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A priori results in linear-quadratic optimal control theory
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Abstract

In the present paper we shall see that philosophizing on the specific nature of Linear-Quadratic optimal Control Problems (LQCPs) yields several a priori statements that are valid for the entire set of these problems. For instance, the real symmetric matrix that represents the optimal cost for a particular LQCP necessarily is a rank minimizing solution of the dissipation inequality (DI). Since, in case of a positive definite input weighting matrix, the set of these solutions of the DI is equivalent to the set of real symmetric solutions of the algebraic Riccati equation (ARE), our result thus covers both the regular and the singular case. In addition, we will provide a characterization of the afore-mentioned set of solutions of the DI.

Next, a serious attempt is made at reducing general (indefinite) LQCPs to nonnegative definite LQCPs. Moreover, a distributional framework for singular LQCPs is proposed.

Keywords

System theory, optimal control, stability, dissipation inequality, matrix algebra, numerical methods.

1. Preliminaries

In this paper we will consider the linear time-invariant finite-dimensional system $\Sigma$:

\begin{align}
\dot{x} &= Ax + Bu, \quad x(0) = x_0, \quad (1.1a) \\

\text{where} \quad x(t) &\in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m \quad \text{for all} \quad t \geq 0, \quad \text{together with the quadratic form in} \quad (x, u) \in \mathbb{R}^{n+m} \\
w(x, u) &= x'Qx + 2u'Sx + u'Ru, \quad (1.1b)
\end{align}

with $Q = Q'$, $R = R'$. All matrices involved are real and constant.
The allowed inputs are assumed to be elements of $C^m_{\text{sm}}$:

$$\{ u : \mathbb{R}^+ \rightarrow \mathbb{R}^m | \exists \varepsilon > 0 \exists \psi \in C^\infty((-\varepsilon, \infty) \rightarrow \mathbb{R}^m) \forall t \geq 0 : u(t) = \psi(t) \} \tag{1.2}$$

the space of controls that are smooth on $[0, \infty)$. Now we introduce the infinite horizon cost criterion

$$J(x_0, u) := \int_0^\infty w(x, u) \, dt \tag{1.3}$$

and here $\int_0^\infty w(x, u) \, dt$ is understood to be $\lim_{T \to \infty} \int_0^T w(x, u) \, dt$. The class of $x_0$-dependent elements of $C^m_{\text{sm}}$ for which this limit exists in $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, is denoted by $U(x_0)$.

With $x = x(x_0, u)$ we indicate the dependence of $x$ on $x_0$ and $u$. Then, let $T \subset \mathbb{R}^n$ be an arbitrary subspace. We define the distance from $x(x_0, u)$ to $T$ at infinity by

$$d_\infty(x(x_0, u), T) := \lim_{t \to \infty} d(x(x_0, u)(t), T), \tag{1.4}$$

if this limit exists. Here $d(x, T), x \in \mathbb{R}^n$, denotes the (Euclidean) distance from $x$ to $T$. Without loss of generality, we may assume that

$$[B' S R]' \text{ is of full column rank}. \tag{1.5}$$

The general infinite horizon Linear-Quadratic optimal Control Problem with stability modulo $T$ (LQCP)$_T$ now is defined as follows:

For all $x_0 \in \mathbb{R}^n$, determine

$$J_T(x_0) := \inf \{ J(x_0, u) \mid u \in U(x_0) \text{ such that } d_\infty(x(x_0, u), T) = 0 \} \tag{1.6}$$

and, if for all $x_0 J_T(x_0)$ is finite, then characterize, if one exists, all optimal controls $u^* \in U(x_0)$ (i.e., all inputs $u^* \in U(x_0)$ for which $J(x_0, u^*) = J_T(x_0)$).

Next, we introduce the dissipation matrix

$$F(K) := \begin{bmatrix} Q + A'K + KA & KB + S' \\ B'K + S & R \end{bmatrix}, \tag{1.7}$$

where $K$ denotes any $n \times n$ real symmetric matrix. If $F(K) \succeq 0$, then $K$ is said to satisfy the Dissipation Inequality (Willems, 1971), abbreviated DI. We will define

$$\Gamma := \{ K \in \mathbb{R}^{n \times n} | K = K', F(K) \succeq 0 \} \tag{1.8}$$

the set of solutions of the DI.
If \((s_{1,2} \in \mathcal{C})\)
\[
H(s_{1},s_{2}) := R + \left(\begin{bmatrix} l s_{1} - A' \end{bmatrix}^{-1} S' + S \left(\begin{bmatrix} l s_{2} - A \end{bmatrix}^{-1} B
\right.
\]
\[
\left.\left. + B' \left(\begin{bmatrix} l s_{1} - A' \end{bmatrix}^{-1} Q \left(\begin{bmatrix} l s_{2} - A \end{bmatrix}^{-1} B
\right.\right.\right)\right)
\]
then Willems (1971) showed that for almost all \(\omega \in \mathbb{R}\), \(H(-i\omega, i\omega)\geq 0\) and we may set
\[
\rho := \text{normal rank } (H(-s, s)).
\]
Then Schumacher (1983) established that

**Lemma 1.1**

If \(K \in \Gamma\), then rank \((F(K))\) \(\geq \rho\).

Hence we are invited to define
\[
\Gamma_{\text{min}} := \{K \in \Gamma : \text{rank } (F(K)) = \rho\}, \tag{1.11}
\]
the set of rank minimizing solutions of the DI.

For every \(K \in \Gamma\) it is possible to find real constant matrices \(C_K\) and \(D_K\) such that \([C_K D_K]\) is of full row rank and such that \(F(K) = [C_K D_K]'[C_K D_K]\). If, in addition, we define the linear system \(\Sigma_K\) by the system equation (1.1a) and the artificial output equation
\[
y_K = C_K x + D_K u \tag{1.12}
\]
\((u \in C_{\infty}^m)\), then it is readily seen (Willems, 1971) that for every \(x_0\), every \(T > 0\) and every smooth \(u\),
\[
\int_0^T w(x, u) dt + x'(T)Kx(T) = \int_0^T y_K'y_K dt + x_0'Kx_0 . \tag{1.13}
\]
with \(x(T) = x(x_0, u)(T)\), of course. For further use, we set \((K \in \Gamma)\)
\[
J_K(x_0, u) := \int_0^\infty y_K'y_K dt \tag{1.14}
\]
(and we admit that this might cause some slight confusion). Moreover, we note that \([B' D_K]'\) is of full column rank (if \(Bu = 0\) and \(D_Ku = 0\), then \(Ru = 0\) and \(0 = C_K'D_Ku = (KB + S')u = S'u\), whence \(u = 0\)). Finally, we mention that, if
\[
T_K(s) := D_K + C_K(sI - A)^{-1}B
\]  
(1.15)

\(s \in \mathcal{C}\), then (1.10) \(p = \text{normal rank } (T_K(s))\) (Willems, 1971). The relation (1.13) will be of paramount significance in the sequel, as it has been before in e.g. (Willems, 1971; Brockett, 1970).

Now we make the following

**Standing Assumption**

\((A, B)\) is stabilizable and \(\exists K^* \in \Gamma : K^0 \leq 0\).

Note that thus, in particular, \(R \geq 0\) and that \(K^0\) is not necessarily required to be in \(\Gamma_{\text{min}}\). Furthermore, we observe that

\[
0 \in \Gamma \iff \begin{bmatrix} Q & S' \\ S & R \end{bmatrix} \succeq 0 \iff \forall x \forall u : w(x, u) \succeq 0
\]  
(1.16)

and LQCPs with a nonnegative definite integrand will be called *nonnegative definite* LQCPs. The remaining ones will be called *indefinite*.

**Proposition 1.2**

For every subspace \(T\) and every \(x_0\), \(U(x_0) \neq \emptyset\). Moreover, there exist real symmetric matrices \(M^+\) and \(M^-\) such that, for all subspaces \(T\) and all \(x_0\),

\[
x_0'M^-x_0 \leq J_T(x_0) \leq x_0'M^+x_0.
\]

**Proof.** Let \(F \in \mathbb{R}^{m \times n}\) be such that \(A_F := A + BF\) is asymptotically stable. By applying the feedback law \(u = Fx\), we get that the solution of (1.1a) equals \(\exp(A_F t)x_0\) and thus \(x(t) \to 0 (t \to \infty)\). Hence, for all \(x_0\), \(J(x_0, u) = x_0'M^+x_0\) with

\[
M^+ = \int_0^\infty (\exp(A_F t)(Q + F'S + SF + F'RF)\exp(A_F t))dt
\]

and \(M^+\) is clearly real and symmetric. We establish that \(U(x_0) \neq \emptyset\) and that for all \(T\), \(J_T(x_0) \leq J_\text{T}^*(x_0) \leq x_0'M^+x_0\). On the other hand, it follows from (1.13) that for any \(T > 0\) and any \(u\), \(\int_0^T w(x, u)dt \geq x_0'K^0x_0\), since \(K^0 \leq 0\). Hence for all \(x_0\) and all \(T\), \(J_T(x_0) \geq J_{\text{IR}^+}(x_0) \geq x_0'M^-x_0\) with \(M^- = K^0\).
Corollary 1.3 (Molinari, 1975; 1977)

Consider \((LQC_P)\). There exists a unique \(K_T \in \{K \in \mathbb{R}^{n \times n} \mid K = K'\}\) such that, for all \(x_0\), \(J_T(x_0) = x_0'K_Tx_0\). Moreover, \(K_T \in \Gamma\).

In Theorem 2.1 we will confirm an old conjecture concerning \(K_T\) (Willems, 1971).

2. A general determination of \(\Gamma_{\text{min}}\)

Theorem 2.1

Consider \((LQC_P)\). There exists a unique \(K_T \in \Gamma_{\text{min}}\) such that, for all \(x_0\), \(J_T(x_0) = x_0'K_Tx_0\).

Proof. See Theorem 2.1 in (Geerts, 1989c).

If \(R > 0\) (the regular case), then we can define the quadratic matrix function

\[
\phi(K) := Q + A'K + KA - (KB + S')R^{-1}(B'K + S) \tag{2.1}
\]

\((K \text{ an } n \times n \text{ real symmetric matrix})\), and it is immediately seen (Willems, 1971) that then

\[
\Gamma = \{K \in \mathbb{R}^{n \times n} \mid K = K', \phi(K) \geq 0\}, \tag{2.2}
\]

\[
\Gamma_{\text{min}} = \{K \in \Gamma \mid \phi(K) = 0\}.
\]

In other words, in the regular case the elements of \(\Gamma_{\text{min}}\) are the real symmetric solutions of the algebraic Riccati equation (ARE) \(\phi(K) = 0\).

In the singular case (\(R\) not positive definite) \(\phi(K)\) is not defined. However, we will present a representation of \(\Gamma_{\text{min}}\) that captures both the regular and the singular case.

For this we will need the following concepts. Let \(K \in \Gamma\) and \(\Sigma_K\) be the system described by (1.1a) and (1.12). Then the weakly unobservable subspace associated with \(\Sigma_K\) is defined by

\[
V_K = V(\Sigma_K) := \{x_0 \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m : y_K(x_0, u) = 0\} \tag{2.3}
\]

and it is the largest subspace \(L\) for which there exists an \(F \in \mathbb{R}^{m \times n}\) such that \((A + BF)L \subseteq L\), \((C_K + DKF)L = 0\) (Hautus & Silveman, 1983). Dually, \(W_K = W(\Sigma_K)\) is the smallest subspace \(S\) for which there exists a \(G \in \mathbb{R}^{n \times n}\) such that \((A + GC_K)S \subseteq S\), \(\dim(B + GD_K) \subseteq S\). Here \(r_K = \text{rank}(F(K)) = \text{rank}([C_K D_K])\). We state without proof that \(W = 0\) if and only if \(\text{ker}(D_K) = 0\). Finally, we introduce \(R_K := V_K \cap W_K\). Set \(W := W_{K^*}\), \(R := R_{K^*}\). In
Section 2.3 of (Geerts, 1989c) it is proven by direct computation that

**Proposition 2.2**

For every $K \in \Gamma$, we have that $W_K = W$, $R_K = R$ and $(K - K^0)W = 0$.

Next, if $R^+$ denotes the Moore-Penrose inverse of $R \geq 0$, then for any real symmetric matrix $K$ of dimension $n$ we may define

$$
\phi_0(K) := Q + A'K + KA - (KB + S')R^+(B'K + S) .
$$

(2.4)

If $(K \in \Gamma) C_K^{-1} \text{im}(D_K) := \{ u \in \mathbb{R}^m \mid C_Ku \in \text{im}(D_K) \}$, then it is obvious that

$$
C_K^{-1} \text{im}(D_K) = \text{ker}(\phi_0(K))
$$

(2.5a)

and hence, if

$$
W_{K_1} := W_K \cap (C_K^{-1} \text{im}(D_K)), \ W_2 := W \cap (C_K^{-1} \text{im}(D_K^*)) .
$$

(2.5b)

then, by Proposition 2.2, for every $K \in \Gamma$,

$$
W_{K_1} = W_2 .
$$

(2.5c)

We arrive at one of our main results.

**Theorem 2.3**

Let $\tilde{W}_1$ be any left invertible matrix such that $\text{im}(\tilde{W}_1) \oplus W_2 = W$. Then

$$
\Gamma = \{ K \in \mathbb{R}^{n \times n} \mid K = K', (K - K^0)W = 0, \psi(K) \geq 0 \}
$$

and

$$
\Gamma_{\text{min}} = \{ K \in \mathbb{R}^{n \times n} \mid K = K', (K - K^0)W = 0, \psi(K) = 0 \}
$$

with, for every $n \times n$ real symmetric matrix $K$ that satisfies $(K - K^0)W = 0,$

$$
\psi(K) := \phi_0(K) - (\phi_0(K)) \tilde{W}_1 (\tilde{W}_1' (\phi_0(K)) \tilde{W}_1)^{-1} \tilde{W}_1' (\phi_0(K)) .
$$

and it holds that $W \subset \text{ker}(\psi(K))$.

Proof. Theorem 2.34 in (Geerts, 1989c).
For one thing, Theorem 2.3 expresses that \( \psi(K) \) is independent of the choice for \( \tilde{W}_1 \). If \( R > 0 \), then \( \mathbf{W} = 0 \) and we reobtain the results in (2.2). If \( \begin{bmatrix} Q & S' \\ S & R \end{bmatrix} \succeq 0 \), i.e. if \( 0 \in \Gamma (1.16) \), then Theorem 2.3 transforms into Theorem 3.3 of (Geerts, 1989b). Theorem 2.3 can also be given in a form which is independent of \( K^0 \); in Section 2.3 of (Geerts, 1989c) the author describes in full detail a sequence of matrix computations, to be applied to the matrices \( A, B, Q, S \) and \( R \). In fact, this technique is nothing else than the application of the generalized dual structure algorithm (Geerts, 1989) to a system \( \Sigma_K (K \in \Gamma) \), without actually knowing the matrices \( C_K \) and \( D_K \). This technique leads to matrices \( \tilde{B}, \tilde{S}' \) and \( \tilde{B}, \tilde{S}' \) and \( \tilde{R} \), where \( \tilde{R} \) is invertible, \( \text{rank}(\tilde{R}) = p (1.10) \). Then, if for any real symmetric \( K \) of dimension \( n \),

\[
\tilde{\phi}(K) := Q + A'K + KA - (KB + \tilde{S}')\tilde{R}^{-1}(\tilde{B}'K + \tilde{S})
\]

and

\[
\tilde{L}(K) := K\tilde{B} + \tilde{S}',
\]

it follows that \( K \in \Gamma \) if and only if \( \tilde{L}(K) = 0 \) and \( \tilde{\phi}(K) \succeq 0 \). Moreover, if \( \tilde{L}(K) = 0 \) then \( (\tilde{\phi}(K)) \tilde{B} = 0 \). In addition, \( K \in \Gamma_{\text{min}} \) if \( \tilde{L}(K) = 0, \tilde{\phi}(K) = 0 \) (see Proposition 2.31 (h) - (i) in (Geerts, 1989c)). Of course, if \( R > 0 \), then \( \tilde{B}, \tilde{S}' \) are not appearing, \( \tilde{B} = B, \tilde{S}' = S', \tilde{R} = R \). Hence, if for some real symmetric \( K^0 \succeq 0 \), \( \tilde{L}(K^0) = 0 \) and \( \tilde{\phi}(K^0) \succeq 0 \), then, apparently, there exists a negative semi-definite element of \( \Gamma \).

So much for the computational aspects of this paper. Now it is time for some analysis.

3. Linear-quadratic control problems in a broad perspective

Let \( K \) be any real symmetric matrix of dimension \( n \). Then, due to Theorem 2.1, there exists a unique \( \hat{K} \in \Gamma_{\text{min}} \) such that, for all \( x_0 \), \( J_{\text{ker}(K)}(x_0) = x_0' \hat{K}x_0 \).

This defines a function

\[
\eta: \{ K \in \mathbb{R}^{n \times n} \mid K = K' \} \to \Gamma_{\text{min}}
\]

with \( \eta(K) := \hat{K} \).

Lemma 3.1

Let \( K \in \Gamma \). Then \( \eta(K) \succeq K \).
Proof. Take any \( x_0 \in \mathbb{R}^n \) and let \( u = U(x_0) \) be such that \( d_\omega(x(x_0, u), \ker(K)) = 0 \) (such a control \( u \) exists!). Then (1.13) - (1.14) \( J(x_0, u) = J_K(x_0, u) + x_0'Kx_0 \) and thus \( \eta(K) \geq K \).

If \( K \) is real and symmetric, but \( K \not\in \Gamma \), then we cannot say that \( \eta(K) \geq K \). Recall (Theorem 2.1) that every subspace \( T \) generates an element \( K_T \) of \( \Gamma_{\text{min}} \). Note that \( \eta(0) = K_{\mathbb{R}^n}, \eta(I_n) = K_0 \). More generally, let \( T \) be a given subspace, and let the matrix \( T \) (of full row rank) be such that \( \ker(T) = T \). Then \( \ker(T) = \ker(K_T) = T \) with \( K_T := T'T \), and hence \( \eta(K_T) = K_T \). From this observation we derive directly that

Lemma 3.2

\[ \forall K \in \{ K \in \mathbb{R}^{n \times n} \mid K = K' \} : \eta(K) = \eta(K) \iff \forall T \subseteq \mathbb{R}^n : \eta(T) = K_T. \]

We introduce

\[ \Gamma_{\text{min}}^{\text{eq}} := \{ K \in \Gamma_{\text{min}} \mid \eta(K) = K \} \quad (3.2) \]

and note from the above that

\[ \Gamma_{\text{min}}^{\text{eq}} = \{ K \in \mathbb{R}^{n \times n} \mid K = K', \eta(K) = K \}. \quad (3.3) \]

If, from now on,

\[ K^- := K_{\mathbb{R}^n}, K^+ := K_0 \quad (3.4) \]

then we find that \( \Gamma_{\text{min}}^{\text{eq}} \neq \emptyset \), since \( K^+ \geq \eta(K^+) \) \((0 \subset \ker(K^+))\) and \( \eta(K^+) \geq K^+ \) (Lemma 3.1). It follows easily from Lemma 3.1 that \( K^+ \) is the largest element of \( \Gamma \) and thus \( K^+ \) is the largest element of \( \Gamma_{\text{min}}^{\text{eq}} \).

Now suppose that we are able to prove that for every \( T \subseteq \mathbb{R}^n \), \( K_T \in \Gamma_{\text{min}}^{\text{eq}} \) (i.e., that \( \eta^2 = \eta \), by Lemma 3.2). Then, clearly,

\[ K^- \text{ is the smallest element of } \Gamma_{\text{min}}^{\text{eq}}. \]

If this turns out to be true, then it is the set \( \Gamma_{\text{min}}^{\text{eq}} \) rather than the set \( \Gamma_{\text{min}} \) which appears to be the pivot in linear-quadratic optimal control theory:

Every \( K_T \in \Gamma_{\text{min}}^{\text{eq}} \) and \( K^+ \) and \( K^- \) then are the largest and smallest element of this set, respectively.

But first, for something completely different. Recall (1.12) - (1.14) and read \( K_T \) instead of \( K \) there.
Theorem 3.3

Let \( u \in U(x_0) \) be such that \( d_\infty(x(x_0, u), T) = 0 \). Then

(a) \( J(x_0, u) \geq J_{K_T}(x_0, u) + x_0'K_TX_0 \).

Now assume that \( J(x_0, u) \) is finite. Then the next statements are valid.

(b) The limit \( (x'(\cdot)K_Tx(\cdot))_\infty := \lim_{T \to \infty} (x'(T)K_Tx(T)) \) exists and it is smaller than or equal to zero.

(c) \( J(x_0, u) = x_0'K_TX_0 \iff (x'(\cdot)K_Tx(\cdot))_\infty = 0 \) and \( y_{K_T} = 0 \).

(d) \( \inf \{ J_{K_T}(x_0, u) \mid u \in \mathbb{C}_n^m \text{ such that } d_\infty(x(x_0, u), T) = 0 \} = 0 \).

(e) If \( \bar{K} \in \{ K \in \Gamma \mid KT = 0 \} \), then \( \bar{K} \leq K_T \).

Proof. Let \( u \in U(x_0) \) be such that \( d_\infty(x(x_0, u), T) = 0 \). If \( J(x_0, u) = +\infty \), then (a) is trivial. Since always \( J(x_0, u) \geq x_0'K^0x_0 \), we now assume that \( J(x_0, u) \) is finite. Let \( T > 0 \), then (Corollary 1.3)

\[
x'(T)K_Tx(T) \leq \int_0^T w(x, u)dt \quad x(T) = x(x_0, u)(T),
\]

and hence, by (1.13),

\[
J(x_0, u) \geq \int_0^T y_{K_T}y_{K_T}dt + x_0'K_TX_0.
\]

This yields (a). Next, from (a), \( J_{K_T}(x_0, u) < \infty \), and thus (1.13) \( (x'(\cdot)K_Tx(\cdot))_\infty \) exists. From the above it must be \( \leq 0 \) and we have (b) and

\[
J(x_0, u) + (x'(\cdot)K_Tx(\cdot))_\infty = J_{K_T}(x_0, u) + x_0'K_TX_0.
\]

Since \( J_{K_T}(x_0, u) \geq 0 \), we now establish (c), and (d) is immediate from (a). Finally, if \( \bar{K}T = 0 \) and \( u \in U(x_0) \) is such that \( d_\infty(x, T) = 0 \), then \( x'(T)\bar{K}x(T) \to 0 (t \to \infty) \), and hence \( J(x_0, u) = J_{\bar{K}}(x_0, u) + x_0'\bar{K}_x0 \) (1.13). Thus, \( K_T \geq \bar{K} \) and if, moreover, \( T \subset \ker(K_T) \) then \( K_T \in \{ K \in \Gamma \mid KT = 0 \} \).

Consider Theorem 3.3 (e). It is clear that the first claim is a generalization of Lemma 3.1. Since \( 0 \in K \) for every \( K \in \Gamma \), we reobtain the well-known fact that \( K^+ \geq K \) for all \( K \in \Gamma \) from the second claim.

If \( R > 0 \), then there exists an invertible matrix \( D \) such that

\[
F(K_T) = [C_{K_T} \quad D]'[C_{K_T} \quad D]
\]

with \( C_{K_T} = (D^{-1})'(B'K_T + S) \), because (2.1) - (2.2) \( \phi(K_T) = 0 \). It follows that
and hence, by Theorem 3.3 (c), that

Corollary 3.4

If \( R > 0 \) and for a given \( x_0 \) there exists an optimal input for \( (LQC_P)_T \), then this input is unique and it can be given by the state feedback law

\[
  u = -R^{-1}(B'K_T + S)x
\]

The corresponding state trajectory \( x(t) = \exp(A_K t)x_0 \) (\( t \geq 0 \), with

\[
  A_K := A - BR^{-1}(B'K_T + S)
\]

is such that \( x'(t)K_TX(t) \to 0 (t \to \infty) \).

Hence, every optimal control for a regular LQCP can be implemented as a state feedback. This is in accordance with our expectations (e.g. Willems, 1971; Brockett, 1970).

If for some \( T, K_T \geq 0 \), then Theorem 3.3 yields us

Corollary 3.5

Let \( K_T \geq 0 \). Then, for all \( x_0 \), \( J_{(ker(K_T)\cap T)}(x_0) = J_{ker(K_T)}(x_0) = J_T(x_0) \). In particular, \( K_T \in \Gamma_{\text{fin}} \).

Proof. Let \( x_0 \) be given and \( u \in U(x_0) \) such that \( d(x, T) = 0 \) and \( J(x_0, u) \) is finite (and \( \geq 0 \)).

Then (Theorem 3.3 (b)) \( d_{\infty}(x, \ker(K_T)) = 0 \) and hence \( J_{(ker(K_T)\cap T)}(x_0) = J_T(x_0) \). On the other hand, \( J_{(ker(K_T)\cap T)}(x_0) \geq J_{ker(K_T)}(x_0) \geq J_T(x_0) \) by Lemma 3.1.

Thus, if \( 0 \in \Gamma (1.16) \), then for all \( T, K_T \in \Gamma_{\text{fin}} \). Now we are going to consider the general case. Analogously to the proof of Theorem 3.3, we can establish that if \( u \in U(x_0) \) is such that \( J(x_0, u) \) is finite, then \( J_{K^0}(x_0, u) < \infty \) and

\[
  (x'(\cdot)K^0x(\cdot))_{\infty} := \lim_{T \to \infty} x'(T)K^0x(T)
\]

exists and it is \( \leq 0 (K^0 \leq 0!) \). In addition,
and thus we are motivated to investigate the nonnegative definite LQCP associated with $\Sigma_K$: For all $x_0$, determine $\hat{J}_K(x_0) :=$

$$\inf_{T \to \infty} \left\{ \lim_{T \to \infty} \int_0^T y_K^T y_K \, dt - x'(T)K_0^T x(T) \mid u \in C_\text{adm}^m \right\} .$$

(3.7)

Due to $(A, B)$-stabilizability, the optimal cost for this problem is finite for every $x_0$ and it can be proven (compare Lemmas 1, 3 in (Molinari, 1977)) that there exists a real symmetric matrix $\hat{L}$ such that (for all $x_0$) $\hat{J}_K(x_0) = x_0'\hat{L}x_0$.

Moreover, if

$$F_K(L) := \begin{bmatrix} C_K^T C_K + A'L + LA & LB + C_K^T D_K \\ B'L + D_K^T C_K & D_K^T D_K \end{bmatrix} ,$$

(3.8)

with $L$ any $n \times n$ real symmetric matrix,

$$\Gamma_K := \{ L \in \mathbb{R}^{n \times n} \mid L = L', F_K(L) \succeq 0 \} ,$$

(3.9a)

and

$$\Gamma_{K_0} := \{ L \in \Gamma_K \mid \text{rank}(F_K(L)) = \text{normal rank } (T_K(s)) \} ,$$

(3.9b)

then it follows from (Schumacher, 1983) (or Theorem 2.1) that $\hat{L} \in \Gamma_{K_0}$.

But then, of course,

$$K^- = \hat{L} + K_0$$

(3.10)

and $\hat{L} + K_0 \in \Gamma_{\text{min}} (1.11), (1.15)!$ In fact, we have much more than that,

**Proposition 3.6**

$$K \in \Gamma \iff L = K - K_0 \in \Gamma_K ,$$

$$K \in \Gamma_{\text{min}} \iff L = K - K_0 \in \Gamma_{K_0} .$$

Now we make the following
Assumption 3.7

For every subspace $T$ and every $x_0$,
\[
\inf \{ \lim_{T \to \infty} \left( \int_0^T y^* y' dt - x'(T) K^0 x(T) \right) \mid u \in C^m_{m} \text{ such that } d_\infty(x, T) = 0 \} = \\
\inf \{ J_K(x_0, u) \mid u \in C^m_{m} \text{ such that } d_\infty(x(x_0, u), (\ker(K^0) \cap T)) = 0 \}.
\]

The author believes that Assumption 3.7 is generally true, but he has not (yet) been able to prove this. Actually, he conjectures that even the next assumption is satisfied.

Assumption 3.8

Let the system $\Sigma$ be described by $\dot{x} = Ax + Bu$, $x(0) = x_0$, and $y = Cx + Du$. The inputs are assumed to be smooth on $\mathbb{R}^+$, $J(x_0, u) = \int_0^\infty y' y dt$ and $M_0 \geq 0$ is a given real symmetric matrix. Then, for all subspaces $T$ and for all $x_0$,
\[
\inf \{ \lim_{T \to \infty} \left( \int_0^T y' y dt + x'(T) M_0 x(T) \right) \mid u \in C^m_{m} \text{ such that } d_\infty(x(x_0, u), T) = 0 \} = \\
\inf \{ J(x_0, u) \mid u \in C^m_{m} \text{ such that } d_\infty(x(x_0, u), (\ker(M_0) \cap T)) = 0 \}.
\]

Anyway, let Assumption 3.7 be satisfied. Then, from (3.6) - (3.7), for every subspace $T$ and every $x_0$,
\[
J_T(x_0) = \inf \{ J_K(x_0, u) \mid u \in C^m_{m} \text{ such that } d_\infty(x(x_0, u), (\ker(K^0) \cap T)) = 0 \}
+ x_0' K^0 x_0
\]
(3.11)

(and thus, by definition (3.1), $\eta(K^0) = K^-$).

Suppose that for all $x_0$,
\[
\inf \{ J_K(x_0, u) \mid u \in C^m_{m} \text{ such that } d_\infty(x(x_0, u), (\ker(K^0) \cap T)) = 0 \} = x_0' L_T^0 x_0
\]
(3.12)

with $L_T^0 \geq 0$ and $L_T^0 \in \Gamma_{K^0_{2m}}$ (3.9b). Then, apparently,
\[
K_T = L_T^0 + K^0,
\]
(3.13)
i.e., we have the optimal cost for the general (LQCP)$_T$ if the optimal cost for the nonnegative definite LQCP with stability modulo $(\ker(K^0) \cap T)$ is known.

Next, we observe that $\ker(K^0) \cap \ker(K_T) = \ker(K^0) \cap \ker(L_T^0)$. Now if $u$ is such that $J_K(x_0, u) < \infty$ and $d_\infty(x(x_0, u), (\ker(K^0) \cap T)) = 0$, then (Theorem 3.3 (b)) also
If Assumption 3.7 (or 3.8) is valid, then the above yields us a method for reducing indefinite LQCPs to nonnegative definite LQCPs. The idea runs as follows. Let the subspace $T$ be given and assume for the moment that we can find the optimal cost for the nonnegative definite LQCP with stability modulo $(\ker(K^0) \cap T)$ associated with $\Sigma^\ast (1.1a), (1.12)$. Let this optimal cost be denoted by $L^0 \in \Gamma^\ast (3.9b), L^0 \geq 0$. Then (3.13) $K^0 = L^0 + K^0$.

Next, let $x_0 \in \mathbb{R}^n$ be given. If $u \in C_{mn}^m$ is such that $d_{\omega}(x_0, u, (\ker(K^0) \cap T)) = 0$, then (3.6) $J(x_0, u) = J_K^s(x_0, u) + x_0'K^0x_0$. However, if $R$ is not positive definite, then optimal controls within $C_{mn}^m$ need not exist (see Example 2.11 in (Hautus & Silverman, 1983)). A reformulation in the style of (Hautus & Silverman, 1983) is needed incorporating distributions as allowed inputs. An appropriate distributional extension of $C_{mn}^m$ is the input class $C_{imp}^m$, the space of impulsive-smooth distributions on $R$ with support on $[0, \infty)$. Here an impulsive distribution is a linear combination of the Dirac $\delta$ distribution and its derivatives. If $U_{\Sigma^\ast}$ ($K \in \Gamma$, see (1.1a), (1.12)) denotes the space of controls $u \in C_{imp}^m$ for which $y_K$ is smooth (i.e., has no impulsive component), then it turns out (Proposition 2.31 (e) in (Geerts 1989c)) that for every $K \in \Gamma$, $U_{\Sigma^\ast} = U_{\Sigma^\ast} =: U$ (compare with Proposition 2.2). Now if we define

\[ J_T(x_0) := \inf \{ J_K^s(x_0, u) \mid u \in U \text{ such that } d_{\omega}(x, (\ker(K^0) \cap T)) = 0 \} + x_0'K^0x_0 \]
for every $x_0$, then this definition coincides with (3.11) if $R > 0$ and it is a reasonable extension of (3.11) if $R$ is merely $\geq 0$.

Note that if we would have chosen any other negative semi-definite element $\tilde{K}^0$ of $\Gamma$, then the space of allowed distributional inputs remains the same.

Next, it is well known (see e.g. (Geerts, 1989c)), that the existence of optimal controls for nonnegative definite LQCPs associated with $\Sigma_{K^+}$, say, is related to the question whether the intersection of the imaginary axis $\mathcal{C}^0$ and $\sigma^*(\Sigma_K^+)$ is empty or not. Here the set $\sigma^*(\Sigma_K^+)$ denotes the set of invariant zeros associated with $\Sigma_K^+$ (Wonham, 1979). In Proposition 2.37 of (Geerts, 1989) it is shown that if $K \in \Gamma$ then $\sigma^*(\Sigma_K) \cap \mathcal{C}^0 = \emptyset$ if and only if $\sigma^*(\Sigma_K^+) \cap \mathcal{C}^0 = \emptyset$. Hence if for all $x_0$, optimal controls exists for the LQCP with stability $(T = 0)$ associated with $\Sigma_{K^+}$, then for all $x_0$ there exist optimal controls for the (LQCP)$_0$ associated with $\Sigma^+_K$ as well, and vice versa. Moreover (Proposition 2.2), $R_{K^+} = 0 \iff R_{\tilde{K}^+} = 0$ and hence (Hautus & Silverman, 1983) optimal controls are unique for the former problem if and only if they are unique for the latter problem (if $R_K = 0$, then $\Sigma^+_K$ is called left invertible).

The reader will agree with the author, that the above-given strategy looks promising if (at least) we can solve nonnegative definite LQCPs with arbitrary stability requirements. These problems have been investigated in depth in (Geerts, 1989c). Related material can be found in (Geerts, 1989).

Briefly, our approach thus consists of the following steps. First, we must try to verify whether Assumption 3.7 (or 3.8) is valid or not. Then, we must find a negative semi-definite solution of the DI. Recall that at the end of Section 2 we mentioned that $K \in \Gamma \iff \{\tilde{L}(K) = 0$ and $\tilde{\phi}(K) \geq 0\}$, with $\tilde{L}(K)$ and $\tilde{\phi}(K)$ a certain linear and a certain quadratic matrix function, respectively. Finally, with (Geerts, 1989c), the LQCP with stability modulo $T$ is solvable.

Of course, many issues are not yet fully understood. To name but a few:

Suppose that, if $J_{K^+}(x_0, u) < \infty$, then automatically $x'(t)K^0x(t) \to 0$. Hence, apparently, $L_{K^+}^0$ is the smallest positive semi-definite element of $\Gamma_{K^+}^\infty$ (3.9b), by (Geerts, 1989). Thus $K^-$ is the smallest element of $\Gamma_{\min} \cap \{K \in \Gamma \mid K \geq K^0\}$, i.e., $K^-$ is the smallest element $K$ of $\Gamma_{\min}$ that satisfies $K \geq K^0$ (if $K^0 = 0$, then we reobtain Corollary 6.4 of (Geerts, 1989)).

If $J_{K^+}(x_0, u) < \infty$, but $x'(t)K^0x(t)$ does not automatically converge to zero, then one might ask oneself whether the choice of $K^0$ matters or not. Assume that $K^0_1 \leq K^0_2 \leq 0$ and $K^0_{1,2} \in \Gamma$, is it then sensible to choose $K^0_2$ instead of $K^0_1$ or is the choice irrelevant?

Yes, still a lot of work has to be done. Nevertheless the author has faith in the approach described above, not in the least because the easiest LQCP, the one with stability $(T = 0)$, has been solved along the lines of the above in Section 2.3 of (Geerts, 1989c).
Let us summarize the most relevant observations made in this paper.

Conclusions

The real symmetric matrix that represents the optimal cost for any LQCP is necessarily a rank minimizing solution of the dissipation inequality.

The set of these solutions can be characterized in an elegant way.

If Assumption 3.8 holds, then for every subspace $T$, $K_T \in \Gamma_{\text{eq}}^{\text{min}}$.

If $K_T \geq 0$, then $K_T \in \Gamma_{\text{eq}}^{\text{min}}$.

Optimal controls for regular problems can always be implemented as state feedbacks.

If $T \subset \ker(K_T)$, then $K_T$ is the largest element of the set $\{K \in \Gamma \mid K'T = 0\}$.

Indefinite LQCPs can be reduced to nonnegative LQCPs.

References


