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Non-Hamiltonian symmetries of a Boussinesq equation

H. M. M. Ten Eikelder

Department of Mathematics and Computing Science, Eindhoven University of Technology, P. O. Box 513, Eindhoven, The Netherlands

L. J. F. Broer

Department of Physics, Eindhoven University of Technology, P. O. Box 513, Eindhoven, The Netherlands

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For a class of Hamiltonian systems there exist infinite series of non-Hamiltonian symmetries. Some properties of these series are illustrated using a Boussinesq equation. It is shown that the recursion operators generated by these non-Hamiltonian symmetries are powers of the original recursion operator. A class of recursion formulas for the constants of the motion (not for the corresponding symmetries!) is given.

I. INTRODUCTION

For a certain class of Hamiltonian systems there exist so-called recursion operators for symmetries. Repeated application of such a recursion operator yields a series of symmetries. Often it is possible to construct in this way infinite series of Hamiltonian symmetries (corresponding to constants of the motion) and infinite series of non-Hamiltonian symmetries. The most well-known example is the Korteweg-de Vries equation, where the Lénard operator generates an infinite series of Hamiltonian symmetries and an infinite series of non-Hamiltonian symmetries. In this paper we use a Boussinesq equation to illustrate some properties of these series, in particular the series of non-Hamiltonian symmetries. Similar results can be obtained for various other equations, see Ten Eikelder.^{1,2} In this paper we work within the framework of differential geometry. For definitions of various concepts (symmetry, recursion operator for symmetries, etc.) see, for instance, Ref. 2, where also notations and conventions are given.

II. SYMMETRIES OF A BOUSSINESQ EQUATION

We study a Boussinesq equation of the form

$$\begin{aligned} v_t &= w_x, \\ w_t &= vv_x + \lambda v_{xxx}, \end{aligned} \quad -\infty < x < \infty, \quad t > 0. \quad (1)$$

We consider (1) as an evolution equation in a topological vector space $\mathcal{W}$ of pairs of smooth functions (v, w) , which decay, together with their x derivatives, sufficiently fast for $|x| \rightarrow \infty$. The spaces $\mathcal{W}$ and $\mathcal{W}^*$ are constructed such that

$$\Lambda = \begin{pmatrix} \partial(2w \partial^{-1} + \partial^{-1}w) & \partial(v \partial^{-1} + \partial^{-1}v) + 8\lambda \partial^2 \\ 2v \partial v \partial^{-1} + 2\lambda \partial(\partial^2 v \partial^{-1} + \partial^{-1}v \partial^2) & \partial(w \partial^{-1} + 2 \partial^{-1}w) \\ + 3\lambda \partial(\partial v + v \partial) + 8\lambda^2 \partial^4 & \end{pmatrix}.$$

Three infinite series of symmetries now can be defined by

$$X_k = \Lambda^k X_0, \quad Y_k = \Lambda^k Y_0, \quad Z_k = \Lambda^k Z_0, \quad k = 0, 1, 2, \dots$$

It is shown by Fokas and Anderson⁴ that the Nijenhuis tensor of Λ vanishes (in their terminology, Λ is a hereditary symmetry). So for all vector fields A we have $\mathcal{L}_{\Lambda A} \Lambda$

their duality map $\langle \cdot, \cdot \rangle$ is the L_2 inner product. A possible choice is $\mathcal{W} = \mathcal{S}_p \times \mathcal{S}_p$ and $\mathcal{W}^* = \mathcal{U}_p \times \mathcal{U}_p$, where the function spaces $\mathcal{S}_p$ and $\mathcal{U}_p$ are described in Ref. 1. In terms of $u = (v, w) \in \mathcal{W}$ we can write (1) as

$$\dot{u} = X(u) \quad \left(\dot{u}(t) = \frac{d}{dt} u(t) \right).$$

A Hamiltonian form of (1) is well known. Let the function (functional) F_0 on $\mathcal{W}$ be given by

$$F_0 = \int_{-\infty}^{\infty} \left(\frac{1}{6} v^3 - \frac{1}{2} \lambda v_x^2 + \frac{1}{2} w^2 \right) dx$$

and let the symplectic form Ω on $\mathcal{W}$ be (represented by the linear mapping $\Omega: \mathcal{W} \rightarrow \mathcal{W}^*$) given by

$$\Omega = \begin{pmatrix} 0 & \partial^{-1} \\ \partial^{-1} & 0 \end{pmatrix}.$$

Then the vector field X can be written as $X = \Omega^{-1} dF_0$ (d = exterior derivative), so (1) is a Hamiltonian system.

The invariance of (1) for translations along the t and x axis and for a scale transformation yields the following elementary symmetries:

$$\begin{aligned} X_0 &= X = \begin{pmatrix} w_x \\ vv_x + \lambda v_{xxx} \end{pmatrix}, \\ Y_0 &= \begin{pmatrix} v_x \\ w_x \end{pmatrix}, \quad Z_0 = \begin{pmatrix} 2v + xv_x \\ 3w + xw_x \end{pmatrix} + 2tX_0. \end{aligned} \quad (2)$$

A recursion operator for symmetries of (1), written in terms of the "coordinates" of a modified Boussinesq equation, has been given by Fordy and Gibbons.³ In terms of the "original coordinates" v and w this operator reads

$\Lambda = \mathcal{L}_A \Lambda$ ($\mathcal{L}_A$ = Lie derivative in direction of A). This also can be verified by a straightforward computation.

Let A and B be vector fields on $\mathcal{W}$ such that $\mathcal{L}_A \Lambda = a\Lambda$ and $\mathcal{L}_B \Lambda = b\Lambda$ for $a, b \in \mathbb{R}$. Define $A_k = \Lambda^k A$ and $B_k = \Lambda^k B$, for $k = 0, 1, 2, \dots$. Using the fact that the Nijenhuis tensor of Λ vanishes, it is easily shown (see, for instance, Ref.

2) that the Lie bracket $[A_k, B_l]$ is given by

$$[A_k, B_l] = laB_{k+l} - kbA_{k+l} + \Lambda^{k+l}[A, B]. \quad (3)$$

A simple computation shows that

$$\mathcal{L}_{X_0}\Lambda = 0, \quad \mathcal{L}_{Y_0}\Lambda = 0, \quad \mathcal{L}_{Z_0}\Lambda = 3\Lambda, \quad (4)$$

$$[X_0, Y_0] = 0, \quad [Z_0, X_0] = 2X_0, \quad [Z_0, Y_0] = Y_0.$$

Substitution in (3) yields that the only nonvanishing Lie brackets between the elements of the series X_k , Y_k , and Z_k are given by

$$\begin{aligned} [Z_k, X_l] &= (3l+2)X_{k+l}, & [Z_k, Y_l] &= (3l+1)Y_{k+l}, \\ [Z_k, Z_l] &= 3(l-k)Z_{k+l}. \end{aligned} \quad (5)$$

Since the Nijenhuis tensor of Λ vanishes, it immediately follows that

$$\begin{aligned} \mathcal{L}_{X_k}\Lambda &= 0, & \mathcal{L}_{Y_k}\Lambda &= 0, & \mathcal{L}_{Z_k}\Lambda &= 3\Lambda^{k+1}, \\ k &= 0, 1, 2, \dots \end{aligned} \quad (6)$$

The first relation corresponds to the well-known fact that Λ is also a recursion operator for symmetries of the "higher-order Boussinesq equations" $\dot{u} = X_k$. The second relation shows that Λ is also a recursion operator for the equations $\dot{u} = Y_k$.

Next we discuss some properties of the series of symmetries Z_k . For every non-Hamiltonian symmetry Z a nonvanishing recursion operator for symmetries is given by $\Omega^{-1}\mathcal{L}_Z\Omega$. If Z is a Hamiltonian symmetry this expression yields 0 [because $\mathcal{L}_Z\Omega = d(\Omega Z) = 0$]. Note that the recursion operators obtained in this way are always the product of a canonical operator Ω^{-1} (also called Hamiltonian operator or implectic operator) and a closed operator $\mathcal{L}_Z(\Omega)$ (also called symplectic operator). Most interesting recursion operators have such a factorization, see, for instance, Magri,⁵ Fuchssteiner and Fokas,⁶ or Gel'fand and Dorfman.⁷ In Ref. 2, we computed recursion operators for the massive Thirring model by this method.

The symmetries Z_0 and Z_1 turn out to be non-Hamiltonian. The corresponding recursion operators are found to be

$$\Omega^{-1}\mathcal{L}_{Z_0}\Omega = 3I \quad (I = \text{identity mapping: } \mathcal{W} \rightarrow \mathcal{W}),$$

$$\Omega^{-1}\mathcal{L}_{Z_1}\Omega = 6\Lambda. \quad (7)$$

So the recursion operator Λ can be reconstructed from the symmetry Z_1 . From (6) and (7) it is easily shown by induction that

$$\mathcal{L}_{Z_k}\Omega = 3^k(k+1)!\Omega\Lambda^k, \quad k = 0, 1, 2, \dots \quad (8)$$

Since the Lie derivatives and the exterior derivative commute, this relation yields a very simple proof of the well-known fact that all the two-forms $\Omega\Lambda^k$ are closed. This property implies that

$$\begin{aligned} L_{Z_k}\Omega &= d(\Omega Z_k) = d(\Omega\Lambda^k Z_0) = \mathcal{L}_{Z_0}(\Omega\Lambda^k) \\ &= (\mathcal{L}_{Z_0}\Omega)\Lambda^k + \Omega\mathcal{L}_{Z_0}(\Lambda^k) = (3k+3)\Omega\Lambda^k \neq 0, \\ k &= 0, 1, 2, \dots \end{aligned} \quad (9)$$

Thus we have proved that all the symmetries Z_k are non-Hamiltonian and that the corresponding recursion operators are powers of Λ (up to a multiplicative constant).

Because X_0 is a Hamiltonian symmetry $\mathcal{L}_{X_0}\Omega = 0$. A simple computation shows that $Y_0 = \Omega^{-1}dG_0$ with

$G_0 = \int_{-\infty}^{\infty} vw \, dx$, so Y_0 is also a Hamiltonian symmetry. A computation similar to (9) shows that all the symmetries X_k , Y_k ($k = 0, 1, 2, \dots$) are Hamiltonian vector fields, i.e., there exist two series of constants of the motion F_k and G_k such that

$$X_k = \Omega^{-1}dF_k, \quad Y_k = \Omega^{-1}dG_k, \quad k = 0, 1, 2, \dots \quad (10)$$

The corresponding symmetries commute, so all these constants of the motion are in involution. The existence of the series F_k is a standard property in this case, see, for instance, Ref. 6. It follows from (10) that

$$\Omega\Lambda^k X = dF_k,$$

which can be considered as "pre-Hamiltonian" forms for $X = X_0$. The original Hamiltonian form is obtained for $k = 0$, while formally $k = -1$ with $F_{-1} = \int_{-\infty}^{\infty} \frac{1}{2}w \, dx$ yields the second Hamiltonian form of the Boussinesq equation.

We now give a class of recursion formulas for the constants of the motion F_k and G_k . The Hamiltonian vector field corresponding to the function $\mathcal{L}_{Z_l}F_k$ on $\mathcal{W}$ is

$$\begin{aligned} \Omega^{-1}d\mathcal{L}_{Z_l}F_k &= \Omega^{-1}\mathcal{L}_{Z_l}dF_k \\ &= \mathcal{L}_{Z_l}(\Omega^{-1}dF_k) - (\mathcal{L}_{Z_l}\Omega^{-1})dF_k \\ &= [Z_l, X_k] + \Omega^{-1}(\mathcal{L}_{Z_l}\Omega)\Omega^{-1}dF_k \\ &= (3k+2)X_{k+l} + (3l+3)\Lambda^l X_k \\ &= (3k+3l+5)\Omega^{-1}dF_{k+l}, \end{aligned}$$

where we used (5) and (9). This yields the recursion formulas

$$F_{k+l} = \frac{1}{3k+3l+5} \mathcal{L}_{Z_l}F_k \equiv \frac{1}{3k+3l+5} \langle dF_k, Z_l \rangle. \quad (11)$$

In a similar way we get

$$G_{k+l} = \frac{1}{3k+3l+4} \mathcal{L}_{Z_l}G_k \equiv \frac{1}{3k+3l+4} \langle dG_k, Z_l \rangle. \quad (12)$$

Note that in these recursion formulas it is not necessary to reconstruct a functional from its derivatives. The part of Z_l with "coefficient" t is $2X_l$ [see (2)], so this term can be omitted in (11) and (12).

The symmetry Z_1 is given by

$$Z_1 = \begin{pmatrix} Z_{1,1} \\ Z_{1,2} \end{pmatrix} + 2tX_1,$$

where

$$\begin{aligned} Z_{1,1} &= 12vw + 2w_x \partial^{-1}v + 2v_x \partial^{-1}w \\ &\quad + 40\lambda w_{xx} + x(4(vw)_x + 8\lambda w_{xxx}), \\ Z_{1,2} &= 4v^3 + 2vv_x \partial^{-1}v + 2\lambda v_{xxx} \partial^{-1}v + 58\lambda vv_{xx} \\ &\quad + 45\lambda v_x^2 + 48\lambda^2 v_{xxxx} + 9w^2 + 2w_x \partial^{-1}w \\ &\quad + x(4v^2v_x + 12\lambda vv_{xxx} \\ &\quad + 24\lambda v_x v_{xx} + 8\lambda^2 v_{xxxx} + 4wv_x). \end{aligned}$$

The part of Z_1 with coefficient x turns out to be Y_1 . So $Z_1 = C_1 + xY_1 + 2tX_1$, where C_1 contains (also nonlocal) terms not depending explicitly on x and t . Similar relations

turn out to hold for the other symmetries Z_k . For $l = 1$ we obtain from (11) and (12) the recursion formulas

$$F_{k+1} = \frac{1}{3k+8} \int_{-\infty}^{\infty} \left(\frac{\delta F_k}{\delta v} Z_{1,1} + \frac{\delta F_k}{\delta w} Z_{1,2} \right) dx, \quad (13)$$

$$G_{k+1} = \frac{1}{3k+7} \int_{-\infty}^{\infty} \left(\frac{\delta G_k}{\delta v} Z_{1,1} + \frac{\delta G_k}{\delta w} Z_{1,2} \right) dx. \quad (14)$$

Starting with F_0 and G_0 these relations enable us to generate the series F_k and G_k . In fact it is also possible to begin with F_{-1} and $G_{-1} = \int_{-\infty}^{\infty} v dx$.

A constant of the motion that depends explicitly on t is $J = \int_{-\infty}^{\infty} (xv + tw) dx$. Constants of the motion of this type always exist if a conserved density (in this case v) has a flux that is also conserved, see Broer and Backerra.⁸ The Hamiltonian symmetry corresponding to J is formally given by

$$Z_{-1} = \Omega^{-1} dJ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It can be shown that (11) and (12) also hold for $l = -1$ and $k \geq 0$. This yields the relations

$$F_{k-1} = \frac{1}{3k+2} \mathcal{L}_{Z_{-1}} F_k \equiv \frac{1}{3k+2} \int_{-\infty}^{\infty} \frac{\delta F_k}{\delta w} dx,$$

$$G_{k-1} = \frac{1}{3k+1} \mathcal{L}_{Z_{-1}} G_k \equiv \frac{1}{3k+1} \int_{-\infty}^{\infty} \frac{\delta G_k}{\delta w} dx, \\ k = 0, 1, 2, \dots$$

While (13) and (14) allow us to go upwards in the series of constants of the motion, these two relations allow us to go downwards.

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