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Abelian and Tauberian theorems for the Laplace transform of functions in several variables
by
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ABELIAN AND TAUBERIAN THEOREMS FOR THE LAPLACE TRANSFORM OF FUNCTIONS IN SEVERAL VARIABLES

E. Omey * and E. Willekens **

ABSTRACT

Using two kinds of multivariate regular variation we prove several Abel-Tauber theorems for the Laplace transform of functions in several variables. We generalize some power series results of Alpar and apply our results in multivariate renewal theory.

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1. Introduction

In $\mathbb{R}_+$, Karamata’s Abel-Tauber theorem for Laplace transforms is well-known and reads as follows:

**THEOREM K** [4, p. 445].

Let $U$ be a measure with Laplace transform $U(s) = \int_0^\infty e^{-sx} dU(x) = s \int_0^\infty e^{-sx} U(x) dx$, defined for $s > 0$. If $L$ is slowly varying at infinity and $\delta \geq 0$, then each of the relations

$$\hat{U}(\frac{1}{x}) \sim x^\delta L(x) \quad (x \to \infty)$$

and

$$U(x) \sim \frac{1}{\Gamma(1+\delta)} x^\delta L(x) \quad (x \to \infty)$$

implies the other.

In this paper we generalize theorem K to functions of several variables. To this end we use two types of regular variation in dimension $d \geq 2$. For convenience we only state and prove the results for functions of two variables. Results of this type may be useful in a variety of problems. We mention applications in number theory [1], renewal theory [9,10,11]; generalized renewal theory [11] and in characterizing domains of attraction of multivariate stable laws [6,7].

In section 2 we consider two possible generalizations to dimension 2 of the classical definition of one-dimensional regular variation. Then we prove an Abel-Tauber theorem for the Laplace transform $f$ of $U$, defined as

$$\hat{f}(u,v) = uv \int_0^\infty \int_0^\infty e^{-ux-\nu y} f(x,y) dx dy .$$

In section 3 we apply our results to power series of several variables, hereby generalizing some results of Alpar [1,2]. In section 4 an application to multidimensional renewal theory is given.
2. Regular variation in dimension 2 and Abel-Tauber theorems

The first class of functions which we consider has been introduced by Omey [11] and De Haan et al. [7].

**Definition.** A measurable function \( f : \mathbb{R}^2_+ \to \mathbb{R}_+ \) is regularly varying with auxiliary functions \( r \) and \( s : \mathbb{R}_+ \to \mathbb{R}_+ \) if for some positive function \( h \) and all \( x, y > 0 \),

\[
\lim_{t \to \infty} \frac{f(r(t)x, s(t)y)}{h(t)} = \lambda(x, y)
\]

exists and is finite. Notation \( f(x, y) \in RVF(r, s, \lambda) \).

This class of functions has been useful to characterize domains of attraction of stable laws in \( \mathbb{R}^d_+ \) (cf. [6], [7]. If \( r(t) = s(t) = t \), the class has been studied by Stam [13], De Haan, Omey and Resnick [5], [6].

Apart from (2.1) we shall also consider measurable functions \( f \) for which

\[
\limsup_{t \to \infty} \frac{f(r(t)x, s(t)y)}{h(t)} < \infty
\]

for all \( x, y > 0 \). Notation \( f(x, y) \in O-RVF(r, s, h) \).

In the theorems below we assume that \( f \) is monotone in each variable separately and that the auxiliary functions are regularly varying in \( \mathbb{R}_+ \). If \( r \in RV_\alpha, s \in RV_\beta (\alpha, \beta > 0) \) and if \( f \) is monotone then the limitfunction \( \lambda \) in (2.1) is continuous. If \( \lambda \neq 0 \) then \( \lambda \in RV_\delta (\delta \in \mathbb{R}) \) and \( \lambda \) satisfies the functional equation \( \lambda(a^\alpha x, a^\beta y) = a^{\delta} \lambda(x, y) \) \((a, x, y > 0)\). For further properties of \( RVF \) we refer to [7], [11].

The second type of regular variation is defined as follows.

**Definition.** [11, p. 25].

A measurable function \( f : \mathbb{R}^2_+ \to \mathbb{R}_+ \) is weakly regularly varying if for some positive function \( h \) and all \( x, y > 0 \),

\[
\lim_{\min(a, b) \to \infty} \frac{f(ax, by)}{h(a, b)} = \lambda(x, y)
\]

exists and is finite. Notation \( f(x, y) \in WRV(h) \).

If (2.2) holds and if \( \lambda(x_0, y_0) \neq 0 \) for some \( x_0, y_0 > 0 \), it follows that

\[
\lim_{\min(a, b) \to \infty} \frac{f(ax, by)}{f(a, b)} = \mu(x, y)
\]

for all \( x, y > 0 \). Using the identity \( f(ax, by) = \frac{f(ax, by)}{f(ax, by)} f(ax, by) \) it follows that \( \mu(x, y) = \mu(u, v) \mu(x, y) \). Hence \( \mu(x, y) = x^{\alpha} y^{\beta} \) for some real numbers \( \alpha \) and \( \beta \) [11, lemma 2.4.1]. If (2.3) holds with \( \mu(x, y) = x^{\alpha} y^{\beta} \) we use the notation \( f \in WRV(\alpha, \beta) \). Obviously if (2.2) holds, then \( \lambda(x, y) = Cx^{\alpha} y^{\beta} \) with \( C \geq 0 \).

If (2.2) is replaced by
\[
\limsup_{\min(a,b) \to \infty} \frac{f(ax, by)}{h(a, b)} < \infty
\]

we use the notation \( f \in O-WRV(h) \).

Our first result is the following two-dimensional analogue of theorem \( K \).

**Theorem 2.1**

Suppose that \( f: \mathbb{R}_+^2 \to \mathbb{R}_+ \) is nondecreasing and that \( f(u, v) < \infty \) for \( u, v > 0 \).

(i) Let \( r \in RV, s \in RV \) and \( h \in RV \) \((\alpha, \beta > 0, \delta \geq 0) \). Then for some \( \lambda \geq 0 \) we have \( f(x, y) \in RVF(r, s, h, \lambda) \) if and only if for some \( \phi \geq 0 \), \( f\left(\frac{1}{x}, \frac{1}{y}\right) \in RVF(r, s, h, \phi) \). Moreover both imply that \( \phi\left(\frac{1}{x}, \frac{1}{y}\right) = \lambda(x, y) \).

(ii) Let \( h \in WRV(\alpha, \beta) \) \((\alpha, \beta \geq 0) \). Then we have \( f(x, y) \in WRV(h) \) if and only if \( f\left(\frac{1}{x}, \frac{1}{y}\right) \in WRV(h) \). Moreover if the limitfunction of \( f \) is \( \lambda(x, y) = Cx^\alpha y^\beta \) then

\[
\lim_{\min(a,b) \to \infty} \frac{f(x, y)}{h(a, b)} = C \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{x^\alpha y^\beta}.
\]

**Proof.**

(i) See [7, Theorem 2.4].

(ii) First suppose \( f(x, y) \in WRV(h) \). We will prove that there exist positive constants \( t_0, \gamma \) and \( C \geq 1 \) such that for all \( a, b \geq t_0 \),

\[
(2.4) \quad \frac{f(ax, by)}{h(a, b)} \leq \begin{cases} 
C & \text{if } x \leq e, y \leq e \\
C \max(x, y)^\gamma & \text{if } x \geq e \text{ or } y \geq e.
\end{cases}
\]

To prove (2.4), note that for \( x \leq e \) and \( y \leq e \) we have

\[
\frac{f(ax, by)}{h(a, b)} \leq \frac{f(ae, be)}{h(a, b)} \leq C, \quad \forall a, b \geq t_0
\]

and

\[
\frac{h(ae, be)}{h(a, b)} \leq C, \quad \forall a, b \geq t_0
\]

where \( C \geq 1 \). If \( x \) and \( y \) are such that \( e^a \leq \max(x, y) \leq e^{a+1} \) we have

\[
\frac{f(ax, by)}{h(a, b)} \leq \frac{f(ae^{a+1}, be^{a+1})}{h(ae^a, be^a)} \prod_{k=0}^{n-1} \frac{h(ae^{k+1}, be^{k+1})}{h(ae^k, be^k)}
\]

\[
\leq C \cdot C^n.
\]

By the choice of \( n \) we obtain (2.4). Now by (2.4) and Lebesgues theorem on dominated convergence we obtain
\[
\lim_{\min(a,b) \to \infty} \frac{f \left( \frac{s}{a}, \frac{t}{b} \right)}{h(a,b)} = \lim_{\min(a,b) \to \infty} \int_0^\infty \int_0^\infty e^{-sx-ty} \frac{f(ax,by)}{h(a,b)} \, dx \, dy
\]

\[
= C \Gamma(1 + \alpha) \Gamma(1 + \beta) s^{-\alpha} t^{-\beta}.
\]

Next assume that \( f \left( \frac{1}{x}, \frac{1}{y} \right) \in \text{WRV}(h) \). For \( a_n, b_n \) such that \( \min(a_n, b_n) \to \infty \) define \( F_n \) as

\[
F_n(x, y) = \frac{f(a_n x, b_n y)}{h(a_n, b_n)}.
\]

Then \( F_n \) is nondecreasing and for all \( u, v > 0 \) we have

\[
\lim_{n \to \infty} \hat{F}_n(u, v) = Cu^{-\alpha} v^{-\beta}.
\]

It follows from the continuity theorem for Laplace transforms [12, lemma 4] that

\[
\lim_{n \to \infty} F_n(u, v) = \lambda(u, v) \quad \text{a.e.}
\]

and that \( \hat{\lambda}(u, v) = Cu^{-\alpha} v^{-\beta} \). Since this limit is independent of the sequences \( \{a_n\}_{\mathbb{N}} \) and \( \{b_n\}_{\mathbb{N}} \), we obtain that \( f \in \text{WRV}(h) \).

For \( O \)-regularly varying functions we have the following

**Theorem 2.2**

Suppose that \( f : \mathbb{R}_+^2 \to \mathbb{R}_+ \) is nondecreasing and that \( \hat{f}(u, v) < \infty \) for \( u, v > 0 \).

(i) If \( r \in RV_\alpha, s \in RV_\beta \) \((\alpha, \beta > 0)\), \( h \) nondecreasing and

\[
\limsup_{t \to \infty} \frac{h(xt)}{h(t)} < \infty, \quad \forall x \geq 1
\]

then \( f(x, y) \in O-RVF(r, s, h) \) if and only if \( \hat{f}(\frac{1}{x}, \frac{1}{y}) \in O-RVF(r, s, h) \)

(ii) If \( h \in O-WRV(h) \), then \( f(x, y) \in O-WRV(h) \) if and only if \( \hat{f}(\frac{1}{x}, \frac{1}{y}) \in O-WRV(h) \).

**Proof.** The only if parts of both (i) and (ii) follow from the inequality \( \hat{f}(u, v) \geq e^{-au-by} f(a, b) \).

The if part of (i) follows in a similar way as the proof of Theorem 2.4 in [7]. The if part of (ii) follows from (2.4) since to obtain (2.4) we only used the boundedness of \( \frac{f(ax,by)}{h(a,b)} \) and \( \frac{h(ax,by)}{h(a,b)} \) as \( \min(a, b) \to \infty \). \( \square \)

**Remark.** If we replace (2.2) by
\begin{equation}
\lim_{\min(a,b) \leq \frac{b}{a} \leq C \to \infty} \frac{f(ax, by)}{h(a, b)} = \lambda(x, y), \; x, y > 0
\end{equation}

where $0 < c < C < \infty$, then again $\lambda$ is of the form $\lambda(x, y) = Cx^a y^B$. The analogue of Theorem 2.1 for functions satisfying (2.5) is easily established.
3. Abel-Tauber theorems for power series

Let \( \{a_{n,m}\}_{n \times m} \) denote a sequence of nonnegative real numbers and suppose that its generating function \( A(x, y) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,m} x^n y^m \) is finite for \( 0 \leq x, y < 1 \).

Clearly \( A(e^{-u}, e^{-v}) \) is the Laplace transform of the monotone function \( S(x, y) := \sum_{n \leq x} \sum_{m \leq y} a_{n,m} \). An application of the results of section 2 now yields

**Corollary 3.1**

(i) If \( r \in RV_\alpha, s \in RV_\beta, h \in RV_\delta(\alpha, \beta > 0, \delta \geq 0) \) then for some \( \lambda \geq 0 \) and \( \phi \geq 0 \) we have 
\[
S(x, y) \in RVF(r, s, h, \lambda) \text{ if and only if for all } x, y > 0,
\]
\[
\lim_{t \to \infty} \frac{A(1-x/r(t), 1-y/s(t))}{h(t)} = \phi(x, y).
\]

Moreover, if (3.1) holds then \( \phi = \hat{\lambda} \).

(ii) If \( h \in WRF(\alpha, \beta) (\alpha, \beta \geq 0) \) then for \( C \geq 0 \) we have 
\[
\lim_{\min(a,b) \to \infty} \frac{S(ax, by)}{h(a,b)} = Cx^\alpha y^\beta
\]
if and only if
\[
\lim_{\min(a,b) \to \infty} \frac{A(1-x/a, 1-y/b)}{h(a,b)} = C \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{x^\alpha y^\beta}.
\]

**Proof.**

(i) From Theorem 2.1 it follows that regular variation of \( S(x, y) \) is equivalent to the existence of

\[
\lim_{t \to \infty} \frac{A(\exp(-x/r(t)), \exp(-y/s(t)))}{h(t)} = \phi(x, y), \forall x, y > 0.
\]

Since \( \phi \) is continuous this implies (3.1). Conversely, if (3.1) holds, then \( \phi \) is continuous [7] and (3.2) follows.

(ii) Similar, now using Theorem 2.1 (ii).

The previous corollary (ii) generalizes Theorem 1 of Alpar [2] in which \( h(a, b) = a^ab^b \). Result (i) generalizes Theorems 1 to 4 of Alpar [1] in which \( h(t) = t \) or \( h(t) = t^2 \).

**Remarks.**

1. Note that \( S(x, y) \) can be interpreted as the measure \( M(\cdot) \) of the rectangle \( L = \{(u, v) | 0 \leq u \leq x, 0 \leq v \leq y \} \). Now weak regular variation of \( S \) is equivalent to

\[
\lim_{\min(a,b) \to \infty} \frac{A(\exp(-x/a), \exp(-y/b))}{h(a,b)} = C \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{x^\alpha y^\beta}.
\]
\[ (3.3) \quad \lim_{\min(a,b) \to \infty} \frac{M(\{(u,v) \mid (\frac{u}{a}, \frac{v}{b}) \in L\})}{h(a,b)} = m(L) \]

where \( m(L) \) is measure with distribution function \( Cx^a y^b \). It follows as in Alpar [2, p. 172] that (3.3) remains valid if \( F \) is a general Jordan measurable subset of \( \mathbb{R}^2 \).

Similarly, regular variation of \( S \) is equivalent to
\[ (3.4) \quad \lim_{t \to \infty} \frac{M(\{(u,v) \mid (\frac{u}{r(t)}, \frac{v}{s(t)}) \in L\})}{h(t)} = m(L) \]

where \( m(\cdot) \) is the measure with distribution function \( \lambda(x, y) \).

2. If the sequence \( \{a_{n,m}\}_{N \times N} \) is monotone then regular variation (resp. weak regular variation) of \( S \) implies that the function \( f(x, y) := a_{[x, y]} \in RVF(r, s, h/\delta x^\beta) \) (resp. \( f \in WRV(h(x, y)) \)). The proof of both results follows as in the proof of Theorem 2.3 of De Haan et al. [6].

Our next application is devoted to the convolution product of sequences. Let \( \{a_{n,m}\}_{N \times N}, \{b_{n,m}\}_{N \times N}, \{c_{n,m}\}_{N \times N} \) be sequences of nonnegative real numbers related by
\[ c_{n,m} = \sum_{k=0}^{n} \sum_{i=0}^{m} a_{n-k,m-i} b_{k,i} . \]
If the generating functions \( A(x, y) \) and \( B(x, y) \) are finite for \( 0 \leq x, y < 1 \), then also \( C(x, y) \) is finite for \( 0 \leq x, y < 1 \) and
\[ C(x, y) = A(x, y) B(x, y) . \]

In order to formulate our next result, we define \( S^a(x, y) = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} a_{n,m} \) and similarly \( S^b \) and \( S^c \).

**Corollary 3.2**

(i) Suppose \( r \in RV\alpha, s \in RV\beta, h \in RV\delta, g \in RV\eta \) (\( \alpha, \beta, \delta, \eta \geq 0 \)) and suppose \( S^a(x, y) \in RVF(r, s, h, \lambda_a) \). Then \( S^b(x, y) \in RVF(r, s, g, \lambda_b) \) if and only if \( S^c(x, y) \in RVF(r, s, gh, \lambda_c) \). Moreover, both statements imply that
\[ \lambda_c(x, y) = \int_0^x \int_0^y \lambda_a(x-u, y-v) \lambda_a(du, dv) . \]

(ii) Suppose \( h \in WRV(\alpha, \beta), g \in WRV(\gamma, \eta) \) (\( \alpha, \beta, \gamma, \eta \geq 0 \)) and suppose
\[ \lim_{\min(u,v) \to \infty} \frac{S^a(u,v)}{h(u,v)} = N, \quad \text{where} \quad N > 0 . \]
Then \( S^b(x, y) \in WRV(g) \) if and only if \( S^c(x, y) \in WRV(gh) \). Moreover, if \( \lambda_b(x, y) = Mx^\alpha y^\beta (M \geq 0) \) then \( \lambda_c(x, y) = \frac{MN}{\Gamma(1+\alpha+\gamma) \Gamma(1+\beta+\eta)}\Gamma(1+\alpha+\gamma+\beta+\eta) \).

Corollary 3.2 is applicable to obtain some results in connection with \( (C, \xi, \eta) \)-summability of double series (cf. [2, p. 166], [8, p. 209-213]). For \( \eta, \xi \in \mathbb{N} \) let
The quotient $C_{n,m}^{\xi,\eta} = \frac{S_{n,m}^{\xi,\eta}}{A_{n,m}^{\xi,\eta}}$ is called the $(n,m)$th Cesaro mean of order $(\xi, \eta)$ of the sequence \{(a_{n,m})\}_{n \times m}.

The sequence \{(a_{n,m})\}_{n \times m} is called \((C, \xi, \eta)-summable\) if the limit

\[
\lim_{\min(n,m) \to \infty} C_{n,m}^{\xi,\eta} = C_{n,m}^{\xi,\eta}
\]

exists and is finite. It is easy to see that $S_{n,m}^{0,0} = S(n,m)$ and that $S_{n,m}^{\xi,\eta} = \sum_{k=0}^{n} \sum_{l=0}^{m} S_{k,l}^{\xi-1,\eta-1}$ for $\xi, \eta \in \mathbb{N}_0$.

Also

\[
\lim_{\min(n,m) \to \infty} \frac{A_{n,m}^{\xi,\eta}}{n^{\xi}m^{\eta}} = \frac{1}{\Gamma(1+\xi) \Gamma(1+\eta)}
\]

and

\[
A_{n,m}^{\xi,\eta} = \sum_{k=0}^{n} \sum_{l=0}^{m} A_{k,l}^{(\xi-1,\eta-1)}
\]

for $\xi, \eta \in \mathbb{N}_0$.

We now prove that \((C, 0, 0)\) summability is equivalent to \((C, \xi, \eta)\) summability for all $\xi, \eta \in \mathbb{N}_0$.

**Corollary 3.3**

For each $\xi, \eta \in \mathbb{N}_0$ we have

\[
\lim_{\min(n,m) \to \infty} S(n,m) = \lim_{\min(n,m) \to \infty} C_{n,m}^{0,0} = C_{n,m}^{0,0}
\]

if and only if

\[
\lim_{\min(n,m) \to \infty} C_{n,m}^{\xi,\eta} = C_{n,m}^{0,0}.
\]

**Proof.** Let $a_{n,m}' = A_{n,m}^{\xi-1,\eta-1}$, $b_{n,m}' = a_{n,m}$ and $c_{n,m}' = S_{n,m}^{\xi-1,\eta-1}$. Then from (3.5) we have

\[
c_{n,m}' = \sum_{k=0}^{n} \sum_{l=0}^{m} a_{n-k,m-l} b_{k,l}
\]

and from (3.6) we have

\[
S_{(n,m)}^{0,0} = A_{n,m}^{\xi,\eta} - \frac{n^{\xi}m^{\eta}}{\Gamma(1+\xi) \Gamma(1+\eta)} (\min(n,m) \to \infty).
\]
An application of Corollary 3.2 shows that

$$\lim_{\min(n,m) \to \infty} S^{b'}(n, m) = C^{0,0}$$

holds if and only if

$$\lim_{\min(n,m) \to \infty} \frac{S^c(n, m)}{n^\xi m^\eta} = \lim_{\min(n,m) \to \infty} \frac{S^{c, \eta}_{n,m}}{n^\xi m^\eta} = \frac{C^{0,0}}{\Gamma(1+\xi)\Gamma(1+\eta)}.$$
4. Application in multidimensional renewal theory

Let \( \{(X_i, Y_i)\}_i \) denote a sequence of i.i.d. non-negative random vectors with common distribution function \( F \) and let \( (S^1_n, S^2_n) = (\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i) \). Following Hunter [9] we define

\[
N^1(x) := \max \{ n : S^1_n \leq x \}
\]

\[
N^2(y) := \max \{ n : S^2_n \leq y \}
\]

\[
N(x, y) := \max \{ n : S^1_n \leq x, S^2_n \leq y \} = \min(N^1(x), N^2(y)).
\]

The counting processes \( N^1 \) and \( N^2 \) are the (univariate) renewal counting processes; the vector \( (N^1(x), N^2(y)) \) is called the bivariate renewal counting process and \( N(x, y) \) is the two-dimensional renewal counting process. It is well known that

\[
H^1(x) := E(N^1(x) + 1) = \sum_{n=0}^{\infty} P\{S^1_n \leq x\}
\]

\[
H^2(y) := E(N^2(y) + 1) = \sum_{n=0}^{\infty} P\{S^2_n \leq y\}
\]

\[
H(x, y) := E(N(x, y) + 1) = \sum_{n=0}^{\infty} P\{S^1_n \leq x, S^2_n \leq y\}
\]

It is also easily seen that

\[
K(x, y) := E(N^1(x)N^2(y)) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P\{S^1_n \leq x, S^2_m \leq y\}.
\]

In univariate renewal theory the following is well known (see e.g. Feller [4]): let \( \mu_1 = E(X_1) \) and \( \mu_2 = E(Y_1) \)

(4.1) \( \text{if } \mu_1 < \infty, \text{ then } \lim_{t \to \infty} \frac{H^1(t)}{t} = \frac{1}{\mu_1} \)

(4.2) \( \text{if } \mu_2 < \infty, \text{ then } \lim_{t \to \infty} H^1(t) - \frac{t}{\mu_1} = \frac{\mu_2}{2\mu_1^2} \)

(4.3) \( \text{if } \mu_2 < \infty, \text{ then } \lim_{t \to \infty} \frac{1}{t} \int_0^t (H^1(x) - \frac{x}{\mu_1}) \, dx = \frac{\mu_2}{2\mu_1^3} \)

(4.4) \( \text{if } \mu_2 < \infty, \text{ then } \lim_{t \to \infty} \frac{\text{var}(N^1(t))}{t} = \frac{\mu_2 - \mu_1^2}{\mu_1^3} \).

We first prove the two-dimensional analogue of (4.1) for the functions \( H(x, y) \) and \( K(x, y) \).

**Theorem 4.1**

Assume \( \mu_1 = E(X_1) < \infty \) and \( \nu_1 = E(Y_1) < \infty \). Then
(i) \[ \lim_{t \to \infty} \frac{H(tx, ty)}{t} = \min \left( \frac{x}{\mu_1}, \frac{y}{\nu_1} \right) \quad (x, y \leq \infty) \]

(ii) \[ \lim_{t \to \infty} \frac{K(tx, ty)}{t^2} = \frac{xy}{\mu_1 \nu_1} \quad (x, y < \infty). \]

Proof.

(i) It is easy to see that \[ \hat{H}(s, t) = \int_0^\infty \int_0^\infty e^{-s \cdot x - t \cdot y} dH(x, y) = \frac{1}{1-F(s, t)} \]
where \[ \hat{F}(s, t) = E(e^{-sX_1 - tY_1}). \]
Since \[ \lim_{a \to 0} \frac{1-F(as, at)}{a} = s \mu_1 + t \nu_1 \] it follows that
\[ \lim_{a \to 0} \frac{H(tx, ty)}{t} = \phi(x, y) \quad \text{where} \quad \phi(x, y) = \frac{1}{s \mu_1 + t \nu_1}. \] It is easily seen that
\[ \phi(x, y) = \min \left( \frac{x}{\mu_1}, \frac{y}{\nu_1} \right). \]

(ii) Using Laplace transforms we obtain
\[ \hat{K}(s, t) = \sum_{k=1}^\infty \sum_{r=0}^\infty \hat{F}^k(s, t) \hat{F}^r(0, t) + \sum_{k=1}^\infty \sum_{r=0}^\infty \hat{F}^k(s, t) \hat{F}^r(s, 0) \]
\[ = \frac{\hat{F}(s, t)}{(1-\hat{F}(s, t))} \left[ \frac{\hat{F}(0, t)}{1-\hat{F}(0, t)} + \frac{\hat{F}(s, 0)}{1-\hat{F}(s, 0)} + 1 \right]. \]
Since \( \mu_1 < \infty \) and \( \nu_1 < \infty \) it follows that
\[ \lim_{a \to 0} a^2 \hat{K}(as, at) = \frac{1}{\mu_1 s + \nu_1 t} \left( \frac{1}{\mu_1 s} + \frac{1}{\nu_1 t} \right) = \frac{1}{\mu_1 \nu_1 s t}. \]
Since \( K \) is monotone, an application of Theorem 2.1 (i) yields the desired result.

Being interested in the difference \( H(x, y) - \min \left( \frac{x}{\mu_1}, \frac{y}{\nu_1} \right) \) (cf. (4.2) and (4.3)) we now estimate
\[ W(x, y) = \int_0^x \int_0^y (H(u, v) - \min \left( \frac{u}{\mu_1}, \frac{v}{\nu_1} \right)) dudv. \]

Theorem 4.2
Assume that \( W(x, y) \) is nondecreasing and assume that \( \mu_2 + \nu_2 = EX_1^2 + EY_1^2 < \infty \) then
\[ \lim_{t \to \infty} \frac{W(tx, ty)}{t^2} = \begin{cases} \frac{\mu_2}{2 \mu_1^2} xy - \left( \frac{\mu_2 \nu_1}{\mu_1} - E(X_1 Y_1) \right) \frac{x^2}{2 \mu_1^2} & \text{if } \frac{x}{\mu_1} \leq \frac{y}{\nu_1} \\ \frac{\nu_2}{2 \nu_1^2} xy - \left( \frac{\nu_2 \mu_1}{\nu_1} - E(X_1 Y_1) \right) \frac{y^2}{2 \nu_1^2} & \text{if } \frac{x}{\mu_1} \geq \frac{y}{\nu_1} \end{cases} \]
Proof. We have \( \hat{W}(s, t) = \frac{1}{st}(\hat{H}(s, t) - \frac{1}{\mu_1 s + v_1 t}) \) so that
\[
\lim_{a \to 0} a^2 \hat{W}(as, at) = \frac{1}{st} \frac{E(sX_1 + tY_1)^2}{(\mu_1 s + v_1 t)2} = \frac{\mu_2}{2} \frac{s}{t(\mu_1 s + v_1 t)^2} + \frac{v_2}{2} \frac{t}{s(\mu_1 s + v_2 t)^2}
\]
\[
+ \frac{EX_1 Y_1}{2(\mu_1 s + v_1 t)^2}.
\]
Now let \( g(x, y) = \begin{cases} y(x-y) & x \geq y, f(x, y) = g(y, x) \text{ and } h(x, y) = \min(x^2, y^2). \end{cases} \) It is easily seen that \( \hat{g}(s, t) = \frac{t}{s(s+t)^2} \) and that \( \hat{h}(s, t) = \frac{2}{(s+t)^2}. \)
Hence \( \lim_{a \to 0} a^2 \hat{W}(as, at) = \hat{p}(s, t) \) where
\[
\hat{p}(\frac{s}{\mu_1}, \frac{t}{v_1}) = \frac{\mu_2 v_1}{2\mu_1} \hat{f}(s, t) + \frac{v_2 \mu_1}{2v_1} \hat{g}(s, t) + \frac{EX_1 Y_1}{2} \hat{h}(s, t).
\]
Since by assumption \( W \) is monotone, an application of Theorem 2.2 (i) yields
\[
\lim_{t \to \infty} \frac{W(x_t, y_t)}{t^2} = p(x, y)
\]
where
\[
p(\frac{x_1}{\mu_1}, \frac{v_1}{v_1}) = \frac{\mu_2 v_1}{2\mu_1} f(x, y) + \frac{v_2 \mu_1}{2v_1} g(x, y) + \frac{EX_1 Y_1}{2} h(x, y) \text{ and the result follows.}
\]

The limit function in Theorem 4.2 is continuous but not differentiable on the expectation line \( y = \frac{v_1}{\mu_1} x \). As one can expect it may be difficult to obtain the asymptotic behaviour of \( H(tx, ty) - t \min(\frac{x}{\mu_1}, \frac{y}{v_1}) \) in this case. If \( y \neq \frac{v_1}{\mu_1} x \) however the limit function in Theorem 4.2 behaves nice and we have the following refinement.

Lemma 4.3

If \( \mu_2 + v_2 < \infty \), then \( \lim_{t \to \infty} H(tx, ty) - t \min(\frac{x}{\mu_1}, \frac{y}{v_1}) = \begin{cases} \frac{\mu_2}{2\mu_1} & \text{if } y > \frac{v_1}{\mu_1} x \\ \frac{v_2}{2v_1} & \text{if } y < \frac{v_1}{\mu_1} x \end{cases} \)

Proof.

Suppose that \( \frac{x}{\mu_1} < \frac{y}{v_1} \) (similarly if \( \frac{x}{\mu_1} > \frac{y}{v_1} \)). From Theorem 2.6 of Bickel and Yahav [3] it follows that for such \( x \) and \( y \),
Using (4.2) we obtain the desired result.

If, on the other hand, \( \frac{\mu_2 v_1}{\mu_1} = \frac{\nu_2}{v_1} = E X_1 Y_1 \), then the limit function in Theorem 4.2 is differentiable everywhere. We show that in this case, the r.v. \( X_1 \) and \( Y_1 \) have a correlation \( \rho = 1 \). To see this, note that the equality \( \frac{\nu_2}{v_1} = \frac{\mu_2 v_1}{\mu_1} \) implies that \( \frac{\sigma_1 v_1}{\mu_1} = \sigma_2 \), and that the equality \( \frac{\mu_2 v_1}{\mu_1} = E X_1 Y_1 \) implies that \( \rho = \frac{\sigma_1 v_1}{\sigma_1 \sigma_2 \mu_1} \) whence \( \rho = 1 \).

This implies that \( Y_1 = a X_1 + b \). Using the identities

\[
\sigma_2^2 = a^2 \sigma_1^2 \quad \text{and} \quad \nu_2 = a^2 \nu_2 + 2ab \mu_1 + b^2
\]

together with the previous equalities leads to the solution \( a = \frac{\nu_1}{\mu_1} \) and \( b = 0 \). That this case is trivial may be seen from the following

**Lemma 4.4**

If \( Y_1 = \frac{\nu_1}{\mu_1} X_1 \), then for all \( x, y > 0 \),

\[
\lim_{t \to \infty} H(t x, ty) - t \min\left( \frac{x}{\mu_1}, \frac{y}{v_1} \right) = \frac{\mu_2}{2\mu_1^2}
\]

**Proof.**

If \( Y_1 = \frac{\nu_1}{\mu_1} X_1 \) and hence \( S_n^2 = \frac{\nu_1}{\mu_1} S_n^1 \) we have

\[
H(x, y) = \sum_{n=0}^{\infty} P\{S_n^1 \leq \min(x, \frac{y \mu_1}{v_1})\} = H^1(\min(x, \frac{y \mu_1}{v_1}))
\]

Using (4.2) yields the desired result.

In Theorem 4.5 below, we show that on the expectation line, the result of Lemma 4.3 drastically changes. First we need the following result, interesting in its own right. In the result we estimate \( C(x, y) := \text{Cov}(N^1(x), N^2(y)) \) and \( \rho(x, y) := \text{Corr}(N^1(x), N^2(y)) \), the covariance (resp. correlation) between \( N^1(x) \) and \( N^2(y) \).

**Lemma 4.5**

(i) \( C(x, y) = 0 \) if and only if \( X_1 \) and \( Y_1 \) are independent.
(ii) If $\mu_2 + \nu_2 < \infty$ and if $+C(x, y)$ is nondecreasing, then
\[
\lim_{t \to \infty} \frac{C(t \cdot t)}{t} = \frac{\text{Cov}(X_1, Y_1)}{\mu_1 \nu_1} \min\left(\frac{x}{\mu_1}, \frac{y}{\nu_1}\right)
\]
and
\[
\lim_{t \to \infty} p(t \cdot t) = p(X_1, Y_1) \left(\frac{x}{\mu_1} \cdot \frac{y}{\nu_1}\right)^{-\frac{3}{2}} \min\left(\frac{x}{\mu_1}, \frac{y}{\nu_1}\right).
\]

(iii) If $\mu_2 + \nu_2 < \infty$, then $C(t \cdot t) = 0(t)$ ($t \to \infty$).

**Proof.**

(i) See Hunter [9, Theorem 3.5].

(ii) Using Laplace transforms we obtain
\[
\hat{C}(s, t) = \hat{F}(s, 0) - \frac{\hat{F}(s, t) \cdot \hat{F}(0, t)}{1 - \hat{F}(s, t)} - \frac{\hat{F}(s, t) - \hat{F}(s, 0) \cdot \hat{F}(0, t)}{(1 - \hat{F}(s, t))(1 - \hat{F}(s, 0))(1 - \hat{F}(0, t))}.
\]

Since $\mu_2 + \nu_2 < \infty$ it follows that
\[
\lim_{s \to 0} a\hat{C}(as, at) = \frac{\text{Cov}(X_1, Y_1)}{\mu_1 \nu_1 (s \mu_1 + t \nu_2)}
\]
and the result for $C$ follows. The result for $p$ follows from this result and from (4.4).

(iii) First note that by definition
\[
S_{11}^{1}(x) \leq x < S_{11}^{1}(x) + 1 \text{ and } S_{22}^{2}(y) \leq y < S_{22}^{2}(y) + 1
\]
so that
\[
E(S_{11}^{1}(x) \cdot S_{22}^{2}(y)) \leq xy \leq E(S_{11}^{1}(x) + 1 \cdot S_{22}^{2}(y) + 1)
\]
Some straightforward calculations show that for $n, m \geq 0$, $E(S_{n+1}^{1} \cdot S_{m+1}^{2}) = E(S_{1}^{1} \cdot S_{1}^{2}) + (n + m) \mu_1 \nu_1 + E(X_1, Y_1)$ whence
\[
E(S_{11}^{1}(x) + 1 \cdot S_{22}^{2}(y) + 1) = E(S_{11}^{1}(x) \cdot S_{22}^{2}(y) + 1) + \mu_1 \nu_1 (E(N_1) + E(N_2)) + E(X_1 Y_1).
\]
Using $E(S_{11}^{1}(x)) = \mu_1 E(N_1)$ it follows that
\[
\text{Cov}(S_{11}^{1}(x), S_{22}^{2}(y)) \leq xy - \mu_1 \nu_1 E(N_1) E(N_2)
\]
\[
\leq \text{Cov}(S_{11}^{1}(x), S_{22}^{2}(y)) \leq xy - \nu_1 \mu_1 (E(N_1) + E(N_2)) + E X_1 Y_1.
\]
Using (4.2) and (4.1) it follows that as $t \to \infty$
\[
\text{(4.5) } \text{Cov}(S_{11}^{1}(x), S_{22}^{2}(y)) = 0(t).
\]
To finish the proof note that $E(S_{n+1}^{1}S_{m+1}^{2}) = \min(n, m) \text{ Cov}(X, Y) + nm \mu_1 \nu_1$ so that
\[
E(S_{11}^{1}(x) \cdot S_{22}^{2}(y)) = E(\min(N_1(x), N_2(y))) \text{ Cov}(X, Y)
\]
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\[+ \mu_1 \nu_1 E(N_1(x)N_2(y))\]

and hence

\[\text{Cov}(S_{N_1(x)}, S_{N_2(y)}) = E(\min(N_1(x), N_2(y)) \text{Cov}(X, Y)) + \mu_1 \nu_1 C(x, y).\]

Using Theorem 4.1 (i) and (4.5) we obtain that \(C(tx, ty) = 0(t)\) as \(t \to \infty\).

We are now ready to complete lemma 4.3 and lemma 4.4 and we estimate \(H(tx, ty) - t \min(\frac{x}{\mu_1}, \frac{y}{\nu_1})\) on the expectation line.

**Theorem 4.6**

If \(\rho = \rho(X_1, Y_1) \neq 1\), then as \(t \to \infty\),

\[H(t \mu_1, t \nu_1) = \frac{\mu_2}{4\mu_1^2} + \frac{\nu_2}{4\nu_1^2} - \frac{\sqrt{D}}{\sqrt{2\pi}} \sqrt{t} + o(\sqrt{t})\]

where \(D = \frac{\sigma_1^2}{\mu_1} + \frac{\sigma_2^2}{\nu_1} - 2 \frac{\rho \sigma_1 \sigma_2}{\sqrt{\mu_1 \nu_1}}\).

**Proof.**

From the central limit theorem for the vector \((N_1(x), N_2(y))\) (cf. [10, p. 551-552]) we deduce that as \(t \to \infty\)

\[\frac{N_1(t \mu_1) - N_2(t \nu_1)}{\sqrt{t}} \overset{d}{\to} Z\]

where \(Z\) has a normal distribution with mean 0 and variance \(D\).

Now \(E(N_1(t \mu_1) - N_2(t \nu_1))^2 = \text{var}(N_1(t \mu_1)) + \text{var}(N_2(t \nu_1)) + (H_1(t \mu_1) - H_2(t \nu_1))^2 - 2 \text{cov}(N_1(t \mu_1), N_2(t \nu_1))\). Using (4.4), (4.2) and lemma 4.5 we obtain that as \(t \to \infty\)

\[E(N_1(t \mu_1) - N_2(t \nu_1))^2 = 0(t).\] It follows from e.g. Feller [4, p. 252] that

\[(4.6) \quad \lim_{t \to \infty} E | \frac{N_1(t \mu_1) - N_2(t \nu_1)}{\sqrt{t}} | = E | Z | = \frac{2\sqrt{D}}{\sqrt{2\pi}} .\]

Now \(E(\min(N_1(x), N_2(y))) = \frac{E(N_1(x) + E(N_2(y))}{2} - \frac{1}{2} E | N_1(x) - N_2(y) | \) from which it follows that

\[H(t \mu_1, t \nu_1) = \frac{H_1(t \mu_1) + H_2(t \nu_1)}{2} - \frac{1}{2} E | N_1(t \mu_1) - N_2(t \nu_1) | .\]

Using (4.2) and (4.5) we obtain the desired expression.

These results generalize some results obtained by Hunter for the case where \((X_1, Y_1)\) has a double exponential distribution.
Example [Hunter 9,10]

Suppose \((X_1, Y_1)\) has a double exponential distribution defined by its Laplace transform

\[
\hat{F}_p(s, t) = \left(1 + \mu_1 s (1 + \nu_1 t) - \rho \mu_1 \nu_1 s t\right)^{-1}.
\]

It is easily seen that \(X_1\) and \(Y_1\) are exponentially distributed with means \(\mu_1\) and \(\nu_1\) respectively and that \(\text{corr}(X_1, Y_1) = \rho\). Some straightforward calculations give

\[
\hat{H}_p(s, t) = \left(\nu_1 s + \mu_1 t + (1 - \rho)\mu_1 \nu_1 s t\right)^{-1} + 1
\]

and

\[
\hat{C}(s, t) = \rho(\hat{H}_p(s, t) - 1).
\]

It follows that \(\hat{H}_p(s, t) = (1 - \rho) \hat{H}_0((1 - \rho) s, (1 - \rho) t) + \rho\) so that

\[
H_p(x, y) = (1 - \rho) H_0\left(\frac{x}{1 - \rho}, \frac{y}{1 - \rho}\right) + \rho
\]

and

\[
C(x, y) = \rho(H_p(x, y) - 1).
\]

Theorems 4.2-4.4 are applicable.
References


