Constraint classification for mixed integer programming formulations
Nemhauser, G.L.; Savelsbergh, M.W.P.; Sigismondi, G.C.

Published: 01/01/1991

Document Version
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.
Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.
• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal?

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Download date: 04. Jan. 2019
Memorandum COSOR 91-30
Constraint Classification for Mixed
Integer Programming Formulations
G.L. Nemhauser
M.W.P. Savelsbergh
G.C. Sigismondi

Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
5600 MB Eindhoven
The Netherlands

Eindhoven, November 1991
The Netherlands
1. Introduction
The success of branch-and-cut algorithms for combinatorial optimization problems [Hoffman and Padberg 1985, Padberg and Rinaldi 1989] and large scale 0-1 linear programming problems [Crowder, Johnson, and Padberg, 1983] has lead to a renewed interest in mixed integer programming.

The key idea of the branch-and-cut approach is reformulation. Problems are reformulated so as to make the difference in the objective function values between the solutions to the linear programming relaxation and the integer program as small as possible.

There are various ways to tighten the linear programming relaxation of an integer program. Preprocessing techniques [Hoffman and Padberg, 1991] try, among others things, to reduce the size of coefficients in the constraint matrix and to reduce the size of bounds on the variables. Constraint generation techniques [Crowder, Johnson and Padberg, 1983, Van Roy and Wolsey, 1986] try to generate strong valid inequalities.

Reformulation techniques should make the best possible use of the problem structure. It is beneficial to distinguish two modes of operation. General reformulation techniques, which are embedded in mixed integer programming systems such as ABC_OPT [Hoffman and Padberg 1989], MINTO [Savelsbergh, Sigismondi and Nemhauser 1991], MPSARX [Van Roy and Wolsey 1986], and OSL [IBM Corporation, 1990] try to identify problem structure based on an analysis of the constraint matrix. Problem specific reformulation techniques are based on an a priori investigation of the polyhedron associated with the set of feasible solutions.

The first step in the analysis of the constraint matrix is the classification of each constraint. The set of constraints is partitioned into a number of general types. The partition should be based on the specific structures that a system uses for its preprocessing, constraint generation, and branching strategies. As a result, there is no consensus on terminology and classification scheme. Generalized upper bound constraints, for instance, are defined as \( \sum_{j \in S} x_j = 1 \) in Nemhauser and Wolsey [1988] and as \( \sum_{j \in S} x_j \leq 1 \) in Wolsey [1990]
In this note, we define a classification scheme that is used in our system MINTO [Savelsbergh, Sigismondi and Nemhauser 1991]. Its purpose is to identify important classes to be used in preprocessing, constraint generation, branching, etc. We propose it as a general scheme to be evaluated, modified and then, hopefully, adopted, by the mixed integer programming community.

2. Constraint classification

A general mixed integer programming problem is of the form

\[
\begin{align*}
\text{max} & \sum_{j \in B} c_j x_j + \sum_{j \in I} c_j x_j + \sum_{j \in C} c_j x_j \\
\text{subject to} & \sum_{j \in B} a_{i,j} x_j + \sum_{j \in I} a_{i,j} x_j + \sum_{j \in C} a_{i,j} x_j - b_i = 1, \ldots, m \\
& 0 \leq x_j \leq 1 \\
& l_j \leq x_j \leq u_j \\
& x_j \in \mathbb{Z} \\
& x_j \in \mathbb{R}
\end{align*}
\]

where \( B \) is the set of binary variables, \( I \) is the set of integer variables, \( C \) is the set of continuous variables, the sense of a constraint can be \( \leq, \geq, \) or \( = \), and the lower and upper bounds may be plus or minus infinity. See Nemhauser and Wolsey [1988] for a general treatment of the subject.

To classify constraints, we first distinguish binary variables from integer and continuous ones. Note that this is different from a variable classification that surely would separate integer and continuous variables if for no other reason than the need to do so in branching. However, we have not yet found a significant use of constraints that distinguish between integer and continuous variables, e.g., we do not use Gomory mixed integer cuts. We use the symbol \( y \) to indicate integer and continuous variables and \( x \) to indicate binary variables. Each constraint class will be an equivalence class with respect to complementing binary variables, i.e., if a constraint with term \( a_j x_j \) is in a given class then the constraint with \( a_j(1-x_j) \) is also in the class. Consequently, the most general constraint that can appear in a mixed integer programming formulation can be represented as follows

\[
\sum_{j \in B} a_{i,j} x_j + \sum_{j \in I \cup C} a_{i,j} y_j - b_i
\]

where \( a_j \) for \( j \in B \) and \( b \) are positive, and \( a_j \) for \( j \in I \cup C \) are nonzero.

Furthermore, we distinguish variable bounds from simple bounds. In a constraint with a variable bound, there is a distinct binary variable that bounds all others if it is set to either 0 or 1. The most general constraint with a variable bound can be represented as follows

\[
\sum_{j \in B} a_{i,j} x_j + \sum_{j \in I \cup C} a_{i,j} y_j - a_k x_k
\]

where \( a_j \) for \( j \in B, a_j \) for \( j \in I \cup C, \) and \( a_k \) are all positive. Whenever we consider a constraint with a variable bound, we will assume that the distinct binary variable appears in the right-hand-side of the constraint and all other variables appear in the left-hand-side of the constraint.

The language we propose to define constraint classes uses six fields. The first field describes the type of variables that occur in the constraint. The second field specifies the number of variables in the constraint. The third field characterizes the coefficients of the variables in the constraint. The fourth field describes the type of bound. The fifth field specifies the sense of the bound. The sixth field characterizes the value of the bound.
The classification language consists of a set of rules that define allowable structures. Each rule defines a nonterminal symbol (classification or field) in terms of other nonterminal symbols and terminal symbols (values of fields, or 'tokens'); the symbol $\vee$ is used to represent an exclusive or. Each nonterminal symbol is enclosed in angular brackets. Each token is followed by a comment on its interpretation between square brackets. The token $\circ$ indicates the empty symbol; it is used to indicate a default value, which is usually either the simplest or the most appropriate value.

Each constraint class under consideration is defined by a number of tokens, some of which may be equal to $\circ$. For notational convenience, the tokens are represented as follows:

\[
\langle\text{field 1}\rangle\langle\text{field 2}\rangle\langle\text{field 3}\rangle\langle\text{field 4}\rangle\langle\text{field 5}\rangle\langle\text{field 6}\rangle
\]

The partitioning of the set of constraints into a number of classes, as done by mixed integer programming systems, is in fact nothing more than the identification of interesting special cases.

\[
\langle\text{classification}\rangle ::= \\
\langle\text{type of variables}\rangle \\
\langle\text{number of variables}\rangle \\
\langle\text{coefficients of variables}\rangle \\
\langle\text{type of bound}\rangle \\
\langle\text{sense of bound}\rangle \\
\langle\text{value of bound}\rangle
\]

2.1. Type of variables
The first field specifies the type of variables that appear in the left-hand-side of the constraint, i.e., whether there are both binary and non-binary variables, just binary variables, or just non-binary variables.

\[
\langle\text{type of variables}\rangle ::= \circ \vee \text{BIN} \vee \text{MIX}
\]

- $\circ$ [no binary variables]
- BIN [all binary variables]
- MIX [both binary and non-binary variables]

2.2. Number of variables
The second field specifies the number of variables that appear in the left-hand-side of the constraint, i.e., whether there is only a single variable or whether there is more than one variable. Note that in the case of a left-hand-side with both binary and non-binary variables there can never be a single variable.

\[
\langle\text{number of variables}\rangle ::= \circ \vee 1
\]

- $\circ$ [an arbitrary number of variables]
- 1 [a single variable]
2.3. Coefficients of variables
The third field specifies the coefficients of the variables that appear in the left-hand-side of the constraint, i.e., whether they all have coefficient \( c \) or whether they have arbitrary coefficients.

\[
<\text{coefficients of variables}> ::= \circ \vee c
\]

\( \circ \) \hspace{1em} \text{[arbitrary coefficients]}

\( c \) \hspace{1em} \text{[all coefficients equal to} \ c\text{]}

2.4. Type of bound
The fourth field specifies the type of bound. The type of bound can be either simple or variable. In constraints with a variable bound all variables in the left-hand-side have positive coefficients and there is a single binary variable with a positive coefficient in the right-hand-side.

\[
<\text{type of bound}> ::= \circ \vee \text{VAR}
\]

\( \circ \) \hspace{1em} \text{[simple bound]}

\( \text{VAR} \) \hspace{1em} \text{[variable bound]}

2.5. Sense of bound
The fifth field specifies the sense of bound. The sense of bound is determined by the sense of the constraint, which can be either \( \leq, =, \text{ or } \geq \).

\[
<\text{sense of bound}> ::= \text{UB} \vee \text{EQ} \vee \text{LB}
\]

\( \text{UB} \) \hspace{1em} \text{[upper bound]}

\( \text{EQ} \) \hspace{1em} \text{[equal bound]}

\( \text{LB} \) \hspace{1em} \text{[lower bound]}

2.6. Value of bound
The sixth field specifies the value of the bound, i.e., whether it is equal to a specific value \( c \) or whether it can be an arbitrary value.

\[
<\text{value of bound}> ::= \circ \vee c
\]

\( \circ \) \hspace{1em} \text{[arbitrary bound]}

\( c \) \hspace{1em} \text{[bound equal to} \ c\text{]}

The language presented above defines a hierarchical structure of constraint classes. Figure 1 explicitly shows this hierarchical structure for the constraints with sense \( \leq \). We now define for each constraint a unique constraint class by assigning it to the smallest class in the hierarchical structure that contains it. Table 1 illustrates the classification scheme by presenting classes for several types of constraints encountered in the literature.
Figure 1. Hierarchical structure of less-than-or-equal constraints.

<table>
<thead>
<tr>
<th>Classification</th>
<th>Inequality</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$BIN_1EQ_1$</td>
<td>$\sum_{j \in B} x_j = 1$</td>
<td>generalized upper bound</td>
</tr>
<tr>
<td>$BIN_1UB_1$</td>
<td>$\sum_{j \in B} x_j \leq 1$</td>
<td>special ordered set</td>
</tr>
<tr>
<td>$BIN_1UB$</td>
<td>$\sum_{j \in B} x_j \leq a_k x_k$</td>
<td>invariant knapsack</td>
</tr>
<tr>
<td>$BIN_1VARUB$</td>
<td>$\sum_{j \in B} x_j \geq a_k x_k$</td>
<td>plant-location</td>
</tr>
<tr>
<td>$BIN_UB$</td>
<td>$\sum_{j \in B} a_j x_j \leq a_k$</td>
<td>reverse plant-location</td>
</tr>
<tr>
<td>$1VARUB$</td>
<td>$a_j x_j \leq a_k x_k$</td>
<td>knapsack</td>
</tr>
<tr>
<td>$1VARLB$</td>
<td>$a_j x_j \geq a_k x_k$</td>
<td>variable upper bound</td>
</tr>
<tr>
<td>$1UB$</td>
<td>$a_j x_j \leq a_k$</td>
<td>variable lower bound</td>
</tr>
<tr>
<td>$1LB$</td>
<td>$a_j x_j \geq a_k$</td>
<td>simple upper bound</td>
</tr>
<tr>
<td></td>
<td></td>
<td>simple lower bound</td>
</tr>
</tbody>
</table>

Table 1. Examples of classifications.

References


### List of COSOR-memoranda - 1991

<table>
<thead>
<tr>
<th>Number</th>
<th>Month</th>
<th>Author</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>91-01</td>
<td>January</td>
<td>M.W.I. van Kraaij, W.Z. Venema, J. Wessels</td>
<td>The construction of a strategy for manpower planning problems.</td>
</tr>
<tr>
<td>91-03</td>
<td>January</td>
<td>M.W.P. Savelsbergh</td>
<td>The vehicle routing problem with time windows: minimizing route duration.</td>
</tr>
<tr>
<td>91-04</td>
<td>January</td>
<td>M.W.I. van Kraaij</td>
<td>Some considerations concerning the problem interpreter of the new manpower planning system formasy.</td>
</tr>
<tr>
<td>91-06</td>
<td>March</td>
<td>R.J.G. Wilms</td>
<td>Properties of Fourier-Stieltjes sequences of distribution with support in [0,1).</td>
</tr>
<tr>
<td>91-07</td>
<td>March</td>
<td>F. Coolen, R. Dekker, A. Smit</td>
<td>Analysis of a two-phase inspection model with competing risks.</td>
</tr>
<tr>
<td>91-08</td>
<td>April</td>
<td>P.J. Zwietering, E.H.L. Aarts, J. Wessels</td>
<td>The Design and Complexity of Exact Multi-Layered Perceptrons.</td>
</tr>
<tr>
<td>91-09</td>
<td>May</td>
<td>P.J. Zwietering, E.H.L. Aarts, J. Wessels</td>
<td>The Classification Capabilities of Exact Two-Layered Perceptrons.</td>
</tr>
<tr>
<td>91-10</td>
<td>May</td>
<td>P.J. Zwietering, E.H.L. Aarts, J. Wessels</td>
<td>Sorting With A Neural Net.</td>
</tr>
<tr>
<td>91-11</td>
<td>May</td>
<td>F. Coolen</td>
<td>On some misconceptions about subjective probability and Bayesian inference.</td>
</tr>
<tr>
<td>Memo Number</td>
<td>Date</td>
<td>Authors</td>
<td>Title</td>
</tr>
<tr>
<td>------------</td>
<td>----------</td>
<td>-----------------------------</td>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>91-12</td>
<td>May</td>
<td>P. van der Laan</td>
<td>Two-stage selection procedures with attention to screening.</td>
</tr>
<tr>
<td>91-14</td>
<td>June</td>
<td>J. Korst, E. Aarts, J.K. Lenstra, J. Wessels</td>
<td>Periodic assignment and graph colouring.</td>
</tr>
<tr>
<td>91-16</td>
<td>July</td>
<td>P. Deheuvels, J.H.J. Einmahl</td>
<td>Approximations and Two-Sample Tests Based on P - P and Q - Q Plots of the Kaplan-Meier Estimators of Lifetime Distributions.</td>
</tr>
<tr>
<td>91-19</td>
<td>August</td>
<td>P. van der Laan</td>
<td>The efficiency of subset selection of an almost best treatment.</td>
</tr>
<tr>
<td>91-20</td>
<td>September</td>
<td>P. van der Laan</td>
<td>Subset selection for an -best population: efficiency results.</td>
</tr>
<tr>
<td>91-23</td>
<td>September</td>
<td>P. van der Laan</td>
<td>Distribution theory for selection from logistic populations.</td>
</tr>
<tr>
<td>COSOR-MEMORANDA (3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---------------------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>91-24 October</td>
<td>I.J.B.F. Adan</td>
<td>Matrix-geometric analysis of the shortest queue problem with threshold jockeying.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>J. Wessels</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>W.H.M. Zijm</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>J. Wessels</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>W.H.M. Zijm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>91-26 October</td>
<td>E.E.M. van Berkum</td>
<td>D-optimal designs for an incomplete quadratic model.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>P.M. Upperman</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>P.H.M. Ruys</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>S. Jilin</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>J. Wessels</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>W.H.M. Zijm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>91-29 November</td>
<td>J. Wessels</td>
<td>Tools for the Interfacing Between Dynamical Problems and Models within Decision Support Systems.</td>
<td></td>
</tr>
<tr>
<td>91-30 November</td>
<td>G.L. Nemhauser</td>
<td>Constraint Classification for Mixed Integer Programming Formulations.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>M.W.P. Savelsbergh</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>G.C. Sigismondi</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>