Eulerian short-time statistics of turbulent flow at large Reynolds number

Citation for published version (APA):

Brouwers, J. J. H. (2004). Eulerian short-time statistics of turbulent flow at large Reynolds number. *Physics of Fluids*, 16(7), 2300-2308. <https://doi.org/10.1063/1.1737788>

DOI:

[10.1063/1.1737788](https://doi.org/10.1063/1.1737788)

Document status and date:

Published: 01/01/2004

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

Eulerian short-time statistics of turbulent flow at large Reynolds number

J. J. H. Brouwers

*Technische Universiteit Eindhoven, Faculty of Mechanical Engineering, P.O. Box 513,
5600 MB Eindhoven, The Netherlands*

(Received 21 November 2003; accepted 11 March 2004; published online 20 May 2004)

An asymptotic analysis is presented of the short-time behavior of second-order temporal velocity structure functions and Eulerian acceleration correlations in a frame that moves with the local mean velocity of the turbulent flow field. Expressions in closed-form are derived which cover the viscous and inertial subranges. They apply to general anisotropic turbulence at a large Reynolds number obeying the Kolmogorov theory. Previously published results for isotropic turbulence emerge as special cases. In the derivation use is made of the approximation of temporarily frozen turbulence proposed by Tennekes. It is shown to be valid under conditions not other than those for which the Kolmogorov hypotheses hold. The effects of intermittency appear to be marginal. © 2004 American Institute of Physics. [DOI: 10.1063/1.1737788]

I. INTRODUCTION

Eulerian time spectra of turbulent velocities and accelerations in a frame of zero-mean-flow have contracted quite some attention over the last decades.^{1–7} Work is inspired by the wish to extend understanding of the statistical structure of turbulence. A specific motivation is the relevance of these parameters for Lagrangian models of turbulence: e.g., Refs. 8 and 9.

Kolmogorov's similarity hypotheses^{10,11} are the cornerstones of many statistical approaches of turbulence. But as pointed out by Kraichnan,¹ the convection of small scales by large scales makes the implementation of the Kolmogorov hypotheses in Eulerian time spectra rather cumbersome. A step forward was made by Tennekes² introducing the Taylor hypothesis of frozen turbulence for the case of random convection velocity in a frame of zero-mean-flow. In this way he was able to formulate the scaling rules for viscous and inertial subranges of Eulerian time spectra of isotropic turbulence being significantly different from Lagrangian Kolmogorov-based scaling rules.

Central in the random sweeping hypothesis is the dominance at large Reynolds numbers of the convective accelerations being almost entirely canceled by local accelerations.^{3–7} Total or Lagrangian accelerations are comparatively small and obey Kolmogorov scaling. The dominance of the mean square values of convective and local accelerations is confirmed by DNS results of stationary isotropic turbulence.⁷

Despite the many advances made, there are questions outstanding. Theoretical analyses and numerical evaluations are almost entirely concerned with stationary isotropic turbulence. The anisotropy, inhomogeneity, or nonstationarity of the large scales of real turbulence have not been addressed. Another point of concern are the approximations adopted in the theoretical analyses. This includes the hypothesis of frozen turbulence in case of fluctuating convection velocity. Accuracy and range of applicability involved with these approximations have remained unspecified.

Aim of the present analysis is to resolve the aforementioned issues. The analysis comprises the derivation of expressions in closed-form of temporal velocity structure functions and Eulerian acceleration correlations appropriate for the viscous and inertial subrange. Results derived are valid for general forms of turbulence, stationary homogeneous isotropic turbulence forms a special case. The results follow from systematic and consistent application of asymptotic procedures based on large Reynolds number and Kolmogorov theory.

II. ACCELERATIONS IN A FRAME MOVING WITH THE MEAN FLOW

The total or Lagrangian acceleration of a fluid particle a_μ can be described in terms of local and convective accelerations as

$$a_\mu = \frac{\partial u_\mu}{\partial t} + u_i \frac{\partial u_\mu}{\partial x_i}, \quad (1)$$

where u_μ is the fluid velocity, t is time, and x_i is a space coordinate. Throughout the paper summation over repeated Latin indices is implied, not over Greek. We are concerned with incompressible, turbulent flow in which the turbulent velocity field has a mean value $u_\mu^0(\mathbf{x}, t)$ and a stochastically varying component $u'_\mu(\mathbf{x}, t)$, $u_\mu = u_\mu^0 + u'_\mu$, where $u_\mu^0 = \langle u_\mu \rangle$, angular brackets being used to denote moments or ensemble mean values. Mean $\langle a_\mu \rangle$ and fluctuating a'_μ accelerations can be described by

$$\langle a_\mu \rangle = \frac{\partial u_\mu^0}{\partial t} + u_i^0 \frac{\partial u_\mu^0}{\partial x_i} + \frac{\partial}{\partial x_i} \langle u'_i u'_\mu \rangle, \quad (2)$$

$$a'_\mu = \frac{\partial u'_\mu}{\partial t} + u_i^0 \frac{\partial u'_\mu}{\partial x_i} + u'_i \frac{\partial u_\mu^0}{\partial x_i} + u'_i \frac{\partial u'_\mu}{\partial x_i} - \frac{\partial}{\partial x_i} \langle u'_i u'_\mu \rangle. \quad (3)$$

In case of large mean flow Eulerian accelerations will be dominated by a Doppler shift due to convection of fluctuations by mean flow represented by the second term on the right-hand side (rhs) of Eq. (3). The way to eliminate this

effect for practically all cases of turbulent flow is to consider fluctuations in a frame that moves with the local mean flow velocity. This is the system within which Eulerian time spectra are usually analyzed.¹² Transforming according to $x_\mu^* = x_\mu - u_\mu^0(\mathbf{x}_0, t_0)(t - t_0)$, $t^* = t$, where $\mathbf{x}_0$ is the coordinate of some conveniently chosen fixed point and t_0 is some conveniently chosen moment in time, one has

$$\frac{\partial u'_\mu}{\partial t} + u_i^0 \frac{\partial u'_\mu}{\partial x_i} = \frac{\partial u'_\mu}{\partial t^*} + \Delta u_i^0 \frac{\partial u'_\mu}{\partial x_i^*}, \quad (4)$$

$$\Delta u_i^0 = u_i^0(\mathbf{x}^* + \mathbf{u}^0(\mathbf{x}_0, t_0)(t - t_0), t) - u_i^0(\mathbf{x}_0, t_0).$$

The objective is to assess statistical moments of flow variables at a fixed position in the moving frame and to choose this frame such that the effect of the Doppler shift is absent, i.e., $\Delta u_i^0 = 0$. In case of unidirectional stationary mean flow, that is flow where the streamlines of the mean flow field are parallel and the magnitude of the mean velocity is constant along the streamline, this is achieved as follows: Let $\mathbf{x}^* = \mathbf{x}_0^*$ be the position where one wants to evaluate statistical moments. Now take $\mathbf{x}_0$ such that at some specific time, say $t = t_0$, $\mathbf{x}_0 = \mathbf{x}_0^*$. Then in case of unidirectional flow $\Delta u_i^0 = 0$ at any time. Examples of unidirectional flow are turbulence in uniform flow behind a grid, uniform shear flow⁸ and developed turbulent flow in pipes.

For uniform flow behind a grid $\Delta u_i^0 = 0$ is obvious: u_i^0 is the same everywhere. For uniform shear flow and developed pipe flow the mean flow changes in magnitude in directions perpendicular to the mean flow. Here $\Delta u_i^0 = 0$ if one chooses for the velocity of the coordinate transformation $\mathbf{u}^0$ the value of the mean velocity along the streamline where one wants to evaluate statistical averages: i.e., $\mathbf{x}_0^* = \mathbf{x}_0$ at $t = t_0$. The coordinate transformation can then be different at different positions in a direction perpendicular to the streamlines of the mean flow. This forms no limitation for the statistical evaluation of accelerations at a fixed point $\mathbf{x}_0^*$ because for each i $\Delta u_i^0 = 0$ at such point at any time. But in the evaluation of the spatial structure functions of Secs. V and VI a spatially varying Δu_i^0 will yield a contribution. Here spatial distances are considered which extend into the viscous and inertial subrange. These distances decrease with increasing value of the Reynolds number, Re . Adopting an iterative perturbation scheme it is shown in Sec. VI that the contribution of a spatially varying Δu_i^0 is negligibly small in comparison to the leading order terms of the expansion whenever $Re \gg 1$. In this sense, the mean flow can be considered as locally homogeneous.

Another situation is that of diverging flow encountered in boundary layers, jets and wakes. Here, choosing $\mathbf{x}_0 = \mathbf{x}_0^*$ at $t = t_0$, Δu_i^0 will gradually obtain values different from zero as $t > t_0$. To the same extent the stochastic process in the moving frame is nonstationary although it is stationary in the fixed frame. But nonstationarity typically occurring on the time scales of the large scales of turbulence is weak compared to the time scales of the accelerations which are governed by turbulence at the smallest viscous scales. The latter time scales become smaller and smaller with increasing value of Re . For $Re \gg 1$, time averaging to assess statistical

moments of accelerations can be performed over relatively short time intervals such that drift due to non-stationarity and nonzero value of Δu_i^0 is negligibly small. Also in this case a transformation according to (4) setting $\Delta u_i^0 = 0$ is useful and makes sense. In general it can be concluded that in the evaluation of short-time statistics at large Reynolds numbers the mean flow can be treated as locally homogeneous and quasistationary. This conclusion will be substantiated in the last paragraph of Sec. VI where the effect of inhomogeneity and non-stationarity will be quantified in terms of the Eulerian parameters

$$\Gamma_{ij}^0 = \frac{\partial u_i^0}{\partial x_j}(\mathbf{x}_0, t_0), \quad \gamma_i^0 = \frac{\partial u_i^0}{\partial t}(\mathbf{x}_0, t_0). \quad (5)$$

Substituting Eq. (4) in Eq. (3), setting $\Delta u_i^0 = 0$ and henceforth omitting the stars we obtain for the acceleration fluctuations in the moving frame after some rearrangement of terms, the equations

$$\frac{\partial u'_\mu}{\partial t} = -u'_i \frac{\partial u'_\mu}{\partial x_i} + \delta'_\mu,$$

$$\delta'_\mu = a'_\mu + \gamma'_\mu, \quad (6)$$

$$\gamma'_\mu = \frac{\partial}{\partial x_i} (\langle u'_i u'_\mu \rangle - u'_i u_\mu^0).$$

The equations for fluctuating acceleration are in a form which makes them suitable for order of magnitude analysis of mean square values of the various terms.

III. MEAN SQUARE ACCELERATIONS

The magnitude of the various terms in Eq. (6) can be inferred from their mean square values (msv). For the msv of the convective acceleration, one can write

$$\begin{aligned} \left\langle \left(u'_i \frac{\partial u'_\mu}{\partial x_i} \right)^2 \right\rangle &= \left\langle u'_i u'_j \frac{\partial u'_\mu}{\partial x_i} \frac{\partial u'_\mu}{\partial x_j} \right\rangle \\ &= \left\langle \left\langle u'_i u'_j \frac{\partial u'_\mu}{\partial x_i} \frac{\partial u'_\mu}{\partial x_j} \right\rangle \right\rangle + \langle u'_i u'_j \rangle \left\langle \frac{\partial u'_\mu}{\partial x_i} \frac{\partial u'_\mu}{\partial x_j} \right\rangle \\ &\quad + \frac{\partial}{\partial x_i} \langle u'_i u'_\mu \rangle \frac{\partial}{\partial x_j} \langle u'_j u'_\mu \rangle + \left\langle u'_i \frac{\partial u'_\mu}{\partial x_j} \right\rangle \\ &\quad \times \left\langle u'_j \frac{\partial u'_\mu}{\partial x_i} \right\rangle, \end{aligned} \quad (7)$$

where double angular brackets denote correlation or cumulant.^{13,14} Moments of two or three random variables of zero mean are equal to their correlation or cumulant, but moments of four variables are not. More specifically, the moment of the square of the convective acceleration consists of a fourth order correlation and three combinations of second order moments or correlations. The order of magnitude of each of these correlations can be assessed by adopting the principle that in the limit of large Reynolds number small and large scales become uncorrelated. Therefore, when assessing the magnitude of each of the correlations and second-order moments on the rhs of Eq. (7) one can take for each of

the velocities and their derivatives the values characteristic for either small scales or large scales, whatever leads to the largest total value. In this way, one obtains an upper bound for the order of magnitude of each term. Given that a correlation is found to be governed by small scale turbulence, its value can be specified in further detail by implementing the results of classical Kolmogorov (K41) theory for locally isotropic turbulence.¹⁰ The effects of corrections on this theory by intermittency are discussed in a separate section at the end of this paper (Sec. VIII).

The first term on the rhs of Eq. (7) is governed by the small scales: i.e., it becomes largest by taking for each variable the magnitude which is appropriate for the small viscous scales. According to Kolmogorov (K41) scaling¹⁵ for velocity this is $\sigma \text{Re}^{-1/4}$ where σ is a typical value for root mean square fluctuating velocity and Re is the Reynolds number based on fluctuating velocity, $\text{Re} = \sigma L_0 / \nu$, L_0 being a typical value for the size of the large scales and ν is the kinematic viscosity. For the spatial derivative of fluctuating velocity, one can take $\sigma L_0^{-1} \text{Re}^{1/2}$ as characteristic value at the small scales. The net result is

$$\left\langle \left\langle u'_i u'_j \frac{\partial u'_\mu}{\partial x_i} \frac{\partial u'_\mu}{\partial x_j} \right\rangle \right\rangle \sim \frac{\sigma^4}{L_0^2} \text{Re}^{1/2}. \quad (8)$$

In connection with this result it is noted that fourth (and higher) order cumulants or correlations of Gaussian random variables are equal to zero. But turbulence is not Gaussian, in particular the small-scale components which yield more and more the dominant contribution in the earlier correlation with increasing value of the Reynolds number. Neither is the correlation zero because of isotropy or homogeneity. The rhs of Eq. (8) represents the correct order of magnitude in case of turbulence at large Reynolds number obeying K41 theory.

For the second term on the rhs of Eq. (7) the situation is rather different. While the first moment is governed by the large scales, the second is dominated by the small scales, so that

$$\langle u'_i u'_j \rangle \left\langle \frac{\partial u'_\mu}{\partial x_i} \frac{\partial u'_\mu}{\partial x_j} \right\rangle \sim \frac{\sigma^4}{L_0^2} \text{Re}. \quad (9)$$

The two moments of the third term on the rhs of Eq. (7) are governed by the large scales while at first sight the two moments of the fourth term by the small scales. But small scale turbulence is homogeneous and its reflectional symmetry causes these moments to be zero:¹² also the fourth term is governed by large scale turbulence. Hence

$$\begin{aligned} \frac{\partial}{\partial x_i} \langle u'_i u'_\mu \rangle \frac{\partial}{\partial x_j} \langle u'_j u'_\mu \rangle &\sim \frac{\sigma^4}{L_0^2}, \\ \left\langle u'_i \frac{\partial u'_\mu}{\partial x_j} \right\rangle \left\langle u'_j \frac{\partial u'_\mu}{\partial x_i} \right\rangle &\sim \frac{\sigma^4}{L_0^2}. \end{aligned} \quad (10)$$

The conclusion is that in the limit of large Reynolds numbers, the msv of the convective acceleration becomes equal to the second term on the rhs of Eq. (7). With a relative error of $O(\text{Re}^{-1/2})$

$$\left\langle \left(u'_j \frac{\partial u'_\mu}{\partial x_j} \right)^2 \right\rangle = \langle u'_i u'_j \rangle \left\langle \frac{\partial u'_\mu}{\partial x_i} \frac{\partial u'_\mu}{\partial x_j} \right\rangle. \quad (11)$$

In a similar way one can show

$$\langle \gamma_v'^2 \rangle \sim \frac{\sigma^4}{L_0^2}. \quad (12)$$

According to K41 theory one has for the msv of fluid particle acceleration

$$\langle a_v'^2 \rangle \sim \frac{\sigma^4}{L_0^2} \text{Re}^{1/2}, \quad (13)$$

where it is noted that recent work^{16,17} suggests a somewhat higher power law dependency on Re for the variance of fluid particle acceleration, i.e., $\text{Re}^{5/8}$ and $\text{Re}^{3/4}$; recent experimental work at very high Reynolds numbers^{18,19} on the other hand, seems to be more in line with the earlier K41 scaling; effects of intermittency are discussed in Sec. VIII. Conclusion: in the limit of large Reynolds numbers, δ'_v can be represented by the total acceleration a'_v , while the local and convective accelerations dominate over the total or Lagrangian acceleration and cancel each other

$$\frac{\partial u'_\mu}{\partial t} = -u'_i \frac{\partial u'_\mu}{\partial x_i}. \quad (14)$$

With a relative error of $O(\text{Re}^{-1/2})$, the msv of the local acceleration is equal to the rhs of Eq. (11). Invoking the properties of isotropy and homogeneity of small scale turbulence^{15,20} to evaluate the rhs of Eq. (11), one obtains for the msv of Eulerian fluid acceleration in a frame that moves with the mean flow

$$\left\langle \left(\frac{\partial u'_\mu}{\partial t} \right)^2 \right\rangle = \frac{\langle \epsilon \rangle \langle u'^2 \rangle \alpha_\mu}{3\nu}, \quad (15)$$

where $\langle \epsilon \rangle$ is the mean energy dissipation rate and

$$\alpha_\mu = \frac{2\langle u_i'^2 \rangle - \langle u_\mu'^2 \rangle}{5\langle u_\mu'^2 \rangle} \quad (16)$$

anisotropy factor which is unity for isotropic (large-scale) turbulence.

Result (15) applies to general forms of turbulence at large Reynolds numbers. All that was necessary in the earlier derivation were parts of the concepts which underly the Kolmogorov theory: viz., application of the scaling rules appropriate for large and small scale turbulence, the proposition of decorrelation between large and small scales, and implementation of the values of second order moments of spatial velocity derivatives according to locally isotropic turbulence. Important was also the notion that fourth-order moments of zero-mean variables consist of fourth-order correlations and combinations of second-order moments. One of these combinations becomes the dominant term in case of large Reynolds numbers: i.e., the combination of velocity derivatives $(\partial/\partial x_i)u'_\mu$ dominated by small scales with convection velocities u'_i dominated by large scales.

For isotropic turbulence $\alpha_\mu = 1$ in which case Eq. (15) reduces to the well-known result of Tennekes.² Tennekes ob-

tained his result by relying on hypotheses relating to dominance of convective over Lagrangian accelerations and statistical independence of large and small scale turbulence. The present analysis validates this approach on the basis of asymptotic analysis which solely relies on the limit procedures of K41 theory. At the same time results are extended to general anisotropic turbulence.

IV. EULERIAN TIME MICROSCALES

Eulerian time microscales $t_{E\mu}$ can be defined as^{2,12,21}

$$\left\langle \left(\frac{\partial u'_\mu}{\partial t} \right)^2 \right\rangle = \frac{2\langle u'^2_\mu \rangle}{t_{E\mu}^2}. \quad (17)$$

Substitution of Eq. (15) into Eq. (17) yields

$$t_{E\mu} = \left\{ \frac{6\nu}{\langle \epsilon \rangle \alpha_\mu} \right\}^{1/2}. \quad (18)$$

For isotropic turbulence α_μ is unity in which case (18) reduces to the result of Tennekes.² For general anisotropic turbulence the Eulerian time microscales will be different in the three directions in the manner described by Eqs. (16) and (18).

V. STATISTICAL CHARACTERISTICS OF TEMPORAL VELOCITY INCREMENTS

Up to now attention was focused on Eulerian-based statistical averages assessed at a single time. In the present section expressions will be derived for statistical averages involving different times, viz. temporal structure functions defined as

$$D_{\mu\mu}^T(\tau) = \langle [u'_\mu(\mathbf{x}_0, t_0 + \tau) - u'_\mu(\mathbf{x}_0, t_0)]^2 \rangle, \quad (19)$$

where $\mathbf{x}_0$ is a fixed position in a frame that moves with the mean velocity and $\tau = t - t_0$. For large Reynolds numbers local and convective accelerations dominate the dynamic process: cf. Eq. (14). It leads to a state of temporarily frozen turbulence: the velocity at position $\mathbf{x}_0$ and time t , is the same as the velocity at position $\mathbf{x}_0 - \mathbf{u}'_0\tau$ and time t_0 , where $\mathbf{u}'_0 = \mathbf{u}'(\mathbf{x}_0, t_0)$. In the next section it will be shown that this proposition also known as the sweeping hypothesis is valid for times extending over the range: $0 \leq \tau \leq t_c \text{Re}^{-1/2}$ where t_c is a typical value of correlation time of Eulerian velocities in the moving frame; $t_c \sim L_0/\sigma$. Invoking the approximation of frozen turbulence, the rhs of Eq. (19) can be transformed into a spatial velocity difference as

$$\begin{aligned} & \langle [u'_\mu(\mathbf{x}_0, t) - u'_\mu(\mathbf{x}_0, t_0)]^2 \rangle \\ &= \langle [u'_\mu(\mathbf{x}_0 - \mathbf{u}'_0\tau, t_0) - u'_\mu(\mathbf{x}_0, t_0)]^2 \rangle. \end{aligned} \quad (20)$$

While for fixed $\mathbf{u}'_0$, the velocity difference is governed by small scales, $\mathbf{u}'_0$ is governed by large scales. In accordance with the assumptions underlying the Kolmogorov hypotheses, in the limit of large Reynolds numbers small and large scales become independent and this property can be used to split up the averaging process.³ First, averaging is

performed for fixed $\mathbf{u}'_0$; this is denoted by an overbar. Subsequently, averaging over all possible values of $\mathbf{u}'_0$ is executed. Accordingly

$$\begin{aligned} & \langle [u'_\mu(\mathbf{x}_0 - \mathbf{u}'_0\tau, t_0) - u'_\mu(\mathbf{x}_0, t_0)]^2 \rangle \\ &= \int d\mathbf{u}'_0 p(\mathbf{u}'_0) \overline{[u'_\mu(\mathbf{x}_0 - \mathbf{u}'_0\tau, t_0) - u'_\mu(\mathbf{x}_0, t_0)]^2}, \end{aligned} \quad (21)$$

where $p(\mathbf{u}'_0)$ is the probability density of $\mathbf{u}'_0$. Now

$$\overline{[u'_\mu(\mathbf{x} - \mathbf{u}'_0\tau, t_0) - u'_\mu(\mathbf{x}, t_0)]^2} = D_{\mu\mu}(-\mathbf{u}'_0\tau), \quad (22)$$

where $D_{\mu\mu}(-\mathbf{u}'_0\tau)$ is the spatial structure function $D_{\mu\mu}(\mathbf{r})$ ¹⁵ evaluated at $\mathbf{r} = -\mathbf{u}'_0\tau$. The net result is

$$D_{\mu\mu}^T(\tau) = \int d\mathbf{u}'_0 p(\mathbf{u}'_0) D_{\mu\mu}(-\mathbf{u}'_0\tau). \quad (23)$$

Assuming that $\mathbf{u}'(\mathbf{x}_0, t)$ is stationary, Eq. (23) can be transformed into a description of the power density of $\mathbf{u}'(\mathbf{x}_0, t)$. The thus obtained description is identical to that derived by Chen and Kraichnan³ for stationary, homogeneous isotropic turbulence. Result (23), however, is not restricted to stationary, homogeneous isotropic turbulence. It is valid for any turbulence at large Reynolds numbers ($\text{Re} \gg 1$) for which K41 theory applies.

Explicit expressions for the temporal structure function can be obtained by implementing in Eq. (23) the asymptotic results known for the spatial structure function.¹⁵ In the viscous subrange, $0 \leq |\mathbf{u}'_0|\tau \leq \eta$, where η is the Kolmogorov length, $\eta \sim L_0 \text{Re}^{-3/4}$, that is, for $0 \leq \tau \leq t_c \text{Re}^{-3/4}$, one then obtains

$$D_{\mu\mu}^T(\tau) = \frac{\langle \epsilon \rangle \tau^2 \langle u'^2_{\mu 0} \rangle \alpha_{\mu 0}}{3\nu}. \quad (24)$$

In the inertial subrange, that is for $t_c \text{Re}^{-3/4} \leq \tau \leq t_c \text{Re}^{-1/2}$, where the upper bound follows from the range of validity of temporarily frozen turbulence (Sec. VI):

$$D_{\mu\mu}^T(\tau) = \frac{55 \times 2^{1/3} \cdot \Gamma(5/6)}{27\pi^{1/2}} C \beta_{\mu 0} (\tau \langle \epsilon \rangle \sqrt{\langle u'^2_{\mu 0} \rangle})^{2/3}, \quad (25)$$

where C is the Kolmogorov constant,¹⁵ Γ is the gamma function, and $\beta_{\mu 0}$ is the anisotropy factor

$$\beta_{\mu 0} = \frac{4\langle |u'_{i0}|^{2/3} \rangle - \left\langle \frac{u'^2_{\mu 0}}{|u'_{i0}|^{4/3}} \right\rangle}{\frac{55 \times 2^{1/3} \cdot \Gamma(5/6)}{9\pi^{1/2}} \langle u'^2_{\mu 0} \rangle^{1/3}}, \quad (26)$$

$\beta_{\mu 0}$ is equal to unity in case of isotropic Gaussian turbulence.

It is now interesting to compare the above results with the Lagrangian structure function $D_{\mu\mu}^L(\tau)$ based on the differential velocity of a particle at time t with its velocity at position $\mathbf{x}_0$ at time t_0 .¹⁵ In the viscous subrange, $0 \leq \tau \leq t_\eta$, where $t_\eta = t_c \text{Re}^{-1/2}$ is Kolmogorov time¹⁵

FIG. 1. Schematic plot of the derivative of the temporal and Lagrangian structure function (not to scale).

$$D_{\mu\mu}^L(\tau) = \frac{a_0 \langle \epsilon_0 \rangle^{3/2}}{\nu^{1/2}} \tau^2, \quad (27)$$

where a_0 is a constant. In the inertial subrange, $t_\eta \leq \tau \leq t_c$, where it is noted that the typical value of the correlation time of Lagrangian velocities is of the same order of magnitude as that of Eulerian velocities in a frame moving with mean fluid velocity⁹

$$D_{\mu\mu}^L(\tau) = C_0 \langle \epsilon_0 \rangle \tau, \quad (28)$$

where C_0 is a universal constant.¹⁵

Comparing Eq. (24) with Eq. (27) it is seen that the msf of the temporal velocity difference grows much faster than that of the Lagrangian velocity difference. The ratio is proportional to $Re^{1/2}$. The time duration of the faster growth is much less, viscous behavior is limited to $\tau \sim t_c Re^{-3/4}$ for temporal difference as compared to $\tau \sim t_c Re^{-1/2}$ in the Lagrangian case. In the inertial subrange, temporal velocity difference is still much larger. In Fig. 1 an illustrative plot has been given of the derivative of the temporal and Lagrangian msf of velocity difference.

VI. TEMPORARILY FROZEN TURBULENCE

In the derivation of expressions for temporal structure functions of Sec. V use has been made of the approximation of temporarily frozen turbulence. Aim of the analysis of this section is to assess the accuracy of this approximation and its range of applicability. For this purpose, a new coordinate system is introduced which moves with the instantaneous fluctuating velocity at given position $\mathbf{x}_0$, $u'_{\mu 0} = u'_\mu(\mathbf{x}_0, t_0)$. The moving coordinate system is in addition to that introduced in Sec. II which was concerned with mean velocity only

$$x_\mu^* = x_\mu - u'_{\mu 0}(t - t_0), \quad t^* = t - t_0, \quad (29)$$

where x_μ are coordinates in the frame moving with mean velocity u_i^0 . Differential velocities are defined as

$$\Delta u'_\mu = u'_\mu(\mathbf{x}^*, t^*) - u'_{\mu 0}, \quad (30)$$

where $\mathbf{x}^*(t^*)$ and $t(t^*)$ follow from Eq. (29). In terms of the differential velocities in the new frame, Eq. (14) becomes

$$\frac{\partial \Delta u'_\mu}{\partial t^*} = -\Delta u'_i \frac{\partial \Delta u'_\mu}{\partial x_i^*}, \quad (31)$$

where the term on the rhs describes the effect of the change of the transport velocity $u'_i(\mathbf{x}, t)$ in Eq. (14) for $t \neq t_0$ and $|\mathbf{x} - \mathbf{x}_0| \neq 0$. For times and positions close to t_0 and $\mathbf{x}_0$, $\Delta u'_\mu$ is small. The rhs of Eq. (31) is therefore small and as a first approximation $(\partial/\partial t^*)\Delta u'_\mu = 0$: $\Delta u'_\mu$ will behave as being constant with respect to t^* ; it will only be a function of $\mathbf{x}^*$: $\Delta u'_\mu = \Delta u'_\mu(\mathbf{x}^*)$. In the frame moving with the mean velocity, $\Delta u'_\mu$ will then only be a function of $\mathbf{x} - \mathbf{u}'_0 \tau$, $\tau = t - t_0$. In particular, to first order the differential velocity at position $\mathbf{x}_0$ and time t is the same as the differential velocity at position $\mathbf{x}_0 - \mathbf{u}'_0 \tau$ and time t_0 : $\Delta u'_\mu(\mathbf{x}_0, t) = \Delta u'_\mu(\mathbf{x}_0 - \mathbf{u}'_0 \tau, t_0)$. This approximation of frozen turbulence was used to derive Eq. (20). Upon implementing two-stage averaging, cf. Eq. (21), it resulted in analytical expressions for temporal structure functions: Eqs. (23)–(25).

But the approximation of frozen turbulence will only hold for short times. To find how long we incorporate the effect of the so far neglected rhs of Eq. (31) by applying an iterative perturbation scheme. Therefore, substitute the first order solution of $\Delta u'_\mu(\mathbf{x}^*, t^*)$ into the rhs of (31). Noting that this solution is independent on t^* the rhs of (31) can be taken constant with respect to t^* when performing integration in time. Returning to original coordinates $(\mathbf{x}, t)$ one obtains the second order solution, $\Delta u'_\mu(\mathbf{x}_0, t)_{II}$. This is the correction to be added to the first order solution, i.e.:

$$\Delta u'_\mu(\mathbf{x}_0, t) = \Delta u'_\mu(\mathbf{x}_0 - \mathbf{u}'_0 \tau, t_0) + \Delta u'_\mu(\mathbf{x}_0, t)_{II}. \quad (32)$$

The expression found is

$$\Delta u'_\mu(\mathbf{x}_0, t)_{\text{II}} = \tau [u'_i(\mathbf{x}_0 - \mathbf{u}'_0 \tau, t_0) - u'_i(\mathbf{x}_0, t_0)] \times \frac{\partial u'_\mu(\mathbf{x}_0 - \mathbf{u}'_0 \tau, t_0)}{\partial x_i}, \quad (33)$$

which yields for the msv

$$\langle \Delta u'_\mu(\mathbf{x}_0, t)_{\text{II}}^2 \rangle = \tau^2 \left\langle \Delta u'_i(\mathbf{x}_0 - \mathbf{u}'_0 \tau, t_0) \Delta u'_j(\mathbf{x}_0 - \mathbf{u}'_0 \tau, t_0) \times \frac{\partial u'_\mu(\mathbf{x}_0 - \mathbf{u}'_0 \tau, t_0)}{\partial x_i} \frac{\partial u'_\mu(\mathbf{x}_0 - \mathbf{u}'_0 \tau, t_0)}{\partial x_j} \right\rangle. \quad (34)$$

A contribution of leading order to the rhs of Eq. (34) comes from the submoments $\langle \Delta u'_i(\mathbf{x}_0 - \mathbf{u}'_0 \tau, t_0)^2 \rangle$ and $\langle [(\partial/\partial x_i)u'_\mu(\mathbf{x}_0 - \mathbf{u}'_0 \tau, t_0)]^2 \rangle$. The first of these equals the second-order temporal structure function given by Eq. (23), while the second one can be represented by

$$\langle [(\partial/\partial x_i)u'_\mu(\mathbf{x}_0 - \mathbf{u}'_0 \tau, t_0)]^2 \rangle \sim \langle [(\partial/\partial x_i)u'_\mu(\mathbf{x}_0, t_0)]^2 \rangle \sim t_c^{-2} \text{Re}. \quad (35)$$

The net results is that

$$\langle \Delta u'_\mu(\mathbf{x}_0, t)_{\text{II}}^2 \rangle \sim D_{\mu\mu}^T(\tau) \left(\frac{\tau}{t_c} \right)^2 \text{Re}. \quad (36)$$

Comparing this result with the first order solution, cf. Eq. (33), it is seen that in the range $0 \leq \tau \leq t_c \text{Re}^{-1/2}$ the correction on the first order description is small. This range is sufficient to describe the behavior of the temporal structure function by Eq. (23) in the viscous range $\tau \sim t_c \text{Re}^{-3/4}$ and the inertial subrange $t_c \text{Re}^{-3/4} \leq \tau$. The effect of a changing convective transport velocity in Eq. (14) for $t > t_0$ remains thus small for times extending into the range of inertial subrange behavior. But it does not cover the range of times of viscous and inertial subrange behavior of Lagrangian structure functions: i.e., $\tau \sim t_c \text{Re}^{-1/2}$, $t_c \text{Re}^{-1/2} \leq \tau \leq t_c$: see also Fig. 1. Here the effect of a changing transport velocity is so large that the concept of temporarily frozen turbulence no longer holds and solution (23) becomes inaccurate.

Equation (14) which formed the starting point of the earlier analysis is a simplified representation of Eq. (6). The largest term neglected in (14) is the total or Lagrangian acceleration. It can simply be added to the rhs of (31) in order to quantify its contribution to the temporal velocity difference. In a frame that moves with the instantaneous velocity at $\mathbf{x}_0$ and t_0 , the Lagrangian acceleration corresponds to the time derivative of the velocity of a moving marked fluid particle. In other words, the correction on the first order solution by the Lagrangian acceleration term will be the Lagrangian velocity difference. As the time scale of the Lagrangian fluctuations is much slower than that of the first order temporal fluctuations, the two contributions can be assumed to be uncorrelated. The net result is that a description which incorporates the contribution of the Lagrangian acceleration is simply obtained by adding to the msv of the first order solution the Lagrangian structure function

$$D_{\mu\mu}^T(\tau) = \langle [\Delta u'_\mu(\mathbf{x}_0 - \mathbf{u}'_0 \tau, t_0)]^2 \rangle + D_{\mu\mu}^L(\tau). \quad (37)$$

The correction by the Lagrangian term is small over the entire range $0 \leq \tau \leq t_c \text{Re}^{-1/2}$ for which the correction due to changing transport velocity is small. Substituting for the msv of the first order solution the rhs of Eq. (23), viz. the rhs of Eqs. (24) and (25), one finds that for the two following time domains (see also Fig. 1), $0 \leq \tau \leq t_c \text{Re}^{-3/4}$ and $t_c \text{Re}^{-3/4} \leq \tau \leq t_c \text{Re}^{-1/2}$ the relative contribution of the Lagrangian term is $\sim \text{Re}^{-1/2}$ and $\sim (\tau/t_c)^{4/3} \text{Re}^{1/2}$, respectively. For $0 \leq \tau \leq t_c \text{Re}^{-1/2}$ the contribution is small as long as $\text{Re} \gg 1$.

Also the effect of a spatially and temporarily varying mean flow can be shown to be small. Adding to the rhs of Eq. (14) the term appropriate for inhomogeneous instationary mean flow which follows from Eq. (4), one obtains in terms of the moving coordinate system $(\mathbf{x}^*, t^*)$ of Eq. (29)

$$\frac{\partial \Delta u'_\mu}{\partial t^*} = \Delta u_i^0 \frac{\partial \Delta u'_\mu}{\partial x_i^*}, \quad (38)$$

where Δu_i^0 is the difference of the mean velocity with respect to the mean velocity at $\mathbf{x} = \mathbf{x}_0$ and $t = t_0$. To assess the effect of Δu_i^0 we again adopt an iterative perturbation scheme. As first order approximation we neglect the rhs of Eq. (38) yielding the solution corresponding to temporarily frozen turbulence: $\Delta u'_\mu = \Delta u'_\mu(\mathbf{x}^*)$. This solution is subsequently substituted into the rhs of Eq. (38) to arrive at an equation for the correction term $\Delta u'_\mu(\mathbf{x}_0, t)_{\text{III}}$. As we are concerned with small spatial distances from the point $\mathbf{x}_0$ and small time differences relative to $t = t_0$, Δu_i^0 can be approximated by the first term of a Taylor series expansion with respect to $\mathbf{x} - \mathbf{x}_0$ and $t - t_0$:

$$\Delta u_i^0 \approx [x_j - x_{j0} + u'_{j0}(t - t_0)] \Gamma_{ij}^0 + (t - t_0) \gamma_i^0, \quad (39)$$

where Γ_{ij}^0 and γ_i^0 defined by Eq. (5) are the spatial and temporal derivatives of the mean flow in the fixed frame, and where it is noted that in accordance with the convention introduced in the last paragraph of Sec. II, $\mathbf{x}$ is position in the coordinate system moving with the mean velocity. In correspondence with the solution of temporarily frozen turbulence we consider a point moving with the local fluctuating velocity u'_{j0} which is at position $\mathbf{x}_0$ at time t_0 so that Eq. (39) becomes in this moving frame

$$\Delta u_i^0 \approx (u_{j0}^0 + u'_{j0}) t^* \Gamma_{ij}^0 + t^* \gamma_i^0. \quad (40)$$

The equation for the correction then becomes

$$\frac{\partial \Delta u'_{\mu\text{III}}}{\partial t^*} = [(u_{j0}^0 + u'_{j0}) \Gamma_{ij}^0 + \gamma_i^0] t^* \frac{\partial \Delta u'_\mu(\mathbf{x}^*)}{\partial x_i^*}, \quad (41)$$

which upon integration and transforming back to the frame moving with the mean velocity yields

$$\Delta u'_{\mu\text{III}} = \frac{1}{2} [(u_{j0}^0 + u'_{j0}) \Gamma_{ij}^0 + \gamma_i^0] \tau^2 \frac{\partial u'_\mu(\mathbf{x}_0 - \mathbf{u}'_0 \tau, t_0)}{\partial x_i}. \quad (42)$$

In directions perpendicular to the mean flow $u_{j0}^0 = 0$ and one can take $u'_{j0} \Gamma_{ij}^0 \sim \sigma t_c^{-1}$. In case of strong mean flow, u_{j0}^0 can be relatively large; but also in directions of strong mean flow we assume $(u_{j0}^0 + u'_{j0}) \Gamma_{ij}^0 \sim \sigma t_c^{-1}$ because in cases of

FIG. 2. Illustrative plot of power density spectra of temporal and Lagrangian accelerations (not to scale).

main importance, in particular in diverging turbulent flows, the change of the flow in the direction of the mean flow is generally small; an exception is formed by rapidly contracting flow. Furthermore, if the mean flow is nonstationary it is assumed to vary in time such that γ_i^0 is not larger in order of magnitude than σt_c^{-1} . The other terms in Eq. (42) can be estimated according to the previously introduced scaling rules: e.g., Eq. (35). The msv of the rhs of Eq. (42) is then estimated to be

$$\langle \Delta u_{\mu\text{III}}'^2 \rangle \sim (\tau/t_c)^4 \sigma^2 \text{Re}. \quad (43)$$

Compared to the leading order term this becomes vanishingly small for $0 \leq \tau \leq t_c \text{Re}^{-1/2}$ and $\text{Re} \gg 1$. To leading order the effect of nonuniform nonstationary mean flow can thus be disregarded. In general it can be concluded that application of the sweeping hypothesis in the evaluation of the temporal structure function in the viscous and inertial subrange is valid under the limit conditions for which K41 theory holds. Usually the sweeping hypothesis is applied to homogeneous isotropic turbulence. But as the present analysis shows, with specified error it can be applied to practically any turbulence at large Reynolds numbers.

VII. TEMPORAL ACCELERATION CORRELATIONS AND SPECTRA

The acceleration process being governed by small-scale turbulence is stationary; the velocity process is one of stationary increments. Denoting temporal acceleration by $(\partial/\partial t)u'_\mu(\mathbf{x}_0, t) = \dot{u}'_\mu(\mathbf{x}_0, t)$, the autocorrelation of temporal acceleration can be shown to be related to the temporal velocity structure function as¹⁵

$$\langle \dot{u}'_\mu(\mathbf{x}_0, t_0 + \tau) \dot{u}'_\mu(\mathbf{x}_0, t_0) \rangle = \frac{1}{2} \frac{\partial^2 D_{\mu\mu}^T(\tau)}{\partial \tau^2}. \quad (44)$$

The relation enables application of the previously derived expressions for temporal structure functions to construct the temporal acceleration correlation. For $\tau \rightarrow 0$ and implementing Eq. (24) the autocorrelation according to Eq. (44) becomes equal to Eq. (15), as should be. As follows from Eq. (25), the autocorrelation will decay to zero according to $\sim (\sigma^2/t_c^2)(\tau/t_c)^{-4/3}$ for times $\tau \gg t_c \text{Re}^{-3/4}$. The typical correlation time of temporal accelerations $\sim t_c \text{Re}^{-3/4}$, of Lagrangian accelerations $\sim t_c \text{Re}^{-1/2}$. While the correlation time of Lagrangian accelerations is much larger than that of temporal accelerations (in a frame moving with the mean fluid velocity), its value at $\tau=0$ is a factor $\text{Re}^{-1/2}$ smaller.

As the temporal acceleration is a stationary process, a power density spectrum exists being the Fourier transform of the autocorrelation function. In Fig. 2 illustrative plots are presented of the power spectra of temporal and Lagrangian accelerations.

VIII. INTERMITTENCY

So far no attention has been paid to the effects of strongly fluctuating values of the local energy dissipation referred to as intermittency and addressed in Kolmogorov's refined hypotheses.¹¹ In the earlier analysis statistical moments of low order were considered and these are known to be marginally affected by intermittency²² or by other violations of Kolmogorov's local isotropy hypothesis reported in recent literature.^{23–25} A more precise quantification of the effects of intermittency is given later. It begins with consideration of the acceleration statistics of Sec. III.

Using among others multifractal representations of intermittency a correction on the msv of fluid particle acceleration according to K41 theory has been calculated.^{26,27} It involves an increase of the exponent in the Reynolds number dependency of 0.067 in Eq. (13). The correction on K41

theory is thus small; experimental results obtained for msv of particle acceleration until now do not favor any of the two representations.²⁸ Corrections on the fourth order correlation of the square of the convective acceleration, cf. Eq. (8), have not been considered so far. But as the dependency on local energy dissipation is the same as that for fluid particle acceleration, corrections are expected to be small as well. Corrections on Eq. (9) or Eq. (15) are absent as these terms involve linear dependency on local dissipation energy. Hence, the effects of intermittency on mean square acceleration appear only in lower order terms of the acceleration equation. The balance between msv of local and convective acceleration remains the dominant one in the limit of large Reynolds number and this balance is unaffected by intermittency. Results (15)–(18) remain the same.

Corrections on the second-order structure function in the inertial space range, if existing at all, are very small.²² They could amount to a value of 0.03 to be added to the value of $2/3$ and $-1/3$ in the power law dependency of temporal structure function and power spectrum of temporal acceleration: cf. Eq. (25) and Fig. 2. Errors associated with the approximation of temporarily frozen turbulence discussed in Sec. VI involve moments of rather low order of the dissipation energy. Corrections due to intermittency are therefore expected to be small such that the primary balance which underlies the approximation of frozen turbulence, cf. Eq. (14), remains intact in the limit of large Reynolds number.

In general it can be concluded that corrections on the presented results by intermittency, if occurring at all, are marginal. Clearly, this is no longer the case if higher order temporal structure functions were considered. Presented methods enable to assess these and the implications of the refined Kolmogorov hypotheses should then be taken into account.

IX. CONCLUSIONS

Eulerian short-time statistics of turbulent flow have been analyzed in a frame that moves with the local mean velocity. Results derived are the leading terms of asymptotic expansions based on large Reynolds number and classical Kolmogorov theory. Results are valid for general forms of turbulence. Previously published results for homogeneous isotropic stationary turbulence emerge as special cases.

Analogous to the situation encountered in isotropic turbulence, temporal and convective accelerations nearly cancel each other resulting in relatively small Lagrangian accelerations whenever $Re \gg 1$. Expressions obtained for mean square values of Eulerian accelerations and Eulerian time microscales differ from those for isotropic turbulence by the factor α_μ representing the effect of anisotropy: cf. Eqs. (15), (16), and (18).

The concept of temporarily frozen turbulence has been used to derive expressions for second-order temporal velocity structure functions. Again the results differ from those for isotropic turbulence by factors of anisotropy: i.e., the factor α_μ in the viscous subrange, cf. Eq. (24), and the factor β_μ in the inertial subrange, cf. Eqs. (25) and (26). The slope of the temporal structure function is fundamentally different from

that of the Lagrangian structure function for times extending into the inertial subrange and $Re \gg 1$: see Fig. 1. Similarly the power density spectrum of Eulerian accelerations differs fundamentally from that of Lagrangian accelerations for large frequencies appropriate for the inertial and viscous subrange and $Re \gg 1$: see Fig. 2.

The accuracy of the concept of temporarily frozen turbulence has been investigated by an iterative perturbation scheme. In this way it has been found that the error involved with this approximation increases with time; but it remains negligibly small for times $0 \leq \tau \leq t_c Re^{-1/2}$ where t_c is correlation time of large eddies. This range is sufficiently large to cover viscous and inertial subrange behavior. Also deviations caused by inhomogeneity and nonstationarity of the mean flow have been quantified and shown to be small. Corrections resulting from Kolmogorov's refined hypotheses involving intermittency have been specified and if present at all, have been shown to be marginal.

ACKNOWLEDGMENTS

The author wishes to thank J. G. M. Kuerten and C. W. M. van der Geld for discussions on the subject matter.

- ¹R. H. Kraichnan, "Kolmogorov's hypotheses and Eulerian turbulence theory," *Phys. Fluids* **7**, 1723 (1964).
- ²H. Tennekes, "Eulerian and Lagrangian time microscales in isotropic turbulence," *J. Fluid Mech.* **67**, 561 (1975).
- ³S. Chen and R. H. Kraichnan, "Sweeping decorrelation in isotropic turbulence," *Phys. Fluids A* **1**, 2019 (1989).
- ⁴M. Nelkin and M. Tabor, "Time correlation and random sweeping in isotropic turbulence," *Phys. Fluids A* **2**, 81 (1990).
- ⁵T. Sanada and M. Shanmugasundaram, "Random sweeping effect in isotropic numerical turbulence," *Phys. Fluids A* **4**, 1245 (1992).
- ⁶S. Kaneda, "Lagrangian and Eulerian time correlations in turbulence," *Phys. Fluids A* **5**, 2835 (1993).
- ⁷A. Tsinober, P. Vedula, and P. K. Yeung, "Random Taylor hypothesis and the behavior of local and convective accelerations in isotropic turbulence," *Phys. Fluids* **13**, 1974 (2001).
- ⁸B. L. Sawford and P. K. Yeung, "Eulerian acceleration statistics as a discriminator between Lagrangian stochastic models in uniform shear flow," *Phys. Fluids* **12**, 2033 (2000).
- ⁹J. J. H. Brouwers, "On diffusion theory of stochastic inhomogeneous turbulence," *J. Eng. Math.* **44**, 277 (2002).
- ¹⁰A. N. Kolmogorov, "The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers," *Dokl. Akad. Nauk SSSR* **30**, 299 (1941).
- ¹¹A. N. Kolmogorov, "A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds numbers," *J. Fluid Mech.* **13**, 82 (1962).
- ¹²H. Tennekes and J. L. Lumley, *A First Course in Turbulence* (MIT Press, Cambridge, MA, 1994).
- ¹³N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, revised and enlarged ed. (Elsevier, Amsterdam, 1992).
- ¹⁴R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1967), Vol. 1.
- ¹⁵A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1975), Vol. 2.
- ¹⁶R. J. Hill, "Scaling of acceleration in locally isotropic turbulence," *J. Fluid Mech.* **452**, 361 (2002).
- ¹⁷P. Vedula and P. K. Yeung, "Similarity scaling of acceleration and pressure statistics in numerical simulations of isotropic turbulence," *Phys. Fluids* **11**, 1208 (1999).
- ¹⁸G. A. Voth, K. Sathyanarayanan, and E. Bodenschatz, "Lagrangian acceleration measurements at large Reynolds numbers," *Phys. Fluids* **10**, 2268 (1998).
- ¹⁹G. A. Voth, A. M. Crawford, J. Alexander, J. La Porta, and E. Boden-

- schatz, "Fluid particle acceleration in fully developed turbulence," *Phys. Fluids* **10**, 2268 (1998).
- ²⁰A. N. Kolmogorov, "Decay of isotropic turbulence in incompressible viscous fluids," *Dokl. Akad. Nauk SSSR* **31**, 538 (1941).
- ²¹G. Comte-Bellot and S. Corrsin, "Simple Eulerian-time correlation of full- and narrow-band velocity signals in grid generated isotropic turbulence," *J. Fluid Mech.* **48**, 273 (1971).
- ²²M. Nelkin, "Universality and scaling in fully developed turbulence," *Adv. Phys.* **43**, 143 (1994).
- ²³X. Shen and Z. Warhaft, "The anisotropy of the small scale structure in high Reynolds number ($R_\lambda \sim 1000$) turbulent shear flow," *Phys. Fluids* **12**, 2976 (2000).
- ²⁴A. Tsinober, *An Informal Introduction to Turbulence* (Kluwer Academic, Dordrecht, 2001).
- ²⁵W. van de Water and J. A. Herweijer, "Higher-order structure functions of turbulence," *J. Fluid Mech.* **387**, 3 (1999).
- ²⁶M. S. Borgas, "The multifractal Lagrangian nature of turbulence," *Philos. Trans. R. Soc. London Ser. A* **342**, 379 (1993).
- ²⁷M. Nelkin, "Multifractal scaling of velocity derivatives in turbulence," *Phys. Rev. A* **42**, 7226 (1990).
- ²⁸B. L. Sawford, P. K. Yeung, M. S. Borgas, P. Vedula, A. La Porta, A. M. Crawford, and E. Bodenschatz, "Conditional and unconditional acceleration statistics in turbulence," *Phys. Fluids* **15**, 3478 (2003).