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Simons, F.H.; Overdijk, D.A.

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F.H. Simons an D.A. Overdijk
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Generating sweep-out set algebras for Markov processes

F.H. Simons and D.A. Overdijk

1. Introduction

It is not difficult to show that for a dissipative Markov process $P$ on a probability space $(X,\mathcal{E},\mu)$ the recurrent sets are dense in $\mathcal{E}$. Moreover examples have been given of dissipative measurable transformations and dissipative Markov processes for which there exists an algebra of recurrent and even of sweep-out sets generating $\mathcal{E}$ (cf. [5], [6]).

Recently H. Berbee [1] has shown that a Markov process on $((0,1),\mathcal{B},\lambda)$, where $\lambda$ is the Lebesgue measure on the Borel sets $\mathcal{B}$ of the unit interval $(0,1)$, for which there exists a countable sweep-out partition is isomorphic with a process on $((0,1),\mathcal{B},\lambda)$ for which every interval is a sweep-out set. Clearly, this means that for Markov processes on $((0,1),\mathcal{B},\lambda)$ the existence of a countable sweep-out set partition is equivalent with the existence of an algebra of sweep-out sets generating $\mathcal{E}$.

In [7], we gave a necessary and sufficient condition for the existence of arbitrary small sweep-out sets for a Markov process $P$. In this note we shall show that the existence of arbitrary small sweep-out sets is equivalent with the existence of a countable sweep-out set partition, and either of these conditions implies the existence of a generating algebra of sweep-out sets if $\mathcal{E}$ is countably generated.

In particular it follows that for a dissipative Markov process on a probability space $(X,\mathcal{E},\mu)$ where $\mathcal{E}$ is countably generated, there exists a generating algebra of sweep-out sets.

2. Embedded Markov processes and sweep-out sets

We follow the terminology as used e.g. in the book of Foguel [2]. Throughout, $(X,\mathcal{E},\mu)$ will be a fixed probability space and $P$ will be a Markov process on $(X,\mathcal{E},\mu)$, i.e. a mapping of $M^+$, the space of equivalence classes of $\mu$-almost equal non negative extended real valued measurable functions, into itself such that

\begin{align*}
\text{i) } & P(\sum_{n=0}^{\infty} a_n f_n) = \sum_{n=0}^{\infty} a_n P f_n \quad (a_n \geq 0, f_n \in M^+), \\
\text{ii) } & P1 \leq 1.
\end{align*}
Here, as in the sequel, all statements on functions and sets will have to be interpreted modulo \( m \)-null sets. For reasons of convenience we shall suppose \( \Pi = 1 \).

For every \( A \in \Sigma \) and every \( k \) we have

\[
(2.1) \quad \sum_{n=0}^{k} (\Pi_A^n) (\Pi_A')^{k+1} l = p^{k+1} l = 1.
\]

Here \( I_A \) means multiplication by the indicator function \( I_A \) of the set \( A \), and \( A' = X \setminus A \). Hence if we define

\[
(2.2) \quad QAf = \sum_{n=0}^{\infty} (\Pi_{A_n}^n) (\Pi_{A'}^{k+1}) n \quad (f \in \mathcal{M}^+),
\]

then it is easily verified that \( Q_A \) again is a Markov process on \((X, \Sigma, m)\), the embedded process.

Since \( \Pi_A(x) \) can be interpreted as the probability that starting in \( x \) we enter the set \( A \) in one transition, we can interpret \( Q_A(x) \) as the probability that starting in \( x \), we ever visit the set \( A \). We start with two propositions which in this interpretation are obvious.

**Proposition 2.1.** If \( A \subseteq B \) then \( Q_A \leq Q_B \leq 1 \).

If \( A_n \uparrow A \), then \( Q_{A_n} \leq Q_A \) if \( n \to \infty \).

**Proof.** From (2.1) we conclude

\[
1 - Q_A = \lim_{k \to \infty} (\Pi_{A_n}^k) l,
\]

from which the first statement easily follows.

Now suppose \( A_n \uparrow A \). Since \( (\Pi_{A_n}^k) l \) is decreasing both in \( n \) and \( k \), we obtain

\[
1 - Q_A = \lim_{k \to \infty} (\Pi_{A_n}^k) l = \lim_{k \to \infty} \lim_{n \to \infty} (\Pi_{A_n}^k) l = \lim_{n \to \infty} \lim_{k \to \infty} (\Pi_{A_n}^k) l = 1 - \lim_{n \to \infty} Q_{A_n} l,
\]

from which the second statement follows. \( \square \)
Proposition 2.2. If $Q_A \geq \alpha > 0$ on $A'$, then $Q_A = 1$ on $X$.

Proof. From (2.2) we conclude for every $k$

$$Q_A = \sum_{n=0}^{k} (P_A)^n P_A + (P_A')^{k+1} Q_A$$

Hence if $k + \infty$

$$Q_A \geq Q_A + \alpha(1 - Q_A)$$

$$\alpha(1 - Q_A) \leq 0.$$ 

Since $\alpha > 0$ and $1 - Q_A \geq 0$, we obtain $1 - Q_A = 0$ and $Q_A = 1$. □

A set $A \in \Sigma$ for which $Q_A = 1$ on $A$ is said to be a $P$-recurrent set. If $Q_A = 1$ on $B$, then the set $A$ is called a $P$-sweep-out set for $B$, if $B = X$ then we say that $A$ is a $P$-sweep-out set. The next proposition is one of the reasons why in general we speak about sweep-out sets without mentioning the process.

Proposition 2.3. Let $A$ be a $P$-sweep-out set, and $B \subseteq A$ be a $Q_A$-sweep-out set for $A$. Then $B$ is a $P$-sweep-out set.

Proof. By [7] lemma 1 we have for all $f \in M^+$

$$Q_B f = \sum_{n=0}^{\infty} (Q_A B')^n Q_A B f.$$ 

Hence $Q_B = 1$ on $A$, and we obtain

$$Q_B = Q_A B + Q_A B' \sum_{n=0}^{\infty} (Q_A B')^n Q_A B$$

$$= Q_A B + Q_A A \setminus B Q_B$$

$$= Q_A A$$

$$= 1.$$ □
In the next proposition we use the notation $\Sigma | A$ for $\{B \cap A \mid B \in \Sigma\}$.

**Proposition 2.4.** Let $P$ be a conservative Markov process on $(X, \mathcal{E}, m)$, and let $\Sigma_1$ be the $\sigma$-algebra of invariant sets. Then for every $A \in \Sigma$ the conservative part of $X$ with respect to $Q_A$ is the set $A$, and the class of invariant sets for $Q_A$ restricted to $A$ is $\Sigma_1 | A$.

**Proof.** From the relation

$$\sum_{n=1}^{\infty} Q^nf = \sum_{n=1}^{\infty} P^n f$$

for all $f \in M^+$ ([3], theorem 3) we easily deduce that the conservative part of $X$ with respect to $Q_A$ is $A \cap C$, where $C$ is the conservative part of $X$ with respect to $P$.

Since $C = X$, this proves the first statement. The second statement now follows as well from this relation since for any $B \subset A$ the smallest $P$-invariant set containing $B$ is $\{\sum_{n=1}^{\infty} f^n \mid B \geq 0\}$ and the smallest $Q_A$-invariant set containing $B$ is $\{\sum_{n=1}^{\infty} Q^A f \mid B \geq 0\} \cap A$.

**Proposition 2.5.** Let $P$ be a conservative Markov process on $(X, \mathcal{E}, m)$ and let $\Sigma_1$ be the $\sigma$-algebra of invariant sets. If $R \subset \Sigma_1$ and $A \subset R$, then for every $B \in \Sigma$ with $B \cap R = A$ we have $Q_A = Q_B$ on $A$.

**Proof.** Using [7], lemma 4, we have for every $f \in M^+$

$$I_{A} Q_A f = I_{A} \sum_{n=0}^{\infty} (P_{A}^{n})^{n} P_{A} f$$

$$= I_{A} X \sum_{n=0}^{\infty} (P_{A}^{n})^{n} P_{A} f$$

$$= I_{A} X \sum_{n=0}^{\infty} (P_{R \cap A}^{n})^{n} P_{R \cap A} f$$

$$= I_{A} X \sum_{n=0}^{\infty} (P_{R \cap B}^{n})^{n} P_{R \cap B} f$$

$$= I_{A} X \sum_{n=0}^{\infty} (P_{B}^{n})^{n} P_{B} f$$

$$= I_{A} Q_B f.$$
3. Sweep-out set partitions

The existence of arbitrary small sweep-out sets was investigated in [7]. We shall extend the results of that paper by showing that the existence of arbitrary small sweep-out sets is equivalent with the existence of a countable sweep-out set partition. To this end we first have to reconsider some of the preliminaries in [7].

Let P be conservative and let \( \Sigma_i \) be the σ-algebra of P-invariant sets. For every \( R \in \Sigma_i \) with \( m(R) > 0 \) the P-sweep-out set number \( s(R) \) was defined by

\[
    s(R) = \sup \{ n | \text{there exists a partition of } R \text{ into } n \text{ P-sweep-out sets for } R \}.
\]

If no misunderstanding can arise, we shall omit the mentioning of the process P. By [7], theorem 1 there exists a partition \((X_1, \ldots, X_\infty)\) of \( X \) into invariant sets such that for every invariant set \( R \subset X_n \) with \( m(R) > 0 \) we have \( s(R) = n \) (\( 1 \leq n \leq \infty \)). First we show that the converse of [7] lemma 5 also holds.

**Proposition 3.1.** Suppose \( n < \infty \), and let \((A_1, \ldots, A_n)\) be a sweep-out set partition of \( X \) such that \( \Sigma_i |A_j| = \Sigma_i |A_j| \) for \( 1 \leq j \leq n \).

Then \( X = X_n \).

**Proof.** Let \( Y \) be an invariant set of \( X \) with \( m(Y) > 0 \). Then \((A_1 \cap Y, \ldots, A_n \cap Y)\) is a sweep-out set partition for \( Y \) of \( n \) elements, hence \( s(Y) \geq n \).

Suppose that there exists a sweep-out set partition \((B_1, \ldots, B_{n+1})\) of \( Y \). Let \( R_{jk} \) be the smallest invariant set containing \( A_j \cap B_k \), and let \( R \) be an atom of positive measure of the (finite) ring generated by the \( R_{jk} \).

Then both \((R \cap A_1, \ldots, R \cap A_n)\) and \((R \cap B_1, \ldots, R \cap B_{n+1})\) are sweep-out set partitions for \( R \). It follows that there exists a triple \((j,k,\ell)\) with \( k \neq \ell \) such that \( m(R \cap A_j \cap B_k) > 0 \) and \( m(R \cap A_j \cap B_\ell) > 0 \).

From \( k \neq \ell \) we conclude

\[
    (A_j \cap B_k \cap R) \cap (A_j \cap B_\ell \cap R) = \emptyset
\]

Since \( \Sigma_i |A_j| = \Sigma_i |A_j| \), we have

\[
    A_j \cap B_k \cap R = A_j \cap R_{jk} \cap R,
\]

\[
    A_j \cap B_\ell \cap R = A_j \cap R_{j\ell} \cap R,
\]

hence \( A_j \cap (R_{jk} \cap R_{j\ell} \cap R) = \emptyset \).

Since the latter set is invariant and \( A_j \) is a sweep-out set, we obtain
On the other hand, we have $m(R_{jk} \cap R) \geq m(R \cap A_j \cap B_k) > 0$ and $m(R_{j\ell} \cap R) \geq m(R \cap A_j \cap B_{\ell}) > 0$. This contradicts the fact that $R$ is an atom of the ring generated by the $R_{ik}$. Hence $s(R) \leq n$, and therefore $s(R) = n$. 

Proposition 3.2. If $X = X_\omega$, then there exists a sweep-out set partition $(A_1, A_2)$ of $X$ such that the $Q_{A_2}$-sweep-out set number of $A_2$ is $\omega$.

Proof. Since the sweep-out set number of $X$ is $\omega$, there exists a sweep-out set partition $(B_1, B_2)$ of $X$. Because of proposition 2.4 the embedded processes $Q_{B_1}$ and $Q_{B_2}$ have conservative parts $B_1$ and $B_2$ respectively. Hence, by [7], theorem 1, there exist a partition $(Y_1, \ldots, Y_\omega)$ of $B_1$ into $Q_{B_1}$-invariant sets such that every $Q_{B_1}$-invariant subset $R$ of $Y_n$ with $m(R) > 0$ has $Q_{B_1}$-sweep-out set number $n$ ($1 \leq n \leq \omega$), and a partition $(Z_1, \ldots, Z_\omega)$ of $B_2$ into $Q_{B_2}$-invariant sets such that every $Q_{B_2}$-invariant subset $S$ of $Z_m$ of positive measure has $Q_{B_2}$-sweep-out set number $m(1 \leq m \leq \omega)$.

Let us denote by $A^*$ the smallest $P$-invariant set containing the set $A \in \Sigma$. Suppose that for some $n < \omega$, $m < \omega$ we have $m(Y^* \cap Z^*) > 0$. Because of proposition 2.4 the set $(Y^* \cap Z^*) \cap B_1$ is a $Q_{B_1}$-invariant subset of positive measure of $Y_n$. Hence there exists a $Q_{B_1}$-sweep-out set partition $(E_1, \ldots, E_n)$ of $(Y^* \cap Z^*) \cap B_1$ such that for every $E_j$ we have $\Sigma \mid E_j = \Sigma_i (Q_{B_1}) \mid E_j$.

Since $B_1$ is a sweep-out set, by proposition 2.3 every $E_j$ is a sweep-out set. Moreover by proposition 2.4 we have $\Sigma_i (Q_{B_1}) \mid E_j = \Sigma_i \mid E_j$ for every $j$, $1 \leq j \leq n$, and therefore $\Sigma_i \mid E_j = \Sigma \mid E_j$.

Similarly we obtain a partition $(E_{n+1}, \ldots, E_{n+m})$ of $(Y^* \cap Z^*) \cap B_2$ such that every $E_j$ is a $P$-sweep-out set and $\Sigma_i \mid E_j = \Sigma \mid E_j$. By proposition 3.1 the set $Y^* \cap Z^*$ has sweep-out set number $n+m$, which contradicts $X = X_\omega$.

It follows that if $n < \omega$, $m < \omega$, we have $m(Y^* \cap Z^*) = 0$,
and therefore \( Y^n \cap B_2 < Z_\infty \) if \( n < \infty \).

Now put \( A_1 = \bigcup_{1 \leq n < \infty} (Y^n \cap B_1) \cup (Y^n \cap B_2) \)
and \( A_2 = \bigcup_{1 \leq n < \infty} (Y^n \cap B_2) \cup (Y^n \cap B_1) = A'_1 \).

Since \((Y_1^*, \ldots, Y_\infty^*)\) is a partition of \( X \) into invariant sets, and \( B_1 \) and \( B_2 \) are sweep-out sets, both \( A_1 \) and \( A_2 \) are sweep-out sets.

Moreover, by proposition 2.5 we have

\[
Q_{Y^n} \cap B_2 = Q_{A_2} = Q_{B_2} \quad \text{on } Y^n \cap B_2 < Z_\infty \text{ if } n < \infty;
\]

\[
Q_{Y^n} \cap B_1 = Q_{A_2} = Q_{B_1} \quad \text{on } Y^n \cap B_1 < Y_\infty.
\]

Hence the \( Q_{A_2} \)-sweep-out set number of \( A_2 \) is \( \infty \).

**Proposition 3.3.** If \( X = X_\infty \), then there exists a countable sweep-out set partition \((A_1, A_2, \ldots)\) of \( X \).

**Proof** We proceed by induction on \( n \). By proposition 3.2 there exists a sweep-out set partition \((A_1, \tilde{A}_1)\) of \( X \) such that the \( Q_{A_1} \)-sweep-out set number of \( \tilde{A}_1 \) is \( \infty \). Now suppose that we have constructed a sweep-out set partition \((A_1, A_2, \ldots, A_p, \tilde{A}_p)\) of \( X \) such that the \( Q_{A_1} \)-sweep-out set number of \( \tilde{A}_p \) is \( \infty \).

Then by proposition 3.2 there exists a \( Q_{\tilde{A}_p} \)-sweep-out set partition

\((A_{p+1}, \tilde{A}_{p+1})\) of \( \tilde{A}_p \) such that the sweep-out set number of \( \tilde{A}_{p+1} \) with respect to the embedded process of \( Q_{\tilde{A}_p} \) on \( \tilde{A}_{p+1} \) is \( \infty \). However, by [7], lemma 1, the latter process is \( Q_{A_{p+1}} \), and by proposition 2.3 both \( A_{p+1} \) and \( \tilde{A}_{p+1} \) are sweep-out sets. Hence \((A_1, A_2, \ldots, A_{p+1}, \tilde{A}_{p+1})\) is a sweep-out set partition of \( X \) such that the \( Q_{A_{p+1}} \)-sweep-out set number of \( \tilde{A}_{p+1} \) is \( \infty \). In this way we obtain a sequence \( A_1, A_2, \ldots \) of pairwise disjoint sweep-out sets. If this sequence is not a partition, we replace \( A_1 \) by \( A_1^* \equiv A_1 \cup (\bigcup_{n=1}^{\infty} A_n)' \), which by proposition 2.1 also is a sweep-out set.

We now are ready to prove the main result of this section. For measurable transformations instead of Markov processes this result can already be found in [4].
Theorem 3.1. Let $P$ be a Markov process on a probability space $(X, \mathcal{E}, m)$. Then the following statements are equivalent:

i. there exist arbitrary small sweep-out sets.

ii. there exists a countable sweep-out set partition of $X$.

Proof. The implication $\text{ii.} \Rightarrow \text{i.}$ is obvious. In order to prove the reverse implication, we first consider the restriction of $P$ to the conservative part $C$. Since there exist arbitrary small sweep-out sets, by [7] the invariant set $C$ must have sweep-out number $\infty$ and therefore by proposition 3.3 there exists a partition $(C_1, C_2, \ldots)$ of $C$ into sweep-out sets for $C$.

On the dissipative part $D$ there exists a partition $(D_1, D_2, \ldots)$ such that $\forall n \geq 0 \quad P^n_{D_1} < \infty$ on $X$ for every $i$ (cf [2], (2.5)). Then from

$$0 \leq \lim_{n \to \infty} \left( P^n_{D_1} \cup \cdots \cup P^n_{D_{k-1}} \right) \leq \lim_{n \to \infty} P^n_{D_1} \cup \cdots \cup P^n_{D_{k-1}} = 0$$

we conclude by formula (2.1) that for every $k$ the set $C \cup D_1 \cup D_{k+1} \cup \cdots$ is a sweep-out set. But then by [7], proof of lemma 3, for every $i$ and $k$ the set $C_i \cup D_1 \cup D_{k+1} \cup \cdots$ is also a sweep-out set.

We now construct a partition $E_1, E_2, \ldots$ of $X$ in the following way. Put $E_1 = C_1 \cup D_1$. Suppose the disjoint sets $E_1, \ldots, E_p$ have been constructed such that for $1 \leq i \leq p$ we have

$$E_i = C_i \cup D_{n_i} \cup D_{n_i+1} \cup \cdots \cup D_{n_i+1}$$

then the $p+1$ disjoint sets $E_1, \ldots, E_{p+1}$ also satisfy (1) and (2).

Define $A_n = C_{p+1} \cup D_{n+1} \cup \cdots \cup D_n$ and $A = C_{p+1} \cup D_{n+1} \cup \cdots \cup D_n$.

Put $E_{p+1} = A_n$ then the $p+1$ disjoint sets $E_1, \ldots, E_{p+1}$ also satisfy (1) and (2).

Obviously, $(E_1, E_2, \ldots)$ is a partition of $X$. Moreover, let $A$ be any set consisting of infinitely many of the sets $E_i$. Then from (2) and proposition 2.1 it follows that $Q_{A_n} \geq \frac{1}{2}$ on every $E_j$, and therefore by proposition 2.2 the set $A$ is a sweep-out set.

Hence every countable partition of the natural numbers into countable sets yields a sweep-out set partition of $X$. \qed
4. Generating algebra's of sweep-out sets

Let \( P \) be a Markov process on a probability space \((X, \mathcal{E}, m)\), and \( \mathcal{A} \) an algebra of measurable sets. The algebra is said to be generating if \( \mathcal{A} \) is dense in \( E \) with respect to \( m \), and \( \mathcal{A} \) is said to be a sweep-out set algebra if every element of \( \mathcal{A} \) of positive measure is a sweep-out set.

If there exist arbitrary small sweep-out sets, then for every \( n \) there exists a sweep-out set \( B_n \) with \( m(B_n) < \frac{1}{n} \). Then for every \( A \in \mathcal{E} \) by proposition 2.1 the set \( A \cup B_n \) is sweep-out set, and therefore the sweep-out sets are dense in \( E \).

We shall show that, if \( E \) is countably generated, we can select from the class of sweep-out sets a sweep-out set algebra generating \( E \).

**Theorem 4.1.** Let \( P \) be a Markov process on a probability space \((X, \mathcal{E}, m)\) and suppose that \( E \) is countably generated. If there exist arbitrary small sweep-out sets, then there exists a generating algebra of sweep-out sets.

**Proof.** Since there exist arbitrary small sweep-out sets, by theorem 3.1 there also exists a sweep-out set partition \((A_1, A_2, \ldots)\) of \( X \).

For every \( n \) the \( \sigma \)-ring \( E|A_n \) is countably generated. Since for any generating sequence \( B_1, B_2, \ldots \) the sequence \( X, B_1, B_2 \cap B_1, B_2 \cap B_1', B_2 \cap B_1', \ldots \) is also generating, we may suppose that \( E|A_n \) is generated by a system of sets \((A_n; i_1 \ldots i_k)\) with \( i_j = 0 \) or \( 1 \), such that \((A_n; i_1 \ldots i_{k-1}, 0)\) is a partition of \( A_n; i_1 \ldots i_{k-1} \) \((k \geq 1)\).

We now construct a similar system of sweep-out sets of \( X \).

We start by taking a partition \((E_1; 0^* E_1; 1)\) of \( X \) such that \( A_1; 0 \subseteq E_1; 0^* \), \( A_1; 1 \subseteq E_1; 1 \), and for \( n \geq 2 \) every \( A_n \) is contained in \( E_1; 0 \) or \( E_1; 1 \) such that both \( E_1; 0 \) and \( E_1; 1 \) contain infinitely many \( A_n \). Then \((E_1; 0^* E_1; 1)\) is a sweep-out set partition of \( X \).

We now proceed inductively. Suppose that after the \( n \)-th step of the construction we have obtained a partition \((E_k; i_1 \ldots i_n)\), \( 1 \leq k \leq n \), \( i_j = 0 \) or \( 1 \) \((1 \leq j \leq n)\), such that

(i) each \( E_k; i_1 \ldots i_n \) contains infinitely many \( A_j \) with \( j > n \).

(ii) each \( A_j \) with \( j > n \) is contained in one of the sets \( E_k; i_1 \ldots i_n \).

(iii) \( E_k; i_1 \ldots i_n \cap \bigcup_{j=n}^n A_j \).

(iv) the partition \((E_k; i_1 \ldots i_n)\) is a refinement of the partition \((E_k; i_1 \ldots i_{n-1})\).
First note that $A_{n+1}$ is contained in exactly one of the sets $E_k; i_1 \ldots i_n$, say $E_k; i_1 \ldots i_n$. This set is split up into $2^{n+1} + 2$ disjoint subsets,

$$E_k; i_1 \ldots i_n = \bigcup_{i_j = 0 \text{ or } 1}^{n+1} E_{k'; i_1 \ldots i_n}$$

with $i_j = 0$ or $1$ for $1 \leq j \leq n+1$, such that condition (i) and (iii) with $n$ replaced by $n+1$ hold for each of these sets, and every $A_j$ with $j > n+1$ and $A_j \subseteq E_k; i_1 \ldots i_n$ of these sets. Every other set $E_k; i_1 \ldots i_n$ is split up into two subsets

$E_k; i_1 \ldots i_0$ and $E_k; i_1 \ldots i_1$ such that condition (i), (ii) and (iii) hold with $n$ replaced by $n+1$. Condition (iv) is now automatically fulfilled.

Let the class $\mathcal{A}$ consist of finite unions of sets $E_k; i_1 \ldots i_n$, together with $\emptyset$. Obviously the class $\mathcal{A}$ is closed under the operation of taking unions. From (iv) it follows that every element of $\mathcal{A}$ can be written as a finite disjoint union of sets $E_k; i_1 \ldots i_m$ with $m$ fixed. Since the sets $E_k; i_1 \ldots i_m$ form a partition of $X$, the class $\mathcal{A}$ is also closed under the operation of taking complements, and therefore $\mathcal{A}$ is an algebra.

Because of condition (i) every non-empty element of $\mathcal{A}$ is a sweep-out set, and therefore $\mathcal{A}$ is a sweep-out algebra.

In order to show that $\mathcal{A}$ is generating, it suffices to show that every set $A_k; i_1 \ldots i_n$ can be approximated by elements of $\mathcal{A}$.

Take $m \geq \max(k, n)$ and choose $\varepsilon > 0$. Put

$$E = \bigcup_{i_n + 1 = 0}^{1} \bigcup_{i_m = 0}^{1} E_k; i_1 \ldots i_m$$

Then $E \in \mathcal{A}$, and by condition (iii)

$$E \cap \bigcup_{j=1}^{m} A_j = \bigcup_{j=1}^{m} A_k; i_1 \ldots i_m = A_k; i_1 \ldots i_n$$

Hence if we choose $m$ so large that $m(\bigcup A_j) < \varepsilon$, we have $m(A \Delta E) < \varepsilon$. $\Box$

In [7] we have shown that for a dissipative Markov process on $(X, \mathcal{F}, m)$ there exist arbitrary small sweep-out sets. Hence if $P$ is a dissipative Markov process on a countably generated probability space $(X, \mathcal{F}, m)$, then there exists an algebra of sweep-out sets generating $\mathcal{F}$.
5. References


