_n \psi = U[1_Q_n] F \psi = 0 = P_n U[K] \psi \); and if \( \sup \{ |\hat{f}(\omega)| : \omega \in Q_n \} > 0 \) then \( P_n \psi = U[1_Q_n] K \hat{g} \psi = P_n U[K] \psi \). There exists \( \alpha_0 \in M \) such that \( s_K(\phi_\alpha - \phi) < \epsilon \) for all \( \alpha \in M \) with \( \alpha \geq \alpha_0 \). Then for all \( \alpha \geq \alpha_0 \):

\[
\|y_{\phi_\alpha} - y_\phi\|^2 = \| \sum_{n=1}^{N} P_n(y_{\phi_\alpha} - y_\phi)\|^2 + \sum_{n=N+1}^{\infty} \| P_n(y_{\phi_\alpha} - y_\phi)\|^2 \\
= \| U[K] (\phi_\alpha - \phi)\|^2 + \sum_{n=N+1}^{\infty} \| P_n U[F] (x_{\phi_\alpha} - x_\phi)\|^2 \\
\leq (s_K(\phi_\alpha - \phi))^2 + \sum_{n=N+1}^{\infty} \frac{\delta^2}{b_n^2} \| P_n(x_{\phi_\alpha} - x_\phi)\|^2 \\
\leq \epsilon^2 + \sum_{n=N+1}^{\infty} \frac{\delta^2}{b_n^2} \| x_{\phi_\alpha}\| + \| x_\phi\|^2 \\
\leq \epsilon^2 + \delta^2 \epsilon^2 4 c^2 \\
= \epsilon^2 (1 + 4 \delta^2 c^2).
\]

This proves the assertion.

THEOREM 3.10.

Suppose property P4 holds. Let \( B \) be a bounded subset of \( S_{U, C} \). Then there exist \( f \in C \) and a bounded subset \( B_0 \) of \( N_f^T \) such that \( U(f) |_{B_0} : (B_0, \| \|) \to (B, \sigma_{ind} \|_B) \) is a homeomorphism.

Proof. Lemmas 3.8, 3.9 and Corollary 2.31.

COROLLARY 3.11.

Suppose property P4 holds. Let \( B \) be a subset of \( S_{U, C} \). Then:
I. $B$ is bounded in $S_{U,C}$ if and only if there exist $f \in C$ and a bounded $B_0 \subset H$ such that $B = U(f)(B_0)$.

II. $B$ is compact in $S_{U,C}$ if and only if there exist $f \in C$ and a compact $B_0 \subset H$ such that $B = U(f)(B_0)$.

III. $B$ is sequentially compact in $S_{U,C}$ if and only if there exist $f \in C$ and a sequentially compact $B_0 \subset H$ such that $B = U(f)(B_0)$.

IV. $B$ is sequentially compact in $S_{U,C}$ if and only if $B$ is compact in $S_{U,C}$.

V. Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence in $S_{U,C}$. Then $(\phi_n)_{n \in \mathbb{N}}$ is a Cauchy sequence if and only if there exist $f \in C$ and a Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in $H$ such that $\phi_n = U(f)x_n$ for all $n \in \mathbb{N}$.

VI. Every bounded sequence in $S_{U,C}$ has a weakly convergent subsequence.

Proof. Similar to the proof of Corollary 2.33.

We continue with a theorem in the spirit of Theorem 3.4.

THEOREM 3.12.

The following conditions are equivalent

I. Property P4 holds.

II. For all $\ell \in H$ holds: if for all $F \in C^*_p$, $\phi \in D(U[F])$, then $\phi \in S_{U,C}$, i.e.

$$S_{U,C} = \bigcap_{F \in C^*_p} D(U[F]).$$

III. $S_{U,C}$ is complete.

IV. $S_{U,C}$ is sequentially complete.

V. For every bounded subset $B$ of $S_{U,C}$ there exist $f \in C$ and a bounded subset $B_0$ of $H$ such that $B = U(f)(B_0)$.

VI. For every countable bounded subset $B$ of $S_{U,C}$ there exist $f \in C$ and a bounded subset $B_0$ of $H$ such that $B = U(f)(B_0)$.

VII. $S_{U,C}$ is reflexive.

VIII. $S_{U,C}$ is weakly sequentially complete.

IX. Every bounded sequence in $S_{U,C}$ has a weakly convergent subsequence.

Proof.

I $\Rightarrow$ II. This is a special case of Lemma 3.8, because every one-point set of $S$ is bounded.

II $\Rightarrow$ I. We prove that $(T_{U,C}, \tau_{\text{proj}})$ is homological. Then by Theorem 3.4 III $\Rightarrow$ I, property P4 holds.

Let $l : T_{U,C} \to C$ be linear and suppose $l$ is bounded on $\tau_{\text{proj}}$-bounded subsets of $T_{U,C}$.

We shall prove that $l$ is continuous. Define $\alpha : H \to C$ by $\alpha := l \circ \text{emb}$. Since
emb : $H \to (T_U, C, \tau_{\text{proj}})$ is continuous (see Theorem 1.11), the map $\alpha$ is bounded on bounded on bounded sets, so $\alpha$ is continuous. There exists $y \in H$ such that $\alpha(x) = (x, y)$ for all $x \in H$. Let $F \in C^\#$. The map $x \mapsto \alpha(U[F] \cdot x)$ from $H$ into $C$ is bounded on bounded sets (see Lemma 2.16 VI) and hence continuous. For all $x \in D(U[F])$ we have by Lemma 2.16 II: $(U[F] \cdot x, y) = l(\text{emb}(U[F] \cdot x)) = l(U[F] \cdot x)$, so $y \in D(U[F]^\#) = D(U[F])$. Hence $y \in \bigcap_{F \in C^\#} D(U[F]) = S_{U,C}$ by assumption.

Let $\Phi \in (T_U, C)$. By Corollary 2.32 there exist $F \in C^\#$ and $x \in H$ such that $\Phi = U[F] \cdot x$.

Define $\beta_1, \beta_2 : H \to C$

$$\beta_1(x) := l(U[F] \cdot x),$$

$$(z \in H)$$

$$\beta_2(z) := <y, U[F] \cdot z>.$$ 

The map $\beta_1$ is continuous. By Lemma 2.16 VI and Theorem 1.18 the map $\beta_2$ is also continuous. For all $\phi \in S_{U,C}$ it follows from Lemma 2.16 II and III that $\beta_1(\Phi) = \alpha(U[F] \cdot \Phi) = \alpha(U[F] \cdot \phi) = (U[F] \cdot \phi, y) = <y, U[F] \cdot \phi > = \beta_2(\phi)$. Since $S_{U,C}$ is a dense subset of $H$, $\beta_1 = \beta_2$. In particular:

$l(\Phi) = <y, \Phi >$. Hence $l$ is continuous by Theorem 1.18.

So $(T_U, C, \tau_{\text{proj}})$ is bornological and by Theorem 3.4 III $\Rightarrow$ I, property P4 holds.

II $\Rightarrow$ III. Let $(\Phi_\alpha)_{\alpha \in M}$ be a Cauchy net in $(S_{U,C}, \sigma_{\text{proj}})$. For all $F \in C^\#$, the net $(U[F] \Phi_\alpha)_{\alpha \in M}$ is a Cauchy net in $H$, so there exists $\Phi_F \in H$ such that $\lim_{\alpha} U[F] \Phi_\alpha = \Phi_F$. Let $\Phi := \Phi_{1,0}$. For all $F \in C^\#$, we obtain in the Hilbert space $H \times H : \lim_{\alpha} [\Phi_\alpha, U[F] \Phi_\alpha] = [\Phi, \Phi_F]$. Since $U[F]$ is a closed operator, $\phi \in D(U[F])$ and $\Phi_F = U[F] \cdot \Phi$. By assumption: $\Phi \in S_{U,C}$.

For all $F \in C^\#$, we obtain $\lim_{\alpha} s_F(\Phi_\alpha - \Phi) = \lim_{\alpha} \| U[F] \Phi_\alpha - U[F] \Phi \| = \lim_{\alpha} \| U[F] \Phi_\alpha - \Phi_F \| = 0$. So $\lim_{\alpha} \Phi_\alpha = \Phi$ in $(S_{U,C}, \sigma_{\text{proj}}) = (S_{U,C}, \sigma_{\text{ind}})$.

III $\Rightarrow$ IV. Trivial.

IV $\Rightarrow$ II. Let $\phi \in H$ and suppose $\phi \in D(U[F])$ for all $F \in C^\#$. For all $N \in \mathbb{N}$ let $\phi_N := \sum_{n=1}^{N} P_n \phi$ with $P_n := U[1_{D_n}]$, $n \in \mathbb{N}$. Then $\phi_N \in S_{U,C}$ by Corollary 2.26. Let $F \in C^\#$. Then $\lim_{N \to \infty} U[F] \phi_N = \lim_{N \to \infty} \sum_{n=1}^{N} U[F] P_n \phi = \lim_{N \to \infty} \sum_{n=1}^{N} U[F] \phi = U[F] \Phi$ in the Hilbert space $H$. So $\phi_1, \phi_2, \ldots$ is a Cauchy sequence in $(S_{U,C}, \sigma_{\text{proj}}) = (S_{U,C}, \sigma_{\text{ind}})$. Let $\psi \in S_{U,C}$ be its limit. Since the identity map from $S_{U,C}$ into $H$ is continuous, we obtain in the Hilbert space $H : \psi = \lim_{N \to \infty} \phi_N = \phi$. So $\phi = \psi \in S_{U,C}$. 
I \Rightarrow V. Theorem 3.10.
V \Rightarrow VI. Trivial.
VI \Rightarrow IV. Let \( \phi_1, \phi_2, \ldots \) be a Cauchy sequence in \( S_{U,C} \). Then the countable set 
\[ B := \{ \phi_n : n \in \mathbb{N} \} \]
is bounded in \( S_{U,C} \), so by assumption there exist \( f \in C \) and a bounded set \( B_0 \subset H \) such that \( B = U(f)(B_0) \). By Lemma 3.9 there exist \( g \in C \) and a bounded set 
\[ B_1 \subset N_{g}^{1} \]
such that \( U(g) \mid_{B_1} : (B_1, \| \cdot \|) \rightarrow (B, \sigma_{\text{ind}} \mid_{B}) \) is a homeomorphism. For all \( n \in \mathbb{N} \) let \( x_n \in B_1 \) be such that \( U(g)x_n = \phi_n \). Then \( x_1, x_2, \ldots \) is a Cauchy sequence in \( H \), hence \( x := \lim_{n \to \infty} x_n \) exists in \( H \). Then \( \lim_{n \to \infty} \phi_n = U(g)x \) in \( S_{U,C} \).

I \Rightarrow VII. By Theorem 3.4 I \Rightarrow VI, \( (T_{U,C}, \tau_{\text{proj}}) \) is reflexive, so the dual pair \( (T_{U,C}, S_{U,C}) \) is reflexive. Hence the dual pair \( (S_{U,C}, T_{U,C}) \) is reflexive (see [Wil] Theorem 10-3-2), so \( (S_{U,C}, \sigma(S_{U,C}, T_{U,C})) \) is reflexive. Since \( (S_{U,C}, \sigma_{\text{ind}}) \) is homological, the topology \( \tau(S_{U,C}, T_{U,C}) \) is equal to \( \sigma_{\text{ind}} \). So \( (S_{U,C}, \sigma_{\text{ind}}) \) is reflexive.

VII \Rightarrow VIII. By [Wil] Theorem 10-2-4, \( (S_{U,C}, \sigma(S_{U,C}, T_{U,C})) \) is boundedly complete, hence sequentially complete.

VIII \Rightarrow IV. Any weak sequentially complete locally convex topological vector space is sequentially complete. (See [Wil] problem 8-5-5.)

I \Rightarrow IX. Corollary 3.11 VI.

IX \Rightarrow VIII. Let \( \phi_1, \phi_2, \ldots \) be a Cauchy sequence in \( (S_{U,C}, \sigma(S_{U,C}, S_{U,C}')) \). Then 
\[ \{ \phi_n : n \in \mathbb{N} \} \]
is weakly bounded and hence bounded in \( S_{U,C} \). By assumption the sequence \( \phi_1, \phi_2, \ldots \) has a weakly convergent subsequence with weak limit \( \phi \in S_{U,C} \).

Then \( \lim_{n \to \infty} \phi_n = \phi \) in \( (S_{U,C}, \sigma(S_{U,C}, S_{U,C}')) \).

III.3. \( S_{U,C} \) and \( T_{U,C} \) spaces which are Montel or nuclear

In this section necessary and sufficient conditions will be given for spaces of type \( S_{U,C} \) and \( T_{U,C} \) to be a Montel space or a nuclear space. If \( \dim H < \infty \), then clearly \( S_{U,C} \) and \( T_{U,C} \) are isomorphic which \( C^\infty \) with \( n = \dim H \). So by definition of Montel and nuclearity, \( S_{U,C} \) and \( T_{U,C} \) are Montel spaces and nuclear spaces. Therefore we suppose in this section that the Hilbert space is not finite dimensional.

THEOREM 3.13.

For \( n \in \mathbb{N} \) let \( P_n := U[1_{Q_n}] \). The following conditions are equivalent.

I. \( T_{U,C} \) is a semi-Montel space.

II. For all \( f \in C \) and \( F \in C^\infty_p \) the operator \( U[F \hat{f}] \) from \( H \) into \( H \) is compact.
III. For all \( f \in C \) the operator \( U(f) \) from \( H \) into \( H \) is compact.

IV. For all \( n \in \mathbb{N} \) the operator \( P_n \) has finite rank.

V. There exist \( F \in \text{Bor}(\mathcal{G}, \mathbb{R}) \) and \( \varepsilon > 0 \) such that \( F \geq \varepsilon \) uniform and the operator \( U[F^{-1}] \) from \( H \) into \( H \) is compact. Moreover, the map \( F K \hat{f} \) is bounded for all \( f \in C \) and \( K \in C_p^\# \).

VI. There exists a countable orthonormal basis \( e_1, e_2, \cdots \) in \( H \) and a sequence \( \gamma_1, \gamma_2, \cdots \) in \( \mathcal{G} \) such that for all \( x \in G, n \in \mathbb{N} \) and \( f \in C \) holds \( U_x e_n = \gamma_n(x) e_n \) and \( \lim_{k \to \infty} \hat{f}(\gamma_k) = 0 \).

VII. Every weakly convergent sequence in \( T_{U,C} \) is \( \tau_{\text{proj}} \)-convergent.

Proof.

I \( \Rightarrow \) II. Let \( f \in C \) and \( F \in C_p^\# \). Let \( B := \{ x \in H : \|x\| \leq 1 \} \). The set \( U[F] \cdot B \) is bounded in \( T_{U,C} \). Let \( B_1 \) be the closure of \( U[F] \cdot B \) in \( (T_{U,C}, \tau_{\text{proj}}) \). Since \( T_{U,C} \) is a semi-Montel space, the set \( B_1 \) is compact in \( T_{U,C} \). The map \( \Phi : \rightarrow \Phi(f) \) from \( T_{U,C} \) into \( H \) is continuous by Definition 1.8, so the set \( \{ \Phi(f) : \Phi \in B_1 \} \) is compact in \( H \). This set contains the set \( U[F] \cdot \hat{f} \cdot (B) \). So \( U[F] \cdot \hat{f} \cdot (B) \) is compact.

Then also \( U[F] \cdot \hat{f} = U[F] \cdot \hat{f}^* \cdot \) is compact.

II \( \Rightarrow \) I. Let \( B \subset T_{U,C} \) be closed and bounded. By Theorem 2.30 there exist \( F \in C_p^\# \) and a bounded set \( B_0 \subset H \) such that \( B = U[F] \cdot B_0 \). Let \( \Phi_1, \Phi_2, \cdots \) be a sequence in \( B \). For all \( n \in \mathbb{N} \) there exists \( x_n \in B_0 \) such that \( \Phi_n = U[F] \cdot x_n \). The sequence \( x_1, x_2, \cdots \) is bounded in \( H \), so it contains a weakly convergent subsequence \( (x_n)_k \in \mathbb{N} \) with weak limit \( x \). For all \( f \in C \) the operator \( U[F] \cdot \hat{f} = U[F] \cdot \hat{f}^* \) is compact, so

\[
\lim_{k \to \infty} (U[F] \cdot x_n) (f) = \lim_{k \to \infty} U[F] \cdot \hat{f} \cdot x_n = U[F] \cdot \hat{f} \cdot x = (U[F] \cdot x) (f).
\]

Hence

\[
\lim_{k \to \infty} U[F] \cdot x_n = U[F] \cdot x.
\]

Since \( B \) is closed, \( U[F] \cdot x \in B \). So \( B \) is sequentially compact and hence compact in \( T_{U,C} \) by Corollary 2.33 IV.

II \( \Rightarrow \) III. Trivial since \( 1_\mathcal{G} \in C_p^\# \).

III \( \Rightarrow \) IV. Let \( n \in \mathbb{N} \). We may suppose that \( P_n \neq 0 \). By Lemma 2.25 there exist \( f \in C \) and \( \varepsilon > 0 \) such that \( \|f(\gamma)\| \geq \varepsilon \) for all \( \gamma \in Q_n \).

Define \( F : \mathcal{G} \to C \)

\[
F(\gamma) = \begin{cases} 
\frac{1}{f(\gamma)} & \text{if } \gamma \in Q_n, \\
(\gamma \in \mathcal{G}) \\
0 & \text{if } \gamma \notin Q_n.
\end{cases}
\]

Then \( F \) is bounded, so \( P_n = U[F] \cdot U(f) \) is compact, and has finite rank.
IV \Rightarrow V. Define $F \in \text{Bor} (\hat{G}, L^R)$ by $F := \sum_{n=1}^{\infty} (1 + b_n) 1_{Q_n}$. Then $F \geq 1$ and $U [F^{-1}] = \sum_{n=1}^{\infty} (1 + b_n)^{-1} P_n$ is a compact operator from $H$ into $H$. Let $f \in C$ and $K \in C_p^\beta$. By property P3 there exist $g \in C$ and $\delta > 0$ such that

\[
|F(\gamma)(\gamma)\hat{f}(\gamma)| \leq \delta \| K \hat{f} \|_\infty + \delta \| K \hat{g} \|_\infty, \quad \text{so } F K \hat{f} \text{ is bounded.}
\]

V \Rightarrow II. Let $f \in C$ and $K \in C_p^\beta$. Then $U[K] = U[F^{-1}] \circ U[F K \hat{f}]$ is a compact operator from $H$ into $H$.

III \Rightarrow VI. Let $n \in \mathbb{N}$. We have already proved that $\dim P_n(H) < \infty$. Let $x \in G$.

Define $L_x : L^1(G) \to L^1(G)$

\[
(L_x f)(y) := f(x^{-1} y) \quad (f \in L^1(G), \text{ a.e. } y \in G).
\]

By property P2, for every $u \in P_n(H)$, $U_x u = \lim_{\lambda} U_x (U(f_\lambda) u) = 0$. Hence $P_n(H)$ reduces the representation $U$. Because all irreducible representation of an Abelian group are one dimensional, there exist $k_n \in \mathbb{N}$ and an orthonormal basis $a_1, \ldots, a_{k_n}$ for $P_n(H)$ such that $U_x a_{ni} \in \text{span}(\{a_{ni}\})$ for all $x \in G$ and all $i \in \{1, \ldots, k_n\}$. Since the set span $\{P_n u : u \in H\}$ is dense in $H$, the set $\{a_{ni} : n \in \mathbb{N}, P_n \neq 0, i \leq k_n\}$ determines an orthonormal set $\{e_1, e_2, \ldots\}$ in $H$ such that $U_x e_n \in \text{span}(\{e_n\})$ for all $n \in \mathbb{N}$ and $x \in G$. Hence there exists $\gamma_n \in \hat{G}$ such that $U x e_n = \gamma_n(x) e_n$ for all $x \in G$.

Let $f \in C$ and $n \in \mathbb{N}$. For all $u \in H$ holds

\[
(U(f) e_n, u) = \int_G f(x) (U_x e_n, u) \, dx = \int_G f(x) \gamma_n(x) (e_n, u) \, dx = (f(\gamma_n) e_n, u), \quad \text{so } U(f) e_n = f(\gamma_n) e_n.
\]

Since $U(f)$ is a compact operator, it follows that $\lim_{n \to \infty} f(\gamma_n) = 0$.

VI \Rightarrow III. Let $f \in C$. Then $U(f) e_n = f(\gamma_n) e_n$ for all $n \in \mathbb{N}$. Since $e_1, e_2, \ldots$ is an orthonormal basis for $H$ and $\lim_{n \to \infty} f(\gamma_n) = 0$, we obtain that $U(f)$ is a compact operator from $H$ into $H$. (See [Hal] problem 132.)

II \Rightarrow VII. Let $\Phi_1, \Phi_2, \ldots$ be a weakly convergent sequence in $T_{U,C}$ with weak limit $\Phi$. Let $B := \{\Phi\} \cup \{\Phi_n : n \in \mathbb{N}\}$. Then $B$ is weakly bounded and hence $\tau_{\text{proj}}$-bounded in $T_{U,C}$. By Theorem 2.30 there exist $F \in C_f^\beta$ and a bounded subset $B_0 \subset H$ such that $U[F] \tau_{\text{proj}} B_0 := (B_0, I, I) \to (B, \tau_{\text{proj}})$ is a homeomorphism. Let $x, x_n \in B_0$ be such that $\Phi = U[f] x$ and $\Phi_n = U[F] x_n (n \in \mathbb{N})$. Since weak $\lim_{n \to \infty} \Phi_n = \Phi$, also weak $\lim_{n \to \infty} x_n = x$. Let $f \in C$. By assumption the map $U[F \hat{f}] = U[\hat{f}]$ is compact, so in the
Hilbert space \( H : \lim_{n \to \infty} \Phi_n(f) = \lim_{n \to \infty} U[F \tilde{f}] x_n = U[F \tilde{f}] x = \Phi(f) \). Hence
\[
\lim_{n \to \infty} \Phi_n = \Phi \text{ in } (T_\mu, c, \tau_{\text{proj}}).
\]

VII \( \Rightarrow \) III. Let \( f \in C \). Let \( x, x_1, x_2, \ldots \in H \) and suppose weak \( \lim_{n \to \infty} x_n = x \). Since the map \( \text{emb} \) from \( H \) into \( T_\mu, c \) is continuous, also weak \( \lim_{n \to \infty} \text{emb} x_n = \text{emb} x \). By assumption,
\[
\lim_{n \to \infty} \text{emb} x_n = \text{emb} x \text{ in } (T_\mu, c, \tau_{\text{proj}}).\]
In particular, \( \lim_{n \to \infty} U(f)^* x_n = \lim_{n \to \infty} (\text{emb} x_n)(f) = (\text{emb} x)(f) = U(f)^* x \). So \( U(f)^* \) is a compact operator from \( H \) into \( H \).

REMARK. If \( T_\mu, c \) is a semi-Montel space, then by Theorem 3.4, \( T_\mu, c \) is a Montel space if and only if property P4 holds. This is true because by definition a Montel space is a barrelled semi-Montel space.

The following theorem can be seen as a continuation of Theorem 3.13.

**THEOREM 3.14.**

The following conditions are equivalent.

I. \( S_\mu, c \) is a Montel space.

II. \( S_\mu, c \) is a semi-Montel space.

III. Property P4 holds and the operator \( U[F \tilde{f}] \) from \( H \) into \( H \) is compact for all \( f \in C \) and \( F \in C_p^\# \).

IV. Property P4 holds and every weakly convergent sequence in \( S_\mu, c \) is \( \sigma_{\text{ind}} \)-convergent.

**Proof.**

I \( \iff \) II. Trivial since \( S_\mu, c \) is barrelled.

II \( \Rightarrow \) III. Let \( \phi_1, \phi_2, \ldots \) be a Cauchy sequence in \( S_\mu, c \). Let \( B := \{ \phi_n : n \in \mathbb{N} \} \) and let \( B_1 \) be the closure of \( B \) in \( S_\mu, c \). Then \( B_1 \) is bounded and closed in \( S_\mu, c \), hence compact and in particular complete ([Wil] Lemma 6-1-18). So the Cauchy sequence \( \phi_1, \phi_2, \ldots \) is convergent in \( B_1 \) and also in \( S_\mu, c \). By Theorem 3.12 IV \( \Rightarrow \) I property P4 holds.

Let \( f \in C \) and \( F \in C_p^\# \). Let \( B := \{ x \in H : \|x\| \leq 1 \} \). By Corollary 3.11 I, the set \( U(f)(B) \) is bounded in \( S_\mu, c \). Let \( B_1 \) be the closure of \( U(f)(B) \) in \( S_\mu, c \). Since \( S_\mu, c \) is a semi-Montel space, the set \( B_1 \) is compact in \( S_\mu, c \). Because the operator \( U[F] \) from \( S_\mu, c \) into \( H \) is continuous (Lemma 2.12 I), the set \( U[F](B_1) \) is compact in \( H \). But \( U[F \tilde{f}](B) \) is a subset of \( U[F](B_1) \), so the operator \( U[F \tilde{f}] \) from \( H \) into \( H \) is compact.
III \Rightarrow II. Let \( B \) be a closed and bounded subset of \( S_{U, C} \). By Theorem 3.10 there exist \( f \in C \) and a bounded subset \( B_0 \) of \( H \) such that \( B = U(f)(B_0) \). Let \( \phi_1, \phi_2, \cdots \) be a sequence in \( B \). For each \( n \in \mathbb{N} \) there exists \( x_n \in B_0 \) such that \( \phi_n = U(f)x_n \). The sequence \( x_1, x_2, \cdots \) is bounded in \( H \), so it contains a weakly convergent subsequence \( (x_n_k)_{k \in \mathbb{N}} \) with weak limit \( x \). For all \( F \in C_p^\# \), the operator \( U[F, \hat{f}] \) from \( H \) into \( H \) is compact, so 
\[
\lim_{k \to \infty} U[F] U(f)x_n_k = U[F] U(f)x \quad \text{and hence} \quad \lim_{k \to \infty} s_F(\phi_n_k - U(f)x) = 0.
\]
So \( \lim_{k \to \infty} \phi_n_k = U(f)x \) in \( S_{U, C} \) and, since \( B \) is closed, also in \( B \) with respect to the relative topology. Therefore the set \( B \) is sequentially compact and also compact by Corollary 3.11 IV.

III \Rightarrow IV. Let \( \phi_1, \phi_2, \cdots \) be a weakly convergent sequence in \( S_{U, C} \) with weak limit \( \phi \in S_{U, C} \). Let \( B := \{ \phi \} \cup \{ \phi_n : n \in \mathbb{N} \} \). Then \( B \) is weakly bounded and hence \( \sigma_{\text{ind}} \)-bounded. By Theorem 3.10 there exist \( f \in C \) and a bounded set \( B_0 \subset H \) such that \( U(f) l_{\theta \sigma} : (B_0, \| \|) \to (B, \sigma_{\text{ind}} l_B) \) is a homeomorphism. For \( n \in \mathbb{N} \) let \( x, x_n \in B_0 \) be such that \( \phi = U(f)x \) and \( \phi_n = U(f)x_n \). Then weak \( \lim_{n \to \infty} x_n = x \). Let \( F \in C_p^\# \). By assumption the operator \( U[F, \hat{f}] \) is compact, so in the Hilbert space \( H : \lim_{n \to \infty} U[F] \phi_n = \lim_{n \to \infty} U[F, \hat{f}]x_n = U[F, \hat{f}]x = U[F] \phi \). Hence \( \lim_{n \to \infty} s_F(\phi_n - \phi) = 0 \). So \( \lim_{n \to \infty} \phi_n = \phi \) in \( (S_{U, C}, \sigma_{\text{ind}}) \).

IV \Rightarrow III. Let \( f \in C \). We prove that the operator \( U(f) \) from \( H \) into \( H \) is compact. Let \( x, x_1, x_2, \cdots \in H \) and suppose weak \( \lim_{n \to \infty} x_n = x \). The operator \( U(f) \) from \( H \) into \( S_{U, C} \) is continuous, so \( \lim_{n \to \infty} U(f)x_n = U(f)x \) in \( (S_{U, C}, \sigma(S_{U, C}, S_{U, C}')) \). By assumption, \( \lim_{n \to \infty} U(f)x_n = U(f)x \) in \( (S_{U, C}, \sigma_{\text{ind}}) \). Since the identity map from \( S_{U, C} \) into \( H \) is continuous, \( \lim_{n \to \infty} U(f)x_n = x \) in \( H \). So \( U(f) \) is compact.

By Theorem 3.13 III \Rightarrow II, \( U[F, \hat{f}] \) is a compact operator from \( H \) into \( H \) for all \( f \in C \) and \( F \in C_p^\# \).

We need a lemma for the characterization of nuclear spaces.

**Lemma 3.15.**

Suppose there exist an orthonormal basis \( e_1, e_2, \cdots \) in \( H \) and a sequence \( \gamma_1, \gamma_2, \cdots \) in \( \hat{G} \) such that \( U_x e_n = \gamma_n(x) e_n \) for all \( x \in G \) and \( n \in \mathbb{N} \). Let \( F \in \text{Bor}_b(\hat{G}, C) \). Then \( U[F] e_n = F(\gamma_n) e_n \) for all \( n \in \mathbb{N} \).

**Proof.** First we construct a Stone-representant for \( U \). Let \( A := \mathbb{N} \times \hat{G} \) with product topology,
where \( \mathbb{N} \) carries the discrete topology. This defines a locally compact Hausdorff topology for \( A \).

Let \( I := \mathbb{N} \). For \( n \in \mathbb{N} \) let \( A_n := \{ n \} \times \hat{G} \) and define \( \tau_n : \hat{G} \to A_n \) by \( \tau_n(\gamma) := (n, \gamma) \) \( (\gamma \in \hat{G}) \). Then \( A_n \) is open and \( \tau_n \) is a topological homeomorphism. For every Borel set \( Z \) of \( A \) define \( m(Z) := \sum_{n=1}^{\infty} 1_Z(n, \gamma_n) \). Then \( m \) is a regular measure on \( A \).

Define a unitary operator \( W \) from \( H \) onto \( L^2(m) \) by

\[
(W e_n)(k, \gamma) := \begin{cases} 1 & \text{if } (k, \gamma) = (n, \gamma_n), \\ 0 & \text{if } (k, \gamma) \neq (n, \gamma_n). \end{cases}
\]

Then \( W U(f) e_n = W \hat{f}(\gamma_n) e_n = \hat{f} \circ W e_n \) for all \( f \in C \) and \( n \in \mathbb{N} \), so it follows that \( (A, m, I, A_1, \tau_1, W) \) is a Stone-representant for \( U \).

Let \( F \in \text{Bor}(\hat{G}, C) \) and \( n \in \mathbb{N} \). Then \( U[F] e_n = W^{-1}(F \circ W e_n) = W^{-1}(F(\gamma_n) W e_n) = F(\gamma_n) e_n \).

REMARK. Careful reading of [HRII] Remark 33.6 shows that in fact the following stronger result is valid: let \( u \in H \), \( \gamma \in \hat{G} \) and suppose \( U x u = \gamma(x) u \) for all \( x \in G \). Then \( U[F] u = F(\gamma) u \) for all \( F \in \text{Bor}(\hat{G}, C) \). Here we need only Lemma 3.15.

For the definition of nuclear maps and nuclear spaces we refer to [Sch] and appendix \( B \).

**THEOREM 3.16.**

The following conditions are equivalent.

I. \( S_{U,C} \) is nuclear.

II. For all \( f \in C \) and \( K \in C^\#_p \) the operator \( U[K] \) from \( H \) into \( H \) is nuclear.

III. There exists \( F \in \text{Bor}(\hat{G}, R) \), \( F \geq 1 \) such that for all \( f \in C \) and \( K \in C^\#_p \) the function \( F K \hat{f} \) is bounded and the operator \( U[F] \) from \( H \) into \( H \) is nuclear.

IV. There exist a countable orthonormal basis \( e_1, e_2, \ldots \) in \( H \) and a sequence \( \gamma_1, \gamma_2, \ldots \) in \( \hat{G} \) such that \( U x e_n = \gamma_n(x) e_n \) for all \( x \in G \) and all \( n \in \mathbb{N} \). Moreover, for every sequence \( h_1, h_2, \ldots \in [0, \infty) \) with the property that for all \( f \in C \) the set \( \{ h_n \hat{f}(\gamma_n) : n \in \mathbb{N} \} \) is bounded we have \( \sum_{n=1}^{\infty} h_n \| \hat{f}(\gamma_n) \| < \infty \) for all \( f \in C \).

**Proof.** The proof of the equivalence of I, II and III is inspired by the proof of [Mar] Theorem 2.2.

I \( \Rightarrow \) II. The Hilbert space \( H \) is a Banach space. The map \( U[K] \) from \( S_{U,C} \) into \( H \) is continuous by Lemma 2.12 I, hence nuclear. (See [Sch] Theorem III 7.2.) The map \( U(f) \) from \( H \) into \( S_{U,C} \) is continuous by Lemma 1.6, so the map \( U[K] \hat{f} = U[K] \circ U(f) \) is nuclear.
II $\Rightarrow$ III. By Theorem 3.13 II $\Rightarrow$ IV the operator $P_n = U[I_{Q_n}]$ has finite rank, hence $a_n := \dim P_n(H) < \infty$ for all $n \in \mathbb{N}$. Let $F := 1 + \sum_{n=1}^{\infty} a_n b_n^2 1_{Q_n}$. Then $F \in \text{Bor}(\mathcal{G}, \mathcal{R})$ and $F \geq 1$. Let $f \in C$ and $K \in C_p^f$. By property P3 there exist $g \in C$ and $\delta > 0$ such that $b_n \sup \{ |\hat{f}(\gamma)| : \gamma \in Q_n \} \leq \delta \inf \{ |\hat{g}(\gamma)| : \gamma \in Q_n \}$ for all $n \in \mathbb{N}$.

Define $K_1 : \mathcal{G} \to \mathcal{R}$

$$K_1(\gamma) := \begin{cases} b_n \sup \{ |K(\omega)| : \omega \in Q_n \} & \text{if } P_n \neq 0, \\ 1 & \text{if } P_n = 0. \end{cases} (n \in \mathbb{N}, \gamma \in Q_n)$$

By Lemma 3.3 II, $K_1 \in C_p^f$. Denote by $\text{Tr}$ the trace of a compact positive Hermitian operator $V$. For all $n \in \mathbb{N}$ and $\gamma \in Q_n$ we have

$$|F(\gamma) K(\gamma) \hat{f}(\gamma)| \leq \|K\|_\infty + a_n b_n^2 \cdot b_n^{-1} \inf \{ |K(\omega)| : \omega \in Q_n \} + \delta b_n^{-1} \inf \{ |\hat{g}(\omega)| : \omega \in Q_n \}$$

$$\leq \|K\|_\infty + \delta \text{Tr}(U[I_{Q_n}]) \cdot \inf \{ |(K_1 \hat{g})(\omega)| : \omega \in Q_n \}$$

$$\leq \|K\|_\infty + \delta \text{Tr}(U[I_{Q_n}] U[I_{Q_n} \circ U[I_{Q_n}]])$$

$$\leq \|K\|_\infty + \delta \text{Tr}(U[I_{Q_n}] U[I_{Q_n}]) < \infty.$$ 

So $FKf$ is bounded.

It is trivial that the operator $U[F^{-1}]$ from $H$ into $H$ is positive, Hermitian and compact.

Since $\text{Tr} U[F^{-1}] = \sum_{n=1}^{\infty} \text{Tr} \left( \frac{1}{1 + a_n b_n^2} P_n \right) \leq \sum_{n=1}^{\infty} \frac{1}{b_n^2} < \infty$, the operator $U[F^{-1}]$ is nuclear.

III $\Rightarrow$ I. Since every nuclear map is compact, we obtain by Theorem 3.13 V $\Rightarrow$ VI that there exist an orthonormal basis $e_1, e_2, \ldots$ in $H$ and a sequence $\gamma_1, \gamma_2, \ldots$ in $\mathcal{G}$ such that $U_x e_n = \gamma_n(x) e_n$ for all $x \in G$ and $n \in \mathbb{N}$.

Let $p$ be a seminorm on $S_{U, C}$. Let $M_p := \{ \phi \in S_{U, C} : p(\phi) = 0 \}$. $M_p$ is a linear subspace of $S_{U, C}$. Let $S_p$ be the quotient space $S_{U, C}/M_p$ and let $\pi_p$ be the quotient map from $S_{U, C}$ into $S_p$. $S_p$ is a normed space with norm $\bar{p}$ defined by $\bar{p}(\pi_p(\phi)) := p(\phi), \phi \in S_{U, C}$.

Let $\hat{S}_p$ be the completion of $(S_p, \bar{p})$.

We have to prove that there exists a set $\Gamma$ of seminorms on $S_{U, C}$ which generates the topology for $S_{U, C}$ and such that for every $p \in \Gamma$ the canonical map from $S_{U, C}$ into $\hat{S}_p$ is nuclear. Let $\Gamma := \{ s_K : K \in C_p^f \}$. By Corollary 2.31, $\Gamma$ generates the topology for $S_{U, C}$. Let $K \in C_p^f$ and let $p := s_K$. Since $FKf$ is bounded for all $f \in C$ and since $F \geq 1$, the function $FK$ is an element of $C_p^f$. Let $q := s_{FK}$. Then $q \geq p$ and $q$ is a continuous seminorm on $S_{U, C}$. We have proved the implication III $\Rightarrow$ I if we have shown that the natural map from $\hat{S}_q$ into $\hat{S}_p$ is nuclear.
The heuristic idea of the proof can be sketched as follows. Note that $p$ and $q$ are norms. So $S_p$ and $S_q$ can be identified with $S_{U, C}$ as subset of the Hilbert space $H$. Hence for every $z \in S_q \subseteq H$ we obtain the formal identity, which is not proved:

$$z = \sum_{n=1}^{\infty} (z, e_n) e_n = \sum_{n=1}^{\infty} (F(y_n))^{-1} (K(y_n) F(y_n) z, e_n) (K(y_n))^{-1} e_n = \sum_{n=1}^{\infty} \lambda_n h_n(z) y_n, \quad \text{with}$$

$$\lambda_n \in \mathbb{C}, h_n \in \hat{S}_q', y_n \in \hat{S}_p' \quad \text{for all} \quad n \in \mathbb{N}; \sum_{n=1}^{\infty} |\lambda_n| < \infty, \{h_n : n \in \mathbb{N}\} \text{ equicontinuous and the set} \{y_n : n \in \mathbb{N}\} \text{ bounded in} \hat{S}_p.$$

We return to the proof. The normed space $\hat{S}_p$ is in fact a Hilbert space. Denote by $(,)$ the corresponding inner product in the space $\hat{S}_p$. Let $n \in \mathbb{N}$. By property P2,

$$e_n = \lim_{\lambda \to 0} U(f_{\lambda}) e_n, \quad \text{so there exists} \quad f \in C \quad \text{such that} \quad U(f) e_n \neq 0. \quad \text{Hence}$$

$$0 \neq U(f) e_n = (\hat{f}(y_n)) e_n \quad \text{and} \quad e_n = (\hat{f}(y_n))^{-1} U(f) e_n \in S_{U, C}. \quad \text{Let} \quad y_n = (K(y_n))^{-1} e_n \in S_{U, C}, \quad \text{by Lemma 3.15. It is nearly trivial that the set}$$

$$\{\pi_p(y_n) : n \in \mathbb{N}\} \text{ is an orthonormal basis in} \hat{S}_p. \quad \text{In particular the set} \{\pi_p(y_n) : n \in \mathbb{N}\} \text{ is bounded in} \hat{S}_p.$$

Let $n \in \mathbb{N}$. Define $f_n : S_{U, C} \to \mathbb{C}$

$$f_n(\phi) := (U[F K] \phi, e_n) \quad (\phi \in S_{U, C}).$$

Then $|f_n(\phi)| \leq \|U[F K] \phi\| = q(\phi)$ for all $\phi \in S_{U, C}$. So there exists a unique continuous linear functional $f_n'$ from $\hat{S}_q$ into $\mathbb{C}$ such that $f_n'(\pi_q(\phi)) = f_n(\phi)$ for all $\phi \in S_{U, C}$. Then $\|f_n'\| \leq 1$. Hence the set $\{f_n' : n \in \mathbb{N}\}$ is equicontinuous.

For $n \in \mathbb{N}$ let $\lambda_n := (F(y_n))^{-1}$. Then $\lambda_n > 0$ and since $U(F^{-1}) e_n = \lambda_n e_n \text{ (Lemma 3.15)}$ and $U(F^{-1})$ is nuclear, we have $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Define $E : \hat{S}_q \to \hat{S}_p$

$$E(z) := \sum_{n=1}^{\infty} \lambda_n f_n'(z) \pi_p(y_n) \quad (z \in \hat{S}_q).$$

Then $E$ is a nuclear map. (See [Sch] Theorem III 7.1.)

Let $\psi$ be the canonical map from $\hat{S}_q$ into $\hat{S}_p$. Assertion: $E = \psi$.

Let $\phi \in S_{U, C}$. Then $E(\pi_q(\phi)) = \sum_{n=1}^{\infty} \lambda_n (U[F K] \phi, e_n) \pi_p(y_n) = \sum_{n=1}^{\infty} (U[F K] \phi, U[F^{-1}] e_n) \pi_p(y_n) = \sum_{n=1}^{\infty} (U[F^{-1}] U[F K] \phi, e_n) \pi_p(y_n) =$
\[
\sum_{n=1}^{\infty} (U[K] \phi, U[K] y_n) \pi_p(y_n) = \\
\sum_{n=1}^{\infty} (\pi_p(\phi), \pi_p(y_n)) = \pi_p(\phi) = \pi_p(\psi(\phi)).
\]

So \( E \mid _S = \psi \mid _S \). Since \( E \) and \( \psi \) are continuous, \( E = \psi \).

So the canonical map \( \psi \) is nuclear. We have proved the implication III \( \Rightarrow \) I.

II \( \Rightarrow \) IV. Since every nuclear map is compact, we obtain by Theorem 3.13 II \( \Rightarrow \) VI that there exist an orthonormal basis \( e_1, e_2, \ldots \) in \( H \) and a sequence \( \gamma_1, \gamma_2, \ldots \) in \( \hat{G} \) such that \( U_x e_n = \gamma_n(x) e_n \) for all \( x \in G \) and \( n \in \mathbb{N} \).

Let \( h_1, h_2, \ldots \in [0, \infty) \) be such that for all \( f \in C \) the set \( \{h_n \hat{f}(\gamma_n) : n \in \mathbb{N}\} \) is bounded. Let \( \gamma \in \{\gamma_n : n \in \mathbb{N}\} \) and suppose there exists \( f \in C \) such that \( \hat{f}(\gamma) \neq 0 \). Then \( \sup \{h_n : n \in \mathbb{N}, \gamma_n = \gamma\} = |\hat{f}(\gamma)|^{-1} \sup \{h_n \hat{f}(\gamma_n) : n \in \mathbb{N}, \gamma = \gamma_n\} < \infty \).

Define \( K : \hat{G} \to \mathbb{R} \)

\[
K(\gamma) = \begin{cases} 
1 & \text{if } \gamma \notin \{\gamma_n : n \in \mathbb{N}\}, \\
1 & \text{if } \hat{f}(\gamma) = 0 \text{ for all } f \in C, \quad (\gamma \in \hat{G}) \\
1 + \sup \{h_n : n \in \mathbb{N}, \gamma_n = \gamma\} & \text{else}.
\end{cases}
\]

Then \( K \in \text{Bor}(\hat{G}, \mathbb{R}) \), \( K \geq 1 \) and \( K \ast \hat{f} \) is bounded for all \( f \in C \). Hence \( K \in C_p^{b} \). By assumption the operator \( U[\hat{f}K] \) from \( H \) into \( H \) is nuclear. Then by Lemma 3.15:

\[
\sum_{n=1}^{\infty} h_n \hat{f}(\gamma_n) \leq \sum_{n=1}^{\infty} K(\gamma_n) \hat{f}(\gamma_n) = \sum_{n=1}^{\infty} |(U[K] \cdot e_n, e_n)| < \infty.
\]

IV \( \Rightarrow \) II. Let \( f \in C \) and \( K \in C_p^{b} \). For \( n \in \mathbb{N} \) let \( \hat{g}_n := K(\gamma_n) \). Then the set \( \{h_n \hat{g}(\gamma_n) : n \in \mathbb{N}\} \) is bounded for all \( g \in C \), so \( \sum_{n=1}^{\infty} h_n \hat{g}(\gamma_n) < \infty \). By Lemma 3.15,

\[
U[|\hat{f}K|] e_n = |\hat{f}(\gamma_n)| K(\gamma_n) e_n = h_n \hat{f}(\gamma_n) e_n.
\]

So \( U[|\hat{f}K|] = U[|\hat{f}K|]^{*} \) is nuclear and \( U[\hat{f}K] \) is nuclear.

\[\Box\]

**THEOREM 3.17.**

Suppose property P4 holds. The following conditions are equivalent. (Cf. Theorem 3.16.)

I. \( T_U, C \) is nuclear.

II. For all \( f \in C \) and \( K \in C_p^{b} \) the operator \( U[\hat{f}K] \) is nuclear.

**Proof.**

I \( \Rightarrow \) II. Let \( f \in C \) and \( K \in C_p^{b} \). The map \( \Phi \mapsto \Phi(f) \) from \( T_U, C \) into \( H \) is continuous, hence nuclear. The map \( x \mapsto U[\hat{K}] \ast x \) from \( H \) into \( T_U, C \) is continuous, so the map \( x \mapsto (U[\hat{K}] \ast x)(f) \) from \( H \) into \( H \) is nuclear. Hence the map \( U[K \hat{f}] = U[\hat{K} \hat{f}]^{*} \) is nuclear.
This proof is essentially similar to the proof of Theorem 3.16 III \(\Rightarrow\) I. But in this case there are some little difficulties.

By Theorem 3.16 II \(\Rightarrow\) III there exists \(F \in \text{Bor}(\hat{G}, \mathcal{R})\), \(F \geq 1\) such that the function \(FK\hat{f}\) is bounded for all \(f \in C\) and \(K \in C\) and the operator \(U[F^{-1}]\) from \(H\) into \(H\) is nuclear. Since every nuclear operator is compact, we obtain from Theorem 3.13 V \(\Rightarrow\) VI that there exist and orthonormal basis \(e_1, e_2, \ldots\) in \(\hat{G}\) such that \(U\) is bounded for all \(f \in C\) and \(n \in \mathcal{N}\). For \(n \in \mathcal{N}\) let \(\lambda_n := (F(\eta_n))^{-1}\).

Then \(\lambda_n > 0\) and since \(U[F^{-1}]\) is nuclear we obtain that \(\sum_{n=1}^{\infty} \lambda_n < \infty\) by Lemma 3.15.

For every seminorm \(p\) on \(T_U, c\) let \(M_p := \{\Phi \in T_U, c : p(\Phi) = 0\}\) and let \(T_p\) be the normed space \(T_U, c/M_p\) with norm \(\hat{p}\) defined by \(\hat{p}(\pi_p(\Phi)) := p(\Phi), \Phi \in T_U, c\) and \(\pi_p\) the quotient map from \(T_U, c\) into \(T_p\). Let \(\hat{T}_p\) be the completion of \((T_p, \hat{p})\).

Let \(f \in C\) and let \(p := t_f\). The function \(F\hat{f}K\) is bounded for all \(K \in C\), so by property P4 there exist \(g \in C\) and \(c > 0\) such that \(U(1, 2) = 0\) with \(Z = \{\gamma \in \hat{G} : |(F\hat{f})(\gamma)| > c \mid \hat{g}(\gamma) \mid\}\). Let \(q := c t_g\). Then \(q\) is a continuous seminorm on \(T_U, c\).

Define \(K : \hat{G} \rightarrow C\)

\[
K(\gamma) := \begin{cases} \frac{F(\gamma)\hat{f}(\gamma)}{\hat{g}(\gamma)} & \text{if } \hat{g}(\gamma) \neq 0 \text{ and } \gamma \in Z, \\ 0 & \text{else.} \end{cases} \quad (\gamma \in \hat{G})
\]

Then \(K \in \text{Bor}(\hat{G}, C)\), \(\hat{g}K = F\hat{f}1_Z\), and \(\|K\|_{\infty} \leq c\). For all \(x \in H\) we obtain \(q(\text{emb} x) = c \|U(g)^* x\| = c \|U(g)x\| \geq \|U(K_f)\| \|U[F\hat{f}1_Z]\| \|x\| = \|U[F\hat{f}]x\| \geq \|U[F\hat{f}]\| \|x\| = p(\text{emb} x)\). Since \(\text{emb} H\) is dense in \(T_U, c\), \(q \geq p\). We shall prove that the canonical map from \(\hat{T}_p\) into \(\hat{T}_p\) is nuclear.

In fact the normed space \(\hat{T}_p\) is a Hilbert space. Denote by \((\cdot, \cdot)\) the inner product in \(\hat{T}_p\).

We construct an orthonormal basis in \(\hat{T}_p\). For \(n \in \mathcal{N}\) define

\[
y_n := \begin{cases} (\hat{f}(\gamma_n))^{-1} e_n & \text{if } \hat{f}(\gamma_n) \neq 0, \\ 0 & \text{if } \hat{f}(\gamma_n) = 0. \end{cases}
\]

Then \(U(f)^* y_n = e_n\) if \(\hat{f}(\gamma_n) \neq 0\), \(n \in \mathcal{N}\). Assertion: \(\pi_p(\text{emb} y_n) : n \in \mathcal{N}, y_n \neq 0\) is an orthonormal basis in \(\hat{T}_p\). Proof of the assertion. It is trivial that this set is orthonormal. Let \(x \in H\) and suppose for all \(n \in \mathcal{N}\) holds \(\pi_p(\text{emb} x), \pi_p(\text{emb} y_n)\) = 0.

Let \(\alpha_1, \alpha_2, \ldots \in C\) be such that \(x = \sum_{n=1}^{\infty} \alpha_n e_n\). Then for all \(n \in \mathcal{N}\) such that \(\hat{f}(\gamma_n) \neq 0\) we obtain \(0 = (\pi_p(\text{emb} x), \pi_p(\text{emb} y_n)) = (U(f)^* x, U(f)^* y_n) = (U(f)^* x, e_n) = (x, U(f)^* e_n) = \hat{f}(\gamma_n) \alpha_n\). So \(U(f)^* x = 0\) and \(\pi_p(\text{emb} x) = 0\) in \(\hat{T}_p\). Since \(T_p\) is dense in...
\(
\hat{T}_p
\), the assertion is proved. In particular, the set \(\{\pi_p(\text{emb} y_n) : n \in \mathbb{N}\}\) is bounded in \(\hat{T}_p\).

For \(n \in \mathbb{N}\) define \(h_n : T_{U,C} \to \mathcal{C}\)

\[
h_n(\Phi) := (\Phi(g), U[K] e_n) \quad (\Phi \in T_{U,C}).
\]

Then \(\|h_n(\Phi)\| \leq \|U[K]\| \|e_n\| \|\Phi(g)\| \leq c \ell_g(\Phi)\) for all \(\Phi \in T_{U,C}\). So there exists a unique continuous linear function \(h_n'\) from \(\hat{T}_q\) into \(\mathcal{C}\) such that \(h_n'(\pi_q(\Phi)) = h_n(\Phi)\) for all \(\Phi \in T_{U,C}\). Then \(\|h_n'\| \leq 1\).

So the set \(\{h_n' : n \in \mathbb{N}\}\) is equicontinuous.

Define \(E : \hat{T}_q \to \hat{T}_p\)

\[
E(z) := \sum_{n=1}^{\infty} \lambda_n h_n'(z) \pi_p(\text{emb} y_n) \quad (z \in \hat{T}_q).
\]

The map \(E\) is nuclear. (See \[Sch\] Theorem III 7.1.) Let \(\phi\) be the canonical map from \(\hat{T}_q\) into \(\hat{T}_p\). We have proved the theorem if we can show that \(E = \phi\). Let \(x \in H\). Then

\[
E(\pi_q(\text{emb} x)) = \sum_{n=1}^{\infty} \lambda_n (U(g)^* x, U[K] e_n) \pi_p(\text{emb} y_n)
\]

\[
= \sum_{n=1}^{\infty} \lambda_n (U[K]^* U(g)^* x, e_n) \pi_p(\text{emb} y_n)
\]

\[
= \sum_{n=1}^{\infty} (U[F \tilde{f}] \mathbf{1}_Z x, U[F^{-1}] e_n) \pi_p(\text{emb} y_n)
\]

\[
= \sum_{n=1}^{\infty} (U[F^{-1}] U[F \tilde{f}] x, e_n) \pi_p(\text{emb} y_n)
\]

\[
= \sum_{n=1}^{\infty} (U(f)^* x, U(f)^* y_n) \pi_p(\text{emb} y_n)
\]

\[
= \sum_{\substack{n=1 \quad \ell(f_n) \neq 0}} (\pi_p(\text{emb} x), \pi_p(\text{emb} y_n))_Z \pi_p(\text{emb} y_n)
\]

\[
= \pi_p(\text{emb} x)
\]

\[
= \phi(\pi_q(\text{emb} x)).
\]

So \(E |_{\tau_q} = \phi |_{\tau_q}\) and \(E = \phi\).

If \(C \subset C \ast C\), then there exists another characterization of nuclear \(T_{U,C}\) spaces. (Cf. Theorems 3.16 II and 3.17 II.)
THEOREM 3.18.
Suppose $C \subset C \ast C$. Then the following conditions are equivalent.

I. For all $f \in C$ and $K \in C\hat{\circ}$ the operator $U[fK]$ from $H$ into $H$ is nuclear.

II. The operator $U(f)$ from $H$ into $H$ is nuclear for all $f \in C$.

Proof.

I $\Rightarrow$ II. Trivial since $1_{0} \in C\hat{\circ}$.

II $\Rightarrow$ I. Let $f \in C$ and $K \in C\hat{\circ}$. There exist $g, h \in C$ such that $f = g \ast h$. Then $U(g)$ is nuclear and $U[K\hat{h}]$ is continuous from $H$ into $H$. So the operator $U[K\hat{f}] = U(g) \circ U[K\hat{h}]$ from $H$ into $H$ is nuclear.

$\square$
Chapter IV. Examples

Example 1.

Let $A$ be a positive self-adjoint operator on a Hilbert space $H$. For all $t > 0$ the continuous operator $e^{-itA}$ is injective, so there exists a unique norm $\| \cdot \|$ on the vector space $e^{-itA}(H)$ such that $e^{-itA}(H)$ becomes a Hilbert space and the map $e^{-itA}$ from $H$ onto $e^{-itA}(H)$ is a unitary map. The analyticity space $S_{H,A}$ is defined by $S_{H,A} := \bigcup_{t > 0} e^{-itA}(H)$. The topology $\sigma$ for $S_{H,A}$ is the inductive limit topology generated by the Hilbert spaces $e^{-itA}(H)$. The trajectory space $T_{H,A}$ consists of all maps $F$ from the open interval $(0, \infty)$ into $H$ such that

$$\forall t_1, t_2 > 0 \left[ F(t_1 + t_2) = e^{-itA} F(t_2) \right].$$

The topology $\tau$ for $T_{H,A}$ is the locally convex Hausdorff topology for $T_{H,A}$ generated by the seminorms $q_t, t > 0$, defined by $q_t(F) := \| F(t) \|, F \in T_{H,A}$.

In the monograph [EG2] the spaces $S_{H,A}$ and $T_{H,A}$ are introduced for every separable Hilbert space $H$ and every positive self-adjoint operator on $H$. A lot of examples are included in chapter II of the book [EG2]. The assumed separability of the Hilbert space $H$ is an unnecessary restriction which we do not assume here. We shall prove that there exist a locally compact Abelian group $G$, a representation $U$ of $G$ and a subset $C$ of $L^1(G)$ such that the pair $(C, U)$ satisfies properties P1', P2', P3 and P4 and such that the spaces $S_{H,A}$ and $T_{H,A}$ are equal to $S_{U, C}$ and $T_{U, C}$, respectively, as topological vector spaces.

Let $G := \mathbb{R}$ with the Lebesgue measure as its Haar measure. For $y \in \mathbb{R}$ define $\gamma_y : \mathbb{R} \rightarrow \mathbb{C}$

$$\gamma_y(x) := e^{ixy} \ (x \in \mathbb{R}).$$

It is well known that $\hat{G} = \{ \gamma_y : y \in \mathbb{R} \}$. (See [HRI] example 23.27e.) For $x \in \mathbb{R}$ let $U_x := e^{ixA}$. By [Wei] Theorem 7.37, $U$ is a continuous representation of $G$ in $H$. Let $t > 0$.

Define $f_t : \mathbb{R} \rightarrow \mathbb{C}$

$$f_t(x) := \frac{1}{\pi} \frac{t}{t^2 + x^2} \ (x \in \mathbb{R}).$$

Then $f_t \in L^1(G)$. Define $C := \{ f_t : t > 0 \}$.

For all $y \in \mathbb{R}$ and all $t > 0$ we have

$$\hat{f_t}(\gamma_y) = \int_{\mathbb{R}} \frac{1}{\pi} \frac{t}{t^2 + x^2} e^{-ixy} dx = \int_{\mathbb{R}} \frac{1}{\pi} \frac{1}{1 + x^2} e^{-ity} dx = e^{-ity}.$$

Let $t_1, t_2 > 0$. Then $\hat{f}(\gamma_y) \hat{f}(\gamma_y) = e^{-(t_1 + t_2)y} = \hat{f}_{t_1 + t_2}(\gamma_y)$ for all $y \in \mathbb{R}$. Hence $f_{t_1} \ast f_{t_2} = f_{t_1 + t_2}$.

Now property P1' follows. Since $f_t \geq 0$, $\| f_t \|_1 = \int_{\mathbb{R}} f_t(x) dx = \lim_{t \downarrow 0} \int_{\mathbb{R}} f_t(x) dx = \lim_{t \downarrow 0} \int_{\mathbb{R}} \frac{1}{\pi} \frac{1}{1 + x^2} dx = 0$.

Let $J := (0, \infty)$. $J$ is a directed set under the ordering $s \leq t$ in $J \iff s \geq t$ in $\mathbb{R}$ ($s, t \in J$). Then
property P2 follows by [HRII] Theorem 28.52. For \( n \in \mathbb{N} \) define \( b_n = n^2 \) and \( Q_n := \{ y : |y-1| < n \} \). Then \( Q_n \) is a Borel measurable subset of \( \mathbb{C} \) and obviously property P3 is satisfied. For all \( t > 0 \) there exists \( n \in \mathbb{N} \) such that \( n^{-1} < t \) and hence \( f_1 = f_{n^{-1}} \ast f_{n^{-1}} \). By Lemma 1.14 we obtain that the topology \( \tau_{\text{proj}} \) for \( T_{U,C} \) is generated by the set of seminorms \( \{ t_{f_1} : n \in \mathbb{N} \} \). Hence \( (T_{U,C} , \tau_{\text{proj}}) \) is metrizable and property P4 holds by Corollary 3.6.

The equality \( S_{H,A} = S_{U,C} \) as topological vector spaces follows easily from the next theorem.

**THEOREM 4.1.**
Let \( t > 0 \). Then \( U(f_1)u = e^{-tA}u \) for all \( u \in H \).

**Proof.**

By the spectral theorem ([MP] Theorem A7, page 497) there exist a measure space \( (Y, B, m) \), a real valued measurable function \( h \) on \( Y \) and a unitary map \( W \) from \( H \) onto \( L^2(m) \) such that \( A = W^{-1} M_h W \) with \( M_h \) the multiplication operator by \( h \) on \( L^2(m) \). Then \( h \geq 0 \) a.e. because \( A \) is a positive operator. Without loss of generality, \( H = L^2(m) \) and \( W \) is the identity map. Let \( \xi \in L^2(m) \). For \( n \in \mathbb{N} \) let \( Y_n : = \{ y \in Y : |\xi(y)| > \frac{1}{n} \} \). Then \( m(Y_n) < \infty \) for all \( n \in \mathbb{N} \). By Lebesgue’s dominated convergence theorem and Fubini’s theorem for all \( \eta \in L^2(m) \) we have:

\[
(U(f_1) \xi, \eta) = \int_{\mathbb{R}} f_1(x) (U_\xi \xi, \eta) \, dx
\]

\[
= \int_{\mathbb{R}} \lim_{n \to \infty} 1_{Y_n}(y) f_1(x) e^{i \chi(y)} \xi(y) \eta(y) \, dm(y) \, dx
\]

\[
= \lim_{n \to \infty} \int_{\mathbb{R}} f_1(x) e^{i \chi(y)} \xi(y) \eta(y) \, d(1_{Y_n} \, m)(y) \, dx
\]

\[
= \lim_{n \to \infty} \int_{\mathbb{R}} f_1(x) e^{i \chi(y)} \xi(y) \eta(y) \, dx \, d(1_{Y_n} \, m)(y)
\]

\[
= \lim_{n \to \infty} \int_{\mathbb{R}} e^{-it\chi(y)} \xi(y) \eta(y) \, 1_{Y_n}(y) \, d m(y)
\]

\[
= \int_{\mathbb{R}} e^{-it\chi(y)} \xi(y) \eta(y) \, d m(y)
\]

\[
= (e^{-tA} \xi, \eta).
\]

The theorem is proved.

Finally we prove that \( T_{U,C} \) is isomorphic with \( T_{H,A} \) as topological vector spaces.
Define $E : T_{U, C} \rightarrow T_{H, A}$

$$[E(\Phi)](t) := \Phi(f_t) \quad (\Phi \in T_{U, C}, \ t \in (0, \infty)).$$

Obviously, $E$ is an injective linear map. Note that $f_s = f_t$ if and only if $s = t$ for all $s, t \in (0, \infty)$. Let $F \in T_{X, A}$. Define $\Phi : C \rightarrow H$ by $\Phi(f_t) := F(t)$ ($t \in (0, \infty)$). We prove that $\Phi$ is a $C$-trajectory. Let $s, t \in (0, \infty)$, $g \in L^1(G)$ and suppose $f_s = g * f_t$. Then $\tilde{f}_s = \tilde{g} \tilde{f}_t$ and $\tilde{g}(y) = e^{-(y-t)y}$ for all $y \in \mathbb{R}$. Since $\tilde{g} \in C_0(\hat{G})$ we obtain that $s > t$ and $\tilde{g} = \tilde{f}_{s-t}$. So $g = f_{s-t}$ in $L^1(G)$. Hence $\Phi(g * f_t) = F(s) = e^{-(s-t)A} F(t) = U(g) \Phi(f_t)$ and $\Phi \in T_{U, C}$. Then $E(\Phi) = F$. So $E$ is a bijective map. Now it is trivial that $E$ is a homeomorphism.

Example 2.

Let $\Delta$ be the positive self-adjoint operator $-\frac{d^2}{dx^2}$ on the Hilbert space $L^2[0, 2\pi]$ and let $v > 0$. The topological vector space $S_{L^2[0,2\pi],\Delta}$ (see Example 1) has been studied in [EG2] section II.1 case II.d and in [EG1] section 3.1 in the case $v = \frac{1}{2}$. In the following we give another description of the space $S_{L^2[0,2\pi],\Delta}$ for $v > 0$.

Let $G := T = \{ z \in C : |z| = 1 \}$ with the usual multiplication. $T$ is a compact group and we normalize the Haar measure $\mu$ on $G$ such that $\mu(G) = 1$. For $k \in \mathbb{Z}$ define $\gamma_k : G \rightarrow C$

$$\gamma_k(z) := z^k \quad (z \in G).$$

It is well known that $\hat{G} = \{ \gamma_k : k \in \mathbb{Z} \}$ ([HRI] 23.27a) and that $\hat{G}$ is an orthonormal basis for $L^2(G)$. (See [HRI] Lemma 23.19 and [HRII] Lemma 31.4.) Let $F$ be the Fourier transform from $L^2(G)$ onto $L^2(\hat{G})$. Define a self-adjoint operator $A$ on $L^2(G)$ by $A \gamma_k := k^2 \gamma_k$, $k \in \mathbb{Z}$.

Define $V : L^2(G) \rightarrow L^2[0, 2\pi]$

$$(V u)(\alpha) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(e^{i\alpha}) \ (u \in L^2(G), \ \alpha \in [0, 2\pi]).$$

Then $V$ is a unitary map between two Hilbert spaces and $A = V^{-1} \Delta V$. Hence $S_{L^2[0,2\pi],\Delta} = V(S_{L^2(G),\Delta})$.

Let $t > 0$. For every $u \in L^2(G)$ we obtain: $e^{-tA^*} u = e^{-tA^*} \sum_{k \in \mathbb{Z}} (IF u)(\gamma_k) \gamma_k =$

$$\sum_{k \in \mathbb{Z}} (IF u)(\gamma_k) e^{-i|k|^2} \gamma_k = \sum_{k \in \mathbb{Z}} (IF u)(\gamma_k) (\sum_{l \in \mathbb{Z}} e^{-i|l|^2} \gamma_l) \gamma_k =$$

$$\sum_{k \in \mathbb{Z}} (IF [(\sum_{l \in \mathbb{Z}} e^{-i|l|^2} \gamma_l) * u]) (\gamma_k) \gamma_k = (\sum_{l \in \mathbb{Z}} e^{-i|l|^2} \gamma_l) * u,$$ with $*$ the convolution product

between $L^1(G)$ and $L^2(G)$. Define $f_t : G \rightarrow C$

$$f_t(z) := \sum_{l \in \mathbb{Z}} e^{-i|l|^2} z^l \quad (z \in G).$$

Then $f_t \in C(G) \subset L^1(G)$. Let $U$ be the regular representation of $G$ in $L^2(G)$. Then $e^{-tA^*} = U(f_t)$.

Let $C := \{ f_t : t > 0 \}$. Because $\tilde{f}_t(\gamma_k) = e^{-i|k|^2}$ for all $k \in \mathbb{Z}$ and $t > 0$, it is trivial that properties P1' and P3 are satisfied. Property P2 follows from the equality $e^{-tA^*} = U(f_t), \ t > 0$. Then
$SL_2(G,\mathbb{A}^\infty) = SU, C$ as topological vector spaces. Since $SL_2(G,\mathbb{A}^\infty)$ is complete by Example 1 and Theorem 3.12, it follows that property P4 is satisfied by the same Theorem.

In case $\nu = 1/2$ we can give a closed form for $f_t, t > 0$. Let $t > 0$ and $z \in T$. Then

$$f_t(z) = \sum_{k \in \mathbb{Z}} e^{t|k|} z^k$$

$$= \frac{1}{2} + \sum_{k=1}^{\infty} (z e^{-t})^k + \frac{1}{2} + \sum_{k=1}^{\infty} (\bar{z} e^{-t})^k$$

$$= \frac{1}{2} \cdot \frac{1 + z e^{-t}}{1 - z e^{-t}} + \frac{1}{2} \cdot \frac{1 + \bar{z} e^{-t}}{1 - \bar{z} e^{-t}}$$

$$= \frac{1 - e^{-2t}}{1 - \bar{z} e^{-t} - z e^{-t} + e^{-2t}}$$

$$= \frac{\sinh t}{\cosh t - \text{Re} z}.$$

Example 3.

Let $G$ be a locally compact Abelian topological group. Let $U$ be a representation of $G$ in a Hilbert space $H$. Let

$$C := \{ f \in L^1(G) : \hat{f} \in C_c(\hat{G}) \}.$$

THEOREM 4.2.

I. The pair $(C, U)$ satisfies properties P1' and P2''.

II. If $G$ is metrizable then the pair $(C, U)$ satisfies properties P3 and P4'.

Proof.

I. P1'. Let $f, g \in C$. Let $K_1 := \text{supp} \hat{f}$ and $K_2 := \text{supp} \hat{g}$. Then $K := K_1 \cup K_2$ is a compact sub-

set of $\hat{G}$, so by [HRlI] Theorem 31.34 there exists a function $h \in L^1(G)$ such that

$$\hat{h} \in C_c(\hat{G})$$

and $1_K \leq h \leq 1$. Then $h \in C$ and

$$(h * f) = \hat{h} \ast \hat{f} = \hat{f},$$

so $h * f = f$. Similarly $h * g = g$.

P2''. By [HRlI] Theorem 33.12 there exists a net $(f_j)_{j \in J}$ in $C$ such that

$$\lim_{j \to \infty} f_j = g$$

for all $g \in L^1(G)$ and $\|f_j\|_1 = 1$ for all $j \in J$.

II. Suppose $G$ is metrizable. By [HRlI] (24.48) the dual group $\hat{G}$ is $\sigma$-compact, hence there exists a sequence $A_1, A_2, \cdots$ of compact subsets of $\hat{G}$ such that $A_1 \subset A_2 \subset \cdots$ and

$\hat{G} = \bigcup_{n=1}^{\infty} A_n$. Also there exists an open neighbourhood $W$ of the identity of $\hat{G}$ such that $W$ is
compact. Inductively we define a sequence $B_1, B_2, \ldots$ of compact subsets of $\hat{G}$ and a sequence $V_1, V_2, \ldots$ of open subsets of $\hat{G}$ such that $B_1 \subset V_1 \subset B_2 \subset V_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} B_n = \hat{G}$. Let $B_1 := A_1$ and $V_1 := B_1 W$. Then $V_1 \subset B_1 \overline{W}$ is compact, so $\overline{V}_1$ is compact. Let $N \in \mathbb{N}$ and suppose $B_1, B_2, \ldots, B_N, V_1, V_2, \ldots, V_N$ are defined and $\overline{V}_N$ is compact. Since $\overline{V}_N \cup A_{N+1}$ is a compact subset of $\bigcup_{n=1}^{\infty} A_n W$ and all $A_n W$ are open $(n \in \mathbb{N})$, there exist $n \in \mathbb{N}$ such that $\overline{V}_N \cup A_{N+1} \subset A_n W$.

Define $B_{N+1} := A_n \overline{W}$ and $V_{N+1} := B_{N+1} W$.

**P3.** Define $Q_1 := B_1$ and $Q_n := B_n W_{n-1}$ ($n \geq 2$). Let $b_n := n^2 (n \in \mathbb{N})$. Let $f \in C$. Since supp $\hat{f}$ is compact and $V_1, V_2, \ldots$ cover supp $\hat{f}$, there exists $k \in \mathbb{N}$ such that supp $\hat{f} \subset V_k$. Using again [HRH] Theorem 31.34, there exists $g \in C$ such that $1_{B_{k+1}} \preceq \hat{g} \preceq 1$. Let $\delta := 1 + (k+1)^2 \| \hat{f} \|_{\infty}$. Then $n^2 \sup \{ |\hat{f}(\gamma)| : \gamma \in Q_n \} \leq \delta \inf \{ |\hat{g}(\gamma)| : \gamma \in Q_n \}$ for all $n \in \mathbb{N}$.

**P4'.** Let $F \in \text{Bor}(\hat{G}, C)$ and suppose that $F \ast K$ is bounded for all $K \in C^g$. Then $F$ is bounded, since $1_{\hat{G}} \in C^g$. Assertion: supp $F$ is compact.

Proof of the assertion. Suppose supp $F$ is not compact. Let $B_1, B_2, \ldots, V_1, V_2, \ldots$ be as above. Inductively we define a sequence $\gamma_1, \gamma_2, \ldots$ in $\hat{G}$ such that $\gamma_n \notin B_n$ and $F(\gamma_n) \neq 0$ for all $n \in \mathbb{N}$ and $\gamma_i \neq \gamma_j$ for all $i, j \in \mathbb{N}, i \neq j$. Since $\{ \gamma \in \hat{G} : F(\gamma) \neq 0 \}$ is not a subset of the compact set $B_1$, there exists $\gamma_1 \in \hat{G}$ such that $F(\gamma_1) \neq 0$ and $\gamma_1 \notin B_1$. Let $n \in \mathbb{N}$ and suppose $\gamma_1, \ldots, \gamma_n$ are defined. Because $\{ \gamma \in \hat{G} : F(\gamma) \neq 0 \}$ is not a subset of the compact set $B_{n+1} \cup \{ \gamma_1, \ldots, \gamma_n \}$, there exists $\gamma_{n+1} \in \hat{G}$ such that $F(\gamma_{n+1}) \neq 0$ and $\gamma_{n+1} \notin B_{n+1} \cup \{ \gamma_1, \ldots, \gamma_n \}$.

Define $K : \hat{G} \to C$

$$K(\gamma) = \begin{cases} n(F(\gamma_n))^{-1} & \text{if there exists } n \in \mathbb{N} \text{ such that } \gamma = \gamma_n, \\ 0 & \text{if } \gamma \notin \{ \gamma_n : n \in \mathbb{N} \}. \end{cases}$$

Then $K \in \text{Bor}(\hat{G}, C)$. Let $f \in C$. Since $V_1, V_2, \ldots$ is an open cover of supp $\hat{f}$, there exist $n \in \mathbb{N}$, such that supp $f \subset V_n \subset B_{n+1}$. Then

$$|\hat{f} \hat{K}(\gamma)| \leq n \| \hat{f} \|_{\infty} \max \{ (F(\gamma_k))^{-1} : k \in \{1, \ldots, n\} \} < \infty$$

for all $\gamma \in \hat{G}$. Hence $K \in C^g$. But $F \ast K$ is not bounded. Contradiction. This proves the assertion.

Again: there exists $f \in C$ such that $1_{\text{supp}F} \leq \hat{f}$. Then $|F| \leq (1 + \| F \|_{\infty}) |\hat{f}|$.

\[ \square \]

The next theorem shows that not all spaces $S_{U,C}$ are trivial.

**THEOREM 4.3.**

Let $G$ and $C$ as above. The following conditions are equivalent.
I. $G$ is discrete.

II. $S_{U,C} = H$ as sets for every representation $U$ of $G$ in a Hilbert space $H$.

III. $C \ast L^2(G) = L^2(G)$.

Proof.

I $\Rightarrow$ II. Suppose $G$ is discrete. Then $\hat{G}$ is compact. Let $f := 1_{\{e\}}$ with $e$ the identity of $G$. Then $\hat{f} = 1$ and $f \in C$. Let $U$ be a representation of $G$ in a Hilbert space $H$. Then $H = U[\hat{f}] (H) = U(f)(H) \subseteq S_{U,C} \subset H$.

II $\Rightarrow$ III. Let $U$ be the regular representation of $G$ in $L^2(G)$. Then $S_{U,C} = \bigcup_{f \in C} U(f)(L^2(G)) = \bigcup_{f \in C} f \ast L^2(G) = C \ast L^2(G)$. By assumption, $L^2(G) = S_{U,C} = C \ast L^2(G)$.

III $\Rightarrow$ I. Suppose $G$ is not discrete. In order to prove that $C \ast L^2(G) \neq L^2(G)$ we consider two cases.

Case 1. Suppose $G$ is compact. Then $\hat{G}$ is discrete. Because $G$ is not discrete, $g$ is not finite, hence $\hat{G}$ is not finite. So there exists $g \in L^2(\hat{G})$ such that $\{\gamma \in \hat{G} : g(\gamma) \neq 0\}$ is not finite. Let $\mathcal{F}$ be the Fourier transformation from $L^2(G)$ onto $L^2(\hat{G})$. Then $\mathcal{F}^{-1} g \in L^2(G)$. On the other hand, let $f \in C$ and $h \in L^2(G)$. Then $\hat{f} \in C_c(\hat{G})$, so $\{\gamma \in \hat{G} : \hat{f}(\gamma) \neq 0\}$ is finite. Then $\{\gamma \in \hat{G} : \hat{f}(\gamma) \ast (\mathcal{F} h)(\gamma) \neq 0\}$ is finite and $\mathcal{F}(f \ast h) = \hat{f} \ast \mathcal{F} h \neq g$. So $f \ast h \neq \mathcal{F}^{-1} g$ and $\mathcal{F}^{-1} g \notin C \ast L^2(G)$.

Case 2. Suppose $G$ is not compact. Then $C \ast L^2(G) \subset L^2(G) \ast L^2(G) \subset C_0(G)$. We shall prove that $L^2(G) \cap C_0(G) \neq L^2(G)$. Let $e$ be the identity of $G$. Since $G$ is not compact, by [HRI] Example 11.43 there exist a sequence $x_1, x_2, \ldots$ in $G$ and a neighbourhood $V$ of $e$ such that the sets $x_1 V, x_2 V, \ldots$ are pairwise disjoint. Let $\mu$ be the Haar measure on $G$. Since $G$ is nondiscrete and $\mu$ is regular, $0 = \mu(\{e\}) = \inf \{\mu(B) : B \text{ open in } G, e \in B\}$. Hence there exists a sequence $B_1, B_2, \ldots$ of open neighbourhoods of $e$ such that $\mu(B_n) < n^{-4}$ and $B_n \subset V$ for all $n \in \mathbb{N}$. Let $f := \sum_{n=1}^{\infty} n 1_{x_n B_n}$. Then $f \in L^2(G)$ because $\sum_{n=1}^{\infty} n^2 \mu(x_n B_n) = \sum_{n=1}^{\infty} n^2 \mu(B_n) < \infty$ and $(x_n B_n)_{n \in \mathbb{N}}$ are pairwise disjoint. But $f(x_n) = n$ for all $n \in \mathbb{N}$, so $f$ is not bounded and $f \notin C_0(G)$.

THEOREM 4.4.

Let $G$ be a locally compact metrizable Abelian group. Let $C$ be as above and let $U$ be the regular representation of $G$ in $L^2(G)$. The following conditions are equivalent.
I. $G$ is compact.

II. $S_{U,C}$ is nuclear.

III. $S_{U,C}$ is a Montel space.

Proof.

I $\Rightarrow$ II. Suppose $G$ is compact. Then $\hat{G}$ is discrete. Let $f \in C$. Then $\hat{f} \in C_c(\hat{G})$, so $\text{supp} \, \hat{f}$ is compact and hence finite. So $\sum_{\gamma \in \hat{G}} |\hat{f}(\gamma)| < \infty$ and the operator $U(f)$ from $L^2(G)$ into $L^2(G)$ is nuclear. Since $C \subset C \ast C$ (see the proof of property P1'), the space $S_{U,C}$ is nuclear by Theorems 3.16 and 3.18.

II $\Rightarrow$ III. Trivial by Theorems 3.16 I $\Rightarrow$ II and 3.14 III $\Rightarrow$ I.

III $\Rightarrow$ II. Suppose $S_{U,C}$ is a Montel space. Let $V$ be an open neighbourhood of the identity of $G$ such that $\hat{V}$ is compact. By [HRII] Theorem 31.34 there exists $f \in C$ such that $1_{\hat{V}} \leq f \leq 1$. Then the operator $U(f)$ is compact from $L^2(G)$ into $L^2(G)$. (See Theorem 3.14.) Let $\mathcal{F}$ be the Fourier transformation from $L^2(G)$ onto $L^2(\hat{G})$. Then $\mathcal{F}^{-1}(1_{\hat{V}} g : g \in L^2(\hat{G})) \subset U(f)(L^2(G))$ and $\mathcal{F}^{-1}(1_{\hat{V}} g : g \in L^2(\hat{G}))$ is closed, so $N := \dim \{1_{\hat{V}} g : g \in L^2(\hat{G})\} = \dim \mathcal{F}^{-1}(1_{\hat{V}} g : g \in L^2(\hat{G})) < \infty$. Suppose there exist $\gamma_1, \ldots, \gamma_{N+1} \in V$, such that $\gamma_i \neq \gamma_j$ if $i \neq j$. Since $\hat{G}$ is Hausdorff and $V$ is open, by Urysohn's lemma there exist $g_1, \ldots, g_{N+1} \in C_c(\hat{G})$ such that $\gamma_i \in \text{supp} \, g_i \subset V$ ($i \leq N + 1$) and $\text{supp} \, g_i \cap \text{supp} \, g_j = \varnothing$ if $1 \leq i < j \leq N + 1$. Then $g_1, \ldots, g_{N+1}$ are orthogonal in $L^2(\hat{G})$ and non-zero. So $\dim \{1_{\hat{V}} g : g \in L^2(\hat{G})\} \geq N + 1$. Contradiction. Hence $V$ contains only finite many elements. Therefore $\hat{G}$ is discrete and $G$ is compact.

LEMMA 4.5.

Let $G$ be a compact Abelian topological group and let $U$ be the regular representation of $G$ in $L^2(G)$. Let $\mathcal{F}$ be the Fourier transformation from $L^2(G)$ onto $L^2(\hat{G})$. For $F \in \text{Bor}(\hat{G}, C)$ let $M_F$ be the multiplication operator by $F$ on $L^2(\hat{G})$. Then $U[F] = \mathcal{F}^{-1} M_F \mathcal{F}$ for all $F \in \text{Bor}_b(\hat{G}, C)$.

Proof.

Since $G$ is compact, $\hat{G}$ is discrete, so the Haar measure on $\hat{G}$ is the counting measure, which is locally finite.

Let $f \in L^1(G)$. Then $U[\hat{f}] = U(f) = \mathcal{F}^{-1} M_{\hat{f}} \mathcal{F}$.

Let $F_1, F_2, \cdots$ be a uniformly bounded sequence in $\text{Bor}_b(\hat{G}, C)$ and suppose $F(\gamma) := \lim_{n \to \infty} F_n(\gamma)$ exists for every $\gamma \in \hat{G}$. Then by Lebesque's dominated convergence theorem, $\mathcal{F}^{-1} M_F \mathcal{F} = \lim_{n \to \infty} \mathcal{F}^{-1} M_{F_n} \mathcal{F}$. 


Let \( V \) be a subset of \( \hat{G} \). Let
\[
X := \text{clo} \{ \mathbf{L}^{-1} M F \mathbf{L}(L^2(G)) : F \in C_0(\hat{G}), \ 0 \leq F \leq 1_V \}.
\]

Assertion: \( \mathbf{L}^{-1} M 1_V \mathbf{L} \) is the projection of \( L^2(G) \) onto \( X \). Proof of the assertion. Let \( X_0 := \text{clo} \{ \mathbf{L}^{-1} M 1_V \mathbf{L}(L^2(G)) \} \). Then \( X_0 \) is closed in \( L^2(G) \) and \( X \subset X_0 \). Let \( f \in X_0 \) and let \( \varepsilon > 0 \). Let \( g := \mathbf{L} f \). Since the measure on \( \hat{G} \) is the counting measure, there exists \( h \in L^2(G) \) such that
\[
\| h - g \| < \varepsilon, \text{ supp } h = \{ \gamma \in \hat{G} : h(\gamma) \neq 0 \} \subset \text{ supp } g \subset V \text{ and supp } h \text{ is finite}. \]
Let \( F := 1_{\text{ supp } h} \). Then \( F \in C_c(\hat{G}) \subset C_0(\hat{G}) \), \( 0 \leq F \leq 1_V \) and \( \mathbf{L}^{-1} M F \mathbf{L}(\mathbf{L}^{-1} h) - f \| = \| h - g \| < \varepsilon \). So \( f \in X \). This proves the assertion.

The remaining part of the proof is similar to the proof of Theorem 2.6. \( \Box \)

The following theorem is not vacuous. Let \( R_d \) be the Abelian group \( R \) with discrete topology. Then the dual group \( \hat{R}_d \) is a compact, non-metrizable, Abelian group (See [HRI] (24.48).)

**THEOREM 4.6.**

Let \( G \) be a compact, not metrizable, Abelian group, let \( C \) be as above and let \( U \) be the regular representation of \( G \) in \( L^2(G) \). Then there exists \( \Phi \in T_{U,C} \) such that \( \Phi \not\in \bigcup_{F \in C^*} U[F] \ast L^2(G) \).

**Proof.**

Let \( \mathbf{L} \) be the Fourier transformation from \( L^2(G) \) onto \( L^2(\hat{G}) \). For all \( f \in C \), \( \tilde{f} \in C_c(\hat{G}) \subset L^2(\hat{G}) \), so \( f \in L^2(G) \). Define
\[
\Phi : C \to L^2(G)
\]
\[
\Phi(f) := \tilde{f} \quad (f \in C).
\]
It is trivial that \( \Phi \in T_{U,C} \).

Suppose there exist \( F \in C^* \) and \( g \in L^2(G) \) such that \( \Phi = U[F] \ast g \). Since \( G \) is compact, we have \( \hat{G} \) is discrete and the Haar measure on \( \hat{G} \) is equal to the counting measure. Let \( \gamma \in \hat{G} \). By [HRI] Theorem 31.34 there exists \( f \in C \) such that \( \hat{f}(\gamma) = 1 \). Then \( \hat{f} = \Phi(f) = U[F \hat{f}] g \), so by Lemma 4.5 we obtain that
\[
1 = \hat{f}(\gamma) = [\mathbf{L}(U[F \hat{f}]) g](\gamma) = (F \hat{f} \ast \mathbf{L} g)(\gamma) = F(\gamma) \ast (\mathbf{L} g)(\gamma). \]
So
\[
(\mathbf{L} g)(\gamma) \neq 0 \text{ for all } \gamma \in \hat{G}. \text{ But } G \text{ is not metrizable, so } \hat{G} \text{ is not } \sigma\text{-compact. (See [HRI] (24.48).)} \]
Hence \( \hat{G} \) is not countable and \( L^2(\hat{G}) \). Contradiction. \( \Box \)

**COROLLARY 4.7.**

Let \( G \), \( C \), \( U \) be as in Theorem 4.6. Then \( (S_{U,C} \sigma_{\text{ind}}) \neq (S_{U,C} \sigma_{\text{proj}}) \).

**Proof.**

Theorem 2.21. \( \Box \)
COROLLARY 4.8.
Let $G, C, U$ as in Theorem 4.6. Then the pair $(C, U)$ does not satisfy property $P3$. 

Proof.
Theorem 2.30. 

Example 4.
Let $D(\mathbb{R}) := \{f \in C_c(\mathbb{R}) : f$ is infinitely differentiable $\}$. Let $G = \mathbb{R}, U$ is representation of $\mathbb{R}$ and let $C := \{f : f \in D(\mathbb{R})\}$. Since for all compact intervals $K \subset \mathbb{R}$ and all open intervals $U \subset \mathbb{R}$ such that $K \subset U,$ there exists $f \in D(\mathbb{R})$ such that $\int_K f = 1, \int_{\mathbb{R} \setminus U} f = 0$ and $f(\mathbb{R}) \subset [0, 1)$, it follows similarly to Example 3 that properties $P1', P2'$, $P3$ and $P4'$ are satisfied.

Example 5.
Let $G$ be a locally compact Abelian topological group and let $U$ be a representation of $G$ in a Hilbert space $H.$ Let $w$ be a weight function on $G$, i.e. a measurable locally bounded function from $G$ into $[1, \infty)$ such that $w(xy) \leq w(x)w(y)$ for all $x, y \in G.$ Let $L_w^1(G) := \{f \in L^1(G) : f \cdot w \in L^1(G)\}$ be the Beurling algebra defined by $w.$ $L_w^1(G)$ is a subalgebra of $L^1(G).$ Let $C := L_w^1(G).$ Then the pair $(C, U)$ satisfies properties $P1'$ and $P2''$. Indeed. The Banach algebra $L_w^1(G)$ contains a bounded approximate left unit $(g_\alpha)_{\alpha \in M}$ for $L_w^1(G).$ (See [Rei] chapter 3, § 7.2.) Hence by [HR] Corollary 32.24, the pair $(C, U)$ satisfies property $P1'$. Obviously $C_c(G) \subset L_w^1(G)$, so the pair $(C, U)$ has property $P2''$. So by chapter I the space $S_{U, C}$ is defined.

The Hilbert space $H$ is a left Hilbert $L_w^1(G)$-module together with the map $(f, x) \mapsto U(f)x$ from $L_w^1(G) \times H$ into $H$. The range of this map is dense in $H$ because it is equal to $S_{U, C}.$ Because $(g_\alpha)_{\alpha \in M}$ is a bounded approximate left unit for $L_w^1(G)$, actually this range is closed in $H.$ (See [HR] Theorem 32.22.) Hence $S_{U, C} = H$ as sets. Then by Theorem 1.19 $(S_{U, C}, \sigma_{ind}) = H$ as topological vector spaces.

Example 6.
Let $C^1(\mathbb{R})$ be the set of all continuous differentiable functions on $\mathbb{R}.$ Let $\Lambda$ be the set of all functions $f \in C^1(\mathbb{R})$ such that

$f > 0,$

$\forall \alpha > 0 \ [\sup \{f(x) e^{\alpha |x|} : x \in \mathbb{R}\} < \infty],$ 

$\forall \alpha > 0 \ [\sup \{|f'(x)| e^{\alpha |x|} : x \in \mathbb{R}\} < \infty].$
Here the prime denotes differentiation. For \( f \in L^1(\mathbb{R}) \) define \( \hat{f} : \mathbb{R} \to \mathbb{C} \)

\[
\hat{f}(y) := \int_{\mathbb{R}} f(x) e^{-ixy} \, dx \quad (y \in \mathbb{R}).
\]

Let \( U \) be a representation of \( \mathbb{R} \) in a Hilbert space \( H \). Let

\[
C := \{ \mathbb{R} \}
\]

We shall prove that the pair \((C , U)\) satisfies properties P1', P2'' and P3. We need two lemmas.

**Lemma 4.9.**

Let \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap C^1(\mathbb{R}) \) and suppose \( f' \in L^2(\mathbb{R}) \). Then \( f \in (L^1(\mathbb{R}))' \).

**Proof.**

Because \( f' \in L^2(\mathbb{R}) \), the function \( x \mapsto x \hat{f}(x) \) is an element of \( L^2(\mathbb{R}) \). So \( \hat{f} \in L^1(\mathbb{R}) \). By the inversion theorem, \( f \in (L^1(\mathbb{R}))' \).

Clearly every element of \( \Lambda \) satisfies the conditions of Lemma 4.9.

**Lemma 4.10.**

I. Let \( b > 0 \) and define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = e^{-bx^2} \), \( x \in \mathbb{R} \). Then \( f \in \Lambda \).

II. Let \( f, g \in \Lambda \). Then \( f + g \in \Lambda \).

III. Let \( f \in \Lambda \) and \( b > 0 \). Define \( g : \mathbb{R} \to \mathbb{R} \) by \( g(x) = f(x) e^{b(1+x^2)/2} \), \( x \in \mathbb{R} \). Then \( g \in \Lambda \).

IV. Let \( f, g \in \Lambda \). Then there exist \( h \in \Lambda \) and \( f_1, g_1 \in (L^1(\mathbb{R}))' \) such that \( f = h \ast f_1 \) and \( g = h \ast g_1 \).

**Proof.**

I. Trivial.

II. Trivial, because \( \left| \frac{(f+g')^\prime}{f+g} \right| \leq \left| \frac{f'}{f} \right| + \left| \frac{g'}{g} \right| \).

III. Note that \( g'(x) = e^{b(1+x^2)/2} [f'(x) + bx (1 + x^2)^{-1/2} f(x)] \) for all \( x \in \mathbb{R} \). Now it easily follows that \( g \in \Lambda \).

IV. Let \( a > 0 \) be so large that \( \int_{\mathbb{R}} \left| \frac{f(x)}{f(x)} e^{-ax^2} \right|^2 \, dx < \infty \) and \( \int_{\mathbb{R}} \left| \frac{g'(x)}{g(x)} e^{-ax^2} \right|^2 \, dx < \infty \).

Define \( h : \mathbb{R} \to \mathbb{R} \).
\[ h(x) := [f(x) + g(x)] e^{\alpha(1+x^2)^{\beta}} \quad (x \in \mathbb{R}). \]

By II and III, \( h \in \Lambda \). Let \( f_1 := \frac{f}{h} \) and \( g_1 := \frac{g}{h} \). We shall prove that \( f_1 \in (L^1(\mathbb{R}))^* \), similar results hold for \( g_1 \).

For all \( x \in \mathbb{R} \) we have

\[ f_1(x) = \frac{f(x)}{f(x) + g(x)} e^{-\alpha(1+x^2)^{\beta}}, \]

so \( f_1 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap C^1(\mathbb{R}) \). Using the identity

\[ \left[ \frac{f}{f+g} \right]' = \left[ \frac{g'}{g} - \frac{(f+g)'}{f+g} \right] g \]

we obtain that

\[ |f_1'(x)| \leq \left| e^{-\alpha(1+x^2)^{\beta}} \left[ \frac{g(x)}{g(x)} - \frac{f'(x) + g'(x)}{f(x) + g(x)} \right] \frac{g(x)}{f(x) + g(x)} - \frac{f(x)}{f(x) + g(x)} \alpha(1+x^2)^{\beta} \right| \]

\[ \left| \frac{g'(x)}{g(x)} e^{-\alpha x^2} + \frac{f'(x)}{f(x)} e^{-\alpha x^2} \right| \]

for all \( x \in \mathbb{R} \). So \( f_1' \in L^2(\mathbb{R}) \). By Lemma 4.9, \( f_1 \in (L^1(\mathbb{R}))^* \).

**THEOREM 4.11.**

The pair \((C, U)\) satisfies properties P1', P2'' and P3.

**Proof.**

Property P1' follows by Lemma 4.10 IV and the note following Lemma 4.9. By Lemma 4.10 I the pair \((C, U)\) has property P2''. For \( n \in \mathbb{N} \) define \( b_1 := n^2 \) and \( Q_n := \{ y : n - 1 \leq |y| < n \} \), with \( y \) as in Example I. Then property P3 follows by Lemma 4.10 III.

By Stone's theorem ([Wei] Theorem 7.38) there exists a unique self-adjoint operator \( A \) on \( H \) such that \( U_x = e^{-ixA} \) for all \( x \in \mathbb{R} \). Similar to the proof of Theorem 4.1 we obtain that \( U(f) = f(A) \) for all \( f \in C \). So by the second condition on the set \( \Lambda \), \( U(f)(H) \subset D(e^{t|A|}) \) for all \( f \in C \) and \( t > 0 \).

Hence \( S_{U,C} \subset \bigcap_{t>0} D(e^{t|A|}) \). In case \( A \) is an unbounded operator, we see that \( S_{U,C} \neq H \) as sets. We shall prove that \( S_{U,C} = \bigcap_{t>0} D(e^{t|A|}) \) as sets. Having an eye for property P4 we prove a little bit more.

Let \( S := \bigcap_{n=0}^{\infty} D((e^{t|A|})^n) \). The topology for \( S \) is the locally convex Hausdorff topology generated by the seminorms \( p_n \), defined by \( p_n(\phi) := \| e^{n|A|} \phi \|, \phi \in S, n \in \mathbb{N} \cup \{0\} \).

**THEOREM 4.12.**

\( S_{U,C} = S \) as locally convex topological vector spaces.
Proof.
Obviously $S_{U,C}$ is a subspace of $S$ and the inclusion map from $S_{U,C}$ into $S$ is continuous. Clearly
$S$ is metrizable. So the theorem follows if for all $\phi_1, \phi_2, \cdots \in S$ with $\lim_{n \to \infty} \phi_n = 0$ in $S$
there exists $g \in C$ such that $\phi_n \in R_g$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \phi_n = 0$ in $R_g$.

Let $\phi_1, \phi_2, \cdots \in S$ and suppose $\lim_{n \to \infty} \phi_n = 0$ in $S$. By the spectral theorem we may assume that
there exist a measure space $(Y, B, \mu)$ and a real valued measurable function $h$ on $Y$ such that
$H = L^2(\mu)$ and $A$ is equal to the multiplication operator by $h$ on $L^2(\mu)$. Inductively we define a
sequence $N_1, N_2, \cdots$ of real numbers such that $N_{k+1} \geq N_k + 2$ for all $k \in \mathbb{N}$ and
\[
\| 1_{[N_k, \infty)} (A^1) e^{k^1A^1} \phi_n \|^2 \leq k^{-2}
\]
for all $k \in \mathbb{N}$, $k \geq 2$ and all $n \in \mathbb{N}$. Let $N_1 := 0$. Let $k \in \mathbb{N}$, $k \geq 2$ and suppose $N_{k-1}$ is defined.
There exists $M \in \mathbb{N}$ such that $p_k(\phi_n) \leq k^{-1}$ for all $n \in \mathbb{N}$ with $n \geq M$. Since
\[
\lim_{N \to \infty} 1_{[N, \infty)} (A^1) e^{k^1A^1} \phi_n = 0 \text{ in } H \text{ for all } n \in \mathbb{N}, \text{ there exists } N_k \geq N_{k-1} + 2 \text{ such that }
\| 1_{[N_k, \infty)} (A^1) e^{k^1A^1} \phi_n \| \leq k^{-1} \text{ for all } n \in \{1, \cdots, M\}.
\]
Let $\tau$ be an infinite differentiable function from $\mathbb{R}$ onto $[0, 1]$ such that $\tau(x) = 0$ for all $x \leq 0$ and
$\tau(x) = 1$ for all $x \geq 1$.

Define $f : \mathbb{R} \to \mathbb{R}$ by
\[
f(x) = e^{-(k-1+\tau(|x|-N_k)) |x|} \quad (k \in \mathbb{N}, \ 1 \leq 1 \in [N_k, N_{k+1}]).
\]
Assertion: $f \in A$. Proof of the assertion. For all $k \in \mathbb{N}$ and $x \in [N_k, N_{k+1})$ we have
\[
f'(x) = e^{-(k-1+\tau(|x|-N_k))} [1 - k - \tau(x-N_k) - \tau'(x-N_k) x]
\]
(also if $x = N_k$ and $f'(0) = 0$, so $f \in C^1(\mathbb{R})$. Clearly $f > 0$. For all $x \in \mathbb{R}$, $1 \in [N_k, N_{k+1})$, with $k \in \mathbb{N}$ we obtain that $\| f'(x) \| \leq [k+\| \tau' \|_{\infty} \ 1 \ 1] f(x) \leq [k+\| \tau' \|_{\infty} \ 1 \ 1] e^{-(k-1)|x|}$, so
indeed $f \in A$. This proves the assertion.

Let $n \in \mathbb{N}$. Then
\[
\int \frac{1}{Y} (f \circ h)^{-1} \phi_n \|^2 \ dm = \sum_{k=1}^{\infty} \int _{Y} \left| f \circ h(y) \right|^{-1} 1_{(x \in Y : N_k \leq |h(x)| < N_{k+1} \} (y) \phi_n(y) \|^2 \ dm
\]
\[
\leq \sum_{k=1}^{\infty} \int _{Y} e^{k^1h(y)} 1_{(x \in Y : N_k \leq |h(x)| \} (y) \phi_n(y) \|^2 \ dm
\]
\[
= [p_1(\phi_n)]^2 + \sum_{k=2}^{\infty} \| 1_{[N_k, \infty)} (A^1) e^{k^1A^1} \phi_n \|^2.
\]
So
\[
\int \frac{1}{Y} (f \circ h)^{-1} \phi_n \|^2 \ dm \leq [p_1(\phi_n)]^2 + \sum_{k=2}^{\infty} k^{-2} < \infty \quad \text{and} \quad (f \circ h)^{-1} \phi_n \in L^2(\mu).
\]
Let $\xi_n := (f \circ h)^{-1} \phi_n$. Then $\phi_n = (f \circ h) (f \circ h)^{-1} \phi_n = (f \circ h) \xi_n = f(A) \xi_n$ for all $n \in \mathbb{N}$.
Let $g \in C$ be such that $\hat{g} = f$. Then $\phi_n = \hat{g}(A) \xi_n = U(g) \xi_n \in R_g$ for all $n \in \mathbb{N}$. We show that
\[ \lim_{n \to \infty} \phi_n = 0 \text{ in } R_g. \]

Let $\varepsilon > 0$. There exists $K \in \mathbb{N}$ such that
\[ \sum_{k=K+1}^{\infty} k^{-2} \leq \varepsilon. \]
There exists $M \in \mathbb{N}$ such that
\[ |p_k(\phi_n)|^2 \leq \varepsilon K^{-1} \quad \text{for all } k \in \{1, \cdots, K\} \text{ and all } n \geq M. \]
Then for all $n \geq M$ we have
\[ \| \phi_n \|^2 = \int |(f \circ h)^{-1} \phi_n|^2 \, dm \]
\[ \leq |p_1(\phi_n)|^2 + \sum_{k=2}^{\infty} \| 1_{[N_k, \infty)} (1A1) e^{k|A|1} \phi_n \|^2 \]
\[ \leq |p_1(\phi_n)|^2 + \sum_{k=2}^{K} |p_k(\phi_n)|^2 + \sum_{k=K+1}^{\infty} \| 1_{[N_k, \infty)} (1A1) e^{k|A|1} \phi_n \|^2 \]
\[ \leq \varepsilon K^{-1} + \sum_{k=2}^{K} \varepsilon K^{-1} + \varepsilon \]
\[ = 2 \varepsilon. \]

This proves the theorem. \( \Box \)

**COROLLARY 4.13.**

The pair $(C, U)$ satisfies property P4.

**Proof.**

Since all operators $e^{n|A|1}, n \in \mathbb{N} \cup \{0\}$ are closed, the space $S$ is complete. By Theorem 3.12 \( \Rightarrow I \), the pair $(C, U)$ has property P4. \( \Box \)

**REMARK.** In [EGK] Example 3, the space $\tau(H, |A|1)$ is defined by
\[ \tau(H, |A|1) = \cap_{n \in \mathbb{N}} D(e^{n|A|1}). \]
So $\tau(H, |A|1) = S_{U,C}$. Then $\sigma(H, |A|1)$ is isomorphic with $T_{U,C}$ as locally convex topological vector spaces.
Chapter V. Continuous linear maps

In this chapter we characterize continuous linear maps from an $S$-space or a $T$-space into an $S$-space or a $T$-space.

**THEOREM 5.1.**

Let $G$ be a locally compact Abelian group, $U$ a representation of $G$ in a Hilbert space $H$ and let $C \subset L^1(G)$. Suppose properties P1 and P2 are satisfied. Let $X$ be a locally convex separated topological vector space and let $E : S_{U,C} \to X$ be a linear map. The following conditions are equivalent.

I. $E$ is continuous.

II. The map $E \circ U(f) \mid_{N_f^f}$ from $N_f^f$ into $X$ is continuous for all $f \in C$.

III. The map $E \circ U(f)$ from $H$ into $X$ is continuous for all $f \in C$.

IV. For all null sequences $f_1, f_2, \cdots$ in $S_{U,C}$ we have $(E f_n)_{n \in \mathbb{N}}$ is a null sequence in $X$.

**Proof.**

The equivalence of I, II and III follows from [Wil] Theorem 13-1-8 and the definition of $\sigma_{ind}$. Since $(S_{U,C}, \sigma_{ind})$ is bornological, I and IV are equivalent. (See [Sch] Theorem II 8.3.)

**THEOREM 5.2.**

Let $G$ be a locally compact Abelian group, $U$ a representation of $G$ in a Hilbert space $H$ and let $C \subset L^1(G)$. Suppose properties P1-P3 are satisfied. Let $i$ be the identity map from $S_{U,C}$ into $H$. Let $X$ be a Fréchet space and let $E$ be a linear map from $X$ into $S_{U,C}$. The following conditions are equivalent.

I. $E$ is continuous from $X$ into $S_{U,C}$.

II. $i \circ E$ is continuous from $X$ into $H$.

III. $U[F] \circ i \circ E$ is continuous from $X$ into $H$, for all $F \in C^\delta$.

**Proof.**

I $\Rightarrow$ II. Trivial since $i$ is continuous. (Lemma 1.6.)

II $\Rightarrow$ III. Let $F \in C^\delta$. The map $U[F] \circ i \circ E$ has a closed graph. Indeed, let $x_1, x_2, \cdots \in X$, $x \in X$, $y \in H$ and suppose $\lim_{n \to \infty} (x_n, U[F](i E x_n)) = (x, y)$ in $X \times H$. Then by assumption, $\lim_{n \to \infty} i E x_n = i E x$ in $H$, so $\lim_{n \to \infty} (i E x_n, U[F](i E x_n)) = (i E x, y)$ in $H \times H$. Since $U[F]$ is a closed operator, we obtain $y = U[F](i E x)$ and $U[F] \circ i \circ E$ has a closed graph. By the closed graph theorem ([Wil] Theorem 5-3-1), $U[F] \circ i \circ E$ is continuous from $X$ into $H$. 

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III \Rightarrow I. For all \( F \in C^g \) the map \( s_F \circ E \) from \( X \) into \( J \) is continuous. Since \( \sigma_{\text{ind}} = \sigma_{\text{proj}} \), the map \( E \) is continuous. 

From now on we do not write explicitly the map \( i \).

For \( j \in \{1, 2\} \) let \( G_j \) be a locally compact Abelian group, \( U_j \) a representation of \( G_j \) in a Hilbert space \( H_j \) and let \( C_j \subset L^1(G_j) \) be a set. Suppose properties P1-P3 are satisfied for the pair \((C_j, U_j)\). 

Notation: we abbreviate \( S_j := (S \cup C_j, \sigma_{\text{ind}}) \) and \( T_j := (T \cup \sigma_{\text{ind}}) \).

THEOREM 5.3.

Let \( E : S_1 \to S_2 \) be a linear map. The following conditions are equivalent. (Cf. Theorems 5.1 and 5.2.)

I. \( E \) is continuous.

II. The map \( E \circ U_1(f) \) from \( H_1 \) into \( H_2 \) is continuous for all \( f \in C_1 \).

III. The map \( U_2[F] \circ E \circ U_1(f) \) from \( H_1 \) into \( H_2 \) is continuous for all \( f \in C_1 \) and \( F \in C^g_{2p} \).

IV. There exists a dense linear subspace \( D \subset H_2 \) such that for all \( y \in D \) the map \( l_y : S_1 \to C \)

\[
l_y(\phi) := \langle E\phi, y \rangle \quad (\phi \in S_1).
\]

If, in addition, the pair \((C_2, U_2)\) satisfies property P4, then we have the following fifth equivalent condition.

V. For all \( f \in C_1 \) there exists \( g \in C_2 \) such that \( E \circ U_1(f)(H_1) \subset U_2(g)(H_2) \) and \( \Omega_{\sigma_{\text{ind}}} \circ E \circ U_1(f) \) from \( H_1 \) into \( H_2 \) is continuous.

Proof. 

II \iff I \iff III. Theorems 5.1 and 5.2.

I \Rightarrow IV. Take \( D = H_2 \).

IV \Rightarrow II. Let \( f \in C \). Define \( A : H_1 \to H_2 \)

\[
A x := E U_1(f) x \quad (x \in H_1).
\]

Let \( y \in D \). Then \( l_y \in S_1^\prime \), hence there exists \( \Phi_y \in T_1 \) such that \( l_y(\phi) = \langle \phi, \Phi_y \rangle \) for all \( \phi \in S_1 \). Then for all \( x \in H_1 \) we have \( (x, \Phi_y(f)) = \langle U_1(f)x, \Phi_y \rangle = l_y(U_1(f)x) = \langle E U_1(f)x, y \rangle = \langle A x, y \rangle \). So \( y \in D(A^*) \). Therefore \( D \subset D(A^*) \) and \( D(A^*) \) is dense in \( H_2 \). So \( A \) is closable and because \( D(A) = H_1 \) is closed, we conclude that \( A \) is a closed operator and \( A \) is continuous.

III \Rightarrow V. Let \( f \in C_1 \). Let \( B := \{x \in H_1 : \|x\| \leq 1\} \). For all \( F \in C^g_{2p} \), the set

\[
\{s_F(E U_1(f)x) : x \in B\} = \{\|U_2[F] E U_1(f)x\| : x \in B\}
\]

is bounded, so \( E U_1(f)(B) \) is bounded in \( S_2 \). By Theorem 3.10 there exist \( g \in C_2 \) and a bounded set \( B_2 \subset N_{\|\|}^1 \) such that \( E U_1(f)(B) = U_2(g)(B_2) \). Then \( E U_1(f)(H_1) \subset U_2(g)(H_2) \) and

\[
E U_1(f)(H_1) \subset U_2(g)(H_2)
\]
\[ \Omega_{g}^{-1} E U_{1}(f) (B) = B_{2}. \] So \( \Omega_{g}^{-1} E U_{1}(f) \) is continuous from \( H_{1} \) into \( H_{2} \).

**V \Rightarrow I.** Let \( f \in C_{1}. \) Let \( g \in C_{2} \) such that \( E U_{1}(f) (H_{1}) \subset U_{2}(g) (H_{2}) \) and \( \Omega_{g}^{-1} E U_{1}(f) \) from \( H_{1} \) into \( H_{2} \) is continuous. Since \( \Omega_{g} \) is a continuous map from \( N_{g}^{1} \) into \( S_{2} \), the map \( E U_{1}(f) \) from \( H_{1} \) into \( S_{2} \) is continuous. By Theorem 5.1 \( \Rightarrow I, \) \( E \) is continuous. \( \Box \)

**COROLLARY 5.4.**

Let \( E \) a densely defined operator from \( H_{1} \) into \( H_{2} \). Suppose \( S_{1} \subset D(E) \) and for all \( f \in C_{1} \) the operator \( E \circ U_{1}(f) \) from \( H_{1} \) into \( H_{2} \) is continuous and \( E U_{1}(f) (H_{1}) \subset S_{2} \). Then \( E \mid_{S_{1}} \) from \( S_{1} \) into \( S_{2} \) is continuous.

**Proof.**

Theorem 5.3 \( \Rightarrow I. \) \( \Box \)

**COROLLARY 5.5.**

Let \( E \) be a densely defined closable operator from \( H_{1} \) into \( H_{2} \). Suppose \( S_{1} \subset D(E) \) and \( E(S_{1}) \subset S_{2} \). Then \( E \mid_{S_{1}} \) from \( S_{1} \) into \( S_{2} \) is continuous.

**Proof.**

Since \( E \) is closable, \( D(E^*) \) is dense in \( H_{2} \). Let \( f \in C_{1} \). By [Wei] Theorem 4.19a, \( U_{1}(f)^* E^* \subset (E U_{1}(f))^* \), so \( D(E^*) = D(U_{1}(f)^* E^*) \subset D((E U_{1}(f))^*) \). Hence \( (E U_{1}(f))^* \) is densely defined and the operator \( E U_{1}(f) \) from \( H_{1} \) into \( H_{2} \) is closable. But then \( E U_{1}(f) \) is closed and \( E U_{1}(f) \) is continuous from \( H_{1} \) into \( H_{2} \). By Theorem 5.3 \( \Rightarrow I, \) the map \( E \mid_{S_{1}} \) from \( S_{1} \) into \( S_{2} \) is continuous. \( \Box \)

**THEOREM 5.6.**

Let \( E : S_{1} \rightarrow T_{2} \) be a linear map. The following conditions are equivalent. (Cf. Theorem 5.1.)

I. \( E \) is continuous.

II. For all \( f \in C_{1} \) and \( g \in C_{2} \) the map \( A \) is continuous, with \( A : H_{1} \rightarrow H_{2} \)

\[ A x := [E(U_{1}(f)x)](g) \quad (x \in H_{1}). \]

III. For all \( f \in C_{1} \) there exists \( F \in C_{2}^{\|} \) such that \( E U_{1}(f) (H_{1}) \subset U_{2}[F] \ast H_{2} \) and \( (U_{2}[F] \ast)^{-1} E U_{1}(f) \) is continuous from \( H_{1} \) into \( H_{2} \). Here, \( (U_{2}[F] \ast)^{-1} \) is the inverse map of the map \( U_{2}[F] \ast \) from \( H_{2} \) onto \( U_{2}[F] \ast H_{2} \).

IV. For all \( \psi \in S_{2} \) the map \( l_{\psi} \) is continuous, with \( l_{\psi} : S_{1} \rightarrow \mathbb{C} \)

\[ l_{\psi}(\phi) := \langle \psi, E \phi \rangle \quad (\phi \in S_{1}). \]

**Proof.**
I \Rightarrow II. Let \( f \in C \) and \( g \in C_2 \). The map \( U_1(f) \) from \( H_1 \) into \( S_1 \) and the seminorm \( t_e \) from \( T_2 \) into \( R \) are continuous, so II follows.

II \Rightarrow III. Let \( f \in C_1 \). Let \( B := \{ x \in H_1 : \| x \| \leq 1 \} \). For all \( g \in C_2 \), the set 
\[ \{ t_e(E U_1(f)x) : x \in B \} = \{ \| E(U_1(f)x)(g) \| : x \in B \} \] is bounded, so \( E U_1(f)(B) \) is bounded in \( T_2 \). By Theorem 2.30 there exist \( F \in C_2^p \) and a bounded set \( B_0 \subset H_2 \) such that \( E U_1(f)(B) = U_2[F] \cdot B_0 \). Then \( E U_1(f)(H_1) \subset U_2[F] \cdot H_2 \) and \( (U_2[F] \cdot)^{-1} E U_1(f)(B) = B_0 \). So \( (U_2[F] \cdot)^{-1} E U_1(f) \) is continuous from \( H_1 \) into \( H_2 \).

III \Rightarrow I. Let \( f \in C_1 \). Let \( F \in C_2^p \) as in III. Since \( U_2[F] \) from \( H_2 \) into \( T_2 \) is continuous (Lemma 2.16 VI), the map \( E \circ U_1(f) = (U_2[F] \cdot)(U_2[F] \cdot)^{-1} E U_1(f) \) is continuous from \( H_1 \) into \( T_2 \). By Theorem 5.1 III \Rightarrow I, the map \( E \) is continuous.

I \Rightarrow IV. Trivial.

IV \Rightarrow I. By Theorem 1.18 we obtain that \( l \circ E \) is continuous for all \( l \in T_2' \). Since \( S_1 \) is bornological, the map \( E \) is continuous.

\[ \square \]

THEOREM 5.7.

Let \( E : T_1 \rightarrow S_2 \) be a linear map. Suppose the pair \( (C_1, U_1) \) satisfies property P4. The following conditions are equivalent. (Cf. Theorem 5.2.)

I. \( E \) is continuous.

II. There exists a dense linear subspace \( D \subset H_2 \) such that for all \( y \in D \) the map \( l_y \) is continuous, with \( l_y : T_1 \rightarrow C \)
\[ l_y(\Phi) := (E \Phi, y) \quad (\Phi \in T_1). \]

III. The map \( U_2[K] \circ U_1[F] \) from \( H_1 \) into \( H_2 \) is continuous for all \( F \in C_1^p \) and \( K \in C_2^p \).

If, in addition, the pair \( (C_2, U_2) \) satisfies property P4, then we have the following fourth equivalent condition.

IV. For all \( F \in C_1^p \) there exists \( f \in C_2 \) such that \( E(U_1[F] \cdot H_1) \subset U_2(f)(H_2) \) and \( \Omega_f^1 E(U_1[F] \cdot) \) from \( H_1 \) into \( H_2 \) is continuous.

Proof.

I \Rightarrow II. Take \( D := H_2 \).

II \Rightarrow III. Let \( F \in C_1^p \). Define \( A : H_1 \rightarrow H_2 \)
\[ Ax := E(U_1[F] \cdot x) \quad (x \in H_1). \]

Let \( y \in D \). Then \( l_y \in T_1' \), hence there exists \( \phi_y \in S_1 \) such that \( l_y(\Phi) = \langle \phi_y, \Phi \rangle = \langle y, U_1[F] \phi_y \rangle = \langle \Phi(U_1[F] \phi_y), x \rangle = \langle U_1[F] \phi_y, U_1[F] \cdot x \rangle = l_y(U_1[F] \cdot x) = (E(U_1[F] \cdot x), y) = (Ax, y) \). So \( y \in D(A^*) \). Therefore \( D \subset D(A^*) \) and \( D(A^*) \) is dense in \( H_2 \). Then \( A \) is closable and \( A \) is closed hence continuous. Let \( K \in C_2^p \). By Theorem 5.2 II \Rightarrow III, the operator
U_2[K] E U_1[F] \centerdot \text{ from } H_1 \text{ into } H_2 \text{ is continuous.}

III \Rightarrow \text{ I. By Theorem 5.2 III } \Rightarrow \text{ I, } E \circ U_1[F] \centerdot \text{ is continuous from } H_1 \text{ into } S_2 \text{ for all } F \in C^f_{cp}.

Hence ([Wil] Theorem 13-1-8) E \text{ is continuous from } (T_{U_1,C_1}, \tau_{\text{ind}}) \text{ into } S_2. \text{ By Theorem 3.4 I } \Rightarrow \text{ II, } (T_{U_1,C_1}, \tau_{\text{ind}}) = T_1, \text{ so } E \text{ is continuous.}

III \Rightarrow IV. \text{ Let } F \in C^f_{cp}. \text{ Let } B := \{x \in H : \|x\| \leq 1\}. \text{ For all } K \in C^f_{2p}, \text{ the set }
\{k_\varepsilon(E(U_1[F] \centerdot x)) : x \in B\} = \{\|U_2[K] E U_1[F] \centerdot x\| : x \in B\} \text{ is bounded, so }
E(U_1[F] \centerdot B) \text{ is bounded in } S_2. \text{ By Theorem 3.10 there exist } f \in C_2 \text{ and a bounded set }
B_0 \subset N^f_1 \text{ such that } E(U_1[F] \centerdot B) = U_2(f)(B_0). \text{ Then } E(U_1[F] \centerdot H_1) \subset U_2(f)(H_2)
\text{ and } \Omega_f^{-1} E(U_1[F] \centerdot B) = B_0. \text{ So } \Omega_f^{-1} E U_1[F] \centerdot \text{ is continuous from } H_1 \text{ into } H_2.

IV \Rightarrow III. \text{ Trivial, since the map } \Omega_f \text{ is a continuous map from } N^f_1 \text{ into } S_2 \text{ for all } f \in C_2.

THEOREM 5.8.

Let \( E : T_1 \rightarrow T_2 \) be a linear map. Suppose the pair \((C_1, U_1)\) satisfies property P4. The following conditions are equivalent.

I. \( E \) is continuous.

II. \text{ For all } F \in C^f_{cp} \text{ and } f \in C_2 \text{ the map } A \text{ is continuous, with } A : H_1 \rightarrow H_2
\quad A x := [E(U_1[F] \centerdot x)](f) \quad (x \in H_1).

III. \text{ For all } F \in C^f_{cp} \text{ there exists } K \in C^f_{2p} \text{ such that } E(U_1[F] \centerdot H_1) \subset U_2[K] \centerdot H_2 \quad \text{ and }
\quad (U_2[K] \centerdot)^{-1} E U_1[F] \centerdot \text{ is continuous from } H_1 \text{ into } H_2. \text{ Here, } (U_2[K] \centerdot)^{-1} \text{ is the inverse map}
\text{ of the map } U_2[K] \centerdot \text{ from } H_2 \text{ onto } U_2[K] \centerdot H_2.

IV. \text{ For all } \phi \in S_2 \text{ the map } l_\phi \text{ is continuous, with } l_\phi : T_1 \rightarrow C
\quad l_\phi(\Phi) := \langle \phi, E \Phi \rangle \quad (\Phi \in T_1).

Proof.

Note that \((T_{U_1,C_1}, \tau_{\text{ind}}) = T_1\) and that \( T_1 \) is bornological. The proof is similar to the proof of
Theorems 5.6 and 5.7.

In the following we consider the dual map of a continuous linear map.

Let \( E : S_1 \rightarrow S_2 \) be a continuous linear map. Let \( \Phi \in T_2 \). The map \( \phi \mapsto \langle E \phi, \Phi \rangle \) from \( S_1 \)
into \( C \) is continuous and linear, hence there exists a unique \( \Psi \in T_1 \) such that
\( \langle E \phi, \Phi \rangle = \langle \phi, \Psi \rangle \) for all \( \phi \in S_1 \). (See Theorem 1.18.) So we can make the following
definition.

DEFINITION 5.9.

Let \( E : S_1 \rightarrow S_2 \) be a continuous linear map. Define \( E' : T_2 \rightarrow T_1 \) by \( \langle E \phi, \Phi \rangle = \langle \phi, E' \Phi \rangle \)
for all \( \phi \in S_1 \) and \( \Phi \in T_2 \). Similarly define
\( E' : S_2 \rightarrow T_1 \) for every continuous linear map \( E : S_1 \rightarrow T_2 \);
\( E' : T_2 \rightarrow S_1 \) for every continuous linear map \( E : T_1 \rightarrow S_2 \);
$E': S_2 \rightarrow S_1$ for every continuous linear map $E : T_1 \rightarrow T_2$.

It is trivial that the map $E'$ in Definition 5.9 is linear in the four cases.

**THEOREM 5.10.**

I. Let $E : S_1 \rightarrow S_2$ be a continuous linear map and suppose $(C_2, U_2)$ satisfies property P4. Then $E'$ is continuous from $T_2$ into $T_1$.

II. Let $E : S_1 \rightarrow T_2$ be a continuous linear map. Then $E'$ is continuous from $S_2$ into $T_1$.

III. Let $E : T_1 \rightarrow S_2$ be a continuous linear map and suppose $(C_2, U_2)$ satisfies property P4. Then $E'$ is continuous from $T_2$ into $S_1$.

IV. Let $E : T_1 \rightarrow T_2$ be a continuous linear map. Then $E'$ is continuous from $S_2$ into $S_1$.

**Proof.**

We prove only III, the other proofs run similarly.

The space $S_1$ is dense in $H_1$. For every $y \in S_1$ define $l_y : T_2 \rightarrow C$ by

$l_y(\Phi) := \langle E' \Phi, y \rangle$, $\Phi \in T_2$. Let $y \in S_1$. Then $l_y(\Phi) = \langle E' \Phi, \text{emb } y \rangle = \langle E(\text{emb } y), \Phi \rangle$ for all $\Phi \in T_2$, so $l_y$ is continuous. By Theorem 5.7 II $\Rightarrow$ I, $E'$ is continuous from $T_2$ into $S_1$.

Finally we investigate whether a densely defined linear operator from $H_1$ into $H_2$ can be extended to a continuous linear map from $T_1$ into $T_2$.

**THEOREM 5.11.**

Let $E$ be a densely defined operator from $H_1$ into $H_2$. Suppose $(C_1, U_1)$ satisfies property P4. The following conditions are equivalent.

I. There exists a continuous linear map $A$ from $T_1$ into $T_2$ such that $A(\text{emb } x) = \text{emb } (E x)$ for all $x \in D(E)$.

II. The adjoint map $E^*$ from $H_2$ into $H_1$ satisfies $S_2 \subset D(E^*)$ and $E^*(S_2) \subset S_1$.

**Proof.**

I $\Rightarrow$ II. For all $\phi \in S_2$ and $x \in D(E)$ we have $(\phi, E x) = \langle \phi, \text{emb } (E x) \rangle = \langle \phi, A(\text{emb } x) \rangle = \langle A' \phi, \text{emb } x \rangle = (A' \phi, x)$, so $\phi \in D(E^*)$ and $E^* \phi = A' \phi$. Hence $S_2 \subset D(E^*)$ and $E^*(S_2) = A'(S_2) \subset S_1$.

II $\Rightarrow$ I. By Corollary 5.5 the map $E^*\mid_{S_2}$ is continuous from $S_2$ into $S_1$. Let $A := (E^*\mid_{S_2})'$ be its dual map from $T_1$ into $T_2$. Then $A$ is continuous by Theorem 5.10 I. Let $x \in D(E)$. For all $\phi \in S_2$ we have $(\phi, A(\text{emb } x)) = \langle \phi, (E^*\mid_{S_2})'(\text{emb } x) \rangle = \langle E^*\mid_{S_2} \phi, \text{emb } x \rangle = (E^* \phi, x) = (\phi, E x) = \langle \phi, \text{emb } (E x) \rangle$. By Lemma 1.17, $A(\text{emb } x) = \text{emb } (E x)$.
COROLLARY 5.12.

Let $E$ be a linear map from $S_1$ into $S_2$. Suppose $(C_1, U_1)$ satisfies property P4. The following conditions are equivalent.

I. There exists a continuous linear map $A$ from $T_1$ into $T_2$ such that $A(\text{emb}\phi) = \text{emb}(E\phi)$ for all $\phi \in S_1$.

II. The adjoint map $E^*$ from $H_2$ into $H_1$ satisfies $S_2 \subset D(E^*)$ and $E^*(S_2) \subset S_1$.

Proof.

Obvious. \qed
Chapter VI. Conjugate linear homeomorphisms

Let $X$, $Y$ be two (complex) locally convex topological vector spaces and let $E$ be a map from $X$ into $Y$. $E$ is called a conjugate linear map if and only if $E(x + \lambda y) = E x + \overline{\lambda} E y$ for all $x, y \in X$ and $\lambda \in \mathbb{C}$.

VI.1. The strong dual of $S_{U,C}$ and $T_{U,C}$

Let $G$ be a locally compact Abelian group and let $U$ be a representation of $G$ in a Hilbert space $H$. Let $C$ be a subset of $L^1(G)$ and suppose the pair $(C, U)$ satisfies properties P1 and P2.

Let $S_{U,C}'$ be the strong dual of $S_{U,C}$, so $S_{U,C}' = (S_{U,C}', \beta(S_{U,C}', S_{U,C}))$. Similarly let $T_{U,C}' := (T_{U,C}', \beta(T_{U,C}', T_{U,C}))$. Note, the topology for $T_{U,C}$ is the $\tau_{\text{proj}}$-topology unless the contrary is explicitly stated.

Only in this section we need the following two maps. By Theorem 1.18 these maps are well defined.

**DEFINITION 6.1.**

Define $\alpha : S_{U,C} \to T_{U,C}'$

$$[\alpha(\phi)](\Phi) := \langle \phi, \Phi \rangle \quad (\phi \in S_{U,C}, \Phi \in T_{U,C}).$$

Define $\beta : T_{U,C} \to S_{U,C}'$

$$[\beta(\Phi)](\phi) := \langle \phi, \Phi \rangle \quad (\Phi \in T_{U,C}, \phi \in S_{U,C}).$$

**LEMMA 6.2.**

The maps $\alpha$ and $\beta$ are conjugate linear bijections.

**Proof.**

Lemma 1.17 and Theorem 1.18.

**THEOREM 6.3.**

The map $\alpha$ is a conjugate linear homeomorphism.

**Proof.**

We prove that $\alpha$ is continuous. Let $B \subset T_{U,C}$ be bounded and let $V := \{ l \in T_{U,C}' : \| l(\Phi) \| \leq 1 \}$ for all $\Phi \in B \}$. We have to prove that $\alpha^{-1}(V)$ is a neighbourhood of $0$ in $S_{U,C}$. Let $B^*$ be the polar of $B$ in the duality $<S_{U,C} , T_{U,C}>$. By [Wil] Theorem 8-4-12, $B^*$ is absorbing, so $B^*$ is a barrel. Since $S_{U,C}$ is barrelled, $B^*$ is a neighbourhood of $0$ in $S_{U,C}$. Let $\phi \in B^*$. Then for all $\Phi \in B : l(\alpha(\phi))(\Phi) = \| l(\Phi) \| \leq 1 < \phi, \Phi > \leq 1$, so $\alpha(\phi) \in V$. Then $\alpha(B^*) \subset V$ and $\alpha$ is continuous.

Also the inverse map $\alpha^{-1}$ is continuous. Indeed, let $V$ be an absolutely convex closed neighbourhood of $0$ in $S_{U,C}$. Then by [Wil] Theorem 8-3-8, $(V^*)^* = V$ is absorbing. Hence ([Wil] Theorem 8-4-12) $V^*$ is a bounded subset of $T_{U,C}$. Let $W := \{ l \in T_{U,C}' : \| l(\Phi) \| \leq 1 \}$ for all $\Phi \in V^* \}$.

Then $W$ is a neighbourhood of $0$ in $T_{U,C}$. Let $\phi \in S_{U,C}$ and suppose $\alpha(\phi) \in W$. Then
\[ l < \phi, \Phi > = 1 \iff \alpha(\phi)(\Phi) \leq 1 \text{ for all } \Phi \in V^*, \text{ so } \phi \in V^* = V. \text{ So } \alpha^{-1}(W) \subset V \text{ and } \alpha^{-1} \text{ is continuous.} \]

**THEOREM 6.4.**

I. The map \( \beta^{-1} \) is continuous.

II. Suppose the pair \((C, U)\) satisfies property P3. Then the map \( \beta \) is a conjugate linear homeomorphism if and only if property P4 holds.

**Proof.**

I. Let \( f \in C \) and let \( \varepsilon > 0 \). Let \( V := \{ \Phi \in T_{U, C} : t_\beta(\Phi) < \varepsilon \} \). We prove that \( \beta(V) \) is a neighbourhood of 0 in \( S_{U, C'} \). Let \( B_0 := \{ x \in H : \|x\| \leq 1 \} \) and let \( B := U(f)(B_0) \). Then \( B \) is a bounded subset of \( S_{U, C} \). Let \( W := \{ l \in S_{U, C'} : \|l(\phi)\| < \varepsilon \text{ for all } \phi \in B \} \). Then \( W \) is a neighbourhood of 0 in \( S_{U, C'} \).

Let \( l \in W \) and let \( \Phi := \beta^{-1}(l) \). Then, by Theorem 1.16 II, \( \| (x, \Phi(f)) \| = \| U(f)x \cdot \Phi \| = \| U(f)x \| < \varepsilon \text{ for all } x \in B_0 \).

So \( \Phi(f) \| < \varepsilon \) and \( \Phi \in V \).

II. \( \Rightarrow \). Suppose \( \beta \) is a conjugate linear homeomorphism. Then \( \beta \) is continuous. Let \( F \in \text{ Bor}(\hat{G}, C) \) and suppose for every \( K \in C^\# \) the function \( F \cdot K \) is bounded on \( \hat{G} \).

Then \( F \) is bounded. Let \( S_0 := \text{ span}\{ P_n x : n \in \mathbb{N}, x \in H \} \), with \( P_n := U[1_{Q_n}], n \in \mathbb{N} \).

Then \( S_0 \) is a subset of \( S_{U, C} \) by Corollary 2.26. Let \( B := U[F]\phi \in S_0, \| \phi \| \leq 1 \).

Then \( B \subset S_0 \subset S_{U, C} \). For every \( K \in C^\# \) we obtain for every \( \phi \in S_0, \| \phi \| \leq 1 \):

\[ s_K(U[F] \phi) = \| U[K] U[F] \phi \| \leq \| U[K]\| \| \phi \| \leq \| F K \|_\infty < \infty. \]

So \( B \) is bounded in \( S_{U, C} \).

Let \( V := \{ l \in S_{U, C'} : \|l(\psi)\| \leq 1 \text{ for all } \psi \in B \} \). \( V \) is a neighbourhood of 0 in \( S_{U, C'} \) and, by assumption, \( \beta^{-1}(V) \) is a neighbourhood of 0 in \( (T_{U, C}, \tau_{proj}) \).

By Lemma 1.14 there exist \( f \in C \) and \( c > 0 \) such that \( \{ \Phi \in T_{U, C} : t_\beta(\Phi) \leq c \} \subset \beta^{-1}(V) \).

Then \( \text{ emb } x \in \beta^{-1}(V) \) for all \( x \in H \) with \( t_\beta(\text{ emb } x) \leq c \).

Hence \( \| U[F] \phi, x \| = \| \beta(\text{ emb } x) \| U[F] \phi \| \leq 1 \text{ for all } x \in H \) and \( \phi \in S_0 \).

So \( S_0 \) is dense in \( H \) and \( U[F] \) is continuous, \( \| (U[F] y, x) \| \leq c^{-1} \text{ for all } x, y \in H \) with \( \| (U(f)x) \| \leq 1 \) and \( \| y \| \leq 1 \).

Then \( \beta^{-1}(V) \) is a neighbourhood of 0 in \( (T_{U, C}, \tau_{proj}) \).

We only have to prove that \( \beta \) is continuous. Let \( B \) be a bounded subset of \( S_{U, C} \) and let \( \varepsilon > 0 \). Let \( V := \{ l \in S_{U, C'} : \|l(\phi)\| \leq \varepsilon \text{ for all } \phi \in B \} \). We have proved the theorem if we can prove that \( \beta^{-1}(V) \) is a neighbourhood of 0 in \( (T_{U, C}, \tau_{proj}) \).

Assertion: \( \beta^{-1}(V) \) is a neighbourhood of 0 in \( (T_{U, C}, \tau_{proj}) \). Proof of the assertion: Let \( F \in C^\#_\Phi \). Since \( B \) is bounded in \( S_{U, C} \), there exists \( M > 0 \) such that \( s_P(\phi) < M \) for all \( \phi \in B \).

Then for all \( x \in H \) with \( \| x \| < \varepsilon M^{-1} \) we obtain \( \| [\beta(U[F] \phi)](x) \| = \| \phi \|, U[F] \phi \| \leq M \| U[F] \phi \| \| x \| < \varepsilon \text{ for all } \phi \in B \).

Hence \( U[F] \phi \in \beta^{-1}(V) \). Since \( \beta^{-1}(V) \) is absolutely convex, the assertion is proved by [Wil] Theorem 13-I-11.

By Theorem 3.4 \( \Rightarrow \) II, \( \tau_{ind} = \tau_{proj} \), so \( \beta^{-1}(V) \) is a \( \tau_{proj} \)-neighbourhood of 0.
REMARK. The map $\beta$ is continuous from $(T_{U, C}, \tau_{\text{ind}})$ into $S_{U, C'}$.

VI.2. The spaces $S_{U, C}$ and $T_{U, C}$

Let $G$ be a locally compact Abelian group and let $U$ be a representation of $G$ in a Hilbert space $H$. The Hilbert space $H$ is conjugate linear homeomorphic with its dual space $H'$. This homeomorphism will be used to define a new representation. In a natural way $H'$ is a Hilbert space.

DEFINITION 6.5.

Define the bijection $\Gamma : H \rightarrow H'$.

$$\Gamma_x(y) := (y, x) \quad (x, y \in H).$$

Define an inner product on $H'$ by

$$(\Gamma_x, \Gamma_y)_{H'} := (y, x)_H \quad (x, y \in H).$$

For every $x \in G$ define $\overline{U}_x : H' \rightarrow H'$

$$\overline{U}_x(\Gamma_v) := \Gamma_{U^*_v} \quad (v \in H).$$

LEMMA 6.6.

I. $\|\Gamma_x\| = \|x\|$ for all $x \in H$.

II. $\Gamma$ is conjugate linear.

III. $\overline{U}$ is a (continuous) representation of $G$ in $H'$.

IV. $(\overline{U}(f))(\Gamma_v) = \Gamma_{U(f)v}$ for all $f \in L^1(G)$ and $v \in H$.

Proof.

I and II are trivial.

III. It is trivial that $\overline{U}_x$ is continuous and linear from $H'$ into $H'$ for all $x \in G$ and that $\overline{U}_e$ is equal to the identity operator on $H'$, with $e$ the identity element of $G$. A little computation shows that $\overline{U}_x \overline{U}_y = \overline{U}_{xy} = \overline{U}_{yx}$ for all $x, y \in G$, since $G$ is commutative. Hence $\overline{U}_x$ is a unitary operator for all $x \in G$. For every $v \in H$ the map $x \mapsto \Gamma_{U^*_v}$ is continuous from $G$ into $H'$, so $\overline{U}$ is a continuous representation of $G$ in $H'$.

IV. Let $f \in L^1(G)$ and $v \in H$. Let $\mu$ be the Haar measure on $G$. Then for all $w \in H:

$$(\overline{U}(f))(\Gamma_v) = \int f(x) (\overline{U}_x \Gamma_v, \Gamma_w) d \mu(x) = \int f(x) (\Gamma_{U^*_w} \Gamma_v, \Gamma_w) d \mu(x) = \int f(x) (w, U^*_x v) d \mu(x) = \int f(x) (w, U^*_x v) d \mu(x) = (w, U(f)v) = (U(f)v, \Gamma_w).$$

So $$(\overline{U}(f))(\Gamma_v) = \Gamma_{U(f)v}. \quad \Box$$

COROLLARY 6.7.

Let $f \in L^1(G)$. Then
I. \( l \in H' : \widetilde{U}(f) l = 0 \) = \( \{ \Gamma_x : x \in H , \ U(f)x = 0 \} \).

II. \( \widetilde{U}(f)' (H') = \Gamma_{U(f)H} \).

In a natural manner a Stone-representant for \( \widetilde{U} \) can be constructed from a Stone-representant for \( U \).

**THEOREM 6.8.**

Let \( (A , m , I , A_i , \tau_i , W) \) be a Stone-representant for \( U \).

Define \( \overline{W} : H' \rightarrow L^2(m) \)

\[
\overline{W}(\Gamma_x) := \overline{W}_x \quad (x \in H).
\]

Then \( (A , m , I , A_i , \tau_i , \overline{W}) \) is a Stone-representant for \( \widetilde{U} \).

**Proof.**

It is trivial that \( \overline{W} \) is a unitary map from \( H' \) onto \( L^2(m) \). Let \( f \in L^1(G) \) and \( x \in H \). Then for a.e. \( a \in A \) we obtain with \( a = \tau_i(y) \) \( (i \in I , \gamma \in \hat{G} ) \):

\[
[\overline{W} \ U(f)](a) = [\overline{W} (\Gamma_{U(f)\cdot \gamma})](a) = \overline{[W U(f)\cdot \gamma]}(a) = \overline{\left( W U(f)\cdot \gamma \right)(a)} = \overline{\left( W U(f)\cdot \gamma \right)(a)}
\]

Hence \( \overline{W} \ U(f) \overline{W}^{-1} \xi = \hat{f} \xi \) for all \( f \in L^1(G) \) and \( \xi \in L^2(m) \).

**COROLLARY 6.9.**

\( \overline{U}[F] (\Gamma_x) = \Gamma_{U[F]\cdot \gamma} \) for all \( F \in \text{Bor}_0(\hat{G} , \mathcal{C}) \) and \( x \in H \).

**Proof.**

Let \( A , m , I , A_i , \tau_i , W \) and \( \overline{W} \) be as in Theorem 6.8.

Then \( \overline{U}[F] \Gamma_x = \overline{W}^{-1} (F \cdot \overline{W} \Gamma_x) = \overline{W}^{-1} (F \cdot \overline{W} x) = \overline{W}^{-1} (F \cdot \overline{W} x) = \overline{W}^{-1} (W U(F) \cdot x) = \Gamma_{U[F]\cdot \gamma} \)

Let \( C \) be a subset of \( L^1(G) \) and suppose the pair \( (C , U) \) satisfies properties P1 and P2.

**LEMMA 6.10.**

I. The pair \( (\check{C} , \check{U}) \) satisfies properties P1 and P2.

II. The pair \( (\check{C} , \check{U}) \) satisfies property P3 if and only if the pair \( (\check{C} , U) \) satisfies property P3.

III. The pair \( (\check{C} , \check{U}) \) satisfies property P4 if and only if the pair \( (\check{C} , U) \) satisfies property P4.

Similar results hold for properties P1', P2', P2'' and P4'.
Proof.

I. It is trivial that property P1 is satisfied. Let \((f_\lambda)_{\lambda \in I}\) be the net as in property P2 for the pair \((C, U)\). For every \(x \in H\) we obtain by Lemma 6.6 I: 
\[
\lim_{\lambda} \overline{U}(f_\lambda) \Gamma_x = \lim_{\lambda} \Gamma_{U(f_\lambda)x} = \Gamma_x.
\]

II. Trivial.

III. Corollary 6.9.

THEOREM 6.11.
The map \(\Gamma_{\mid_{SU,c}}\) from \((SU, C, \sigma_{ind})\) into \((T\overline{U}, \dot{C}, \sigma_{ind})\) is a conjugate linear homeomorphism.

Proof.

Corollary 6.7.

We extend the map \(\Gamma\) from \(H\) onto \(H'\) to a map from \(T_U, C\) onto \(T\overline{U}, \dot{C}\).

DEFINITION 6.12.
Define \(\Gamma_{\text{ext}} : T_U, C \to (H')^{\dot{C}}\)
\[
[\Gamma_{\text{ext}}(\Phi)](f) := \Gamma_{\Phi(f)} \quad (\Phi \in T_U, C, f \in \dot{C}).
\]

LEMMA 6.13.

I. Let \(\Phi \in T_U, C\). Then \(\Gamma_{\text{ext}}(\Phi) \in T\overline{U}, \dot{C}\).

II. \(\Gamma_{\text{ext}}(\text{emb} x) = \text{emb}(\Gamma_x)\) for all \(x \in H\).

Proof.

I. Let \(f \in \dot{C}, g \in L^1(G)\) and suppose that \(f \ast g \in \dot{C}\). Then by Lemma 6.6 1V,
\[
[\Gamma_{\text{ext}}(\Phi)](f \ast g) = \Gamma_{\Phi(f \ast g)} = \Gamma_{U(g) \phi(f)} = \overline{U}(g) \Gamma_{\Phi(f)} = \overline{U}(g)^* \Gamma_{\text{ext}}(\Phi)(f).
\]

II. Let \(x \in H\). Then for all \(f \in \dot{C}\) we obtain
\[
[\Gamma_{\text{ext}}(\text{emb} x)](f) = \Gamma_{\text{emb} x(f)} = \Gamma_{U(f)x} = \overline{U}(f) \Gamma_x = [\text{emb} \Gamma_x](f).
\]

The map \(\Gamma_{\text{ext}}\) from \(T_U, C\) into \(T\overline{U}, \dot{C}\) is a conjugate linear homeomorphism onto \(T\overline{U}, \dot{C}\).

Proof.

Trivial.

REMARK. In section 1.3 we defined a pairing \(\langle, \rangle\) between \(SU, C\) and \(T_U, C\). Strictly speaking,
this pairing is not a duality since \(<,>\) is sesquilinear and not bilinear. We should have had to define a duality between \(S_{U,C}\) and \(T_{\bar{U},\bar{C}}\) and a duality between \(S_{U,\bar{C}}\) and \(T_{U,C}\), using the conjugate linear homeomorphisms in Theorems 6.11 and 6.14. These homeomorphisms are natural homeomorphisms and we identified \(S_{U,C}\) with \(S_{U,\bar{C}}\) and \(T_{U,C}\) with \(T_{\bar{U},\bar{C}}\).
Chapter VII. Cartesian products

For \( j \in \{1, 2\} \) let \( G_j \) be a locally compact Abelian group, let \( U_j \) be a representation of \( G_j \) in a Hilbert space \( H_j \) and let \( C_j \) be a subset of \( L^1(G_j) \). Suppose the pair \((C_j, U_j)\) satisfies properties P1 and P2. We can form the topological cartesian products \( S_{U_1, C_1} \times S_{U_2, C_2} \) and \( T_{U_1, C_1} \times T_{U_2, C_2} \). Then the following problem arises. Does there exist a group \( G \), a representation \( U \) and a subset \( C \) of \( L^1(G) \) such that \( S_{U_1, C_1} \times S_{U_2, C_2} \) is isomorphic with \( S_{U_1, C_1} \times S_{U_2, C_2} \) and \( T_{U_1, C_1} \times T_{U_2, C_2} \) is isomorphic with \( T_{U, C} \) as topological vector spaces? The answer is affirmative, if we demand something more: suppose for \( j \in \{1, 2\} \) the pair \((C_j, U_j)\) satisfies properties \( P1', P2 \) and the following property.

PROPERTY 7.1. P5.

\[ \hat{f}(1) = 1 \text{ for all } f \in C. \]

Here \( 1 \) is the identity element of \( \hat{G} \).

REMARK. The set \( C \) satisfies property P5 if and only if the set \( \hat{C} \) satisfies property P5.

In the remaining part of this chapter we require properties P1', P2 and P5. An example is Example 1 in chapter IV. We start with the definition of the tensor product of two functions.

DEFINITION 7.2.

Let \( X \) and \( Y \) be two sets. Let \( f \) and \( g \) be complex valued functions on \( X \) resp. \( Y \).

Define the tensorproduct \( f \otimes g : X \times Y \to \mathbb{C} \)

\[ (f \otimes g)(x, y) := f(x) g(y). \]

Let \( \mu_1 \) and \( \mu_2 \) be a Haar measure on \( G_1 \) resp. \( G_2 \). Then \( \mu := \mu_1 \times \mu_2 \) is a Haar measure on \( G_1 \times G_2 \).

LEMMA 7.3.

Let \( f_1 \) and \( f_2 \) be Borel measurable complex valued and integrable functions on \( G_1 \) resp. \( G_2 \). Then \( f_1 \otimes f_2 \) is integrable and \( \| f_1 \otimes f_2 \|_1 = \| f_1 \|_1 \| f_2 \|_1 \).

Proof.

Without loss of generality, \( f_1 \geq 0 \) and \( f_2 \geq 0 \). Then by [HRI] Theorem 13.9 (Fubini's theorem):

\[
\int_{G_1 \times G_2} |f_1 \otimes f_2| \, d\mu = \int_{G_1} f_1 \, d\mu_1 \cdot \int_{G_2} f_2 \, d\mu_2 = \| f_1 \|_1 \| f_2 \|_1 < \infty.
\]

So \( f_1 \otimes f_2 \) is integrable and \( \| f_1 \otimes f_2 \|_1 = \| f_1 \|_1 \| f_2 \|_1 \).

The following definition makes sense by Lemma 7.3.
DEFINITION 7.4.

Let \( f_1 \in L^1(G_1) \) and \( f_2 \in L^1(G_2) \). Define \( f_1 \otimes f_2 \in L^1(G_1 \times G_2) \) by

\[
(f_1 \otimes f_2)(x, y) := f_1(x) f_2(y) \quad (\text{a.e. } (x, y) \in G_1 \times G_2).
\]

We identify \((G_1 \times G_2)^*\) with \(\hat{G}_1 \times \hat{G}_2\) in the natural way.

LEMMA 7.5.

Let \( f_1, g_1 \in L^1(G) \) and \( f_2, g_2 \in L^1(G) \). Then

I. \((f_1 \otimes f_2)^* = f_1^* \otimes f_2^*\).
II. \((f_1 \otimes f_2)^* = f_1^* \otimes f_2^*\).
III. \((f_1 \otimes f_2) * (g_1 \otimes g_2) = (f_1 * g_1) \otimes (f_2 * g_2)\).

Proof.

I, trivial; II and III, Fubini's theorem.

DEFINITION 7.6.

Let \( G := G_1 \times G_2 \).
Let \( H := H_1 \oplus H_2 \) be the Hilbert space with inner product

\[
((v_1, v_2), (w_1, w_2)) := (v_1, w_1) + (v_2, w_2). \quad ((v_1, v_2), (w_1, w_2) \in H)
\]

For \((x, y) \in G\) define \( U_{(x,y)} : H \to H \)

\[
U_{(x,y)}(v, w) := (U_{1x} v, U_{2y} w) \quad ((v, w) \in H).
\]

Let \( C := \{ f_1 \otimes f_2 : f_1 \in C_1, f_2 \in C_2 \} \).

LEMMA 7.7.

I. \( U \) is a representation of \( G \) in \( H \).
II. Let \( f_1 \in L^1(G_1), f_2 \in L^1(G_2), (v, w) \in H \) and suppose \( \hat{f}_1(1) = \hat{f}_2(1) = 1 \). Then

\[
U(f_1 \otimes f_2)(v, w) = (U_1(f_1)v, U_2(f_2)w).
\]

Proof.

I. Trivial.
II. For all \((v_2, w_2) \in H\) we obtain by Fubini's theorem:

\[
(U(f_1 \otimes f_2)(v, w), (v_2, w_2)) = \int_{G_1} \int_{G_2} f_1(x) f_2(y) [(U_{1x} v, v_2) + (U_{2y} w, w_2)] \, d \mu_2(y) \, d \mu_1(x) =
\]

\[
\int_{G_1} f_1(x) (U_{1x} v, v_2) \, d \mu_1(x) \int_{G_2} f_2(y) \, d \mu_2(y) + \int_{G_1} f_1(x) \, d \mu_1(x) \int_{G_2} f_2(y) (U_{2y} w, w_2) \, d \mu_2(y) =
\]
(U_1(f_1)v, v_2) * \hat{f}_2(1) + \hat{f}_1(1) * (U_2(f_2)w, w_2) = ((U_1(f_1)v, U_2(f_2)w), (v_2, w_2)).
So U(f_1 \otimes f_2)(v, w) = (U_1(f_1)v, U_2(f_2)w). \hfill \square

**LEMMA 7.8.**

I. The pair \((C, U)\) satisfies properties P1', P2 and P5.

II. If the pair \((C_1, U_1)\) and \((C_2, U_2)\) satisfy property P3, then the pair \((C, U)\) satisfies property P3.

**Proof.**

P1'. Lemma 7.5 III and property P1' for the pairs \((C_1, U_1)\) and \((C_2, U_2)\).

P2. For \(j \in \{1, 2\}\) let \((f_j, \lambda_j, \iota_j)\) be the net as in property P2 for the pair \((C_j, U_j)\). Let \(J := J_1 \times J_2\). Define a relation \(\leq\) on \(J\) by
\[
(\lambda_1, \iota_1) \leq (\lambda_2, \iota_2) \iff (\lambda_1 \leq \lambda_2) \land (\iota_1 \leq \iota_2) \iff (\lambda_1, \iota_1) \in J.
\]
So \(J\) is a directed set. For \((\lambda_1, \iota_1) \in J\) let \(f(\lambda_1, \iota_1) := f_{1\lambda_1} \otimes f_{2\iota_1}\). Let \((v, w) \in H\). Since \((C_1, U_1)\) and \((C_2, U_2)\) satisfy property P5, we obtain by Lemma 7.7 II:
\[
\lim_{\lambda \rightarrow \lambda_1} U(f_{1\lambda}(v, w)) = \lim_{\iota \rightarrow \iota_1} U(f_{2\iota}(v, w)) = (v, w).
\]

P5. Trivial. (See Lemma 7.5 II.)

II. For \(j \in \{1, 2\}\) let \(Q_{j1}, Q_{j2}, \ldots \subset \hat{G}_j\) and \(b_{j1}, b_{j2}, \ldots \in (0, \infty)\) as in property P3 for the pair \((C_j, U_j)\). Let \(p\) be a bijection from \(\mathbb{N} \times \mathbb{N}\) onto \(\mathbb{N}\). For \((n, m) \in \mathbb{N} \times \mathbb{N}\) let \(Q_{p(n,m)} := Q_{1n} \times Q_{2m}\) and \(b_{p(n,m)} := b_{1n} b_{2m}\). Then \(Q_1, Q_2, \ldots\) are Borel measurable, pairwise disjoint and \(\bigcup_{n=1}^{\infty} Q_n = \hat{G}_1 \times \hat{G}_2 = \hat{G}\). Moreover, \(b_n > 0\) for all \(n \in \mathbb{N}\) and
\[
\sum_{n=1}^{\infty} b_n^{-1} = \sum_{n=1}^{\infty} b_{1n}^{-1} \sum_{m=1}^{\infty} b_{2m}^{-1} < \infty.
\]
Let \(f_1 \in C_1\) and \(f_2 \in C_2\). For \(j \in \{1, 2\}\) there exist \(g_j \in C_j\) and \(\delta_j > 0\) such that \(b_{jn} \sup \{|f_j(\gamma)| : \gamma \in Q_{jn}\} \leq \delta_j \inf \{|g_j(\gamma)| : \gamma \in Q_{jn}\}\) for all \(n \in \mathbb{N}\). Then \(g_1 \otimes g_2 \in C\) and for all \((m, n) \in \mathbb{N} \times \mathbb{N}\) we obtain by Lemma 7.5 II:
\[
b_{p(n,m)} \sup \{|f_1 \otimes f_2|^2((\gamma_1, \gamma_2)) : (\gamma_1, \gamma_2) \in Q_{p(n,m)}\} =
b_{1n} b_{2m} \sup \{|f_1(\gamma_1)| \cdot |f_2(\gamma_2)| : \gamma_1 \in Q_{1n}, \gamma_2 \in Q_{2m}\}\leq
\delta_1 \delta_2 \inf \{|g_1(\gamma_1)| \cdot |g_2(\gamma_2)| : \gamma_1 \in Q_{1n}, \gamma_2 \in Q_{2m}\} =
\delta_1 \delta_2 \inf \{|g_1 \otimes g_2|^2((\gamma_1, \gamma_2)) : (\gamma_1, \gamma_2) \in Q_{p(n,m)}\}.
\]
So property P3 holds for the pair \((C, U)\). \hfill \square

**THEOREM 7.9.**

\(S_{U, C}\) equals \(S_{U_1, C_1} \times S_{U_2, C_2}\) in the sense of topological vector spaces.
Proof.

Let $f_1 \in C_1$ and $f_2 \in C_2$. By Lemma 7.7 II and property P5, $R_{f_1 \oplus f_2} = R_{f_1} \times R_{f_2}$. Hence

$$S_{U, C} = \bigcup_{f_1 \in C_1} R_{f_1} \oplus \bigcup_{f_2 \in C_2} R_{f_2} = (\bigcup_{f_1 \in C_1} R_{f_1}) \times (\bigcup_{f_2 \in C_2} R_{f_2}) = S_{U_1, C_1} \times S_{U_2, C_2},$$

as sets.

Let $i$ be the identity map from $S_{U, C}$ onto $S_{U_1, C_1} \times S_{U_2, C_2}$. Let $f_1 \in C_1$ and $f_2 \in C_2$. The map $(v, w) \mapsto U_1(f_1) v$ from $H$ into $S_{U_1, C_1}$ is continuous and the map $(v, w) \mapsto U_2(f_2) w$ from $H$ into $S_{U_2, C_2}$ is continuous. So the map $(v, w) \mapsto (U_1(f_1) v, U_2(f_2) w) = (U(f_1 \oplus f_2)(v, w))$ from $H$ into $S_{U_1, C_1} \times S_{U_2, C_2}$ is continuous for all $f_1 \in C_1$ and $f_2 \in C_2$. By Theorem 5.1 III $\Rightarrow$ I, $i$ is continuous from $S_{U, C}$ into $S_{U_1, C_1} \times S_{U_2, C_2}$.

Let $j$ be the identity map from $S_{U_1, C_1} \times S_{U_2, C_2}$ onto $S_{U, C}$. Note that the direct topology on $S_{U_1, C_1} \times S_{U_2, C_2}$ is equal to the product topology on $S_{U_1, C_1} \times S_{U_2, C_2}$. (See [Wil] Lemma 13-2-8.)

First we prove that the map $\phi_1 \mapsto j((\phi_1, 0))$ from $S_{U_1, C_1}$ into $S_{U, C}$ is continuous. Indeed, let $f_1 \in C_1$. Since $C_2 \neq \emptyset$ (property P2), there exist $f_2 \in C_2$. The map $(v, w) \mapsto U(f_1 \oplus f_2)(v, w)$ from $H$ into $S_{U, C}$ is continuous, hence the map $v \mapsto U(f_1 \oplus f_2)(v, 0) = (U(f_1)v, 0) = j((U_1(f_1) v, 0))$ from $H_1$ into $S_{U, C}$ is continuous. By Theorem 5.1 III $\Rightarrow$ I, the map $\phi_1 \mapsto j((\phi_1, 0))$ is continuous from $S_{U_1, C_1}$ into $S_{U, C}$.

Similarly, the map $\phi_2 \mapsto j((0, \phi_2))$ is continuous from $S_{U_2, C_2}$ into $S_{U, C}$. By [Wil] Lemma 13-2-6, the map $j$ is continuous.

COROLLARY 7.10.

Suppose the pairs $(C_1, U_2)$ and $(C_2, U_2)$ satisfy property P3. Then $(C, U)$ satisfies property P4 if and only if $(C_1, U_1)$ and $(C_2, U_2)$ satisfy property P4.

Proof.

By the assumption and Lemma 7.8 II, the pair $(C, U)$ satisfies property P3. Using Theorem 3.12 and [Wil] Theorem 6-1-7 we obtain the following equivalent statements:

$(C, U)$ satisfies property P4.

$S_{U, C}$ is complete.

$S_{U_1, C_1} \times S_{U_2, C_2}$ is complete.

$S_{U_1, C_1}$ and $S_{U_2, C_2}$ are complete.

$(C_1, U_1)$ and $(C_2, U_2)$ satisfy property P4.

In order to prove that $T_{U, C}$ and $T_{U_1, C_1} \times T_{U_2, C_2}$ are isomorphic, we need two lemmas.
LEMMA 7.11.
Let \( f_1, g_1 \in C_1 \) and \( f_2, g_2 \in C_2 \) and suppose \( f_1 \otimes f_2 = g_1 \otimes g_2 \). Then \( f_1 = g_1 \) and \( f_2 = g_2 \).

Proof.
Let \( \gamma_1 \in \hat{G}_1 \). Then \( \hat{f}_1(\gamma_1) = \hat{f}_1(\gamma_1) \hat{f}_2(1) = (f_1 \otimes f_2)(\gamma_1, 1) = (g_1 \otimes g_2)(\gamma_1, 1) = \hat{g}_1(\gamma_1) \).
Hence \( \hat{f}_1 = \hat{g}_1 \) and \( f_1 = g_1 \). Similarly: \( f_2 = g_2 \).

LEMMA 7.12.
Let \( \pi_1 \) be the natural projection of \( H \) onto \( H_1 \). For all \( \gamma \in H \), \( \pi_1, \pi_2 \in C_1 \) and \( \gamma_1, \gamma_2 \in C_2 \) we have \( \pi_1(\Phi(f_1 \otimes f_2)) = \pi_1(\Phi(f_1 \otimes g_2)) \). For all \( \Phi \in T_{U_1,C} \) define \( \Phi^1 \in (H_1)^{C_1} \) by \( \Phi^1(f_1) := \pi_1(\Phi(f_1 \otimes f_2)) \). Then \( \Phi^1 \in T_{U_1,C} \), and the map \( \Phi \mapsto \Phi^1 \) from \( T_{U_1,C} \) into \( T_{U_1,C} \) is continuous.

Proof.
Let \( f_1 \in C_1 \) and \( f_2, g_2 \in C_2 \). Let
\[
E := \{ \Phi \in T_{U_1,C} : \pi_1(\Phi(f_1 \otimes f_2)) = \pi_1(\Phi(f_1 \otimes g_2)) \}.
\]
Then \( \text{emb}H \subset E \). Indeed, let \( (v, w) \in H \). Then \( \pi_1(\text{emb}(v, w))(f_1 \otimes f_2) = \pi_1(U(f_1 \otimes f_2)(v, w)) = \pi_1(U(f_1 \otimes f_2)v, U(f_1 \otimes f_2)w) = \pi_1(\text{emb}(v, w))(f_1 \otimes g_2) \). So \( \text{emb}(v, w) \in E \). Since \( E \) is closed in \( T_{U_1,C} \) and \( \text{emb}H \) is dense in \( T_{U_1,C} \), \( E = T_{U_1,C} \). Hence \( \pi_1(\Phi(f_1 \otimes f_2)) = \pi_1(\Phi(f_1 \otimes g_2)) \) for all \( \Phi \in T_{U_1,C} \).

Let \( \Phi \in T_{U_1,C} \). Let \( f_1 \in C_1 \), \( g_1 \in L^1(G_1) \) and suppose \( f_1 * g_1 \in C_1 \). We prove that \( \Phi^1(f_1 * g_1) = U_1(g_1)^* \Phi^1(f_1) \). Since \( C_2 \neq \emptyset \), there exists \( f_2 \in C_2 \). By property P1', there exist \( h_2 \in C_2 \) and \( g_2 \in L^1(G_2) \) such that \( f_2 = h_2 * g_2 \). Then \( 1 = \hat{f}_2(1) = \hat{h}_2(1) \hat{g}_2(1) = \hat{g}_2(1) \) by property P5. So \( \hat{g}_2(1) = 1 \). Similarly \( \hat{g}_1(1) = 1 \). So by Lemma 7.7 II,
\[
\Phi^1(f_1 * g_1) = \pi_1(\Phi(f_1 * g_1) \otimes h_2) = \pi_1(U_1(g_1)^* \Phi(f_1) \otimes h_2)) = U_1(g_1)^* \Phi^1(f_1). \]
It is trivial that the map \( \Phi \mapsto \Phi^1 \) is continuous from \( T_{U_1,C} \) into \( T_{U_1,C} \).

THEOREM 7.13.
The spaces \( T_{U_1,C} \) and \( T_{U_1,C_1} \times T_{U_2,C_2} \) are isomorphic as topological vector spaces.

Proof.
Let \( \pi_1 \) resp. \( \pi_2 \) be the natural projection from \( H \) onto \( H_1 \) resp. \( H_2 \). For \( \Phi \in T_{U_1,C} \) define \( \Phi^1 \in T_{U_1,C_1} \) and \( \Phi^2 = T_{U_2,C_2} \) by
These definitions make sense because of the previous lemma. The maps $\Phi \mapsto \Phi^1$ resp. $\Phi \mapsto \Phi^2$ are continuous from $T_{U,C}$ into $T_{U_1,C_1}$ resp. $T_{U_2,C_2}$.

Define $i : T_{U,C} \to T_{U_1,C_1} \times T_{U_2,C_2}$

$$i(\Phi) := (\Phi^1, \Phi^2) \quad (\Phi \in T_{U,C}).$$

Then the map $i$ is linear and continuous.

By Lemma 7.11 we can define $j : T_{U_1,C_1} \times T_{U_2,C_2} \to H^C$ by

$$[j(\Phi_1, \Phi_2)] (f_1 \otimes f_2) := (\Phi_1(f_1), \Phi_2(f_2))$$

$$(\Phi_1 \in T_{U_1,C_1}, \Phi_2 \in T_{U_2,C_2}, f_1 \in C_1, f_2 \in C_2).$$

Note that $[j(\text{emb}v, \text{emb}w)] (f_1 \otimes f_2) = (U_1(f_1)^* v, U_2(f_2)^* w) = U(f_1 \otimes f_2)^* (v, w)$ for all $v \in H_1$, $w \in H_2$, $f_1 \in C_1$ and $f_2 \in C_2$.

Assertion: $j(\Phi_1, \Phi_2) \in T_{U,C}$ for all $\Phi_1 \in T_{U_1,C_1}$ and $\Phi_2 \in T_{U_2,C_2}$. Proof of the assertion.

Let $f_1 \in C_1$, $f_2 \in C_2$, $g \in L^1(G)$ and suppose $(f_1 \otimes f_2)^* g \in C$. Let $E := \{ (\Phi_1, \Phi_2) \in T_{U_1,C_1} \times T_{U_2,C_2} : [j(\Phi_1, \Phi_2)] (f_1 \otimes f_2)^* g \in C \}$.

Then $E$ is closed in $T_{U_1,C_1} \times T_{U_2,C_2}$. Let $v \in H_1$ and $w \in H_2$. Then

$$[j(\text{emb}v, \text{emb}w)] (f_1 \otimes f_2)^* g = U((f_1 \otimes f_2)^* g)^* (v, w) = U(g)^* U(f_1 \otimes f_2)^* (v, w) = U(g)^* U(\text{emb}v, \text{emb}w)(f_1 \otimes f_2).$$

So $\text{emb}H_1 \times \text{emb}H_2 \subset E$. Since $\text{emb}H_2 \times \text{emb}H_2$ is dense in $T_{U_1,C_1} \times T_{U_2,C_2}$ we have $E = T_{U_1,C_1} \times T_{U_2,C_2}$ and the assertion is proved.

It is obvious that $j$ is linear and continuous from $T_{U_1,C_1} \times T_{U_2,C_2}$ into $T_{U,C}$.

Let $\Phi \in T_{U,C}$, $f_1 \in C_1$ and $f_2 \in C_2$. Then $([j \circ i](\Phi)) (f_1 \otimes f_2) = [j(\Phi^1, \Phi^2)] (f_1 \otimes f_2) = (\Phi^1(f_1), \Phi^2(f_2)) = (\pi_1(\Phi(f_1 \otimes f_2)), \pi_2(\Phi(f_1 \otimes f_2))) = \Phi(f_1 \otimes f_2)$.

So $j \circ i(\Phi) = \Phi$ for all $\Phi \in T_{U,C}$.

Let $\Phi_1 \in T_{U_1,C_1}$, $\Phi_2 \in T_{U_2,C_2}$, $f_1 \in C_1$ and $f_2 \in C_2$. Then $\pi_1([j(\Phi_1, \Phi_2)] (f_1 \otimes f_2)) = \pi_1(\Phi_1(f_1), \Phi_2(f_2)) = \Phi_1(f_1)$ and $\pi_2([j(\Phi_1, \Phi_2)] (f_1 \otimes f_2)) = \Phi_2(f_2)$. So

$$(i \circ j)(\Phi_1, \Phi_2) = (\Phi_1, \Phi_2).$$

Hence $i$ is a topological homeomorphism and $T_{U,C}$ is isomorphic with $T_{U_1,C_1} \times T_{U_2,C_2}$ as topological vector spaces.
Chapter VIII. Tensor products and kernel theorems

VIII.1. Tensor products

The problem we raise in this section is comparable to the problem raised in chapter VII: Does there exist a group $G$, a representation $U$ of $G$ and a subset $C$ of $L^1(G)$ such that $T_{U,C}$ is homeomorphic with a suitable topological tensor product of $T_{U_1,C_1}$ and $T_{U_2,C_2}$ where $T_{U_1,C_1}$ and $T_{U_2,C_2}$ are given spaces. We start with the investigation of the tensor product of two representations of Abelian groups.

For $j \in \{1, 2\}$ let $G_j$ be a locally compact Abelian group with Haar measure $\mu_j$ and let $U_j$ be a representation of $G_j$ in a Hilbert space $H_j$. Let $C_j$ be a subset of $L^1(G_j)$ and suppose the pair $(C_j, U_j)$ satisfies properties P1' and P2'. We use Definitions 7.2 and 7.4 and Lemmas 7.3 and 7.5. The following definitions are adapted from [HRII] appendix D14 and D15. However we use the notation $\otimes$ instead of the notation $\otimes$ in [HRII].

**Definition 8.1.**

Let $H_0 := H_1 \otimes H_2$ be the tensor product of the spaces $H_1$ and $H_2$. Let $H := H_1 \hat{\otimes} H_2$ be the Hilbert space tensor product of $H_1$ and $H_2$. Let $G := G_1 \times G_2$ and $\mu := \mu_1 \times \mu_2$. For $(x, y) \in G$ define the unitary operator

$$U(x,y) : H \rightarrow H$$

$$U(x,y) := U_1x \otimes U_2y.$$ 

We write $U_1 \otimes U_2 := U$. Let $C := C_1 \otimes C_2 := \{ f_1 \otimes f_2 : f_1 \in C_1, f_2 \in C_2 \}$.

**Remark.** For all $\lambda \in \mathbb{C}$, $\nu \in H_1$, and $w \in H_2$ we have $\lambda(\nu \otimes w) = (\lambda \nu) \otimes w = \nu \otimes (\lambda w)$. Again, we identify $\hat{G}$ with $\hat{G}_1 \times \hat{G}_2$.

**Lemma 8.2.**

I. $U$ is a continuous representation of $G$ in $H$.

II. Let $f_1 \in L^1(G_1)$, $f_1 \in L^1(G_2)$, $\nu \in H_1$ and $w \in H_2$. Then

$$U(f_1 \otimes f_2)(\nu \otimes w) = U_1(f_1) \nu \otimes U_2(f_2)w.$$ 

**Proof.**

I. By [HRII] Theorem D15, $U_{(x_1, y_1)} U_{(x_2, y_2)} = U_{(x_1, y_1, y_2)}$ for all $(x_1, y_1), (x_2, y_2) \in G$. Obviously, the map $(x, y) \mapsto (U_{(x,y)}(v_1 \otimes w_1), v_2 \otimes w_2)$ from $G$ into $C$ is continuous for every $v_1, v_2 \in H_1$ and $w_1, w_2 \in H_2$, so the map $(x, y) \mapsto (U_{(x,y)}(\xi_1), \xi_2)$ is continuous from $G$ into $C$ for all $\xi_1, \xi_2 \in H_0$. Let $\xi_1, \xi_2 \in H$. We prove that the map $(x, y) \mapsto (U_{(x,y)}(\xi_1), \xi_2)$ is continuous from $G$ into $C$. Let $\varepsilon > 0$. Since $H_0$ is dense in $H$, there exist $\xi_3, \xi_4 \in H_0$ such that $\| \xi_1 - \xi_3 \| < \varepsilon$ and $\| \xi_2 - \xi_4 \| < \varepsilon$. There exists open $V \subset G$
such that $e \in V$ ($e$ is the identity element of $G$) and $|| (U_{(x,y)} \xi_3, \xi_4) - (\xi_3, \xi_4)|| < \epsilon$ for all $(x, y) \in V$. Then for all $(x, y) \in V$: $|| (U_{(x,y)} \xi_1, \xi_2) - (\xi_1, \xi_2)|| \leq || (U_{(x,y)} (\xi_1 - \xi_3), \xi_2)|| + || (U_{(x,y)} \xi_3, \xi_2 - \xi_4)|| + || (U_{(x,y)} \xi_3, \xi_4) - (\xi_3, \xi_4)|| + $ $|| (\xi_3 - \xi_1, \xi_4)|| + || (\xi_1, \xi_4 - \xi_2)|| \leq \epsilon || \xi_2|| + \epsilon || \xi_3|| + \epsilon || \xi_4|| + \epsilon || \xi_1|| \leq \epsilon (2 || \xi_1|| + 2 || \xi_2|| + 2 || \xi_3|| + 2 || \xi_4|| + 1)$. So $(x, y) \mapsto (U_{(x,y)} \xi_1, \xi_2)$ is continuous in $e$, and then also continuous on $G$. By [HRI] (22.20), the representation is continuous.

II. Let $v_2 \in H_1$ and $w_2 \in H_2$. Then by Fubini’s theorem:

$$
(U(f_1 \otimes f_2) (v \otimes w), v_2 \otimes w_2) = \int_G \int f_1(x) f_2(y) (U_{1x} v, v_2) (U_{2y} w, w_2) \, d\mu_1(x) \, d\mu_2(y) = (U_{1(f_1)}, v_2) (U_{2(f_2)} w, w_2) = (U_{1(f_1)} v \otimes U_{2(f_2)} w, v_2 \otimes w_2).
$$

Hence $(U(f_1 \otimes f_2) (v \otimes w), \xi) = (U_{1(f_1)} v \otimes U_{2(f_2)} w, \xi)$ for all $\xi \in H_0$. Since $H_0$ is dense in $H$, the lemma follows.

**Lemma 8.3.**

I. The pair $(C, U)$ satisfies properties P1' and P2'.

II. If the pairs $(C_1, U_1)$ and $(C_2, U_2)$ satisfy property P3, then the pair $(C, U)$ satisfies property P3.

**Proof.**

I. The proof of property P1’ is the same as in Lemma 7.8 I. For $j \in \{1, 2\}$ let $(f_{j, \lambda})_{\lambda \in J_j}$ be the net as in property P2’ for the pair $(C_j, U_j)$. Let $M_j > 0$ be such that $|| f_{j, \lambda} ||_1 \leq M_j$ for all $\lambda \in J_j$. Let $J := J_1 \times J_2$. Define a relation $\leq$ on $J$ by

$$(\lambda_1, \lambda_2) \leq (\rho_1, \rho_2) \iff (\lambda_1 \leq \rho_1 \wedge \lambda_2 \leq \rho_2) \quad ((\lambda_1, \lambda_2), (\rho_1, \rho_2) \in J).$$

So $J$ is a directed set. For $(\lambda_1, \lambda_2) \in J$ let $f_{(\lambda_1, \lambda_2)} := f_{1\lambda_1} \otimes f_{2\lambda_2}$. Then $|| f_{\lambda} ||_1 \leq M_1 M_2$ for all $\lambda \in J$.

First we prove that $\lim_{\lambda} U(f_{\lambda}) (x_1 \otimes x_2) = x_1 \otimes x_2$ for all $x_1 \in H_1$ and $x_2 \in H_2$. Let $x_1 \in H_1, x_2 \in H_2$ and $\epsilon > 0$. For all $j \in \{1, 2\}$ there exists $\lambda_j \in J_j$ such that $|| U_j(f_{j, \rho_j}) x_j - x_j || < \epsilon$ for all $\rho_j \in J_j$ with $\rho_j \geq \lambda_j$. Then for all $\rho_1, \rho_2 \in J$ with $(\rho_1, \rho_2) \geq (\lambda_1, \lambda_2)$ we obtain by Lemma 8.2 II: $\| U(f_{(\rho_1, \rho_2)}) (x_1 \otimes x_2) - x_1 \otimes x_2 \| = || U_{1(f_{\rho_1})} x_1 \otimes U_{2(f_{\rho_2})} x_2 - x_1 \otimes x_2 || \leq $ $|| U_{1(f_{\rho_1})} x_1 \otimes (U_{2(f_{\rho_2})} x_2 - x_2) || + || (U_{1(f_{\rho_1})} x_1 - x_1) \otimes x_2 || =$ $|| U_{1(f_{\rho_1})} x_1 || \| U_{2(f_{\rho_2})} x_2 - x_2 \| + \| U_{1(f_{\rho_1})} x_1 - x_1 \| \| x_2 \| \leq $ $(\| x_1 \| + \epsilon) \epsilon + \epsilon \| x_2 \| = \epsilon (\| x_1 \| + \| x_2 \| + \epsilon)$. So $\lim_{\lambda} U(f_{\lambda}) (x_1 \otimes x_2) = x_1 \otimes x_2$.

By linearity, $\lim_{\lambda} U(f_{\lambda}) \xi = \xi$ for all $\xi \in H_0$. 

Finally, let $\xi \in \cal H$. Let $\epsilon > 0$. There exists $\xi_0 \in H_0$ such that $\|\xi - \xi_0\| < \epsilon$. There exists $\lambda_0 \in \cal J$ such that for all $\lambda \in \cal J, \lambda \geq \lambda_0$: $\|U(f_\lambda)\xi_0 - \xi_0\| < \epsilon$. Then for all $\lambda \in \cal J, \lambda \geq \lambda_0$ we obtain: $\|U(f_\lambda)\xi - \xi\| \leq \|U(f_\lambda)(\xi - \xi_0)\| + \|U(f_\lambda)\xi_0 - \xi_0\| + \|\xi_0 - \xi\| < \epsilon \\
\|f_\lambda\|, \|\xi - \xi_0\| + \epsilon + \epsilon \leq (M_1 M_2 + 2)\epsilon$. So $\lim U(f_\lambda)\xi = \xi$ and the pair $(\cal C, U)$ satisfies property $P2'$.

II. This proof is the same as the proof of Lemma 7.8 II.

LEMMA 8.4.

I. Let $f_1 \in C_1$ and $f_2 \in C_2$. Then $R_{f_1} \otimes R_{f_2} \subset R_{f_1 \otimes f_2}$.

II. $S_{U_1, c_1} \otimes S_{U_2, c_2} \subset S_{U_1}, c$. 

III. The map $\phi_2 \mapsto \phi_1 \otimes \phi_2$ from $S_{U_2, c_2}$ into $S_{U_1}, c$ is continuous for all $\phi_1 \in S_{U_1, c_1}$.

Proof.

I. By Lemma 8.2 II, $U_1(f_1)x \otimes U_2(f_2)y = U(f_1 \otimes f_2)(x \otimes y) \in R_{f_1 \otimes f_2}$ for all $x \in H_1$ and $y \in H_2$. Hence $R_{f_1} \otimes R_{f_2} \subset R_{f_1 \otimes f_2}$.

II. By I, $S_{U_1, c_1} \otimes S_{U_2, c_2} = \text{span} \bigcup_{f_1 \in C_1} R_{f_1} \otimes R_{f_2} = \text{span} \bigcup_{f_1 \in C_1} R_{f_1 \otimes f_2} = \text{span} S_{U_1}, c = S_{U_1}, c$.

III. Let $f_1 \in C$ and $x \in H_1$. For all $f_2 \in C_2$ the map $y \mapsto U_1(f_1)X \otimes U_2(f_2)y = U(f_1 \otimes f_2)(X \otimes y)$ from $H_2$ into $S_{U_1}, c$ is continuous. So the lemma follows by Theorem 5.1.

LEMMA 8.5.

I. Let $f_1, g_1 \in C_1 \setminus \{0\}, f_2, g_2 \in C_2 \setminus \{0\}$ and suppose $f_1 \otimes f_2 = g_1 \otimes g_2$. Then there exists $c \in \cal C, c \neq 0$ such that $f_1 = c g_1$ and $g_2 = c f_2$.

II. Let $f_1 \in C_1, f_2 \in C_2$ and suppose $f_1 \otimes f_2 = 0$. Then $f_1 = 0$ or $f_2 = 0$.

III. Let $f \in C_1, c \in \cal C$ and suppose $c f \in C_1$. Then $\Phi(cf) = c \Phi(f)$ for all $\Phi \in T_{U_1, c_1}$. Similar results hold for $T_{U_2, c_2}$.

Proof.

I. Since $f_1 \neq 0$, there exists $\gamma_1 \in \hat{\cal G}_1$ such that $\hat{f}_1(\gamma_1) \neq 0 (j \in \{1, 2\})$. Then $\hat{g}_1(\gamma_1) \hat{g}_2(\gamma_2) = (g_1 \otimes g_2) \gamma_1 \gamma_2 = (f_1 \otimes f_2) \gamma_1 \gamma_2 = \hat{f}_1(\gamma_1) \hat{f}_2(\gamma_2) \neq 0$.

So $\hat{g}_2(\gamma_2) \neq 0$. Let $c := \frac{\hat{g}_2(\gamma_2)}{\hat{f}_2(\gamma_2)}$. Then for all $\gamma \in \hat{\cal G}_1$ $\hat{f}_2(\gamma_1) = \hat{g}_1(\gamma) \hat{g}_2(\gamma_2) = (c g_1) \gamma$.

Hence $f_1 = c g_1$. Similarly $g_2 = c f_2$. 

II. Suppose \( f_1 \neq 0 \). Then there exists \( \gamma_1 \in \hat{G}_1 \) such that \( \hat{f}_1(\gamma_1) \neq 0 \). Hence for all \( \gamma_2 \in \hat{G}_2 : \hat{f}_1(\gamma_1) \hat{f}_2(\gamma_2) = (f_1 \otimes f_2)^* (\gamma_1, \gamma_2) = 0, \) so \( \hat{f}_2(\gamma_2) = 0. \) So \( f_2 = 0. \)

III. Let \( Z := \{ \Phi \in T_{U_1,C_1} : \Phi(c f) = \hat{c} \Phi(f) \} \). Then \( \text{emb}(H_1) \) is a dense subset of the closed set \( Z \), so \( Z = T_{U_1,C_1}. \)

In appendix C we define a so called \( \theta \)-topology for the tensor product of two locally convex topological vector spaces in which there exists sufficiently many continuous semi inner products. Note that each seminorm \( t_f \) for \( T_{U_j,C_j} \) corresponds to a semi inner product. \( j \in \{1, 2\}, f_j \in C_j. \)

**THEOREM 8.6.**

The space \( T_{U_1,C} \) is a completion of the space \( T_{U_1,C_1} \otimes_\theta T_{U_2,C_2} \).

**Proof.**

By Lemma 8.5 we can define the following map

\[
B_0 : T_{U_1,C_1} \times T_{U_2,C_2} \to H^C
\]

\[
[B_0(\Phi_1, \Phi_2)] (f_1 \otimes f_2) := \Phi_1(f_1) \otimes \Phi_2(f_2) \quad (\Phi_1 \in T_{U_1,C_1}, \Phi_2 \in T_{U_2,C_2}, f_1 \in C_1, f_2 \in C_2).
\]

Then \( B_0(\text{emb } x, \text{emb } y) = \text{emb}(x \otimes y) \) for all \( x \in H_1 \) and \( y \in H_2 \) by Lemma 8.2 II. Then \( B_0(\Phi_1, \Phi_2) \in T_{U,C} \) for all \( \Phi_1 \in T_{U_1,C_1} \) and \( \Phi_2 \in T_{U_2,C_2}. \) Indeed, let \( f_1 \in C_1, f_2 \in C_2, \) \( g \in L^1(G) \) and suppose that \( (f_1 \otimes f_2) \ast g \in C. \) Let \( Z := \{ (\Phi_1, \Phi_2) \in T_{U_1,C_1} \times T_{U_2,C_2} : \)

\[
[B_0(\Phi_1, \Phi_2)] ((f_1 \otimes f_2) \ast g) = U(g)^* [B_0(\Phi_1, \Phi_2)] (f_1 \otimes f_2).
\]

Since \( B_0(\text{emb } x, \text{emb } y) = \text{emb}(x \otimes y) \), the set \( Z \) contains \( \text{emb } H_1 \times \text{emb } H_2. \) Because \( Z \) is closed, \( Z = T_{U_1,C_1} \times T_{U_2,C_2}. \) So \( B_0(\Phi_1, \Phi_2) \in T_{U,C} \) for all \( \Phi_1 \in T_{U_1,C_1} \) and \( \Phi_2 \in T_{U_2,C_2}. \) The map \( B_0 \) is bilinear, so there exists a unique linear map \( B \) from \( T_{U_1,C_1} \otimes T_{U_2,C_2} \) into \( T_{U,C} \) such that \( B(\Phi_1 \otimes \Phi_2) = B_0(\Phi_1, \Phi_2) \) for all \( \Phi_1 \in T_{U_1,C_1} \) and \( \Phi_2 \in T_{U_2,C_2}. \)

Assertion, the map \( B \) is a homeomorphism from \( T_{U_1,C_1} \otimes_\theta T_{U_2,C_2} \) onto \( B(T_{U_1,C_1} \otimes T_{U_2,C_2}). \) Proof of the assertion. Let \( f \in C_1 \) and \( g \in C_2. \) (For typographical reasons, here we use \( f \) and \( g \) instead of \( f_1 \) and \( f_2. \)) Let \( (, )_f \) resp. \( (, )_g \) be the semi inner product on \( T_{U_1,C_1} \) resp. \( T_{U_2,C_2} \) corresponding to the seminorm \( t_f \) resp. \( t_g \). Let \( r_{f,g} \) be the seminorm on \( T_{U_1,C_1} \otimes_\theta T_{U_2,C_2} \) as defined in appendix C. Let \( n \in \mathbb{N}, \)

\( \Phi_{1i}, \ldots, \Phi_{1n} \in T_{U_1,C_1} \) and \( \Phi_{2i}, \ldots, \Phi_{2n} \in T_{U_2,C_2}. \) Let \( \xi := \sum_{i=1}^{n} \Phi_{1i} \otimes \Phi_{2i}. \) Then

\[
|t_f \otimes g \left( B(\xi) \right)|^2 = (B(\xi) (f \otimes g), B(\xi) (f \otimes g))
\]

\[
= \sum_{i,j} (B(\Phi_{1i} \otimes \Phi_{2i}) (f \otimes g), B(\Phi_{1j} \otimes \Phi_{2j}) (f \otimes g))
\]

\[
= \sum_{i,j} (\Phi_{1i}(f) \otimes \Phi_{2i}(g), \Phi_{1j}(f) \otimes \Phi_{2j}(g))
\]

\[
= \sum_{i,j} (\Phi_{1i}(f), \Phi_{1j}(f)) (\Phi_{2i}(g), \Phi_{2j}(g))
\]
\[
\sum_{i,j} (\Phi_{1i}, \Phi_{2j})_t (\Phi_{2i}, \Phi_{2j})_s
\]
\[=[r_{Ht}(\xi)]^2.
\]
So \(t_{Ht}(B(\xi)) = [r_{Ht}(\xi)]^2\) for all \(\xi \in T_{U_1,C_1} \otimes_\theta T_{U_2,C_2}\). This proves the assertion.

For all \(x \in H_1\) and \(y \in H_2\), \(B(\text{emb} x \otimes \text{emb} y) = \text{emb}(x \otimes y)\), so \(B(T_{U_1,C_1} \otimes_\theta T_{U_2,C_2}) \supset \text{emb}(H_0)\). Since \(H_0\) is dense in \(H\), \(\text{emb} H\) is dense in \(T_{U,C}\) and the map \(\text{emb}\) is continuous, we obtain that \(B(T_{U_1,C_1} \otimes_\theta T_{U_2,C_2})\) is dense in \(T_{U,C}\). So \(T_{U,C}\) is a completion of \(T_{U_1,C_1} \otimes_\theta T_{U_2,C_2}\).

\[\square\]

**DEFINITION 8.7.**

The map \(B\), as defined in the above proof, is called the **natural map** from \(T_{U_1,C_1} \otimes_\theta T_{U_2,C_2}\) into \(T_{U,C}\).

**THEOREM 8.8.**

Let \(B\) be the natural map from \(T_{U_1,C_1} \otimes_\theta T_{U_2,C_2}\) into \(T_{U,C}\). Let \(\Phi_1 \in S_{U_1,C_1}\), \(\Phi_2 \in S_{U_2,C_2}\), \(\Phi_1 \in T_{U_1,C_1}\) and \(\Phi_2 \in T_{U_2,C_2}\). Then \(<\Phi_1 \otimes \Phi_2, B(\Phi_1 \otimes \Phi_2)> = <\Phi_1, \Phi_1> <\Phi_2, \Phi_2>\).

**Proof.**

Let \(f_1 \in C_1, f_2 \in C_2, x \in H_1\) and \(y \in H_2\) such that \(\Phi_1 = U_1(f_1) x\) and \(\Phi_2 = U_2(f_2) y\). Then by Lemma 8.2 II and Theorem 1.16 II we have
\[
<\Phi_1 \otimes \Phi_2, B(\Phi_1 \otimes \Phi_2)> = <U(f_1) \otimes f_2) (x \otimes y), B(\Phi_1 \otimes \Phi_2)>
\]
\[
= (x \otimes y, [B(\Phi_1 \otimes \Phi_2)] (f_1 \otimes f_2)) = (x \otimes y, \Phi_1(f_1) \otimes \Phi_2(f_2))
\]
\[
= (x, \Phi_1(f_1)) (y, \Phi_2(f_2)) = <U(f_1) x, \Phi_1> <U(f_2) y, \Phi_2> = <\Phi_1, \Phi_1> <\Phi_2, \Phi_2>.
\]

\[\square\]

**VIII.2. Kernel theorems**

For \(j \in \{1, 2\}\) let \(G_j\) be a locally compact Abelian group, \(U_j\) a representation of \(G_j\) in a Hilbert space \(H_j\). Let \(C_j\) be a subset of \(L^1(G)\) and suppose the pair \((C_j, U_j)\) satisfies properties P1' and P2'. In this section we construct several continuous maps from \(S_{U_1,C_1}\) into \(S_{U_2,C_2}\), from \(S_{U_1,C_1}\) into \(T_{U_2,C_2}\), from \(T_{U_1,C_1}\) into \(S_{U_2,C_2}\) and from \(T_{U_1,C_1}\) into \(T_{U_2,C_2}\). We use the spaces as defined in sections VI.2 and VIII.1.

Let \(\Gamma : H_1 \rightarrow H_1'\) be the map as in Definition 6.5 and let \(\Gamma_{ext} : T_{U_1,C_1} \rightarrow T_{U_1,C_1}\) as in Definition 6.12. By Lemma 6.10 the pair \((\tilde{C}_1, \tilde{U}_1)\) satisfies properties P1' and P2', hence we can define \(G := G_1 \times G_2, H := H_1' \otimes H_2, U := \tilde{U}_1 \otimes U_2\) and \(C := \tilde{C}_1 \otimes C_2\). (See section VIII.1.) \(U\) is a representation of the locally compact Abelian group \(G\) in the Hilbert space \(H\) and the pair \((C, U)\) satisfies properties P1' and P2'.

**VIII.2.A. Continuous linear maps from \(S_{U_1,C_1}\) into \(T_{U_2,C_2}\)**
Let $\Phi \in T_{U,C}$ and $\phi_1 \in S_{U_1,C_1}$. By Lemma 8.4 III the map $\phi_2 \mapsto \Gamma_\Phi \otimes \phi_2$ is continuous from $S_{U_2,C_2}$ into $S_{U_2,C}$. By Theorem 1.18 there exists a unique $\Phi_2 \in T_{U_2,C_2}$ such that $\langle \phi_2, \Phi_2 \rangle = \langle \Gamma_\Phi \otimes \phi_2, \Phi \rangle$ for all $\phi_2 \in S_{U_2,C_2}$. So the following definition makes sense.

**DEFINITION 8.9.**

Let $\Phi \in T_{U,C}$. There exists a unique map $\Phi_{\text{ext}}$ from $S_{U_2,C_2}$ into $T_{U_2,C_2}$ such that $\langle \Phi_{\text{ext}}(\phi_1), \phi_2 \rangle = \langle \phi_1, \Phi_2 \rangle$ for all $\phi_1 \in S_{U_1,C_1}$ and $\phi_2 \in S_{U_2,C_2}$.

**THEOREM 8.10.**

I. The map $\Phi_{\text{ext}}$ is a continuous linear map from $S_{U_2,C_2}$ into $T_{U_2,C_2}$ for all $\Phi \in T_{U,C}$.

II. Let $\phi_1 \in S_{U_1,C_1}$. The map $\Phi \mapsto \Phi_{\text{ext}}(\phi_1)$ is a continuous linear map from $T_{U_2,C}$ into $T_{U_2,C_2}$.

**Proof.**

I. Clearly $\Phi_{\text{ext}}$ is linear. For all $\phi_2 \in S_{U_2,C_2}$ the map $\phi_1 \mapsto \langle \phi_2, \Phi_{\text{ext}}(\phi_1) \rangle$ is continuous from $S_{U_1,C_1}$ into $C$. By Theorem 1.18, $l \circ \Phi_{\text{ext}}$ is continuous from $S_{U_1,C_1}$ into $C$ for all $l \in T_{U_2,C'_2}$. Since $S_{U_1,C_1}$ is bornological, the map $\Phi_{\text{ext}}$ is continuous from $S_{U_1,C_1}$ into $T_{U_2,C_2}$.

II. Let $f_1 \in C_1$ and $x \in H_1$ be such that $\phi_1 = U_1(f_1)x$. Let $f_2 \in C_2$. We prove that the map $\Phi \mapsto \| [\Phi_{\text{ext}}(\phi_1)] (f_2) \|$ from $T_{U_2,C}$ into $R$ is continuous. Let $y \in H_2$. Then by Theorem 1.16 II, $\| (y, \{ \Phi_{\text{ext}}(\phi_1) \} (f_2)) \| = \| U_2(f_2)y \| < \langle \Gamma_\Phi \otimes U_2(f_2)y, \Phi \rangle = \| U_1(f_1)\Gamma_2 \otimes U_2(f_2)y, \Phi \rangle = \| U_1(f_1)\Gamma_2 \otimes \Phi_{\text{ext}}(\phi_1) \|$. Hence $\| [\Phi_{\text{ext}}(\phi_1)] (f_2) \| \leq \| x \| \| \Phi_{\text{ext}}(\phi_1) \| = \| x \| \| \Phi_{\text{ext}}(\phi_1) \|$. So the map $\Phi \mapsto \Phi_{\text{ext}}(\phi_1)$ is continuous from $T_{U_2,C}$ into $T_{U_2,C_2}$.

**LEMMA 8.11.**

Let $x \in H_1$, $y \in H_2$ and $\phi_1 \in S_{U_1,C_1}$. Let $\Phi := \text{emb}(\Gamma_x \otimes y)$. Then $\Phi_{\text{ext}}(\phi_1) = (\phi_1, x) \text{emb} y$.

**Proof.**

Obviously $(\phi_1, x) \text{emb} y \in T_{U_2,C_2}$. For all $\phi_2 \in S_{U_2,C_2}$ we obtain: $\langle \phi_2, \Phi_{\text{ext}}(\phi_1) \rangle = \langle \Gamma_\Phi \otimes \phi_2, \text{emb}(\Gamma_x \otimes y) \rangle = \langle \Gamma_\Phi \otimes \phi_2, \Gamma_x \otimes y \rangle = (x, \phi_1) \langle \phi_2, y \rangle = \langle \phi_2, \text{emb}(\phi_1, x) y \rangle$.

So $\Phi_{\text{ext}}(\phi_1) = (\phi_1, x) \text{emb} y$.

We can describe the maps $\Phi_{\text{ext}}$, $\Phi \in T_{U,C}$ in another way. Let $H_{HS}$ be the Hilbert space of all Hilbert-Schmidt operators from $H_1$ into $H_2$. Some authors define the tensor product $H_1 \otimes H_2$ of
two Hilbert spaces $H_1$ and $H_2$ as the set of all Hilbert-Schmidt operators from $H_1$ into $H_2$. There
tensor is not bilinear and hence $(H_1 \hat{\otimes} H_2) \hat{\otimes} H_3$ is not isomorphic with $H_1 \hat{\otimes}(H_2 \hat{\otimes} H_3)$ in a
canonical way, with $H_3$ a third Hilbert space. We shall not use those definitions.

In the following lemma we prove that $H = H_1' \hat{\otimes} H_2$ (our definition) is unitary isomorphic with
$H_{HS}$.

LEMMA 8.12.

There exists a unique unitary map $A$ from $H$ onto $H_{HS}$ such that $[A(l \otimes y)](x) = l(x)y$ for all
$l \in H_1'$, $y \in H_2$ and $x \in H_1$.

Proof.

Define $A_1 : H_1' \times H_2 \rightarrow H_{HS}$

$$[A_1(l, y)](x) = l(x)y \quad (l \in H', y \in H_2, x \in H_1).$$

The map $A_1$ is bilinear, so there exists a unique linear map $A_2$ from $H_1' \otimes H_2$ into $H_{HS}$ such that

$$A_2(l \otimes y) = A_1(l, y) \quad \text{for all } l \in H_1' \text{ and } y \in H_2.$$

Let $\xi \in H_1'$. There exists $n, m \in \mathbb{N}$, an orthonormal set $l_1, \ldots, l_n$ in $H_1'$, an orthonormal set $y_1, \ldots, y_m$ in $H_2$ and for all $i \leq n$ and $j \leq m$ there exists $\lambda_{ij} \in \mathbb{C}$ such that

$$\xi = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij} l_i \otimes y_j.$$

For all $i \leq n$ let $x_i \in H_1$ such that $l_i = \Gamma_{x_i}$.

Then

$$\|A_2(\xi)\|_{H_{HS}}^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij}^2 \|l_i \otimes y_j\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij}^2 = \|\xi\|^2,$$

so $A_2$ is an isometry from $H_1' \otimes H_2$ into $H_{HS}$. Since $H_{HS}$ is complete, there exists a unique
isometry $A$ from $H = H_1' \otimes H_2$ into $H_{HS}$ such that $A(\xi) = A_2(\xi)$ for all $\xi \in H_1' \otimes H_2$. Because

$A(H_1' \otimes H_2)$ is a dense subspace of $H_{HS}$, the map $A$ is surjective, hence unitary.

Obviously the map $A$ is unique.

Let $A$ be the unitary map from $H$ onto $H_{HS}$ as in Lemma 8.12.

THEOREM 8.13.

Let $\Phi \in T_{U, C}, f_1 \in C_1$ and $\Phi_1 \in R_{f_1} \subset S_{U_1, C_1}$. Then $[\Phi_{ext}(\Phi_1)](f_2) = [A(\Phi(f_1 \otimes f_2))](\Omega_{f_1}^{-1}(\Phi_1))$

for all $f_2 \in C_2$.

Proof.

Let $f_1 \in C_1$, $\Phi_1 \in R_{f_1}$ and $f_2 \in C_2$. Let

$$Z := \{\Phi \in T_{U, C} : [\Phi_{ext}(\Phi_1)](f_2) = [A(\Phi(f_1 \otimes f_2))](\Omega_{f_1}^{-1}(\Phi_1)).\}.$$

By Theorem 8.10 II, the set $Z$ is a closed linear subspace of $T_{U, C}$. Let $x \in H_1$ and $y \in H_2$. Let

$\Phi := \text{emb}(\Gamma_x \otimes y)$. Then $[A(\Phi(f_1 \otimes f_2))](\Omega_{f_1}^{-1}(\Phi_1)) = [A \ U(\tilde{f}_1 \otimes f_2)^* (\Gamma_x \otimes y)](\Omega_{f_1}^{-1}(\Phi_1)) = $
In the nuclear case we can prove that \( \{ \Phi_{\text{ext}} : \Phi \in SU,c \} \) contains all continuous linear maps from \( SU,c \) into \( TU,c \). We need some lemmas.

**LEMMA 8.14.**

Let \( g_1 \in L^1(G_1) \), \( g_2 \in L^1(G_2) \) and \( \eta \in H_{HS} \). Then \( U_2(g_2) \eta U_1(g_1) = A(U(g_1 \otimes g_2)A^{-1} \eta) \).

**Proof.**

Let \( g_1 \in L^1(G_1) \) and \( g_2 \in L^1(G_2) \). Let \( x \in H_1 \) and \( y \in H_2 \). Let

\[
Z := \{ \eta \in H_{HS} : (U_2(g_2) \eta U_1(g_1) x, y) = (A(U(g_1 \otimes g_2)A^{-1} \eta) (x), y) \}.
\]

Obviously \( Z \) is a closed subset of \( H_{HS} \). Let \( v \in H_1 \) and \( w \in H_2 \). We prove that \( A(\Gamma, \otimes w) \in Z \).

Indeed,

\[
(U_2(g_2) A(\Gamma, \otimes w) U_1(g_1) x, y) = (U_2(g_2) [(U_1(g_1) x, v) w], y) =
\]

\[
((x, U_1(g_1) v) U_2(g_2) w, y) = (A(\Gamma_U, \otimes w) U_2(g_2) w) (x), y) =
\]

\[
((A(\Gamma_U, \otimes g_2) (\Gamma, \otimes w)) (x), y). \quad \text{So } \ A(\Gamma, \otimes w) \in Z.
\]

Hence \( A(H_0) \subset Z \). Since \( A(H_0) \) is a dense subspace of \( H_{HS} \), \( Z = H_{HS}^\perp \). The lemma follows.

**LEMMA 8.15.**

Let \( f_1 \in C_1 \), \( f_2 \in C_2 \) and \( \xi \in N_{f_1 \otimes f_2}^\perp \). Then \( U_2(f_2) [(A \xi) (x)] = 0 \) for all \( x \in N_{f_1}^\perp \).

**Proof.**

Let \( Z := \{ x \in H_1 : U_2(f_2) [(A \xi) (x)] = 0 \} \). \( Z \) is a closed subset of \( H_1 \). Let \( y \in H_1 \). Then

\[
U_2(f_2) (A \xi) U_1(f_1) x = A [U_1(f_1 \otimes f_2) \xi] (y) = 0 \] by Lemma 8.14. So \( U_1(f_1)(H_1) \subset Z \). Then \( N_{f_1}^\perp = U_1(f_1)^\perp (H_1) = U_1(f_1) (H_1) \subset Z \).

We prove the first kernel theorem. Note that we do not assume properties P3 or P4.

**THEOREM 8.16.**

Suppose \( SU,c_1 \) is nuclear or \( TU,c_2 \) is nuclear. Then \( \{ \Phi_{\text{ext}} : \Phi \in TU,c \} \) is the set of all continuous linear maps from \( SU,c_1 \) into \( TU,c_2 \).

**Proof.**

By Theorem 8.10 I, the map \( \Phi_{\text{ext}} \) is continuous for all \( \Phi \in TU,c \). Let \( E \) be a continuous linear map from \( SU,c_1 \) into \( TU,c_2 \). We prove in five steps that there exists \( \Phi \in TU,c \) such that \( E = \Phi_{\text{ext}} \).
Step 1. Let $f_1 \in C_1$ and $f_2 \in C_2$. Define $E_{f_1f_2} : H_1 \to H_2$

$$E_{f_1f_2}(x) := [E\ U_1(f_1)\ x]\ (f_2) \quad (x \in H_1).$$

The map $E_{f_1f_2}$ is nuclear. Indeed, suppose $S_{U_1,C_1}$ is a nuclear space. The map $\phi \mapsto [E\ \phi](f_2)$ is a continuous linear map from $S_{U_1,C_1}$ into the (Banach) space $H_2$, hence this map is nuclear. (See [Sch] III 7.2b.) The map $x \mapsto U_1(f_1)x$ from $H_1$ into $S_{U_1,C_1}$ is continuous, so the composition map $E_{f_1f_2}$ is a nuclear map from $H_1$ into $H_2$. Suppose $T_{U_3,C_3}$ is a nuclear map. The map $\Psi \mapsto \Psi(f_2)$ from $T_{U_3,C_3}$ into $H_2$ is continuous hence nuclear. The map $x \mapsto E\ U_1(f_1)x$ from $H_1$ into $T_{U_3,C_3}$ is continuous, so again $E_{f_1f_2}$ is nuclear. In both cases the map $E_{f_1f_2}$ is nuclear, hence a Hilbert-Schmidt operator.

By Lemma 8.5 we obtain that $E_{f_1f_2} = E_{g_1g_2}$ for all $f_1, g_1 \in C_1$ and $f_2, g_2 \in C_2$ such that $f_1 \otimes f_2 = g_1 \otimes g_2$. So the following definition makes sense.

Define $\Phi : C \to H$

$$\Phi(\tilde{f}_1 \otimes f_2) := A^{-1}(E_{f_1f_2}) \quad (f_1 \in C_1, f_2 \in C_2).$$

In step 4 we prove that $\Phi$ is a C-trajectory and in step 5 the proof finishes with $E = \Phi_{\text{ext}}$. We have to prove that $\Phi(f\ * g) = U(g)*\ \Phi(f)$ for all $f = f_1 \otimes f_2 \in C$ and $g \in L^1(G)$ such that $f\ * g \in C$. Note that in general it is not possible to write $g = g_1 \otimes g_2$ with $g_1 \in L^1(G_1)$ and $g_2 \in L^1(G_2)$ such that $f_1 \otimes g_1 \in C_1$ and $f_2 \otimes g_2 \in C_2$ ($g$ factorizes). If $g$ factorizes, then we prove in step 3 that $\Phi(f\ * g) = U(g)*\ \Phi(f)$. The technical step 2 is used in step 4 where we show that $\Phi(f\ * g) = U(g)*\ \Phi(f)$ for all $g \in L^1(G)$ such that $f\ * g \in C$.

Step 2. Let $f_1 \in C_1$ and $f_2 \in C_2$. Then $\Phi(\tilde{f}_1 \otimes f_2) \in N_{f_1f_2}^{x}$. Indeed, let $\xi \in N_{f_1f_2}^{x}$. Let $(e_i)_{i \in I}$ be an orthonormal set in $H_1$, $J$ a subset of $I$ such that $(e_i)_{i \in I}$ is a basis for $H_1$ and $(e_i)_{i \in J}$ is a basis for $N_{f_1}^{x}$. Then $U_1(f_1)e_i = 0$ for all $i \in I \setminus J$, so $E_{f_1f_2} e_i = 0$. By Lemmas 8.15 and 11.12 we obtain for all $i \in J$ that $(A\ \xi) (e_i) \in N_{f_1}$ and $[E\ U_1(f_1)e_i]\ (f_2) \in N_{f_1}^{x}$. Hence $((E\ U_1(f_1)e_i)\ (f_2), (A\ \xi) (e_i)) = 0$ for all $i \in J$. Then

$$\Phi(\tilde{f}_1 \otimes f_2), (A\ \xi) = \sum_{i \in J} ((E\ U_1(f_1)e_i)\ (f_2), (A\ \xi) (e_i)) = 0.$$

Step 3. Let $f_1, h_1 \in C_1; f_2, h_2 \in C_2; g_1 \in L^1(G_1); g_2 \in L^1(G_2)$ and suppose that $h_1 = g_1 * f_1$ and $h_2 = g_2 * f_2$. Then $\Phi(\tilde{h}_1 \otimes h_2) = U(\tilde{g}_1 \otimes g_2) * \Phi(\tilde{f}_1 \otimes f_2)$. To prove step 3, we use Lemma 8.14. For all $x \in H_1$, $E_{h_1,h_2}\ (x) = [E\ U_1(h_1)\ x]\ (h_2) = U_2(g_2)^*\ [E\ U_1(f_1)\ U_1(g_1)\ x]\ (f_2) = U_2(g_2)^*\ E_{f_1f_2}\ U_1(g_1)\ x$, so $E_{h_1h_2} = U_2(g_2)^*\ E_{f_1f_2}\ U_1(g_1)$. Hence $\Phi(\tilde{h}_1 \otimes h_2) = A^{-1}(E_{h_1,h_2}) = A^{-1}(U_2(g_2)^*\ E_{f_1f_2}\ U_1(g_1)) = (U(\tilde{g}_1 \otimes g_2) * A^{-1}\ E_{f_1f_2}) = U(\tilde{g}_1 \otimes g_2) * \Phi(\tilde{f}_1 \otimes f_2)$.

Step 4. In this step we prove that $\Phi \in T_{U,C}$. Let $f_1, h_1 \in C_1; f_2, h_2 \in C_2; g \in L^1(G)$ and suppose that $\tilde{h}_1 \otimes h_2 = (\tilde{f}_1 \otimes f_2) * g$. We prove that $\Phi(\tilde{h}_1 \otimes h_2) = U(g)*\ \Phi(\tilde{f}_1 \otimes f_2)$. By property P1' there exists for all $j \in \{1, 2\}$: $k_j \in C_j, f_j', h_j' \in L^1(G_j)$ such that $f_j = k_j * f_j'$ and $h_j = k_j * h_j'$. Let $f := \tilde{f}_1 \otimes f_2', f' := \tilde{f}_1' \otimes f_2', h := \tilde{h}_1 \otimes h_2, h' := \tilde{h}_1' \otimes h_2'$ and $k := \tilde{k}_1 \otimes k_2$. Then $h' * k = h = g * f = (g \ * f') * k$, so $(h' - g \ * f') * k = 0$. In step 2 we have proved that
Now we can use Lemma 1.20 and we obtain that \( U(h' - g * f') * \Phi(k) = 0 \). So \( U(h') * \Phi(k) = U(g) * U(f') * \Phi(k) \). Then by step 3, \( \Phi(h) = U(h') * \Phi(k) = U(g) * U(f') * \Phi(k) = U(g) * \Phi(f) \).

**Step 5.** We finish with the proof that \( E = \Phi_{\text{ext}} \). Let \( \phi_1 \in S_{U_1,c_1} \) and \( f_2 \in C_2 \). Let \( f_1 \in C_1 \) be such that \( \phi_1 \in R_{f_1} \). Then by Theorem 8.13: \( \{\Phi_{\text{ext}}(\phi_1)\}(f_2) = \{A \Phi(f_1 \otimes f_2) \}(\Omega_j f_1(\phi_1)) = E_{f_1, f_2}(\Omega_j f_1(\phi_1)) = \{E U_1(f_1) \Omega_j f_1(\phi_1)\}(f_2) = \{E(\phi_1)\}(f_2) \). So \( E = \Phi_{\text{ext}} \).

In Theorem 8.6 we proved that \( T_{U_1,c} \) is a completion of \( T_{U_1, \tilde{c}} \otimes_\theta T_{U_2,c_2} \) into \( T_{U,c} \). Let \( \Phi_1 \in T_{U_1,c_1} \), \( \Phi_2 \in T_{U_2,c_2} \) and let \( \Phi := B(\Gamma_{\text{ext}}(\Phi_1) \otimes \Phi_2) \). Then \( \Phi_{\text{ext}}(\phi_1) = \Phi_1 \otimes \Phi_2 \) for all \( \phi_1 \in S_{U_1,c_1} \), by Theorem 8.8. (Cf. the proof of Lemma 8.11.) So for every \( \xi \in T_{U_1, \tilde{c}} \otimes_\theta T_{U_2,c_2} \) we can define a continuous linear map \( W(\xi) \) from \( S_{U_1,c_1} \) into \( T_{U_2,c_2} \). Define a topology for \( L(S_{U_1,c_1} , T_{U_1,c_1}) \), the vector space of all continuous linear maps from \( S_{U_1,c_1} \) into \( T_{U_1,c_1} \), by the generating set of seminorms \( p_{f_1, f_2} \).

These seminorms are defined by \( p_{f_1, f_2} \) \( (E) := \sup\{\|E U_1(f_1)x\} (f_2)\| : x \in H_1, \|x\| \leq 1\}, \)

\( f_1 \in C_1, \ f_2 \in C_2, \ E \in L(S_{U_1,c_1} , T_{U_1,c_1}) \). Then the map \( \Phi \mapsto \Phi_{\text{ext}} \) from \( T_{U,c} \) into \( L(S_{U_1,c_1} , T_{U_1,c_1}) \) is continuous, hence the map \( W \) can be extended to a unique continuous linear map \( W_1 \) from \( T_{U_1, \tilde{c}} \otimes_\theta T_{U_2,c_2} \) into \( L(S_{U_1,c_1} , T_{U_2,c_2}) \). By the previous theorem, every continuous linear map from \( S_{U_1,c_1} \) into \( T_{U_2,c_2} \) is equal to some \( W_1(\xi) \), with \( \xi \in T_{U_1, \tilde{c}} \otimes_\theta T_{U_2,c_2} \), if \( S_{U_1,c_1} \) or \( T_{U_2,c_2} \) is a nuclear space.

In the following subsection we define a topology for \( L(T_{U_1,c_1} , T_{U_2,c_2}) \), the set of all continuous linear maps from \( T_{U_1,c_1} \) into \( T_{U_2,c_2} \). Also we define a map from \( S_{T_{U_1,c_1} , T_{U_2,c_2}} \) into \( L(T_{U_1,c_1} , T_{U_2,c_2}) \) in a natural way. In the nuclear case we prove that this map is surjective.

We repeat these investigations for the spaces \( L(S_{U_1,c_1} , S_{U_2,c_2}) \) and \( L(T_{U_1,c_1} , S_{U_2,c_2}) \).

**VIII.2.B. Continuous linear maps from \( T_{U_1,c_1} \) into \( T_{U_2,c_2} \)**

In this subsection we suppose in addition that the pair \((C_1 , U_1)\) satisfies properties P3 and P4.

Let \( L(T_{U_1,c_1} , T_{U_2,c_2}) \) be the vector space of all continuous linear maps from \( T_{U_1,c_1} \) into \( T_{U_2,c_2} \). For every \( F_1 \in C^*_1 \) and \( f_2 \in C_2 \) define the seminorm \( p_{F_1, f_2} \) on \( L(T_{U_1,c_1} , T_{U_2,c_2}) \) by

\[ p_{F_1, f_2}(E) := \sup\{\|E U_1(1_x)(f_2)\| : x \in H_1, \|x\| \leq 1\} \]

\((E \in L(T_{U_1,c_1} , T_{U_2,c_2}))\). By Corollary 2.32, the set \( \{p_{F_1, f_2} : F_1 \in C^*_1, f_2 \in C_2\} \) defines a locally convex Hausdorff topology for \( L(T_{U_1,c_1} , T_{U_2,c_2}) \).

Define \( W_1 : S_{T_{U_1, \tilde{c}}} \times T_{U_2,c_2} \rightarrow L(T_{U_1,c_1} , T_{U_2,c_2}) \)

\[ [W_1(\Gamma_1, \Phi_2)](\Phi_1) := \frac{\Phi_1 \otimes \Phi_2}{\Phi_1 \otimes \Phi_2}. \quad (\Phi_1 \in S_{U_1,c_1}, \Phi_2 \in T_{U_2,c_2}, \Phi_1 \in T_{U_1,c_1}). \]

The map \( W_1 \) is a bilinear map, so there exists a unique linear map \( W_2 \) from \( S_{T_{U_1, \tilde{c}}} \otimes T_{U_2,c_2} \) into \( L(T_{U_1,c_1} , T_{U_2,c_2}) \) such that \( W_2(\Gamma_1 \otimes \Phi_2) = W_1(\Gamma_1, \Phi_2) \) for all \( \phi_1 \in S_{U_1,c_1} \) and \( \Phi \in T_{U_2,c_2} \).
We shall prove that the map $W_2$ is continuous from $S_{U_1, C_1} \otimes \theta T_{U_2, C_2}$ into $L(T_{U_1, C_1}, T_{U_2, C_2})$ and that this map can be extended to a continuous map defined on $S_{U_1, C_1} \otimes \theta T_{U_2, C_2}$. For this purpose we define a topological vector space $Z$ and prove that this space is a completion of $S_{U_1, C_1} \otimes \theta T_{U_2, C_2}$. Recall that $H_{HS}$ is the Hilbert space of all Hilbert-Schmidt operators from $H_1$ into $H_2$ and let $A$ be the map from $H := H_1' \otimes H_2$ onto $H_{HS}$ as in Lemma 8.12.

**Lemma 8.17.**

Let $X \in H_{HS}$ and let $(Y_\alpha)_{\alpha \in M}$ be a bounded net of continuous linear maps from $H_2$ into $H_2$. Let $I$ be the identity operator on $H_2$ and suppose $\lim_{\alpha} Y_\alpha = I$. Then $\lim_{\alpha} Y_\alpha X = X$ in $H_{HS}$.

**Proof.**

Let $\alpha > 0$ be such that $\| Y_\alpha \| < \alpha$ for all $\alpha \in M$. Let $\varepsilon > 0$. There exists a finite rank projection $P$ from $H_2$ into $H_2$ such that $\| PX - X \|_{HS} \leq \frac{\varepsilon}{3(1+\alpha)}$. Clearly $\lim_{\alpha} \| Y_\alpha P - P \|_{HS} = 0$. So there exists $\alpha_0 \in M$ such that $\| Y_\alpha P - P \|_{HS} \leq \frac{\varepsilon}{3(1+\alpha_0)}$ for all $\alpha \in M$ with $\alpha \geq \alpha_0$. Then for all $\alpha \in M$ with $\alpha \geq \alpha_0$ we obtain

$$\| Y_\alpha X - X \|_{HS} \leq \| Y_\alpha X - Y_\alpha P X \|_{HS} + \| Y_\alpha P X - P X \|_{HS} + \| P X - X \|_{HS} \leq \| Y_\alpha \| \| X - P X \|_{HS} + \| Y_\alpha P - P \|_{HS} \| X \| + \| P X - X \|_{HS} \leq \varepsilon.$$ 

**Corollary 8.18.**

Let $X \in H_{HS}$ and let $(Y_\alpha)_{\alpha \in M}$ be a bounded net of continuous normal linear map from $H_1$ into $H_1$. Let $I$ be the identity operator on $H_1$ and suppose $\lim_{\alpha} Y_\alpha = I$. Then $\lim_{\alpha} X Y_\alpha = X$ in $H_{HS}$.

**Proof.**

Let $x \in H_1$. Then $\lim_{\alpha} \| Y_\alpha^* x - x \| = \lim_{\alpha} \| (Y_\alpha - I)^* x \| = \lim_{\alpha} \| (Y_\alpha - I) x \| = 0$, since all $Y_\alpha$ are normal operators. Interchanging the role of $H_1$ and $H_2$ in Lemma 8.17 we obtain that

$$\lim_{\alpha} \| Y_\alpha X - X \|_{HS} = \lim_{\alpha} \| (X Y_\alpha - X) \|_{HS} = \lim_{\alpha} \| Y_\alpha^* X^* - X^* \|_{HS} = 0.$$

**Definition 8.19.**

Let $Z$ be the vector space of all maps $\Phi$ from $C^1 \times C_2$ into $H_{HS}$ such that
\( \Phi(F_1, f_2 * g_2) = U_2(g_2) \Phi(F_1, f_2) \)

for all \( F_1 \in C^d_1, f_2 \in C_2 \) and \( g_2 \in L^1(G_2) \) such that \( f_2 * g_2 \in C_2 \); and

\( \Phi(F_1, K_1, f_2) = \Phi(F_1, f_2) U_1[K_1] \)

for all \( F_1 \in C^d_1, K_1 \in \text{Bor}_c(\hat{G}_1, C) \) and \( f_2 \in C_2 \).

The Hausdorff topology for \( Z \) is the locally convex topology for \( Z \) generated by the seminorms \( z_{F_1, f_2} \), defined by \( z_{F_1, f_2}(\Phi) := \| \Phi(F_1, f_2) \|_{HS} \) \((F_1 \in C^d_1, f_2 \in C_2, \Phi \in Z)\).

**THEOREM 8.20.**

\( Z \) is a completion of \( S_{U_1, c_1} \otimes_\theta T_{U_2, c_2} \).

**Proof.**

\( Z \) is complete. This can be proved similarly to the proof of Theorem 1.13.

Define \( \Lambda_1 : S_{U_1, c_1} \times T_{U_2, c_2} \to Z \)

\[ \{([\Lambda_1(\Gamma_{\Phi_1, \Phi_2}) (F_1, f_2]) (\alpha) = (\alpha, U_1[\bar{F}_1] \Phi_1) \Phi_2 (f_2). \]

\( (\Phi_1 \in S_{U_1, c_1}, \Phi_2 \in T_{U_2, c_2}, F_1 \in C^d_1, f_2 \in C_2, \alpha \in H_1) \).

The map \( \Lambda_1 \) is bilinear, so there exists a unique linear map \( \Lambda_2 \) from \( S_{U_1, c_1} \otimes T_{U_2, c_2} \) into \( Z \) such that \( \Lambda_2(\Gamma_{\Phi_1} \otimes \Phi_2) = \Lambda_1(\Gamma_{\Phi_1, \Phi_2}) \) for all \( \Phi_1 \in S_{U_1, c_1} \) and \( \Phi_2 \in T_{U_2, c_2} \). We show that \( \Lambda_2 \) is a topological homeomorphism from \( S_{U_1, c_1} \otimes T_{U_2, c_2} \) onto \( \Lambda_2(S_{U_1, c_1} \otimes T_{U_2, c_2}) \) with relative topology from \( Z \). Let \( N \in N, \Phi_{I1}, \ldots, \Phi_{IN} \in S_{U_1, c_1}, \Phi_{21}, \ldots, \Phi_{2N} \in T_{U_2, c_2} \) and let

\[ \xi := \sum_{i=1}^N \Gamma_{\Phi_{Ii}} \otimes \Phi_{2i}. \]

Let \( (x_\alpha)_{\alpha \in \mathcal{M}} \) be an orthonormal basis for \( H_1 \). Then for all \( F_1 \in C^d_1 \) and \( f_2 \in C_2 \) we obtain:

\[ z_{F_1, f_2}(\Lambda_2(\xi))^2 = \sum_{\alpha \in \mathcal{M}} \| \{\Lambda_2(\xi) (F_1, f_2)\} (\alpha) \|^2 \]

\[ = \sum_{\alpha} \sum_{i=1}^N (x_\alpha, U_1[\bar{F}_1] \Phi_{Ii}) \Phi_{2i}(f_2))^2 \]

\[ = \sum_{\alpha} \sum_{i=1}^N \sum_{j=1}^N (U_1[\bar{F}_1] \Phi_{Ii}, x_\alpha) (x_\alpha, U_1[\bar{F}_1] \Phi_{Ii}) (\Phi_{2i}(f_2), \Phi_{2j}(f_2)) \]

\[ = \sum_{i=1}^N \sum_{j=1}^N (U_1[\bar{F}_1] \Gamma_{\Phi_{Ii}}, U_1[\bar{F}_1] \Gamma_{\Phi_{Ii}}) (\Phi_{2i}(f_2), \Phi_{2j}(f_2)) \]

\[ = \sum_{i=1}^N \sum_{j=1}^N (U_1[\bar{F}_1] \Gamma_{\Phi_{Ii}}, U_1[\bar{F}_1] \Gamma_{\Phi_{Ii}}) (\Phi_{2i}(f_2), \Phi_{2j}(f_2)) \]
(Cf. appendix C.) So the map $\Lambda_2$ from $S\bar{U}_1, \hat{c}_1 \otimes_0 T_{U_2, C_2}$ onto $\Lambda_2(S\bar{U}_1, \hat{c}_1 \otimes T_{U_2, C_2}) \subset Z$ is a topological homeomorphism.

Finally we have to prove that $Z_1 := \Lambda_2(S\bar{U}_1, \hat{c}_1 \otimes T_{U_2, C_2})$ is dense in $Z$. Let $\Phi \in Z$. Similarly to the proof of Lemma 1.9 we obtain that $U_2(g_2)^* \Phi(F_1, f_2) = U_2(f_2)^* \Phi(F_1, g_2)$ for all $F_1 \in C_1^\#$ and $f_2, g_2 \in C_2$. Let $(f_{2\lambda})_{\lambda \in J}$ be the bounded net in $C_2$ as in property P2'. For $\lambda \in J$ define

$$\Phi_\lambda : C_1^\# \times C_2 \to H_{HS}$$

$$\Phi_\lambda(F_1, f_2) := U_2(f_2)^* \Phi(F_1, f_{2\lambda}) \quad (F_1 \in C_1^\#, f_2 \in C_2).$$

Then $\Phi_\lambda \in Z$ for all $\lambda \in J$. Let $F_1 \in C_1^\#$ and $f_2 \in C_2$. Then $s - \lim \lambda U_2(f_{2\lambda})^* = I$, so by Lemma 8.17:

$$\lim_{\lambda} s_{F_1, f_2}(\Phi - \Phi_\lambda) = \lim_{\lambda} \| \Phi(F_1, f_2) - U_2(f_2)^* \Phi(F_1, f_{2\lambda}) \|_{HS} = 0.$$ 

$$\lim_{\lambda} \| \Phi(F_1, f_2) - U_2(f_2)^* \Phi(F_1, f_{2\lambda}) \|_{HS} = 0. \quad \text{So } \lim_{\lambda} \| \Phi_\lambda \|_{HS} = \Phi \in Z. \quad \text{We are ready if } \Phi_\lambda \in Z_1 \text{ for all } \lambda \in J.$$ 

Let $\lambda \in J$. Let $Q_1, Q_2, \ldots \subset \hat{G}_1$ as in property P3 for the pair $(C_1, U_1)$. For $N \in \mathbb{N}$ define $L_N : \hat{G}_1 \to \mathbb{R}$

$$L_N(\gamma) = \begin{cases} 1 & \text{if } n \leq N \text{ and } U_1[1_{Q_n}] \neq 0, \\ 0 & \text{if } n > N \text{ or } U_1[1_{Q_n}] = 0. \end{cases} \quad (n \in \mathbb{N}, \gamma \in Q_n)$$

Then by Lemma 3.3 II, $L_N F_1$ is bounded for all $F_1 \in C_1^\#$, hence $\Phi_\lambda(L_N F_1, f_2) = \Phi_\lambda(L_N, f_2) U_1[L_N F_1]$ for all $f_2 \in C_2$. For $N \in \mathbb{N}$ define $\Phi_{\lambda N} : C_1^\# \times C_2 \to H_{HS}$

$$\Phi_{\lambda N}(F_1, f_2) := \Phi_\lambda(L_N, f_2) U_1[L_N F_1] \quad (F_1 \in C_1^\#, f_2 \in C_2).$$

Then $\Phi_{\lambda N} \in Z$ for all $N \in \mathbb{N}$. Since $s - \lim_{N \to \infty} U_1[L_N] = I$ and the operator $U_1[L_N]$ is normal and bounded by 1 for all $N \in \mathbb{N}$, by Corollary 8.18 we obtain for all $F_1 \in C_1^\#$ and $f_2 \in C_2$ that

$$\lim_{N \to \infty} s_{F_1, f_2}(\Phi_{\lambda N} - \Phi_\lambda) = \lim_{N \to \infty} \| \Phi_{\lambda N}(F_1, f_2) - \Phi_\lambda(F_1, f_{2\lambda}) \|_{HS} = 0.$$ 

So $\lim_{N \to \infty} \Phi_{\lambda N} = \Phi_\lambda$ in $Z$.

We finish with the proof that $\Phi_{\lambda N} \in Z_1$ for all $N \in \mathbb{N}$. Let $N \in \mathbb{N}$. The set $S\bar{U}_1, \hat{c}_1 \otimes H_2$ is dense in $H_1 \otimes H_2$, $H' \otimes H_2$ is dense in $H$ and $A$ is a unitary map, so the set $A(S\bar{U}_1, \hat{c}_1 \otimes H_2)$ is dense in $H_{HS}$. Hence for every $\varepsilon > 0$ there exists $\xi_\varepsilon \in S\bar{U}_1, \hat{c}_1 \otimes H_2$ such that

$$\| A(\xi_\varepsilon) - \Phi(L_N, f_{2\lambda}) \|_{HS} < \varepsilon. \quad \text{Let } \varepsilon > 0. \text{ There exists } M \in \mathbb{N}, \phi_{11}, \ldots, \phi_{1M} \in S_U, c_1,$$

$$y_1, \ldots, y_M \in H_2 \text{ such that } \xi_\varepsilon = \sum_{i=1}^M \Gamma_{\phi_i} \otimes y_i. \text{ Then } U_1[L_N] \phi_{1i} \in S_U, c_1 \text{ for all } i \leq M. \text{ (See Corollary 2.26.) Define }$$

$$\xi_\varepsilon := \sum_{i=1}^M \Gamma_{U_1[L_N] \phi_{1i}} \otimes \text{emb} y_i. \text{ Then } \xi_\varepsilon \in S\bar{U}_1, \hat{c}_1 \otimes T_{U_2, c_2}. \text{ For all }$$
Let $F_1 \in C^p_1$, $f_2 \in C_2$ and $x \in H_1$ we obtain that

$$\left[\Lambda_2(\zeta)\right](F_1, f_2) = \sum_{i=1}^{M} (\Phi_i) U_2(f_2)^* \sum_{i=1}^{M} (U_1[F_1 L_N] x, \Phi_i) y_i =$$

$$U_2(f_2)^* \left[\Lambda_2(\zeta)\right](U_1[F_1 L_N] x).$$

Hence for all $F_1 \in C^p_1$, $f_2 \in C_2$ and $\varepsilon > 0$ we obtain $z_{F_1, f_2}(\Lambda_2(\zeta) - \Phi_{AN}) = \| \Lambda_2(\zeta) \|_H S \leq \| U_2(f_2)^* \Phi_{AN} \|_H S \leq \| U_1[F_1 L_N] \|_H S \leq \varepsilon \| U_2(f_2)^* \| \| U_1[F_1 L_N] \|_H S.$

So $\lim_{\varepsilon \to 0} z_{F_1, f_2}(\Lambda_2(\zeta) - \Phi_{AN}) = 0$ for all $F_1 \in C^p_1$ and $f_2 \in C_2$. Then $\lim_{\varepsilon \to 0} \Lambda_2(\zeta) = \Phi_{AN}$ in $Z$ and $\Phi_{AN} \in \bar{Z}_1$.

Let $\Lambda_2 : S_{U_1, C_1} \otimes \theta T_{U_2, C_2} \to Z$ be the map as in the proof of Theorem 8.20. For every $\Phi \in Z$ we define a continuous linear map from $T_{U_1, C_1}$ into $T_{U_2, C_2}$. This definition coincides with the definition of $W_2$. (See Lemma 8.22 IV.)

**DEFINITION 8.21.**

Let $\Phi \in Z$. Define $V(\Phi) : T_{U_1, C_1} \to T_{U_2, C_2}$ by $\left[\left[V(\Phi)\right](U_1[F_1] x)\right](f_2) = \left[\Phi(F_1, f_2)\right](x)$ for all $F_1 \in C^p_1$, $x \in H_1$ and $f_2 \in C_2$. (Cf. Lemma 2.18 II, Corollary 2.32 and the definition of $Z$.)

**LEMMA 8.22.**

1. $V(\Phi)$ is continuous for all $\Phi \in Z$.
2. $\left[\left[V(\Phi)\right](U_1[F_1] x)\right](f_2) = \left[\Phi(F_1, f_2)\right](x)$ for all $F_1 \in C^p_1$, $x \in H_1$ and $f_2 \in C_2$.
3. The map $V$ is continuous from $Z$ into $L(T_{U_1, C_1}, T_{U_2, C_2})$.
4. $W_2 = V \circ \Lambda_2$.

**Proof.**

1. The topology for $T_{U_1, C_1}$ is equal to the inductive limit topology $\tau_{ind}$. (See Theorem 3.4 I $\Rightarrow$ II.) So the assertion follows.
2. Let $F_1 \in C^p_1$, $x \in H_1$ and $f_2 \in C_2$. Let $L_1 := |F_1| + 1$. Then $L_1 \in C^p_1$. Let $K_1 := \frac{F_1}{L_1}$.

Then $K_1 \in \text{Bor}_V(G, C)$ and $L_1 K_1 = F_1$. Then by Lemma 2.16 IV,
III. Let $F_1 \in C_1^1$, $f_2 \in C_2$ and $\Phi \in Z$. Then by II,

\[
p_{F_1,f_2} (\Phi (\Psi)) = \sup \{ \| \Phi (F_1 x) \| : x \in H_1, \| x \| \leq 1 \} = \sup \{ \| \Phi (F_1, f_2) x \| : x \in H_1, \| x \| \leq 1 \} = \| \Phi (F_1, f_2) \|_{HS} = \tau_{F_1,f_2} (\Phi).
\]

So $\Psi$ is continuous.

IV. Let $N \in \mathbb{N}$, $\phi_{11}$, $\cdots$, $\phi_{1N} \in S_{U_1,C_1}$, $\Phi_{11}$, $\cdots$, $\Phi_{2N} \in T_{U_2,C_2}$ and let $\xi := \sum_{i=1}^{N} \Gamma_i \Phi_{2i}$. Let $F_1 \in C_1^1$, $x \in H_1$ and $f_2 \in C_2$. Then $\{ [[\Phi (\Psi)] (U_1[F_1] \star x)] (f_2) = \sum_{i=1}^{N} \langle \phi_{1i}, U_1[F_1] \cdot x \rangle \Phi_{2i}(f_2) =\}

\[
\{ [[W_2(\xi)] (U_1[F_1] \star x)] (f_2).
\]

THEOREM 8.23.

The map $W_2$ is a continuous linear map from $S_{U_1,C_1} \otimes_0 T_{U_2,C_2}$ into $L(T_{U_1,C_1}, T_{U_2,C_2})$. There exists a unique continuous extensions $W_3$ of $W_2$, defined from $S_{U_1,C_1} \otimes_0 T_{U_2,C_2}$ into $L(T_{U_1,C_1}, T_{U_2,C_2})$.

Proof.

Since $W_2 = V \circ A_2$ is the composition of two continuous maps, the map $W_2$ is also continuous. Because $V$ is continuous (see Lemma 8.22 III) and $Z$ is a completion of $S_{U_1,C_1} \otimes_0 T_{U_2,C_2}$, the second assertion follows.

Just after the definition of $W_2$ we could give an easy and elementary proof to show that $W_2$ is continuous and we could also prove that $L(T_{U_1,C_1}, T_{U_2,C_2})$ is complete. Hence there exists a continuous linear map from $S_{U_1,C_1} \otimes_0 T_{U_2,C_2}$ into $L(T_{U_1,C_1}, T_{U_2,C_2})$ which extends $W_2$. However in the chosen set up we can easily prove the following kernel theorem.

THEOREM 8.24.

Suppose $T_{U_1,C_1}$ or $T_{U_2,C_2}$ is a nuclear space. Then $\{ W_3(\xi) : \xi \in S_{U_1,C_1} \otimes_0 T_{U_2,C_2} \}$ is the set of all continuous linear maps from $T_{U_1,C_1}$ into $T_{U_2,C_2}$. 
Proof.

Let $E \in L(T_{U_1,c_1}, T_{U_2,c_2})$. We have to prove that there exists $\Phi \in Z$ such that $E = V(\Phi)$. For $F_1 \in C_1$ and $F_2 \in C_2$ define $E_{F_1,F_2} : H_1 \to H_2$ by $E_{F_1,F_2}(x) = [E(U_1[F_1] \cdot x)](F_2)$, $x \in H_1$. Since $T_{U_1,c_1}$ or $T_{U_2,c_2}$ is a nuclear space, it follows that $E_{F_1,F_2}$ is a nuclear map. (The proof is analogous to the proof of step 1 in the proof of Theorem 8.16.) Hence the operator $E_{F_1,F_2}$ is a Hilbert-Schmidt operator.

Define $\Phi : C_1 \times C_2 \to H_{HS}$

$$\Phi(F_1,F_2) := E_{F_1,F_2} \quad (F_1 \in C_1, F_2 \in C_2)$$

By Lemma 2.16 IV, $\Phi \in Z$ and obviously, $V(\Phi) = E$.

VIII.2.C. Continuous linear maps from $S_{U_1,c_1}$ into $S_{U_2,c_2}$

In this subsection we suppose in addition that the pair $(C_2, U_2)$ satisfies properties P3 and P4. Let $L(S_{U_1,c_1}, S_{U_2,c_2})$ be the vector space of all continuous linear maps from $S_{U_1,c_1}$ into $S_{U_2,c_2}$. For all $F_1 \in C_1$ and $F_2 \in C_2$ define the seminorm $p_{f_1,F_2}$ on $L(S_{U_1,c_1}, S_{U_2,c_2})$ by

$$p_{f_1,F_2}(E) := \|U_2[F_2] \circ E \circ U_1(f_1)\|, \quad E \in L(S_{U_1,c_1}, S_{U_2,c_2})$$

(See Theorems 5.1 and 5.2.) The locally convex Hausdorff topology for $L(S_{U_1,c_1}, S_{U_2,c_2})$ is generated by the set of seminorms $\{p_{f_1,F_2}, f_1 \in C_1, F_2 \in C_2\}$.

Define $W_1 : T_{U_1,c_1} \times S_{U_2,c_2} \to L(S_{U_1,c_1}, S_{U_2,c_2})$

$$[W_1(\Gamma_{ext}(\Phi_1), \phi_2)](\phi_1) := \phi_1 \cdot \Phi_1 > \phi_2. \quad (\Phi_1 \in T_{U_1,c_1}, \phi_2 \in S_{U_2,c_2}, \phi_1 \in S_{U_1,c_1})$$

Since $W_1$ is a bilinear map, there exists a unique linear map $W_2$ from $T_{U_1,c_1} \otimes S_{U_2,c_2}$ into $L(S_{U_1,c_1}, S_{U_2,c_2})$ such that $W_2(\Gamma_{ext}(\Phi_1) \otimes \phi_2) = W_1(\Gamma_{ext}(\Phi_1), \phi_2)$ for all $\Phi_1 \in T_{U_1,c_1}$ and $\phi_2 \in S_{U_2,c_2}$. We shall prove that $W_2$ is a continuous linear map from $T_{U_1,c_1} \otimes S_{U_2,c_2}$ into $L(S_{U_1,c_1}, S_{U_2,c_2})$ and that $W_2$ extends to a continuous map defined on $T_{U_1,c_1} \otimes S_{U_2,c_2}$. As in the previous subsection we define a topological vector space $Z$ and prove that this space is a completion of $T_{U_1,c_1} \otimes S_{U_2,c_2}$.

DEFINITION 8.25.

Let $Z$ be the vector space of all maps $\Phi$ from $C_1 \times C_2^g$ into $H_{HS}$ such that

$$\Phi(f_1 \cdot g_1, F_2) = \Phi(f_1, F_2) \cdot U_1(g_1)$$

for all $f_1 \in C_1$, $g_1 \in L^1(G_1)$ and $F_2 \in C_2^g$ such that $f_1 \cdot g_1 \in C_1$; and
\[
\Phi(f_1, F_2 \cdot K_2) = U_2[K_2] \Phi(f_1, F_2)
\]
for all \(f_1 \in C_1\) and \(F_2, K_2 \in C_2^q\) such that \(F_2 \cdot K_2 \in C_2^q\).

The Hausdorff topology for \(Z\) is the locally convex topology for \(Z\) generated by the seminorms \(z_{f_1, F_2}(\Phi) := \|\Phi(f_1, F_2)\|_{HS}\) for \((f_1 \in C_1, F_2 \in C_2^q, \Phi \in Z)\)

**Remark.** The equality \(\Phi(f_1, F_2 \cdot K_2) = U_2[K_2] \Phi(f_1, F_2)\) means in particular that \(\Phi(f_1, F_2) \cdot x \in D(U_2[K_2])\) for all \(x \in H_1\).

**Theorem 8.26.**

\(Z\) is complete. Indeed, let \((\Phi_a)_{a \in M}\) be a Cauchy-net in \(Z\). For every \(f_1 \in C_1\) and \(F_2 \in C_2^q\), the net \((\Phi_a(f_1, F_2))_{a \in M}\) is a Cauchy-net in \(H_{HS}\), so there exists \(\Phi(f_1, F_2) \in H_{HS}\) such that \(\lim_a \Phi_a(f_1, F_2) = \Phi(f_1, F_2)\) in \(H_{HS}\). Let \((f_1, F_2, K_2) \in C_1 \times C_2^q\) and suppose \(F_2 \cdot K_2 \in C_2^q\). Let \(x \in H_1\). Then \(\lim_a \Phi_a(f_1, F_2) x = \Phi(f_1, F_2) x\) and \(\lim_a \Phi_a(f_1, F_2 \cdot K_2) x = \Phi(f_1, F_2 \cdot K_2) x\) in \(H_2\), so in the Hilbert space \(H_2 \times H_2\) we obtain \(\lim_a [\Phi_a(f_1, F_2) x, U_2[K_2] (\Phi_a(f_1, F_2) x)] = \lim_a [\Phi_a(f_1, F_2) x, \Phi_a(f_1, F_2 \cdot K_2) x]\).

Since \(U_2[K_2]\) is a closed operator, \(\Phi(f_1, F_2) x \in D(U_2[K_2])\) and \(\Phi(f_1, F_2 \cdot K_2) x = U_2[K_2] \Phi(f_1, F_2) x\). Similarly, \(\Phi(f_1 \ast g_1, F_2) = \Phi(f_1, F_2) U_1(g_1)\) for all \(f_1 \in C_1, g_1 \in L^1(G_1)\) and \(F_2 \in C_2^q\) such that \(f_1 \ast g_1 \in C_1\). So \(\Phi \in Z\) and \(\lim_a \Phi_a = \Phi\) in \(Z\).

Define \(\Lambda_1 : T_{U_1, C_1} \times SU_2, C_2 \rightarrow Z\)

\[
\{[\Lambda_1(T_{\text{ext}}(\Phi_1), \Phi_2)](f_1, F_2)\} x := (x, \Phi_1(f_1)) U_2[F_2] \Phi_2.
\]

\((\Phi_1 \in T_{U_1, C_1}, \Phi_2 \in SU_2, C_2, f_1 \in C_1, F_2 \in C_2^q, x \in H_1)\). Then \(\Lambda_1\) is a bilinear map, so there exists a unique linear map \(\Lambda_2\) from \(T_{U_1, C_1} \otimes SU_2, C_2\) into \(Z\) such that \(\Lambda_2(T_{\text{ext}}(\Phi_1) \otimes \Phi_2) = \Lambda_1(T_{\text{ext}}(\Phi_1), \Phi_2)\) for all \(\Phi_1 \in T_{U_1, C_1}\) and \(\Phi_2 \in SU_2, C_2\). As in the proof of Theorem 8.20, the map \(\Lambda_2\) is a topological homeomorphism from \(T_{U_1, C_1} \otimes SU_2, C_2\) onto \(\Lambda_2(T_{U_1, C_1} \otimes SU_2, C_2) \subseteq Z\).

Finally we have to prove that \(Z_1 := \Lambda_2(T_{U_1, C_1} \otimes SU_2, C_2)\) is dense in \(Z\). Let \(\Phi \in Z\). As in Lemma 1.9 it follows that \(\Phi(f_1, F_2) U_1(g_1) = \Phi(g_1, F_2) U_1(f_1)\) for all \(f_1, g_1 \in C_1\) and \(F_2 \in C_2^q\). Let \((f_{1, \lambda})_{\lambda \in J}\) be the bounded net in \(C_1\) as in property P2'. For all \(\lambda \in J\) define \(\Phi_\lambda : C_1 \times C_2^q \rightarrow H_{HS}\)

\[
\Phi_\lambda(f_1, F_2) := \Phi(f_{1, \lambda}, F_2) U_1(f_1). \quad (f_1 \in C_1, F_2 \in C_2^q)
\]

Then \(\Phi_\lambda \in Z\) for all \(\lambda \in J\). Let \(f_1 \in C_1\) and \(F_2 \in C_2^q\). Then by Corollary 8.18:
\lim_{\lambda} z_{f_1, f_2} (\Phi_\lambda - \Phi) = \lim_{\lambda} \| \Phi(f_{1\lambda}, F_2) U_1(f_1) - \Phi(f_1, F_2) \|_{HS} = \\

\lim_{\lambda} \| \Phi(f_1, F_2) U_1(f_{1\lambda}) - \Phi(f_1, F_2) \|_{HS} = 0. \text{ So } \lim_{\lambda} \Phi_\lambda = \Phi \text{ in } Z. \text{ We shall prove that } \Phi_\lambda = \tilde{Z}_1 \text{ for all } \lambda \in J.

Let \lambda \in J. \text{ Let } \mathcal{Q}_1, \mathcal{Q}_2, \cdots \subset \hat{G}_2 \text{ as in property P3 for the pair } (C_2, U_2). \text{ For } N \in \mathbb{N} \text{ define } L_N : \hat{G}_2 \to C

\begin{align*}
L_N(\gamma) = \begin{cases} 
1 & \text{if } n \leq N \text{ and } U_2[1_{\mathcal{Q}_n}] \neq 0, \\
0 & \text{if } n > N \text{ or } U_2[1_{\mathcal{Q}_n}] = 0.
\end{cases} 
\end{align*}

Then \( L_N \in \text{Bor}_b(\hat{G}_2, C) \) and \( L_N \) is bounded by 1. Note that \( L_N F_2 \) is bounded for all \( F_2 \in C_2^F \).

For \( N \in \mathbb{N} \) define \( \Phi_{N} (f_1, F_2) := \Phi_{\lambda}(f_1, F_2 L_N) \). \( (f_1 \in C_1, F_2 \in C_2^F) \)

Then \( \Phi_{N} \in Z \) for all \( N \in \mathbb{N} \). Since \( s - \lim_{N \to \infty} U_2[L_N] = I \), by Lemma 8.17 we obtain for all \( f_1 \in C_1 \) and \( F_2 \in C_2^F \) that \( \lim_{N \to \infty} z_{f_1, F_2} (\Phi_{N} - \Phi_\lambda) = \lim_{N \to \infty} \| \Phi_\lambda(f_1, F_2 L_N) - \Phi_{\lambda}(f_1, F_2) \|_{HS} = \)

\lim_{N \to \infty} \| U_2[L_N] \Phi_\lambda(f_1, F_2) - \Phi_\lambda(f_1, F_2) \|_{HS} = 0. \text{ Hence } \lim_{N \to \infty} \Phi_{N} = \Phi_{\lambda} \text{ in } Z.

We finish with the proof that \( \Phi_{N} \in \tilde{Z}_1 \) for all \( N \in \mathbb{N} \). Let \( N \in \mathbb{N} \). Since \( A(H_1^1 \otimes S_{U_2, C_2}) \) is dense in \( H_{HS} \), for every \( \varepsilon > 0 \) there exists \( \xi_\varepsilon \in H_1^1 \otimes S_{U_2, C_2} \) such that 

\( \| A(\xi_\varepsilon) - \Phi(f_{1\lambda}, L_N) \|_{HS} < \varepsilon \). \text{ Let } \varepsilon > 0. \text{ There exist } \( M \in \mathbb{N}, x_1, \cdots, x_M \in H_1, \phi_{21}, \cdots, \phi_{2M} \in S_{U_2, C_2} \) such that \( \xi_\varepsilon = \sum_{i=1}^{M} \Gamma_{xi} \otimes \phi_{2i} \).

Define

\( \zeta_{\varepsilon} := \sum_{i=1}^{M} \Gamma_{\text{ext}(\text{emb} x_i)} \otimes U_2[L_N] \phi_{2i} \). \text{ Then } \zeta_{\varepsilon} \in TU_1 C_1 \otimes S_{U_2, C_2}. \text{ For all } x \in H_1, f_1 \in C_1 \text{ and } F_2 \in C_2^F \text{ we have that } \{ [(A_2(\zeta_{\varepsilon}))(f_1, F_2)](x) = \sum_{i=1}^{M} (x, (\text{emb} x_i)(f_1)) U_2[F_2] U_2[L_N] \phi_{2i} = \)

\( \sum_{i=1}^{M} U_2[F_2 L_N] \{ A(\xi_{\xi\varepsilon}) \} U_1(f_1) x \).

So \( [A_2(\zeta_{\varepsilon})](f_1, F_2) = U_2[F_2 L_N] \circ A(\xi_{\xi\varepsilon}) \circ U_1(f_1) \). \text{ Hence for all } f_1 \in C_1, F_2 \in C_2^F \text{ and } \varepsilon > 0 \text{ we obtain } z_{f_1, F_2} (A_2(\zeta_{\varepsilon}) - \Phi_{\lambda N}) = \| [A_2(\zeta_{\varepsilon})](f_1, F_2) - \Phi_{\lambda N}(f_1, F_2) \|_{HS} = \)

\( \| U_2[F_2 L_N] \circ A(\xi_{\xi\varepsilon}) \circ U_1(f_1) - \Phi(f_{1\lambda}, F_2 L_N) U_1(f_1) \|_{HS} = \\
\| U_2[F_2 L_N] \circ A(\xi_{\xi\varepsilon}) \circ U_1(f_1) - U_2[F_2 L_N] \circ \Phi(f_{1\lambda}, L_N) \circ U_1(f_1) \|_{HS} \leq 
\)
\[ U_2[F_2, L_N] \| A(\xi) - \Phi(f_{1k}, L_N) \|_{HS} \| U_1(f_1) \| \leq \varepsilon \| U_2[F_2, L_N] \| \| U_1(f_1) \|. \]

So \( \lim_{\varepsilon \downarrow 0} z_{f_1, F_2}, (A_2, -\Phi) = 0 \) for all \( f_1 \in C_1 \) and \( F_2 \in C_2^g \). Hence \( \lim_{\varepsilon \downarrow 0} A_2(\xi) = \Phi_{N} \) in \( Z \) and \( \Phi_{N} \in \bar{Z} \).

Let \( A_2 : T_{U_1, C_1} \otimes S_{U_1, C_1} \rightarrow Z \) be the map as in the previous proof. Let \( f_1 \in C_1 \) and \( x \in H_1 \). Then \( \Phi(f_1, F_2)_x = U_2[F_2, C_2^g \otimes \Phi(1, 1_n)_x \otimes x \) for all \( F_2 \in C_2^g \), so \( \Phi(f_1, 1_n)_x \in D(U_2[F_2, C_2^g \otimes \Phi(1, 1_n)_x \) for all \( F_2 \in C_2^g \). Then by Theorem 3.12 I \( \Rightarrow \) II, \( \Phi(f_1, 1_n)_x \in S_{U_1, C_2} \). So the following definition makes sense.

**DEFINITION 8.27.**

Let \( \Phi \in Z. \) Define \( V(\Phi) : S_{U_1, C_1} \rightarrow S_{U_2, C_2} \) by \( [V(\Phi)](\phi_1) := [\Phi(f_1, 1_n)] \cdot \Omega_{f_1}^{-1}(\phi_1) \) for all \( f_1 \in C_1 \) and \( \phi_1 \in R_{f_1} \). (Cf. Lemma 1.4, the definition of \( Z \) and property P1.)

**LEMMA 8.28.**

I. \( V(\Phi) \) is continuous for all \( \Phi \in Z. \)

II. \( [V(\Phi)](U_1(f_1)_x) = \Phi(f_1, 1_n)_x \) for all \( f_1 \in C_1 \) and \( x \in H_1. \)

III. The map \( V \) is continuous from \( Z \) into \( L(S_{U_1, C_1}, S_{U_2, C_2}). \)

IV. \( W_2 = V \circ A_2. \)

**Proof.**

I. Let \( f_1 \in C_1 \) and \( F_2 \in C_2^g \). Then for all \( x \in N_{U_1}^1 \cdot U_2[F_2, C_2^g \otimes \Phi(1, 1_n)_x \),

\[ \| \Phi(f_1, F_2)_x \| \leq \| \Phi(f_1, F_2) \| \| x \|. \]

So \( V(\Phi) \) is continuous by Theorems 5.1 and 5.2.

II. Let \( f_1 \in C_1 \) and \( x \in N_{f_1} \). Let \( (f_{1k})_{k \in \mathbb{N}} \) be the net as in property P2'. Then

\[ \Phi(f_{1k}, 1_n)_x = \lim_{k} \Phi(f_{1k}, 1_n)_x U_1(f_{1k})_x = \lim_{k} \Phi(f_{1k}, 1_n)_x U_1(f_{1k})_x = \lim_{k} 0 = 0. \]

Now let \( f_1 \in C_1 \) and \( x \in H_1. \) There exists \( y \in N_{f_1} \) and \( z \in N_{f_1}^\perp \) such that \( x = y + z. \) Then

\[ [V(\Phi)](U_1(f_1)_x) = [V(\Phi)](U_1(f_1)_z) = \Phi(f_1, 1_n)_z = \Phi(f_1, 1_n)(y + z) = \Phi(f_1, 1_n)_x. \]

III. Let \( f_1 \in C_1 \), \( F_2 \in C_2^g \) and \( \Phi \in Z. \) Then by II,

\[ p_{f_1, F_2, V(\Phi)} = \| U_2[F_2, C_2^g \otimes \Phi(1, 1_n)] \cdot \Omega_{f_1}^{-1}(\phi_1) \| \leq \| U_2[F_2, C_2^g \otimes \Phi(1, 1_n)] \| \| x \|. \]

So \( V \) is continuous.

IV. Let \( N \in \mathbb{N}, \Phi_{1i}, \Phi_{2i} \in T_{U_1, C_1} \otimes S_{U_2, C_2} \) and let

\[ \xi := \sum_{i=1}^{N} \Gamma_{x_{i}}(\Phi_{1i}) \Phi_{2i}: \]

Let \( f_1 \in C_1 \) and \( x \in H_1. \) Then \( [(V \circ A_2)(\xi)](U_1(f_1)_x) = \sum_{i=1}^{N} (x, \Phi_{1i}(f_1)) U_2[1_n] \Phi_{2i} = \sum_{i=1}^{N} <U_1(f_1)_x, \Phi_{1i}> \Phi_{2i} = \sum_{i=1}^{N} \Gamma_{x_{i}}(\Phi_{1i}) \Phi_{2i} \).
THEOREM 8.29.
The map \( W_2 \) is a continuous linear map from \( T_{U_1, C_1} \otimes_\theta S_{U_2, C_2} \) into \( L(S_{U_1, C_1}, S_{U_2, C_2}) \). There exists a unique continuous extension \( W_3 \) of \( W_2 \), defined from \( T_{U_1, C_1} \otimes_\theta S_{U_2, C_2} \) into \( L(S_{U_1, C_1}, S_{U_2, C_2}) \).

Proof.
Similar to the proof of Theorem 8.23.

THEOREM 8.30.
Suppose \( S_{U_1, C_1} \) or \( S_{U_2, C_2} \) is a nuclear space. Then \( \{W_3(\xi) : \xi \in T_{U_1, C_1} \otimes_\theta S_{U_2, C_2} \} \) is the set of all continuous linear maps from \( S_{U_1, C_1} \) into \( S_{U_2, C_2} \).

Proof.
Let \( E \in L(S_{U_1, C_1}, S_{U_2, C_2}) \). We prove that there exists \( \Phi \in Z \) such that \( E = V(\Phi) \). For \( f_1 \in C_1 \) and \( F_2 \in C_2^* \), define \( E_{f_1, F_2} : H_1 \to H_2 \) by \( E_{f_1, F_2} := U_2[F_2] \circ E \circ U_1(f_1) \). Since \( S_{U_1, C_1} \) or \( S_{U_2, C_2} \) is a nuclear space, the map \( E_{f_1, F_2} \) is a nuclear map, hence a Hilbert-Schmidt operator.

Define \( \Phi : C_1 \times C_2^* \to H_{HS} \)
\[
\Phi(f_1, F_2) := E_{f_1, F_2} \quad (f_1 \in C_1, F_2 \in C_2^*)
\]
Then \( \Phi \in Z \) and \( V(\Phi) = E \).

VIII.2.D. Continuous linear maps from \( T_{U_1, C_1} \) into \( S_{U_2, C_2} \)

In this subsection we suppose in addition that both pairs \((C_1, U_1)\) and \((C_2, U_2)\) satisfy properties P3 and P4. Let \( L(T_{U_1, C_1}, S_{U_2, C_2}) \) be the vector space of all continuous linear maps from \( T_{U_1, C_1} \) into \( S_{U_2, C_2} \). For \( F_1 \in C_1^* \) and \( F_2 \in C_2^* \), define the seminorm \( p_{F_1, F_2} \) on \( L(T_{U_1, C_1}, S_{U_2, C_2}) \) by
\[
p_{F_1, F_2}(E) := \sup \{ \| U_2[F_2] E(U_1[F_1] \cdot x) \| : x \in H_1, \| x \| \leq 1 \}.
\]
\( (E \in L(T_{U_1, C_1}, S_{U_2, C_2})) \) The topology for \( L(T_{U_1, C_1}, S_{U_2, C_2}) \) is the locally convex Hausdorff topology generated by the set of seminorms \( \{p_{F_1, F_2} : F_1 \in C_1^*, F_2 \in C_2^* \} \).
Define \( W_1 : S_{U_1, C_1} \times S_{U_2, C_2} \to L(T_{U_1, C_1}, S_{U_2, C_2}) \)

\[ [W_1(\Gamma_1, \phi_2)](\Phi_1) := \langle \phi_1, \Phi_1 \rangle \phi_2. \quad (\Phi_1 \in S_{U_1, C_1}, \phi_2 \in S_{U_2, C_2}, \Phi_1 \in T_{U_1, C_1}) \]

The map \( W_1 \) is bilinear, so there exists a unique linear map \( W_2 \) from \( S_{U_1, C_1} \otimes S_{U_2, C_2} \) into \( L(T_{U_1, C_1}, S_{U_2, C_2}) \) such that \( W_2(\Gamma_1 \otimes \phi_2) = W_1(\Gamma_1, \phi_2) \) for all \( \phi_1 \in S_{U_1, C_1} \) and \( \phi_2 \in S_{U_2, C_2} \). As in the previous subsections it can be proved that \( W_2 \) is a continuous linear map from \( S_{U_1, C_1} \otimes S_{U_2, C_2} \) into \( L(T_{U_1, C_1}, S_{U_2, C_2}) \) and that there exists a continuous extension \( W_3 \) of \( W_2 \), defined from \( S_{U_1, C_1} \otimes \delta\theta S_{U_2, C_2} \) into \( L(T_{U_1, C_1}, S_{U_2, C_2}) \).

The following theorem can be proved similarly.

**THEOREM 8.31.**

Suppose \( T_{U_1, C_1} \) or \( S_{U_2, C_2} \) is a nuclear space. Then \( \{ W_3(\xi) : \xi \in S_{U_1, C_1} \otimes \delta\theta S_{U_2, C_2} \} \) is the set of all continuous linear maps from \( T_{U_1, C_1} \) into \( S_{U_2, C_2} \).
Appendix A. Topological groups

The following definitions are adapted from the book of Hewitt and Ross [HRI]. The only exception is the definition of a regular measure.

A topological group is a group $G$ equipped with a topology such that the multiplication function and inversion function are continuous. We always assume that the topology is a Hausdorff topology. $G$ is $\sigma$-compact if there exists a sequence of compact subsets $K_1, K_2, \cdots$ of $G$ such that $G = \bigcup_{n=1}^{\infty} K_n$.

Let $G$ be a fixed locally compact Abelian topological group. Let $B$ be the smallest $\sigma$-algebra of subsets of $G$ which contains all open subsets of $G$. The elements of $B$ are called Borel measurable sets. Let $\mu$ be a measure defined on $B$. $\mu$ is a regular measure if and only if

$$
\left\{ \begin{array}{l}
\mu(K) < \infty \quad K \subset G \text{ compact}, \\
\mu(V) = \sup\{\mu(K) : K \subset V \text{ compact}\} \quad V \subset G \text{ open}, \\
\mu(E) = \inf\{\mu(V) : V \supset E \text{ open}\} \quad E \in B.
\end{array} \right.
$$

A measure $\mu$ on $B$ is a (left) Haar measure if and only if $\mu$ is a regular measure, $\mu(G) > 0$ and $\mu(xE) = \mu(E)$ for all $x \in G$ and $E \in B$. A Haar measure is unique up to a positive constant.

Let $\mu$ be a fixed Haar measure on $G$. We write $L^1(G)$ resp. $L^2(G)$ for $L^1(G, B, \mu)$ resp. $L^2(G, B, \mu)$. Let $f, g \in L^1(G)$. The convolution product $f \ast g$ of $f$ and $g$ is the $L^1(G)$ function (equivalence class) satisfying

$$
(f \ast g)(x) = \int_G f(y) g(y^{-1}x) \, d\mu(y), \quad \text{a.e. } x \in G.
$$

This convolution product is commutative, since $G$ is commutative. Let $f \in L^1(G)$. The adjoint $\check{f}$ of $f$ is the $L^1(G)$ function with

$$
\check{f}(x) = \overline{f(x^{-1})}, \quad \text{a.e. } x \in G.
$$

(Recall, $G$ is commutative.)

Let $H$ be a Hilbert space. We assume no restrictions about the Hilbert space dimension of $H$. A representation $U$ of $G$ in $H$ is a map $x \mapsto U_x$ from $G$ into the set of all unitary operators on $H$ such that $U_x U_y = U_{xy}$ for all $x, y \in G$. A representation $U$ is continuous if for all $v \in H$ the map $x \mapsto U_x v$ is continuous from $G$ into $H$. Throughout this report all representations of $G$ will be taken to be continuous. A representation $U$ of $G$ in a Hilbert space $H$ is irreducible if $H$ and $\{0\}$ are the only closed subspaces of $H$ which are invariant under $U_x$ for all $x \in G$. Since $G$ is commutative, $U$ is irreducible if and only if $\dim H = 1$.

A character of $G$ is a function $\gamma$ from $G$ into $\{ z \in \mathbb{C} : |z| = 1 \}$ with $\gamma(xy) = \gamma(x) \gamma(y)$ for all $x, y \in G$. The dual group $\hat{G}$ of $G$ is the set of all continuous characters of $G$, equipped with pointwise multiplication. Let $f \in L^1(G)$. The Fourier transform of $f$ is the complex valued function $\hat{f}$ on $\hat{G}$ such that
\hat{f}(\gamma) = \int_{\hat{G}} f(x) \overline{\gamma(x)} \, d\mu(x), \quad \gamma \in \hat{G}.

(\mu is the Haar measure.) The topology for \hat{G} is the weakest topology on \hat{G} such that all functions \hat{f}, f \in L^1(G) are continuous. With this topology \hat{G} is a locally compact Abelian topological group.
Appendix B. Topological vector spaces

Most of the following definitions are adapted from the book of Wilanski [Wil]. Nuclear spaces are defined in [Sch]. The scalar field is \( \mathbb{C} \).

Let \( X \) be a vector space and let \( A \) be a subset of \( X \). Then
\[
\text{span} A := \{ \sum_{n=1}^{N} \lambda_n a_n : N \in \mathbb{N} \cup \{0\}, a_1, \ldots, a_N \in A, \lambda_1, \ldots, \lambda_N \in \mathbb{C} \}
\]
denotes the span of \( A \). \( A \) is convex if \( \lambda A + (1-\lambda) A \subset A \) for all \( \lambda \in [0, 1] \) and \( A \) is balanced if \( \lambda A \subset A \) for all \( \lambda \in \mathbb{C} \) with \( |\lambda| \leq 1 \). \( A \) is called absolutely convex if \( \lambda \sim 0 , A \) is convex and balanced. \( A \) is called absorbing if for every \( x \in X \) there exists \( \varepsilon > 0 \) such that \( \lambda x \in A \) for all \( \lambda \in \mathbb{C} \) with \( |\lambda| < \varepsilon \). Let \( B \) be a subset of \( X \). Then \( A \) absorbs \( B \) if there exists \( M > 0 \) such that \( B \subset \lambda A \) for all \( \lambda \in \mathbb{C} \) with \( |\lambda| > M \).

Let \( X \) be a vector space and let \( p \) be a map from \( X \) into \( \mathbb{R} \). \( p \) is a seminorm if for all \( x, y \in X \) and \( \lambda \in \mathbb{C} \):
\[
\begin{align*}
 p(x) &\geq 0, \\
 p(x + y) &\leq p(x) + p(y), \\
 p(\lambda x) &= |\lambda| p(x).
\end{align*}
\]
\( p \) is a norm if \( p \) is a seminorm and \( p(x) = 0 \iff x = 0, x \in X \). A semi inner product on \( X \) is a sesquilinear map \(( \cdot, \cdot )\) from \( X \times X \) into \( \mathbb{C} \) such that \(( x, x ) \geq 0 \) for all \( x \in X \). An inner product is a semi inner product \(( \cdot, \cdot )\) such that \(( x, x ) = 0 \iff x = 0, x \in X \). Let \(( \cdot, \cdot )\) be a semi inner product on \( X \). Define a map \( p \) from \( X \) into \( \mathbb{R} \) by \( p(x) = (x, x), x \in X \). Then \( p \) is a seminorm. We call \( p \) the seminorm which corresponds with \(( \cdot, \cdot )\) and we call \(( \cdot, \cdot )\) the semi inner product which corresponds to \( p \).

A topological vector space (TVS) is a vector space equipped with a topology such that the vector addition and scalar multiplication are continuous. Two topological vector spaces \( X, Y \) are called isomorphic as topological vector spaces if there exists a bijection from \( X \) onto \( Y \) which is linear and a topological homeomorphism. Let \( P \) be a set of seminorms on a vector space \( X \). The \( P \)-topology for \( X \) is the smallest topology \( t \) for \( X \) such that \((X \times t)\) is a TVS and all elements of \( P \) are continuous seminorms. \( P \) separates the points of \( X \) if for every \( x \in X, x \neq 0 \) there exists \( p \in P \) such that \( p(x) \neq 0 \). A LCSTVS is a locally convex separated topological vector space. Let \( X \) be a LCSTVS. A local base of neighbourhoods of 0 in \( X \) is a set \( B \) of neighbourhoods of 0 such that for each neighbourhood \( U \) of 0 there exists \( V \in B \) with \( V \subset U \).

Let \( X \) be a TVS. A Cauchy net in \( X \) is a net \((x_k)_{k \in I}\) such that for each neighbourhood \( U \) of 0 there exists \( k_0 \in I \) such that for all \( k \geq k_0 \) and \( l \geq k_0 \) implies \( x_k - x_l \in U \). A sequence \((x_k)_{k \in \mathbb{N}}\) is a Cauchy sequence if \((x_k)_{k \in \mathbb{N}}\) is a Cauchy net. \( X \) is called complete if each Cauchy net is convergent and \( X \) is sequentially complete if every Cauchy sequence is convergent. A Fréchet space is a metrizable complete locally convex topological vector space. The dual space \( X' \) of \( X \) is the vector space of all continuous linear maps from \( X \) into \( \mathbb{C} \). The weak topology for \( X \) is the weakest topology \( \tau \) for \( X \) such that \((X, \tau)\) is a TVS and each element of \( X' \) is continuous. A completion of a LCSTV space \( X \) is a pair \((E, j)\) such that \( E \) is a complete LCSTVS and \( j \) is an injective linear map from \( X \) into \( E \) such that \( j \) is a topological homeomorphism from \( X \) onto \( j(E) \), with the restriction
topology of $E$ for $j(E)$. Furthermore $j(X)$ is dense in $E$. We identify $(E, j)$ with $E$. Two completions are isomorphic as topological vector spaces. We write $\hat{X} := E$ if $(E, j)$ is a completion of $X$ and we identify $X$ with $j(X)$.

Let $X$ be a LCSSTVS and let $A$ be a subset of $X$. $A$ is bounded if every neighbourhood of 0 absorbs $A$. $A$ is a bornivore if $A$ absorbs every bounded subset of $X$. $A$ is a barrel if $A$ is an absolutely convex absorbing closed set. $X$ is called bornological; barrelled resp. quasibarrelled if every absolutely convex bornivore; every barrel resp. every bornivore barrel is a neighbourhood of 0. $X$ is a semi Montel space if every closed bounded subset of $X$ is compact. A Montel space is a barrelled semi-Montel space.

Let $X, Y$ be two vector spaces and let $[\ , \ ]$ be a bilinear functional on $X \times Y$. $[\ , \ ]$ is called nondegenerate if for all $x_0 \in X$ and $y_0 \in Y$:

$$
[X_0, y] = 0 \text{ for all } y \in Y \iff x_0 = 0,
$$

$$
[x, y_0] = 0 \text{ for all } x \in X \iff y_0 = 0.
$$

$(X, Y, [\ , \ ])$ is a dual pair, notation $[X, Y]$, if $[\ , \ ]$ is nondegenerate. For every $y \in Y$ define $\hat{y}: X \to \mathcal{C}$ by $\hat{y}(x) := [x, y]$, $x \in X$. Let $\tau$ be a topology for $X$. $\tau$ is called compatible with the dual pair $[X, Y]$ if $(X, \tau)$ is a LCSSTVS and $(X, \tau)' = \{ \hat{y} : y \in Y \}$. Let $\alpha(X, Y)$ denote the smallest compatible topology for $X$. Let $A$ be a subset of $X$. The polar of $A$ is $A^\circ := \{ y \in Y : |[x, y]| \leq 1 \text{ for all } x \in A \}$.

Let $\tau$ be a compatible topology for $X$. The strong topology $\beta(Y, X)$ for $Y$ is the unique locally convex topology for $Y$ such that $A^\circ : A$ is a bounded subset of $(X, \tau)$ is a local base of neighbourhoods of 0. Remark. These definitions can be adapted in a natural way if $[\ , \ ]$ is a sesquilinear map from $X \times Y$ into $\mathcal{C}$.

Let $(X, \tau)$ be a LCSSTVS. Then $(x, f) := f(x), x \in X, f \in X'$ defines a bilinear functional on $X \times X'$. So $(X, X')$ is a dual pair. Let $X'_\beta := (X', \beta(X', X))$. For $x \in X$ define $\hat{x}: X' \to \mathcal{C}$ by $\hat{x}(f) := f(x), f \in X'$. Then $\hat{x} \in (X'_\beta)'$. $X$ is called semireflexive if $\{ \hat{x} : x \in X \} = (X'_\beta)'$ and $X$ is called reflexive if the map $x \mapsto \hat{x}$ is a homeomorphism from $X$ onto $(X'_\beta)'$. $(X, \tau)$ is a Mackey space if $\tau$ is the largest compatible topology with the dual pair $(X, X')$.

Let $(X_\alpha)_{\alpha \in A}$ be a family of LCSSTVR's, let $Y$ be a vector space and for each $\alpha \in A$ let $u_\alpha : X_\alpha \to Y$ be a linear map. Suppose $Y = \text{span}\{ u_\alpha(x) : \alpha \in A, x \in X_\alpha \}$. The inductive limit topology for $Y$ is the finest topology $\tau$ for $Y$ such that $(Y, \tau)$ is a locally convex TVR and each $u_\alpha, \alpha \in A$ is continuous.

Let $E, F$ be two Banach spaces and let $u$ be a continuous linear map from $E$ into $F$. $u$ is called a nuclear map if there exist a sequence $\lambda_1, \lambda_2, \cdots$ in $\mathcal{C}$, a sequence $f_1, f_2, \cdots$ in $E'$ and a sequence $y_1, y_2, \cdots$ in $F$ such that $\sum_{n=1}^\infty |\lambda_n| < \infty$, $\sup\{ \|f_n\| : n \in \mathbb{N} \} < \infty$, $\sup\{ \|y_n\| : n \in \mathbb{N} \} < \infty$ and $u(x) = \sum_{n=1}^\infty \lambda_n f_n(x) y_n$ for all $x \in E$. Let $X$ be LCSSTVS. Let $p$ be a seminorm on $X$. Let $N_p := \{ x \in X : p(x) = 0 \}$, let $X_p$ be the quotient space $X/N_p$ and let $\pi_p$ be the quotient map. Define $\overline{p} : X_p \to \mathbb{R}$ by $\overline{p}(\pi_p(x)) := p(x), x \in X$. Let $\hat{X}_p$ be the completion of the normed space $(X_p, \overline{p})$. Let $q$ be another seminorm on $X$ and suppose $q \geq p$. Then the map
\( \phi : X_q \rightarrow X_p \) defined by \( \phi(q(x)) := \pi_p(x), x \in X \) is continuous, so there exists a unique continuous extension \( \hat{\phi} \) of \( \phi \) from \( \tilde{X}_q \) into \( \tilde{X}_p \). This map \( \hat{\phi} \) is called the canonical map from \( \tilde{X}_q \) into \( \tilde{X}_p \). The space \( X \) is called nuclear if for every continuous seminorm \( p \) on \( X \) there exists a continuous seminorm \( q \) on \( X \) such that \( q \geq p \) and the canonical map from \( \tilde{X}_q \) into \( \tilde{X}_p \) is a nuclear map.

Let \( X, Y \) be two LCSTVR's. A tensor product of \( X \) and \( Y \) is a pair \((E, j)\) with \( E \) a vector space and \( j \) a bilinear map from \( X \times Y \) into \( E \) which has the following universal property: for every vector space \( Z \) and every bilinear map \( f \) from \( X \times Y \) into \( Z \) there exists a unique linear map \( \tilde{f} \) from \( E \) into \( Z \) such that \( \tilde{f} \circ j = f \). (The existence is proved in [HRII] Appendix D.) It is clear that a tensor product of \( X \) and \( Y \) is unique up to a vector space isomorphism. We write \( X \otimes Y := E \) and \( x \otimes y := j(x, y), x \in X, y \in Y \) if \((E, j)\) is a tensor product of \( X \) and \( Y \). The projective tensor product topology \( \tau \) is the finest topology for \( X \otimes Y \) such that \((X \otimes Y, \tau)\) is a LCSTVS and the map \( (x, y) \mapsto x \otimes y \) is continuous from \( X \times Y \) into \( X \otimes Y \). (See [Sch] III 6.1.) We write \( X \otimes_x Y \) for \((X \otimes Y, \tau)\).
Appendix C. A suitable topology for the tensor product of two locally convex topological vector spaces which have additional structure

Let \( E, F \) be two LCSTV spaces and let \( P \) and \( Q \) be two sets of seminorms on \( E \) resp. \( F \). Suppose each seminorm \( p \in P \) corresponds to a semi inner product, called \((\cdot,\cdot)_p\), on \( E \) and the topology for \( E \) equals the \( P \)-topology. Similarly we consider \( Q \) on \( F \).

Let \( p \) resp. \( q \) be a seminorm on \( E \) resp. \( F \) which corresponds to the semi inner product \((\cdot,\cdot)_p \) resp. \((\cdot,\cdot)_q \).

Let \( x_0 \in E \) and \( y_0 \in F \). Define \( f_1 : E \times F \to \mathbb{C} \)

\[
f_1(x, y) = (x, x_0)_p (y, y_0)_q. \quad (x \in E, y \in F).
\]

\( f_1 \) is bilinear, so there exists a unique \( f_2 : E \otimes F \to \mathbb{C} \) such that \( f_2(x \otimes y) = f_1(x, y) \) \( (x \in E, y \in F) \) and \( f_2 \) is linear.

Let \( z = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F \). Define \( f_3 : E \times F \to \mathbb{C} \)

\[
f_3(x_0, y_0) = \sum_{i=1}^n \frac{(x_i, x_0)_p (y_i, y_0)_q}{(x_0, y_0)_q} \quad (x_0 \in E, y_0 \in F).
\]

By the above \( f_3 \) does not depend on the representation of \( z \). \( f_3 \) is bilinear, so there exist a unique \( f_4 : E \otimes F \to \mathbb{C} \) such that \( f_4(x \otimes y) = f_3(x, y) \). So we can define

\[
< , >_{pq} : E \otimes F \times E \otimes F \to \mathbb{C}
\]

\[
< \sum_{i=1}^n x_i \otimes y_i + \sum_{j=1}^m x_j' \otimes y_j' \rangle_{pq} := \sum_{i=1}^n \sum_{j=1}^m (x_i, x_j')_p (y_i, y_j')_q
\]

\[
\left( \sum_{i=1}^n x_i \otimes y_i \in E \otimes F , \sum_{j=1}^m x_j' \otimes y_j' \in E \otimes F \right).
\]

\( < , >_{pq} \) is sesquilinear, \( < z, w >_{pq} = < w, z >_{pq} (z, w \in E \otimes F) \), \( < , >_{pq} \) is semi positive definite, so \( < , >_{pq} \) is a semi inner product.

Define \( r_{pq} : E \otimes F \to \mathbb{C} \)

\[
r_{pq}(z) := \sqrt{< z, z >_{pq}} \quad (z \in E \otimes F).
\]

\( r_{pq} \) is a seminorm.

Assertion: \( \{r_{pq} : p \in P, q \in Q \} \) separates the points of \( E \otimes F \). Indeed, for \( N \in \mathbb{N} \cup \{0\} \) let

\[
A_N := \left\{ \sum_{i=1}^N x_i \otimes y_i : x_i , \cdots , x_N \in E , y_1 , \cdots , y_N \in F \right\}.
\]

Then \( E \otimes F = \bigcup_{N=1}^\infty A_N \). We prove by induction to \( N \) that for all \( z \in A_N , z \not= 0 \) there exist \( p \in P \) and \( q \in Q \) such that \( r_{pq}(z) \not= 0 \). If \( N = 0 \) nothing has to be proved. Let \( N \in \mathbb{N} \) and suppose for all \( z \in A_{N-1} , z \not= 0 \) there exist \( p \in P \) and \( q \in Q \) such that \( r_{pq}(z) \not= 0 \). Let \( z = \sum_{i=1}^N x_i \otimes y_i \in A_N , z \not= 0 \). We may suppose that \( y_1 \not= 0 \) and also
that the set \( \{y_1, \ldots, y_N\} \) is linearly independent. There exist \( q \in Q \) such that \( q(y_1) \neq 0 \). Similar to the Gram-Schmidt orthonormalization procedure there exist linearly independent \( y'_1, \ldots, y'_N \in F \) such that \( (y'_i, y'_j)_q = 0 \) for all \( i, j \leq N, \ i \neq j \) and span \( \{y_1, \ldots, y_i\} = \text{span} \{y'_1, \ldots, y'_i\} \) for all \( i \leq N \). Then there exist \( x'_1, \ldots, x'_N \in E \) such that \( z = \sum_{i=1}^{N} x'_i \otimes y'_i \). If \( x'_1 = 0 \) then \( z = x'_1 \) and the assertion follows by induction hypothesis. If \( x'_1 \neq 0 \), then there exists \( p \in P \) such that \( p(x'_1) \neq 0 \). Then

\[
[r_{pq}(z)]^2 = \sum_{i,j} (x'_i, x'_j)_p (y'_i, y'_j)_q = \sum_i |p(x'_i)|^2 (q(y'_i))^2 \geq |p(x'_1)|^2 |q(y'_1)|^2 > 0.
\]

**DEFINITION C1.**

The \( P \otimes Q \)-topology for \( E \otimes F \) is the \( \{r_{pq} : p \in P, q \in Q\} \)-topology.

**THEOREM C2.**

Let \( P_0 \) and \( P \) be two sets of seminorms on \( E \) which separate the points of \( E \). Each element of \( P_0 \cup P \) induces a semi inner product on \( E \). Suppose the \( P \)-topology for \( E \) equals the \( P_0 \)-topology for \( E \). Similarly let \( Q \) be a set of semi norms on \( F \) as above. Then: the \( P \otimes Q \)-topology for \( E \otimes F \) equals the \( P_0 \otimes Q \)-topology for \( E \otimes F \).

**Proof.**

Let \( p_0 \in P_0, q \in Q \). There exist \( \lambda > 0 \) and \( p_1, \ldots, p_n \in P \) such that \( p_0 \leq \lambda (p_1 \lor \cdots \lor p_n) \).

Define \( p : E \to R \)

\[
p(x) := \left( \sum_{i=1}^{n} [p_i(x)]^2 \right) (x \in E).
\]

Then \( p \) is a seminorm on \( E \), which induces the semi-inner-product \( (\ , \ )_p := \sum_{i=1}^{n} (\ , \ )_{p_i} \) on \( E \). Since \( p \leq \sum_{i=1}^{n} p_i \), the semi norm \( p \) is \( P \)-continuous. For all \( z, w \in E \otimes F \) holds \( <z, w>_p = \sum_{i=1}^{n} <z, w>_p \), \( r_{pq}(z) = \left( \sum_{i=1}^{n} [r_{pq}(z)]^2 \right)^\frac{1}{2} \leq \sum_{i=1}^{n} r_{pq}(z) \) \( (z \in E \otimes F) \). Hence \( r_{pq} \) is \( P \otimes Q \)-continuous. Note: \( p_0 \leq \lambda p \). Let \( z \in E \otimes F \). There exist \( n \in N, x_1, \ldots, x_n \in E, y_1, \ldots, y_n \in F \) such that \( z = \sum_{i=1}^{n} x_i \otimes y_i \). Without loss of generality: \( y_1, \ldots, y_n \) are orthogonal in \( (F, q) \). Then

\[
[r_{pq}(z)]^2 = \sum_{ij} (x_i, y_j)_p (y_i, y_j)_q = \sum_{i=1}^{n} [p_0(x_i)]^2 (q(y_i))^2 \leq 
\]
\[ \lambda^2 \sum_{i=1}^{n} [p(x_i)]^2 [q(y_i)]^2 = \cdots = \lambda^2 [r_{pq}(z)]^2. \]

So \( r_{pq} \leq \lambda r_{pq} \) and \( r_{pq} \) is \( P \otimes Q \)-continuous.

Because of this theorem we can define:

**DEFINITION C3.**

Suppose \( P \) and \( Q \) are sets of seminorms on \( E \) resp. \( F \) which separate the points and correspond to semi-inner products. The \( \theta \)-topology \( E \otimes F \) is defined to be the \( \{ r_{pq} : p \in P, q \in Q \} \)-topology.

Let \( E \otimes_\theta F \) be the LCSTVS \( E \otimes F \) with the \( \theta \)-topology.

**THEOREM C4.**

The \( \pi \)-topology on \( E \otimes F \) is stronger than the \( \theta \)-topology.

**Proof.**

Let \((x_\alpha, y_\alpha)_{\alpha \in A} \) be a net in \( E \times F \) and suppose \( \lim_{\alpha} (x_\alpha, y_\alpha) = 0 \). Let \( p \in P, q \in Q \). Since \( \lim_{\alpha} x_\alpha = 0 \) in \( E \), we have \( \lim_{\alpha} p(x_\alpha) = 0 \). Similarly \( \lim_{\alpha} q(y_\alpha) = 0 \). Then \( \lim_{\alpha} r_{pq}(x_\alpha \otimes y_\alpha) = \lim_{\alpha} p(x_\alpha) q(y_\alpha) = 0 \). So the map \((x, y) \mapsto (x \otimes y)\) from \( E \times F \) into \( E \otimes_\theta F \) is continuous and hence the identical map from \( E \otimes_\pi F \) onto \( E \otimes_\theta F \) is continuous. \( \square \)
Acknowledgement

The author wishes to thank J. de Graaf, S.J.L. van Eijndhoven and F.J.L. Martens for their suggestions and comments.
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List of properties

P1. For all \( f, g \in C \) there exists \( h \in C \) such that 1 and 2 hold:
   1) \( f = h \) or there exists \( f_1 \in L^1(G) \) such that \( f = h \ast f_1 \),
   2) \( g = h \) or there exists \( g_1 \in L^1(G) \) such that \( g = h \ast g_1 \).

P1'. For all \( f, g \in C \) there exist \( h \in C \) and \( f_1, g_1 \in L^1(G) \) such that \( f = h \ast f_1 \) and \( g = h \ast g_1 \).

P2. There exists a net \((f_\lambda)_{\lambda \in I}\) in \( C \) such that for all \( x \in H \) holds \( \lim_{\lambda} U(f_\lambda)x = x \).

P2'. There exists a \( L^1(G) \)-bounded net \((f_\lambda)_{\lambda \in I}\) in \( C \) such that for all \( x \in H \) holds
\[ \lim_{\lambda} f_\lambda x = x. \]

P2''. There exists a \( L^1(G) \)-bounded net \((f_\lambda)_{\lambda \in I}\) in \( C \) such that for all \( g \in L^1(G) \) holds
\[ \lim_{\lambda} f_\lambda \ast g = g \text{ in } L^1(G). \]

P3. There exists a sequence of Borel measurable disjoint sets \( Q_1, Q_2, \ldots \) in \( \hat{G} \) and a sequence of positive real numbers \( b_1, b_2, \ldots \) such that \( \hat{G} = \bigcup_{n=1}^{\infty} Q_n \) and \( \sum_{n=1}^{\infty} b_n < \infty \) and for all \( f \in C \) there exists \( g \in C \) and \( \delta > 0 \) such that for all \( n \in \mathbb{N} \) holds
\[ b_n \sup_{\gamma \in Q_n} |f(\gamma)| \leq \delta \inf_{\gamma \in Q_n} |g(\gamma)|. \]

P3'. There exists a sequence of Borel measurable disjoint sets \( Q_1, Q_2, \ldots \) in \( \hat{G} \), a sequence of positive real numbers \( b_1, b_2, \ldots \) and \( \nu > 0 \) such that \( \hat{G} = \bigcup_{n=1}^{\infty} Q_n \) and \( \sum_{n=1}^{\infty} b_n^{-\nu} < \infty \) and for all \( f \in C \) there exists \( g \in C \) and \( \delta > 0 \) such that for all \( n \in \mathbb{N} \) holds:
\[ b_n \sup_{\gamma \in Q_n} |f(\gamma)| \leq \delta \inf_{\gamma \in Q_n} |g(\gamma)|. \]

P4. \( \forall f \in \text{Bor}(\hat{G}, C) \) \( \{(\forall K \in C \text{ [F \ast K is bounded]} \Rightarrow \exists c_0 \exists c > 0 \left[ U_{1} \left[ \gamma \in \hat{G} \mid |f(\gamma)| > c |f(\gamma)| \right] = 0 \right] \} \).

P4'. \( \forall f \in \text{Bor}(\hat{G}, C) \) \( \{(\forall K \in C \text{ [F \ast K is bounded]} \Rightarrow \exists c_0 \exists c > 0 \left[ |F(\gamma)| \leq c \left| f(\gamma) \right| \right] \} \).

P5. \( \hat{f}(1) = 1 \) for all \( f \in C \).


