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A polynomial characterization of
(A,B)-invariant and reachability subspaces

by

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The Netherlands
1. INTRODUCTION

The geometric approach to linear system theory has proved very successful in solving a variety of problems (see [14] for a detailed account of this theory). The principal concepts in this theory, which are instrumental in the description of many results, are (A,B)-invariant subspaces and reachability (controllability) subspaces. An alternative approach to linear system design has been developed in [11-13]. This theory depends to a large extent on polynomial matrix techniques. It is evident that a method for translating results of one theory to another is very desirable, because such a method would yield a better understanding of the relations between the two different approaches. This would be very useful, in particular since the geometric method may be viewed as exponent of the so-called "modern control theory" and the polynomial matrix method may be considered a generalization of the classical frequency domain methods.

A number of papers with the objective of translating the results of geometric control theory into polynomial matrix terms have appeared (e.g. [1-3], [8-9]). It is the purpose of this paper to show that a very useful link between the two approaches can been based on the work of P. Fuhrmann ([6-8]). Specifically, it will be shown that using the state space model associated with a system matrix, introduced by Fuhrmann, one can give characterizations of the concepts of (A,B)-invariant subspaces and reachability subspaces in terms of polynomial matrices. This will be the subject of sections 3 and 5. An application of the polynomial characterization of (A,B)-invariant subspaces will be given in section 4, where it will be shown that the disturbance decoupling problem (see [14, Ch. 4]) and the exact model matching problem (see [13], [10], [5], [2]) are equivalent problems. In section 6, the concept of row properness defined in [12-13] is used to formulate a necessary and sufficient condition for the existence of a solution of the exact model matching problem and hence of the disturbance decoupling problem in terms of degrees of polynomial matrices. Also in section 6 a constructive characterization of the supremal (A,B)-invariant subspace and reachability space contained in ker C is given.

The preliminary section 2 contains a short description of Fuhrmann's state space model in addition to some auxiliary results.
2. THE STATE SPACE MODEL ASSOCIATED WITH A POLYNOMIAL SYSTEM MATRIX

Let \( K \) be a field. We denote by \( K[s] \) the set of polynomials and by \( K(s) \) the set of rational functions over \( K \). If \( S \) is any set and \( p, q \in \mathbb{N} \), we denote by \( S^p \) the set of \( p \)-vectors with components in \( S \) and by \( S^{p \times q} \) the set of \( p \times q \) matrices with entries in \( S \). If \( A \) is a \( p \times q \) matrix we denote by \( \{A\} \) the \( K \)-linear space generated by the columns of \( A \). If \( U(s) \in K^{q \times r}[s] \) and \( \ell: K^{q}[s] \to K^{p}[s] \) is a linear map, then \( \ell U(s) \) denotes the result obtained by applying \( \ell \) to each of the columns of \( U(s) \).

Let \( x(s) \in K^{p}(s) \). We denote by \((x(s))_{-}\) the strictly proper part of \( x(s) \) and by \((x(s))_{-1}\) the coefficient of \( s^{-1} \) in the expansion of \( x(s) \) in powers of \( s^{-1} \).

\[
\text{(2.1) DEFINITION. Let } T(s) \in K^{p \times q}[s]. \text{ Then } K_{T} = \{x(s) \in K^{p}[s] \mid T^{-1}(s)x(s) \text{ is strictly proper}\}. 
\]

In what follows, \( K_{T} \) plays a fundamental role (compare the closely related concept of right rational annihilator [4]).

In particular, if \( p = q \) and \( T(s) \) is nonsingular then

\[
K_{T} = \{x(s) \in K^{p}[s] \mid T^{-1}(s)x(s) \text{ is strictly proper}\}.
\]

In this particular situation we define the map

\[
\pi_{T}: K^{p}[s] \to K_{T}: x(s) \mapsto T(s)(T^{-1}(s)x(s))_{-}.
\]

(Compare [5] and [7] where further properties of this map are given.)

Following H.H. Rosenbrock ([11]) we consider a system represented by a system matrix

\[
(2.2) \quad P(s) = \begin{bmatrix}
T(s) & U(s) \\
-V(s) & W(s)
\end{bmatrix}
\]

where \( T(s) \in K^{q \times q}[s] \) is nonsingular and \( P(s) \in K^{(q+r) \times (q+r)}[s] \).

We assume that the transfer function matrix

\[
G(s) := V(s)T^{-1}(s)U(s) + W(s)
\]

and the matrix \( T^{-1}(s)U(s) \) are strictly proper. If the latter condition is not satisfied, we can obtain this by strict system equivalence (see [11, § 3.1]). Indeed, if we define

\[
U_{1}(s) := \pi_{T}(U(s))
\]
then

\[ Q(s) := T^{-1}(s)(U(s) - U_1(s)) \]

is a polynomial matrix. Therefore

\[ P_1(s) := \begin{bmatrix} T(s) & U_1(s) \\ V(s) & W(s) + V(s)Q(s) \end{bmatrix} \]

is a polynomial system matrix with the same transfer matrix \( G(s) \).

In the following we consider \( K_T \) as a \( K \)-vector space. Define the linear maps

\[ A: K_T \to K_T: x(s) \mapsto \pi_T(\pi_x(s)) \]
\[ B: K^r \to K_T: u \mapsto U(s)u \]
\[ C: K_T \to K_q: x(s) \mapsto (V(s)T^{-1}(s)x(s))_{-1} \]

Then the following result is proved in [7]:

\[ \text{(2.3) THEOREM. The system } \Sigma := (C,A,B) \text{ with state space } K_T \text{ is a realization of } G(s). \text{ The realization is reachable iff } T(s) \text{ and } U(s) \text{ are left coprime and observable iff } T(s) \text{ and } V(s) \text{ are right coprime.} \]

We will call this realization \( \Sigma \) the state space model associated with \( P(s) \).

By definition, for \( x(s) \in K_T \) we have \( Ax(s) = sx(s) - T(s)c(s) \) for some \( c(s) \in K_T[q] \). Since \( T^{-1}(s)x(s) \) and \( T^{-1}(s)Ax(s) \) are strictly proper it follows that \( c(s) \) must be constant. Hence

\[ (2.4) \quad Ax(s) = sx(s) - T(s)c \]

for some \( c \in K^q \), depending on \( x(s) \).

We will also use the following result of Fuhrmann (see [6, Thms 4.5, 4.7]).

\[ \text{(2.5) LEMMA. Let } T_1(s) \in K^{p \times p}[s] \text{ and } T_2(s) \in K^{q \times q}[s] \text{ be nonsingular. Then a map } \ell: K_T \to K_T \text{ is a } K[s] \text{-module homomorphism iff there exist } L_1(s) \text{ and } L_2(s) \text{ in } K^{p \times p}_T[s] \text{ such that} \]
\[ L_1(s)T_1(s) = T_2(s)L_2(s) \]
and
\[ \ell x(s) = \pi_T^{-1}(L_1(s)x(s)) \]

for every \( x(s) \in K_T \). The map \( \ell \) is an isomorphism iff \( L_1(s) \) and \( T_2(s) \) are left coprime and \( T_1(s) \) and \( L_2(s) \) are right coprime. \[ \square \]
In this lemma $K_1$ and $K_2$ are considered $K[s]$-modules, where the scalar multiplication is defined by

$$p(s) \cdot x(s) := \pi_{T_1} (p(s)x(s))$$

for $x(s) \in K_1$, $p(s) \in K[s]$.

Most of our paper will be concerned with a special case of the above state space model, i.e., with the case $V(s) = I$, in which case $W(s) = 0$. In this situation $\Sigma$ will be an observable realization of the transfer function matrix

$$(2.6) \quad G(s) = T^{-1}(s)U(s).$$

We call $\Sigma$ the $T$-realization of $G(s)$ and (2.6) a left matrix fraction representation of $G(s)$.

It is well known that every (strictly proper) transfer matrix has a factorization of the form (2.6) for which $T(s)$ and $U(s)$ are left coprime, in which case $\Sigma$ is also reachable. For our purpose, it is not necessary that $T(s)$ and $U(s)$ be left coprime.

In the following section we will derive a number of results for the particular system $\Sigma$. The question arises, whether these results are applicable if we are given an arbitrary system. The following lemma states that this is the case if the given system $(C,A,B)$ is observable, for in that situation we can define $T(s)$ and $U(s)$ such that the $T$-realization of $T^{-1}(s)U(s)$ is isomorphic with $(C,A,B)$.

$$(2.7) \text{LEMMA. Let } (C,A,B) \text{ be an observable } n \times 1 \text{ dimensional realization of an } n \times r \text{ transfer function matrix } G(s). \text{ Let } T(s) \text{ and } S(s) \text{ be left coprime matrices such that}$$

$$(2.8) \quad C(sI - A)^{-1} = T^{-1}(s)S(s).$$

Then we have

i) The columns of $S(s)$ form a basis of $K_T$ (considered as a $K$-linear space),

ii) If $U(s) := S(s)B$ and $(C,A,B)$ is the $T$-realization of $G(s)$, then $C,A$ and $B$ are matrix representations of $C$, $A$ and $B$ with respect to the canonical bases of $K$ and $K_T$ and the basis $S(s)$ of $K_T$.

iii) The $K$-linear map $S(s) : K^n + K_T$ provides an $K$-isomorphism between the realizations $(C,A,B)$ and $(C,A,B)$ of $G(s) = T^{-1}(s)U(s)$, i.e.,

$$AS(s) = S(s)A$$
$$B = S(s)B$$
$$CS(s) = C.$$
PROOF.
i) Equation (2.8) is equivalent to
\[ T(s)C = S(s)(sI - A). \]

Therefore, according to lemma (2.3), the map
\[ \ell: K_{sI-A} \rightarrow K_T: x \mapsto S(s)x \]
is a \( K[s] \)-module isomorphism. Since \( K_{sI-A} = K^n \), it follows that every
\[ x(s) \in K_T \]
can uniquely be represented as
\[ x(s) = S(s)v \]
for some \( v \in K^n \), that is, as a linear combination of the columns of \( S(s) \).
Consequently, the columns of \( S(s) \) are independent and form a basis of \( K_T \).

ii) and iii) are obviously equivalent statements.

iii) We have
\[ Bu = U(s)u = S(s)Bu \]
for \( u \in K^r \). Also, for \( x \in K^n \),
\[ A(S(s)x) = \pi_T(S(s)(sI - A)x) + \pi_T(S(s)Ax) = \]
\[ = \pi_T(T(s)Cx) + S(s)Ax = S(s)Ax \]
and
\[ C(S(s)x) = (T^{-1}(s)S(s)x)_{-1} = (C(sI - A)^{-1}x)_{-1} = Cx \]
for \( x \in K^n \). \qed

(2.9) REMARK. If \( \bar{\Sigma} = (\bar{A}, \bar{B}, \bar{C}) \) is an observable realization with an abstract
state space \( X \), then choosing a basis matrix \( \bar{X} \) for \( X \) we obtain an isomorphism
\( \bar{X}: K^n \rightarrow X \). This isomorphism induces an observable realization with state
space \( K^n \), to which we may apply lemma (2.7). Thus we may conclude that \( \bar{\Sigma} \) is
isomorphic to a suitable state space model \( \Sigma \) of the type discussed in this
section. \qed

We conclude this section with two simple results, which will be needed
in the sequel.

(2.10) LEMMA. Let \( Q(s) \in K^{k \times n}([s], A \in K^{n \times n}, B \in K^{n \times r} \). Then
i) \( (Q(s)(sI - A)^{-1})_{-1} = 0 \) implies that \( Q(sI - A)^{-1} \) is a polynomial matrix.
ii) If \( (A,B) \) is reachable and \( Q(s)(sI - A)^{-1}B \) is a polynomial matrix, then
\( Q(sI - A)^{-1} \) is a polynomial matrix.
PROOF. We decompose the rational matrix \( Q(s)(sI - A)^{-1} \) into its polynomial and strictly proper part

\[
Q(s)(sI - A)^{-1} = P(s) + R(s).
\]

Then

\[
R_0 := R(s)(sI - A) = Q(s) - P(s)(sI - A)
\]

is a polynomial of degree zero and hence constant.

i) \((Q(s)(sI - A)^{-1})^{-1} = (P(s))^{-1} + (R_0(sI - A)^{-1})^{-1} = R_0 = 0\)

implies \(Q(s)(sI - A)^{-1} = P(s)\).

ii) If \(Q(s)(sI - A)^{-1}B = P(s)B + R_0(sI - A)^{-1}B\) is a polynomial, then

\(R_0(sI - A)^{-1}B = 0\) (being strictly proper, while \(P(s)B\) is a polynomial).

By reachability it follows that \(R_0 = 0\) and hence \(Q(s)(sI - A)^{-1} = P(s)\).\]

3. \((A,B)\)-INVARIANT SUBSPACES

We give a characterization of the \((A,B)\)-invariant subspaces of the state space model \(\Sigma\) associated with the system matrix \(P(s)\), as defined in the previous section. For the definition of \((A,B)\)-invariant subspaces we refer to [14].

(3.1) THEOREM. Let \(\psi(s)\) be a \(q \times m\) polynomial matrix. Then \(\{\psi(s)\}\) is an \((A,B)\)-invariant subspace of \(K_T^q\) iff there exist \(C_1 \in K^{q \times m}, F_1 \in K^{r \times m}\) and \(A_1 \in K^{m \times m}\) such that

\[
T(s)C_1 + U(s)F_1 = \psi(s)(sI - A_1).
\]

PROOF. Suppose that \(\{\psi(s)\}\) is an \((A,B)\)-invariant subspace, i.e.,

\[
A(\psi(s)) \subseteq \{\psi(s)\} + \text{im } B.
\]

Applying (2.4) to each column of \(\psi(s)\), we find that \(A\psi(s) = \psi_1(s)\), where

\[
\psi_1(s) := s\psi(s) - T(s)C_1
\]

for some \(C_1 \in K^{q \times m}\). On the other hand, (3.3) implies

\[
\psi_1(s) = \psi(s)A_1 + U(s)F_1
\]

for some \(A_1 \in K^{m \times m}\) and \(F_1 \in K^{r \times m}\). Combining (3.4) and (3.5) yields (3.2).
Conversely, if we assume (3.2), then

\[ T^{-1}(s)\mathcal{V}(s) = (C_1 + T^{-1}(s)U(s)F_1)(sI - A_1)^{-1} \tag{3.6} \]

is strictly proper and hence \( \mathcal{V}(s) \subseteq K_T \). Furthermore, if we define \( \mathcal{V}_1(s) \) by (3.4) then (3.5) follows from (3.2) and hence \( \mathcal{V}_1(s) \subseteq K_T \). It follows that

\[ \mathcal{A}(\mathcal{V}(s)) = \pi_T(s\mathcal{V}(s)) = \pi_T(\mathcal{V}_1(s) + T(s)C_1) = \mathcal{V}_1(s). \]

Thus, (3.5) implies (3.3).

The next result gives a characterization of \((A,B)\)-invariant subspaces contained in \( \ker \mathcal{C} \).

**Theorem.** Let \( \mathcal{V}(s) \) be a \( q \times m \) polynomial matrix. Then \( \mathcal{V}(s) \) is an \((A,B)\)-invariant subspace in \( \ker \mathcal{C} \) iff there exist \( C_1 \in K_q^{q \times m}, F_1 \in K_r^{r \times m}, A_1 \in K_m^{m \times m} \) and an \( r \times m \) polynomial matrix \( \phi(s) \) such that

\[\begin{bmatrix} C_1 \\ F_1 \end{bmatrix} = \begin{bmatrix} \mathcal{V}(s) \\ \phi(s) \end{bmatrix} (sI - A_1) \]

where \( P(s) \) is the system matrix (2.2).

**Proof.** By theorem (3.1) \( \mathcal{V}(s) \) is an \((A,B)\)-invariant subspace of \( K_T \) iff for some \( C_1, F_1, A_1 \) we have (3.2) and hence (3.6). But then

\[ \mathcal{C} \mathcal{V}(s) = (V(s)T^{-1}(s)\mathcal{V}(s))^{-1} = \]

\[ = ((V(s)C_1 + (G(s) - W(s))F_1)(sI - A_1)^{-1})^{-1} = \]

\[ = ((V(s)C_1 - W(s)F_1)(sI - A_1)^{-1})^{-1} \]

since \( G(s) \) and \((sI - A_1)^{-1}\) are both strictly proper. Now we may appeal to lemma (2.10) and conclude that

\[ \phi(s) := (-V(s)C_1 + W(s)F_1)(sI - A_1)^{-1} \tag{3.9} \]

is a polynomial iff \( \mathcal{C} \mathcal{V}(s) = 0 \). Combining (3.2) and (3.9) yields the desired result.

In the case \( V(s) = I \), the characterization of theorem (3.7) can be simplified considerably.
(3.10) COROLLARY. Assume that $V(s) = I$ (and $W(s) = 0$). Let $T(s) \in K^{\times m}[s]$. Then $\{T(s)\}$ is an $(A,B)$-invariant subspace contained in $\ker C$ iff there exist matrices $F_1, A_1$ such that

$$U(s)F_1 = T(s)(sI - A_1)$$

PROOF. In this case (3.8) reduces to:

$$T(s)C_1 + U(s)F_1 = T(s)(sI - A_1),$$

and

$$-C_1 = \phi(s)(sI - A_1).$$

The second equation can only hold if $C_1 = 0, \phi(s) = 0$. Hence we must have (3.11).

(3.12) COROLLARY. Under the conditions of corollary (3.10) we have the following: If $\{T(s)\}$ is an $(A,B)$-invariant subspace in $\ker C$, then $\{T(s)\} \subseteq K_U$.

PROOF. According to (3.11) we have

$$T(s) = U(s)F_1(sI - A_1)^{-1}.$$

The results follow immediately from definition (2.1).

The foregoing implies that the set of $(A,B)$-invariant subspaces in $\ker C$ is uniquely determined by the numerator polynomial matrix of the matrix fraction representation of the transfer function matrix:

(3.13) COROLLARY. Let $U(s) \in K^{q \times \ell}[s], T_i(s) \in K^{q \times \ell}[s]$ $(i = 1, 2)$ such that

$$G_i(s) := T_i^{-1}(s)U(s)$$

is strictly proper for $i = 1, 2$. Let $(C_i, A_i, B_i)$ be the state space models associated with the system matrices $P_i(s)$ (where $V_i(s) = I, W_i(s) = 0$). Then $M \subseteq K_U$ is an $(A_1, B_1)$-invariant subspace of $K_{T_1}$ contained in $\ker C_1$ iff $M$ is an $(A_2, B_2)$-invariant subspace of $K_{T_2}$ contained in $\ker C_2$.

Finally, we give a characterization of the maximal $(A,B)$-invariant subspace contained in $\ker C$:

(3.14) COROLLARY. Assume that $V(s) = I$. Then $K_U$ is the largest $(A,B)$-invariant subspace of $K_T$ contained in $\ker C$. 
PROOF. Because of (3.12) it suffices to show that $K_U$ is an $(A,B)$-invariant subspace. Let $K_U = \{\Phi(s)\}$ for some polynomial matrix $\Phi(s)$. By definition (2.1) there exists a strictly proper matrix $Q(s)$ such that $U(s)Q(s) = \Phi(s)$.

Let $(F_1,A_1,B_1)$ be a reachable realization of $Q(s)$, so that

$$U(s)F_1(sI - A_1)^{-1}B_1 = \Phi(s).$$

It follows from lemma (2.10) that

$$\Psi(s) := U(s)F_1(sI - A_1)^{-1}$$

is a polynomial matrix. Since $\Phi(s) = \Psi(s)B$ we have $K_U = \{\Phi(s)\} \subseteq \{\Psi(s)\}$. On the other hand, corollary (3.10) implies that $\{\Psi(s)\}$ is an $(A,B)$-invariant subspace contained in $\ker C$. Hence, by corollary (3.12) $\{\Psi(s)\} \subseteq K_U$, and consequently, $K_U = \{\Psi(s)\}$ is an $(A,B)$-invariant subspace contained in $\ker C$.

The result of corollary (3.14) can be generalized to the situation described in theorem (3.7). We define

$$P: K^{q+r}[s] \to K^q[s]: \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} \mapsto x(s).$$

(3.15) COROLLARY. If $(C,A,B)$ is the realization associated with the system matrix $P(s)$, then the largest $(A,B)$-invariant subspace of $K_T$ contained in $\ker C$ is $P(K_F)$.

The proof is similar to the proof of (3.14) and will be omitted.

(3.16) REMARK. The results may be specialized to the case $U(s) = 0$, that is, $B = 0$. In that case we have a realization of $G(s) = 0$ with the same state space $K_T$ and the same map $C$ as before. An $(A,B)$-invariant subspace of $K_T$ then is just an $A$-invariant subspace. Thus we obtain the following characterization of $A$-invariant subspaces.

PROPOSITION. Let $\Psi(s)$ be a $q \times m$ polynomial matrix. Then, $\{\Psi(s)\}$ is an $A$-invariant subspace of $K_T$ iff there exist $Q_1 \in K^{q\times m}$, $A_1 \in K^{m\times m}$ such that

$$T(s)Q_1 = \Psi(s)(sI - A_1).$$

Furthermore $\{\Psi(s)\}$ is an $A$-invariant subspace of $K_T$ contained in $\ker C$ iff there exist $Q_1 \in K^{q\times m}$, $A_1 \in K^{m\times m}$ such that

$$\begin{bmatrix} T(s) \\ -V(s) \end{bmatrix} Q_1 = \begin{bmatrix} \Psi(s) \\ \Phi(s) \end{bmatrix} (sI - A_1)$$

for some $\Phi(s) \in K^{l\times m}[s]$. □
4. EXACT MODEL MATCHING AND DISTURBANCE DECOUPLING

If we have an observable system \((\mathcal{C}, \mathcal{A}, \mathcal{B})\) with state space \(X\) then we may consider the problem of characterizing the \((\mathcal{A}, \mathcal{B})\)-invariant subspaces contained in \(\ker \mathcal{C}\). Using the isomorphism given in lemma (2.7) (see also remark (2.9)) we transform the problem to the case of a suitable \(T\)-realization. For this case we may appeal to corollary (3.10) by which a complete characterization is given. It is important that, as already noted in corollary (3.13), this characterization depends only on the numerator polynomial \(U(s)\). Consequently, we have the following result

\[(4.1) \text{THEOREM.} \text{ Let } X = (\mathcal{C}, \mathcal{A}, \mathcal{B}) \text{ be a realization with state space } X \text{ of a transfer matrix } G(s) = T^{-1}(s)U(s), \text{ and let } \Sigma = (\mathcal{C}, \mathcal{A}, \mathcal{B}) \text{ be the } T\text{-realization of } G(s). \text{ If } X \text{ and } \Sigma \text{ are isomorphic by the isomorphism } L: X \to X_A, \text{ then } M \subseteq X \text{ is an } (\mathcal{A}, \mathcal{B})\text{-invariant subspace contained in } \ker \mathcal{C} \text{ iff there exist constant matrices } F_1, A_1 \text{ satisfying}

\[U(s)F_1 = \mathcal{V}(s)(sI - A_1)\]

where \(\mathcal{V}(s)\) is a basis matrix of \(L(M)\).

Thus we see how characterizations for \((\mathcal{A}, \mathcal{B})\)-invariant subspaces of the particular state space model \(X\) can be generalized to arbitrary (observable) state space models.

In this section we use the theory developed thus far to show the equivalence of the exact model matching problem and the disturbance decoupling problem.

\[(4.1) \text{PROBLEM (Disturbance decoupling problem (DDP)).} \text{ Given the system}

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Eq(t), \\
y(t) &= Cx(t),
\end{align*}
\]

where \((\mathcal{C}, \mathcal{A})\) is observable, determine a constant matrix \(F\) such that if

\[u(t) = Fx(t) \quad (t \geq 0),\]

the output \(y(t)\) does not depend on \(q(t)\) \((t \geq 0)\).

The following result has been given in [14, Theorem 4.2] in a slightly different but equivalent formulation:
(4.3) **THEOREM.** Problem (4.1) has a solution iff there exists a subspace $M$ of the state space such that

$$AM \subseteq M + \{B\}$$

$$\{E\} \subseteq M \subseteq \ker C .$$

In this paper we will also consider a slightly modified problem (compare also [15]).

(4.4) **PROBLEM** (Modified disturbance decoupling problem (MDDP)). Given system (4.2), determine constant matrices $F$ and $D$ such that if

$$u(t) = Fx(t) + Dq(t) ,$$

the output does not depend on $q(t)$.

In the modified problem one assumes that not only the state but also the disturbance is directly available for measurement. Similarly to (4.3) we have the following result

(4.5) **THEOREM.** Problem (4.4) has a solution iff there exists a subspace $M$ such that

$$AM \subseteq M + \{B\}$$

$$\{E\} \subseteq M + \{B\}$$

$$M \subseteq \ker C .$$

The exact model matching problem is defined as follows

(4.6) **PROBLEM.** Given transfer function matrices $G_1(s)$ and $G_2(s)$ determine a (i) strictly proper or (ii) proper rational matrix $Q(s)$ such that

$$G_1(s)Q(s) = G_2(s) .$$

Problem (4.6)(i) will be called the exact model matching problem (EMMP) and (4.6)(ii) will be called the modified exact model matching problem (MEMMP). It is the purpose of this section to show that the existence of a solution of problem (4.1) is equivalent to the existence of a solution of problem (4.6)(i). Similarly: (4.4) has a solution iff (4.6)(ii) has a solution. We will concentrate on the modified problems. The original problems can be dealt with similarly.
First we have to indicate which MEMMP corresponds to a given MDDP and vice versa. Let us start with system (4.2). The data $G_1(s)$ and $G_2(s)$ of MEMMP are then defined by

$$G_1(s) := C(sI - A)^{-1} B$$
$$G_2(s) := C(sI - A)^{-1} E.$$  

Conversely, if we are given $G_1(s)$ and $G_2(s)$ in MEMMP, we construct an observable realization $(C,A,[B,E])$ of the transfer matrix $[G_1(s),G_2(s)]$. Then $C,A,B,E$ are the data for MDDP. Thus, we have a one to one correspondence between MEMMP's and MDDP's.

Following lemma (2.7), we assume that

$$\frac{T(s)}{S(s)}$$

are relatively prime and $U(s) = S(s)B$ and we consider the $T$-realization $(C,A,B)$ of $G_1(s) = T^{-1}(s)U(s)$. According to lemma (2.7) the map

$$x \mapsto S(s)x: \mathbb{K}^n \rightarrow \mathbb{K}$$

is an isomorphism. Consequently, we introduce the polynomial matrix $R(s) := S(s)E$ as representative of $E$ in $\mathbb{K}_T$. Then we have $G_2(s) = T^{-1}(s)R(s)$ and we can state the following result

(4.7) **THEOREM.** Let $\{\mathbb{V}(s)\}$ be an $(A,B)$-invariant subspace in $\ker C$, so that there exist constant matrices $F_1$ and $A_1$ satisfying

$$U(s)F_1 = \mathbb{V}(s)(sI - A_1)$$

In addition, assume that $\{R(s)\} \subseteq \{\mathbb{V}(s)\} + \{U(s)\}$, so that there exist matrices $B_1$ and $D_1$ such that

$$R(s) = \mathbb{V}(s)B_1 + U(s)D_1.$$  

Then $Q(s) := F_1(sI - A_1)^{-1}B_1 + D_1$ is a solution of MEMMP. Conversely, let $Q(s)$ be a solution of MEMMP and let $(F_1',A_1',B_1',D_1')$ be a reachable realization of $Q(s)$. Then there exists a polynomial matrix $\mathbb{V}(s)$ satisfying (4.8) and (4.9).

**PROOF.** If $\mathbb{V}(s)$ satisfies (4.8) and (4.9) then

$$U(s)Q(s) = \mathbb{V}(s)B_1 + U(s)D_1 = R(s)$$

which implies $G_1(s)Q(s) = G_2(s)$. Conversely, the latter equation implies $U(s)Q(s) = R(s)$. Hence

$$U(s)F_1(sI - A_1)^{-1}B_1 = R(s) - U(s)D_1.$$
Since \((A_1,B_1)\) is reachable it follows from lemma (2.10) that

\[(4.11) \quad \forall(s) := U(s)F_1(sI - A_1)^{-1}\]

is a polynomial. Now (4.10) and (4.11) imply (4.9) and (4.8).

(4.12) COROLLARY. \textit{MEMMP} has a solution iff the corresponding \textit{MDDP} has a solution.

Similarly one proves

(4.13) PROPOSITION. \textit{EMMP} has a solution iff the corresponding \textit{DDP} has a solution.

Thus, if we want to solve \((M)\text{EMMP}\) we may construct the data \(A,B,C,E\) of \((M)\text{DDP}\) and solve the latter problem. Then we do not only obtain a solution \(Q(s)\) of \((M)\text{EMMP}\) but also a realization of this solution. In this respect, it is important to note that the solution of \((M)\text{EMMP}\) only depends on the numerator polynomials \(U(s)\) and \(R(s)\). Consequently, by a suitable choice of \(T(s)\) (not necessarily equal to the original denominator polynomial) we may try to obtain a simple \((M)\text{DDP}\), compare [2]. We will more explicitly formulate this idea in section 6. Also in section 6, we will give existence conditions for a solution of \((M)\text{EMMP}\) and hence of \((M)\text{DDP}\) in terms of \(U(s)\) and \(R(s)\).

The following result states that if disturbance decoupling is at all possible by a (dynamic) control depending causally upon \(q(t)\), then it is possible by a feedback control of the form \(u = Fx + D_1q\).

(4.13) COROLLARY. \textit{Let there exist a proper rational matrix} \(H(s)\) \textit{such that, if the control} \(u = U(s)q\) \textit{is used in (4.2), the output does not depend on} \(q\). \textit{Then MDDP has a solution. If there exists a strictly proper matrix} \(H(s)\) \textit{with this property, then DDP has a solution.}

PROOF. If the control \(u = H(s)q\) is used in (4.2), then the transfer function matrix from \(q\) to \(y\) is \(G_1(s)H(s) + G(s)\). If \(y\) does not depend on \(q\), then this transfer matrix must be zero, hence

\[G_1(s)H(s) = -G_2(s)\]

that is, \(-H(s)\) is a solution of \textit{MEMMP}. Consequently, by corollary (4.12), \textit{MDDP} has a solution.
5. REACHABILITY SUBSPACES

If the matrix $\Psi(s)$ occurring in theorem (3.1) etc. has full column rank, it is possible to give an interpretation to the matrix $A_1, F_1, C_1$. For in that case there exists a $K$-linear map $F: K_T \to K^r$ satisfying

$$F\Psi(s) = F_1.$$ 

Then equation (3.2) implies

$$(A - BF)\Psi(x) = \Psi(s)A_1.$$ 

It follows that $\{\Psi(s)\}$ is $(A - BF)$-invariant and that $A_1$ is the matrix of the restriction of $A - BF$ to $\{\Psi(s)\}$ with respect to the basis matrix $\Psi(s)$. In addition $F_1$ is the matrix (with respect to the basis matrix $\Psi(s)$ of $\{\Psi(s)\}$ and the natural basis in $K^r$) of $F$. In addition if $V = I, W = 0$, we have

$$C\Psi(s) = C_1$$

so that $C_1$ is the matrix of the restriction of $C$ to $\{\Psi(s)\}$ with respect to the basis matrix $\Psi(s)$ of $\{\Psi(s)\}$ and the natural basis of $K^r$ (compare corollary (3.10)).

Now, let $B_1$ be any constant $m \times p$ matrix such that $\{\Psi(s)B_1\} \subseteq \{\Psi(s)\}$, say

$$\Psi(s)B_1 = U(s)L_1.$$ 

Then $B_1$ is the matrix of the (codomain) restriction of $BL_1$ to $\{\Psi(s)\}$. It follows that

$$(A - BF)^kBL_1v = \Psi(s)A_1^kB_1v$$

for every $v \in K^p$. Consequently

$$(5.1) \quad \langle A - BF \mid BL_1 > = \{\Psi(s)[B_1, \ldots, A_1^{m-1}B_1]\}.$$ 

This formula immediately implies the following result:

(5.2) THEOREM. Let $\Psi(s)$ be a (full column rank) basis matrix of an $(A, B)$-invariant subspace. Then

(i) $\{\Psi(s)\}$ is a reachability subspace iff there exists a constant matrix $B_1$ such that $\{\Psi(s)B_1\} \subseteq \{\Psi(s)\}$ and $(A_1, B_1)$ is reachable (here $A_1$ is given by (3.2)).

(ii) If $B_1$ is a constant matrix such that

$$(5.3) \quad \{\Psi(s)B_1\} = \{\Psi(s)\} \cap \{\Psi(s)\}$$

then $\{\Psi(s)[B_1, \ldots, A_1^{m-1}B_1]\}$ is the supremal reachability subspace contained in $\{\Psi(s)\}$. \qed
Let us now consider reachability subspaces contained in \( \ker C \). Let \( \Psi(s) \) be a basis matrix of such a space. According to (3.10), there exists matrices \( F_1 \) and \( A_1 \) such that

\[
(5.4) \quad \Psi(s) = U(s)P_1(sI - A_1)^{-1}.
\]

It follows from (5.2) that there exists \( B_1 \) such that \((A_1,B_1)\) is reachable and \( \{\Psi(s)B_1\} \subseteq \{U(s)\} \), say \( \Psi(s)B_1 = U(s)L_1 \). Hence

\[
(5.5) \quad U(s)Q(s) = U(s)L_1
\]

where \( Q(s) := F_1(sI - A_1)^{-1}B_1 \). Also, since \( \Psi(s) \) has full column rank, \((F_1,A_1)\) is observable, as follows from (5.4). Hence \((F_1,A_1,B_1)\) is a minimal realization of \( Q(s) \).

(5.5) COROLLARY. There exists a nontrivial reachability subspace contained in \( \ker C \) iff

\[
\{U(s)\} \cap K_U \neq \{0\}.
\]

PROOF. If \( \Psi(s) \) is a basis matrix of the \((A,B)\)-invariant subspace \( K_U \) and \( \Psi(s) = U(s)F_1(sI - A_1)^{-1} \), then the supremal reachability subspace contained in \( K_U \) (or, equivalently, in \( \ker C \)) is nontrivial iff \( B_1 \neq 0 \), where \( B_1 \) is a matrix satisfying (5.3).

According to (5.4), \( Q(s) - L_1 \) is a nontrivial right zero matrix of \( U(s) \). Consequently, if the supremal reachability subspace contained in \( C \) is non-zero then \( U(s) \) is not left invertible. The converse, however, is not true. For example, if \( U(s) = [U_1(s),0] \) where \( U_1(s) \) is left invertible, then it is easily seen that \( U(s) \) is not left invertible and \( \{U(s)\} \cap K_U = \{0\} \). In order to give a necessary and sufficient condition for the existence of a maximal reachability subspace contained in \( \ker C \), we consider the \( K[s] \)-module

\[
(5.7) \quad \Delta := \{v(s) \in K^r[s] \mid U(s)v(s) = 0\}.
\]

This module is generated by the columns of a matrix \( M(s) \) (see [5, Thm 3.1]).

(5.8) COROLLARY. There exists a nontrivial reachability subspace contained in \( \ker C \) iff the module \( \Delta \) defined in (5.7) is not generated by a constant matrix.
PROOF. Let $M(s)$ be a generator matrix of $\Delta$ of minimal degree, say $M(s) = M_0 s^k + \ldots + M_k$. Then $s^{-k} M(s) = Q(s) - L_1$ where $Q(s) = M_1 s^{-1} + \ldots + M_k s^{-k}$ and $L_1 = -M_0$. We have

$$U(s)Q(s) = U(s)L_1$$

and $U(s)L_1 \neq 0$, since otherwise $[M(s) - s^k M_0, M_0]$ would be a generator matrix of lower degree than $k$. It follows that $\{U(s) L_1\} \subseteq \{U(s)\} \cap K_u$, so that $\{U(s)\} \cap K_u \neq \{0\}$.

Conversely, suppose that $\Delta$ is generated by constant matrix, say $D$, and that $v \in \{U(s)\} \cap K_u$, say $v = U(s)c = U(s)r(s)$, where $c$ is a constant vector and $r(s)$ is a strictly proper rational vector. It follows that there exists a rational vector $q(s)$ such that $c - r(s) = Dq(s)$. Decomposing $q(s)$ into a polynomial and a strictly proper part $q(s) = q_1(s) + q_2(s)$, we conclude that $c = Dq_1(s)$, so that $v = U(s)c = 0$. Hence $\{U(s)\} \cap K_u = \{0\}$.  

Now we have a procedure for constructing reachability subspaces contained in $\ker C$. Choosing any matrix $L_1$ such that $\{U(s) L_1\} \subseteq K_u$, we have $U(s)Q(s) = U(s)L_1$ for some strictly proper $Q(s)$. If $(F_1, A_1, B_1)$ is a minimal realization of $Q(s)$, it follows that $\Psi(x) := U(s)F_1(sI - A_1)^{-1}$ is a basis matrix of a reachability subspace, provided the columns of $\Psi(x)$ are independent. In general, it seems difficult to formulate conditions upon $L_1$ and $Q(s)$ that guarantee that $\Psi(x)$ has full column rank. A sufficient condition for this is, that $Q(s)$ be a strictly proper rational matrix with minimal McMillan degree satisfying the equation $U(s)Q(s) = U(s)L_1$. Indeed, if in this case $\Psi(x)$ does not have full column rank, there exists $\Phi(s)$ with less columns than $\Psi$ such that $\{\Phi(s)\} = \{\Psi(s)\}$. Since $\{\Phi(s)\}$ is an $(A, B)$-invariant subspace, there exist $F_2, A_2$ such that $\Phi(s) = U(s)F_2(sI - A_2)^{-1}$. Also, there exists $D_1$ such that $\Psi(s) = D_1D_1'$. Hence,

$$U(s)Q(s) = \Psi(s)B_1 = \Phi(s)D_1B_1 = U(s)Q_2(s) = U(s)L_1$$

where $Q_2(s) := F_2(sI - A_2)^{-1}D_1B_1$ has lower McMillan degree than $Q(s)$.

(5.9) THEOREM. Let $L_1$ be a constant matrix such that $\{U(s) L_1\} = \{U(s)\} \cap K_u$. Let $Q(s)$ be a strictly proper rational matrix of minimal McMillan degree, satisfying the equation $U(s)Q(s) = U(s)L_1$. Let $(F_1, A_1, B_1)$ a minimal realization of $Q(s)$. Then $\Psi(s) := U(s)F_1(sI - A_1)^{-1}$ is a basis matrix of the supremal reachability space contained in $\ker C$. 

PROOF. The supremal reachability subspace contained in \( \ker C \) is the (unique) minimal \((A, B)\)-invariant subspace \( V \) satisfying \( \text{im } B \cap W \subseteq V \subseteq W \), where \( W \) is the supremal \((A, B)\)-invariant subspace contained in \( \ker C \). To see this, observe that an \((A, B)\)-invariant subspace \( V \) satisfying \( \text{im } B \cap W \subseteq V \subseteq W \) is \((A - BF)\)-invariant for every \( F \) such that \( W \) is \((A - BF)\)-invariant. Indeed, \((A - BF)V \subseteq (A - BF)W \subseteq W \) and \((A - BF)V \subseteq V + \text{im } B \) imply

\[
(A - BF)V \subseteq W \cap (V + \text{im } B) = V + W \cap \text{im } B \subseteq V .
\]

Since \( \{U(s)\} \cap \mathcal{K}_U = \{U(s)L_1\} \cap \{\psi(s)B_1\} \subseteq \{\psi(s)\} \subseteq \mathcal{K}_U \) and because of the minimal McMillan degree of \( \Omega(s) \) the result follows. \( \square \)

In the next section it will be shown how theorem (5.9) can be used for the explicit construction of the supremal reachability subspace.

6. CONSTRUCTIVE CHARACTERIZATIONS

Conditions for solvability and the characterization of solutions of various problems can be made explicit by the use of row and column proper matrices (see [13]). This will be the subject of this section.

If \( R \in \mathbb{K}^{pxq}[s] \) has rows \( r_1(s), \ldots, r_p(s) \) then \( \deg r_i(s) \) is called the \( i^{th} \) row degree of \( R(s) \). The coefficient vector of \( s^{\nu_i} \) in \( r_i(s) \), where \( \nu_i = \deg r_i(s) \), is called the \( i^{th} \) leading coefficient row vector and is denoted \([r_i]_{\nu_i} \). We denote by \([R]_r\) the matrix of leading coefficient row vectors, that is the constant matrix with rows \([r_1]_{\nu_1}, \ldots, [r_p]_{\nu_p}\) . Similarly, \([R]_c\) denotes the matrix of leading coefficient column vectors, that is \([R]_c = ([R']_r)'\). A matrix is called row (column) proper if \([R]_r \) (\([R]_c\) ) is nonsingular. A row proper matrix is easily seen to be right invertible. Conversely we have (see [13, Th. 2.5.7]).

(6.1) LEMMA. If \( L(s) \in \mathbb{K}^{pxq}[s] \) is right invertible there exists a unimodular matrix \( M(s) \in \mathbb{K}^{pxp}[s] \) such that \( M(s)L(s) \) is row proper with row degrees \( \nu_1, \ldots, \nu_p \) satisfying \( \nu_1 \leq \ldots \leq \nu_p \). If \( L(s) \in \mathbb{K}^{pxq}[s] \) is not right invertible, there exists a unimodular matrix \( M(s) \) such that

\[
M(s)L(s) = \begin{bmatrix}
L_1(s) \\
0
\end{bmatrix}
\]

where \( L_1(s) \) is row proper with row degrees \( \nu_1 \leq \ldots \leq \nu_k \). The number \( k \) of rows of \( L_1(s) \) equals the rank of \( L(s) \).
The row degrees \( v_i \) are independent of \( M(s) \) (which is not unique) and will be called the row indices of \( L(s) \).

The following result (see [12, Prop. 2.2]) states a simple criterion for the properness of a rational matrix \( T^{-1}(s)U(s) \) if the denominator polynomial matrix is row proper.

\begin{equation}
(6.2) \text{Lemma. Let } T(s) \text{ be row proper with row degrees } v_1, \ldots, v_q. \text{ If the row degrees of } U(s) \text{ are } \lambda_1, \ldots, \lambda_q \text{ then } T^{-1}(s)U(s) \text{ is proper iff } \lambda_i \leq v_i \ (i = 1, \ldots, q) \text{ and strictly proper iff } \lambda_i < v_i \ (i = 1, \ldots, q).
\end{equation}

Observe that, if \( T \) is not row proper, there exists a unimodular matrix \( M(s) \) such that \( T_1(s) := M(s)T(s) \) is row proper. If we define \( U_1(s) := M(s)U(s) \) we have \( T^{-1}(s)U(s) = T_1^{-1}(s)U_1(s) \) and we may apply lemma (6.2).

Let us now consider \((M)\text{EMMP}\) as defined in 4.6. Assume that we have a matrix fraction representation \( T^{-1}(s)[U(s), R(s)] \) of \([G_1(s), G_2(s)]\). Then the equation for \( Q(s) \) reads

\begin{equation}
(6.3) \quad U(s)Q(s) = R(s).
\end{equation}

In order that this equation has a (not necessarily proper) rational solution, it is necessary and sufficient that \( \text{rank } U(s) \leq \text{rank } [U(s), R(s)] \).

For the existence of a proper solution additional conditions have to be imposed. Writing down the \( i \)th row of (6.3)

\[ u_i(s)Q(s) = r_i(s) \]

we note that a necessary condition for the existence of a proper solution is \( \deg u_i(s) \geq \deg r_i(s) \). The following result shows that this is also sufficient provided that \( U(s) \) has the form

\[
\begin{bmatrix}
U_1(s) \\
0
\end{bmatrix}
\]

with \( U_1(s) \) row proper. According to lemma (6.1) this can always be obtained by premultiplying (6.3) with a suitable unimodular matrix \( M(s) \).

\begin{equation}
(6.4) \text{Theorem. Let } M(s) \text{ be a unimodular matrix such that}
\end{equation}

\[
M(s)U(s) = \begin{bmatrix}
U_1(s) \\
0
\end{bmatrix}, \quad M(s)R(s) = \begin{bmatrix}
R_1(s) \\
R_2(s)
\end{bmatrix}
\]
where $U_1(s)$ is row proper. Let the row degrees of $U_1(s)$ be $v_1, \ldots, v_k$ and let the row degrees of $R_1(s)$ be $\lambda_1, \ldots, \lambda_k$. Then (6.3) has a proper solution iff $R_2(s) = 0$ and $\lambda_i \leq v_i$ $(i = 1, \ldots, k)$. Equation (6.3) has a strictly proper solution iff $R_2(s) = 0$ and $\lambda_i < v_i$ $(i = 1, \ldots, k)$.

**PROOF.** The conditions are necessary according to the foregoing discussions. Now assume that the conditions hold. Then there exists $L \in \mathbb{K}^{r \times k}$ such that $U_1(s)L$ is a row proper $k \times k$ matrix with row degrees $v_1, \ldots, v_k$. Define

$$Q(s) := L(U_1(s)L)^{-1}R_1(s).$$

Then $Q(s)$ satisfies (6.3). It follows from (6.3) that $Q(s)$ is proper. The proof for the strictly proper solution is similar. \[\square\]

We can express the result of theorem (6.4) in a way not involving explicitly the matrix $M(s)$:

(6.5) **COROLLARY.** Equation (6.3) has a proper solution iff $U(s)$ and $[U(s), R(s)]$ have the same rank and the same row indices. \[\square\]

The set $K_U$ is the largest $(A, B)$-invariant subspace contained in $\text{ker} \ C$. By definition $x(s) \in K_U$ iff the equation

$$U(s)v(s) = x(s)$$

has a strictly proper solution $v(s)$. Therefore, using theorem (6.4) we can give a constructive characterization of $K_U$.

(6.6) **COROLLARY.** Let $M(s)$ be as in theorem (6.4). Then $x(s) \in K_U$ iff

$$y(s) := M(s)x(s)$$

satisfies the conditions

$$\deg y_i(s) < v_i \quad (i = 1, \ldots, k),$$

$$y_i(s) = 0 \quad (i = k+1, \ldots, q).$$

Here $y_i(s)$ denotes the $i$th component of $y(s)$. In particular, if we introduce the row vector $w_k(s) := [s^{-1}, \ldots, 1]$, then $M^{-1}(s)W(s)$ is a basis matrix of $K_U$, where

$$W(s) := \begin{bmatrix} W_1(s) \\ 0 \end{bmatrix},$$

with $W_1(s) := \text{diag}(w_{v_1-1}(s), \ldots, w_{v_k-1}(s))$. 


One way of solving (6.3) already mentioned in section 4, is the reformulation of (6.3) as a (M)DDP. In doing so, it is not necessary to use the original denominator matrix $T(s)$. We rather try to find a new denominator matrix $T_1(s)$ such that $T_1^{-1}(s)U(s)$ is strictly proper and $T_1(s)$ is as simple as possible. If we choose $T_1(s)$ row proper, then according to lemma (6.2), it suffices for the strict properness of $T_1^{-1}U$, that the row degrees of $T_1$ are larger than the row degrees of $U$. If we denote the latter by $\lambda_1, \ldots, \lambda_q$ the simplest choice of $T_1(s)$ is $T_1(s) = \text{diag}(s^{\lambda_1+1}, \ldots, s^{\lambda_q+1})$. We define $n := \sum_{i=1}^{q} (\lambda_i + 1)$ and we may choose $K^N$ as state space for an observable realization of $T_1^{-1}(s)U(s)$. Such a realization will be represented (with respect to the canonical bases of $K^r, K^N, K^Q$) by $(C, A, B)$ where

$$A := \text{diag}(A_1, \ldots, A_q) ,$$

$$A_i := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \in K^{(\lambda_i+1) \times (\lambda_i+1)} \text{ if } \lambda_i > 0 ,$$

$$0 \in K^{1 \times 1} \text{ if } \lambda_i = 0 .$$

Furthermore, if we denote the $i$th row of $U(s)$ by $u_i(s) = \sum_{j=0}^{\lambda_i} u_{ij} s^j$, then

$$B := \begin{bmatrix} B_1 \\ \vdots \\ B_q \end{bmatrix} , \quad B_i := \begin{bmatrix} u_{i1} \\ \vdots \\ u_{i\lambda_i} \\ u_{i0} \end{bmatrix} .$$

Finally, $C := \text{diag}(C_1, \ldots, C_q)$, where

$$C := \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in K^{1 \times (\lambda_i+1)} .$$

A realization of $T_1^{-1}(s)R(s)$ is given by $(C, A, E)$, with the same $C, A, B$, and

$$E := \begin{bmatrix} E_1 \\ \vdots \\ E_q \end{bmatrix} , \quad E_i := \begin{bmatrix} r_{i1} \\ \vdots \\ r_{i1} \end{bmatrix} ,$$

where $r_i(s) := \sum_{j=0}^{\lambda_i} r_{ij} s^j$. Notice that $\deg r(s) \leq \deg u_i(s)$ if equation (6.3) has a solution. For this construction it is not necessary that $U(s)$ is in column proper form. But if we transform $U(s)$ such that it has the form given in theorem (6.4), then the dimension of the state space will be minimal (compare [2], [3]).
We conclude this section with a construction of the supremal reachability subspace contained in \( \ker C \). To this end, we consider the space

\[
\Lambda := \{ v(s) \in K^r(s) \mid U(s)v(s) = 0 \}
\]

and we choose a minimal basis for \( \Lambda \) (see [5]), that is, a basis for \( \Delta \) (see (5.7)) which is column proper. We define \( L_1 := [M]_C \). Furthermore we choose any \( D(s) \in K^{\ell \times \ell} [s] \) which is column proper and has the same column degrees as \( M(s) \). Then we observe (by lemma (6.2)) that, if

\[
N(s) := L_1D(s) - M(s)
\]

then \( Q(s) := N(s)D^{-1}(s) \) is strictly proper. Now we have:

(6.7) THEOREM.

(i) \( \{U(s)L_1\} = K_U \cap \{U(s)\} \),

(ii) \( Q(s) \) is a strictly proper rational matrix of minimal McMillan degree satisfying

(6.8) \[ U(s)Q(s) = U(s)L_1 \]

Hence, if \( \left(P_1, A_1, B_1\right) \) is a minimal realization of \( Q(s) \), then

\[ \Psi(s) := U(s)P_1(sI - A_1)^{-1} \]

is a basis of the supremal reachability subspace contained in \( \ker C \).

PROOF.

(i) Since \( U(s)M(s) = 0 \), it is easily seen that (6.8) is satisfied. This implies that \( \{U(s)L_1\} \subseteq K_U \cap \{U(s)\} \). Suppose that there exists a matrix \( L_1 \) of full column rank such that \( \{U(s)L_1\} \subseteq \{U(s)L_1\} \) and \( U(s)L_1 = U(s)\tilde{Q}(s) \) for some strictly proper \( \tilde{Q}(s) \). Let \( \tilde{N}, \tilde{D} \) be right coprime polynomial matrices such that \( \tilde{Q}(s) = \tilde{N}(s)\tilde{D}^{-1}(s) \) and \( \tilde{D}(s) \) is column proper, with \( [\tilde{D}]_C = I \). Then

\[
U(s)(\tilde{N}(s) - L_1\tilde{D}(s)) = 0.
\]

Since \( \tilde{Q}(s) \) is strictly proper, the columns of \( \tilde{N}(s) - L_1\tilde{D}(s) \) are linearly independent over \( K(s) \). But then \( L_1 \) cannot have more columns than \( L_1 \). Consequently, \( \{U(s)L_1\} = \{U(s)L_1\} \).

(ii) Suppose that \( \tilde{Q}(s) = N(s)\tilde{D}^{-1}(s) \) has a lower McMillan degree than \( Q(s) \) and that \( N(s) \) and \( D(s) \) are relatively prime and that \( \tilde{D}(s) \) is column proper with \( [\tilde{D}]_C = I \). Then we have

\[
U(s)(\tilde{N}(s) - L_1\tilde{D}(s)) = 0.
\]
and hence $\bar{N}(s) - L_1 \bar{D}(s) = M(s)R(s)$. By the "predictable degree property" (see [4, section 3, Remark 3]) this implies that the sum of the column degrees of $\bar{D}(s)$, and hence $\deg \det \bar{D}(s)$ is not less than $\deg \det D(s)$ which contradicts our assumption.

REMARK. The choice of the denominator matrix $D(s)$ in the foregoing construction is free up to the column properness condition and the column degrees. Let these columns degrees be $\mu_1, \ldots, \mu_k$ and satisfy $\mu_1 \geq \ldots \geq \mu_k$. According to Rosenbrock's theorem we can, for any choice of polynomials $\psi_1(s), \ldots, \psi_k(s)$, satisfying the conditions

\begin{align*}
(i) & \quad \psi_{k+1} | \psi_k \\
(ii) & \quad \sum_{j=1}^k \deg \psi_j \geq \sum_{j=1}^k \mu_j \quad (k = 1, \ldots, k) \\
(iii) & \quad \sum_{j=1}^k \deg \psi_j = \sum_{j=1}^k \mu_j
\end{align*}

find a matrix $D(s)$ such that the polynomials $\psi_1(s), \ldots, \psi_k(s)$ are the invariant factors of $D(s)$. Since the invariant factors of $D(s)$ are equal to the invariant factors of the matrix $A_1$ (i.e. of the polynomial matrix $sI - A_1$) it follows that we have a version of Rosenbrock's generalized pole assignment theorem for the supremal reachability subspace.

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REFERENCES


