Solution to Problem 73-8: A polynomial diophantine equation

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providing details and we will investigate your claim.
Here $\varepsilon = +1$ with $\delta = \pm 1$ and $\varepsilon = -1$ with $\delta = -1$ covers all possibilities for

\[
\begin{align*}
t_1 &= \frac{n}{4} - \frac{4 + \varepsilon + \delta}{2}, \\
t_2 &= \frac{n}{4} - \frac{1 - \delta}{2}, \\
t_3 &= \frac{n}{4} - \frac{1 - \varepsilon}{2}, \\
t_4 &= \frac{n}{4}.
\end{align*}
\]

$M$ is the matrix of a 'third' minor. The evaluation of

\[
\text{det}(M^T \cdot M) = \det M_v = \begin{bmatrix} n - 3 & -3 \\ -3 & n - 3 \end{bmatrix},
\]

of order $t_v$, gives

- $\det M = 0$, for $\varepsilon = +1, \delta = +1, n > 8$;
- $\det M = 0$, for $\varepsilon = +1, \delta = -1, n \geq 8$;
- $\det M = \pm 4n^{n/2-3} = \pm 4^{\frac{n}{2}}n^3$, for $\varepsilon = -1, \delta = -1, n \geq 8$.

The problem also shows that the $(n-2)$- and $(n-3)$-order nonsingular submatrices of an $n$-order Hadamard all have inverses whose nonzero entries can be only $+2/n$ or $-2/n$.


Determine all real solutions of the polynomial Diophantine equation

\[
(1) \quad P(x)^2 - P(x^2) = x\{Q(x)^2 - Q(x^2)\}.
\]

Solution by O. P. LOSSERS (Technological University, Eindhoven, the Netherlands).

From the given equation, it follows that

\[
P(x^4) - x^2Q(x^4) = P^2(x^2) - x^2Q^2(x^2)
\]

\[
= \{P(x^2) - xQ(x^2)\}\{P(x^2) + xQ(x^2)\}.
\]

Letting $F(x) = P(x^2) - xQ(x^2)$, we have

\[
(2) \quad F(x^2) = F(x)F(-x).
\]
Conversely, any solution of (1) may be obtained from a solution of (2) by taking
\[ P(x) = \frac{1}{2} \left\{ F(\sqrt{x}) + F(-\sqrt{x}) \right\}, \]
\[ Q(x) = \frac{1}{2x} \left\{ -F(\sqrt{x}) + F(-\sqrt{x}) \right\}. \]

Polynomial solutions of (2) may be written in the form
\[ F(x) = C(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n) \quad (C \text{ is a constant}). \]

Then
\[ F(-x) = (-1)^n C(x + \alpha_1)(x + \alpha_2) \cdots (x + \alpha_n), \]
so that
\[ F(x)F(-x) = (-1)^n C^2(x - \alpha_1)(x + \alpha_1)(x - \alpha_2)(x + \alpha_2) \cdots (x - \alpha_n)(x + \alpha_n). \]

On the other hand, taking \( \beta_i \) such that \( \beta_i^2 = \alpha_i (i = 1, \ldots, n) \), we find
\[ F(x^2) = C(x - \beta_1)(x + \beta_1)(x - \beta_2)(x + \beta_2) \cdots (x - \beta_n)(x + \beta_n). \]

Therefore, in view of (2), excluding the trivial case \( C = 0 \), we obtain \( C = (-1)^n \) and \( (\alpha_i)^n_{i=1} \) is a permutation of \( (\beta_i)^n_{i=1} \).

Finite, squaring-invariant subsets of the complex plane can only contain 0 and roots of unity of odd order. The irreducible polynomials corresponding to these roots are
\[ \lambda_0(x) = x, \quad \lambda_k(x) = \prod_{(2k-1), \text{ odd}} \left[ x - \exp \left( \frac{2\pi i l}{2k-1} \right) \right], \quad k = 1, 2, 3, \ldots, \]
(the cyclotomic polynomials). Since for all \( k = 1, 2, 3, \ldots, \) the set \( \{ \exp \left( \frac{2\pi i l}{2k-1} \right) \}_{l=1}^{2k-1} \) is squaring-invariant and the set of solutions of (2) is closed under multiplication, the general polynomial solution of (2) is
\[ F(x) = (-1)^{\deg F} \prod_{k=0}^{\infty} (\lambda_k(x))^{n_k}, \]
the \( n_k \) being nonnegative integers, \( n_k \neq 0 \), for a finite number of indices \( k \). These polynomials all have integral coefficients.

Also solved by the proposer, who notes that one can give extensions by considering higher order roots of unity. For example, letting \( \omega^3 = 1 \), consider
\[ F(x^3) = F(x)F(\omega x)F(\omega^2 x), \text{ where } F(x) = P(x^3) + \omega x Q(x^3) + \omega^2 x^2 R(x^3). \]