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Abstract

This paper deals with a mixture of $H_2$ and $H_\infty$. We have two inputs and one output. One input signal is a white noise stochastic process, and represents errors e.g. resulting from measurement noise. The other input has a more deterministic character. If one has a reference signal (e.g. a step) as input one can not model this as white noise, but it fits nicely into this new class of inputs. The objective is to minimize the effect of these exogenous signals on the output of the system. We define a cost function which enables us to combine the structural difference between these two exogenous inputs. The analysis of this function leads to a standard $H_\infty$ Riccati equation. We will motivate this cost function by looking at two theoretical applications: the derivation of robust performance bounds and a tracking problem.

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1 Introduction

Since the seventies the progress made in $H_2$ and $H_\infty$ theory is enormous. $H_2$ theory deals with an exogeneous, white noise, disturbance signal entering the system. It gives a measure of the influence that this signal has on the output. On the other hand $H_\infty$ theory deals with square integrable, unknown, disturbance signals, and discusses the worst-case influence of such signals on the output. The $H_\infty$ norm can be used to derive an indication for the sensitivity of the system due to perturbations of the model. It is a good measure for the robustness problem, where we want to guarantee stability for all systems in some neighbourhood of the model. The model uncertainty in this setup is incorporated via some disturbance system in the model, which is bounded in an $H_\infty$ sense. The $H_2$ norm is better suited to treat errors resulting from e.g. measurement noise. The $H_2$ and $H_\infty$ control problems, which minimize these norms, are already solved (see e.g. [2, 4, 9, 10]).

The growing interest for so-called mixed $H_2/H_\infty$ is a logical consequence, and it has already been investigated extensively ([1, 3, 5, 6, 7, 8, 11]). In general we are interested in the $H_\infty$ norm from one input to one output and, simultaneously the $H_2$ norm from another input to another output. We can try to minimize the nominal $H_2$ norm (the influence of exogeneous signals on the output), while keeping the $H_\infty$ norm small (we still want stability if the plant is not equal to our model, but close to it). This is done for example in [1, 5], where they minimize some upperbound (called auxiliary cost) for the $H_2$ norm under an $H_\infty$ constraint.

From a practical point of view it seems more appropriate to consider some kind of worst-case $H_2$ norm, such that for all systems close to the model the $H_2$ norm has a guaranteed upper bound. For example, assume that the plant can be represented by incorporating a disturbance system $\Delta$ in the model, where $\Delta$ has an $H_\infty$ bound. Then we would like to minimize the $H_2$ norm for the worst disturbance system $\Delta$. This disturbance system includes modelling errors as well as discarded non-linearities, time-variations and dynamics. The latter arises because of the desire to get a model of sufficiently simple structure for the design of controllers. Instead of maximizing the $H_2$ norm over all possible disturbance systems $\Delta$, in [8] the approach is taken to maximize instead over all signals, constrained such that they can be connected via a disturbance system $\Delta$. This approach has the disadvantage that it is difficult to incorporate causality of $\Delta$. Moreover, except for SISO systems, the results only yield bounds for worst-case performance.

In this paper we want to combine the $H_2$ and $H_\infty$ norm in one cost function where we have two inputs and one output. The objective is to study a system with two inputs. One input is a white noise input, like in $H_2$, resulting from disturbances which have a clear random character. The other input, resulting e.g. from command signals or from modelling uncertainty, has a more deterministic character. The latter is for instance expressed in the fact that this signal need not have zero expectation. It is not clear what measure we should take on the output space. The $H_\infty$ norm is normally defined via $L_2$-signals and hence is defined as a measure on the transient behaviour. On the other hand, the $H_2$ norm is defined via the steady-state behaviour. However, there is a natural way of defining the $H_\infty$ norm via "power signals" and this shows that the $H_\infty$ norm can also be defined as a measure on the steady state behaviour. Because of the above reasoning we decided to define our cost function as a measure for the steady state behaviour. Our cost function expresses the worst-case effect of the inputs on the steady state behaviour of the output.

In section 2 this will be formulated more explicitly. In section 3 the cost function is evaluated for the finite horizon case, which avoids a number of technicalities. In section 4 this will be
extended to the infinite horizon case, which leads to an expression in terms of a standard $H_\infty$ Riccati equation. In section 5 this cost function is minimized with state feedback using well-known $H_\infty$ techniques. In section 6 an interpretation of the cost function is given, and the conclusions are stated in section 7. Since some of the proofs are rather lengthy and technical, they are given in two appendices.

The notation used is fairly standard. We will use that $<x, y>_T := \int_0^T x'(t)y(t)dt$ and $\|x\|^2_{2,T} :=<x, x>_T$. Moreover, by $z_{w,\beta}$ we denote the output $z$ for initial condition $x(0) = 0$ and inputs $w$ and $v$, where $v$ is a Brownian motion with $\mathcal{E}v(t) = 0$ and $\mathcal{E}v(t)v'(s) = \beta I\min(t, s)$.

2 Problem formulation

To get a somehow natural mixture of the $H_2$ norm and the $H_\infty$ norm, it is important to decide which definitions of these norms to use, how to combine these in some mixed cost function, which class of signals to take, and so on. In this paper we consider the case of 2 inputs and 1 output, as is shown in the following picture:

The objective we have for this paper was already pursued in [11]. In [11] the authors give a cost function which mixes $H_2$ and $H_\infty$. However their work is not very satisfactory since it lacks a solid mathematical foundation. For instance, one of their inputs is a deterministic function which behaves like white noise; a function, the authors admit, that does not exist. We will pursue a cost function which is intuitive, and also fits into a solid mathematical framework. At first we assume $\Sigma$ to be a linear, time-invariant, finite-dimensional system which has the following structure:

$$\Sigma: \begin{cases} \dot{x}(t) &= Ax(t) + Gw(t) + Ed(t), \quad x(0) = 0 \\ z(t) &= Cx(t) \end{cases}$$

with $A$ stable. Here $d$ is a white noise process, which in fact does not exists in continuous time. We can overcome this by writing $d(t)dt$ as $dv(t)$, where $v$ is a Brownian motion. In this way we can get a solid framework by writing $\Sigma$ in terms of a stochastical differential equation:

$$\Sigma: \begin{cases} dx(t) &= Ax(t)dt + Gw(t)dt + Edv(t) \\ z(t) &= Cx(t) \end{cases}$$

Here we assume $v$ to be a Brownian motion with $\mathcal{E}v(t) = 0$ and $\mathcal{E}v(t)v'(s) = \beta I\min(t, s)$. Obviously the direct feedthrough matrix from $v$ to $z$ has to be zero, since otherwise the $H_2$ norm from $v$ to $z$ is infinite. For simplicity we assume the direct feedthrough matrix from $w$ to $z$ to be zero.

For the infinite horizon case we have, as mentioned in the introduction, the problem that the standard definition for the $H_2$ norm investigates the transient behaviour while the standard definition for the $H_2$ norm investigates the steady state behaviour. This makes it difficult to
combine the two. Hence we first investigate the finite horizon case where this problem does not arise.

As in $H_\infty$ we investigate worst-case behaviour. Because we allow the disturbance to depend on the current state we see that in general $w$ will be a stochastic process, albeit not necessarily Gaussian or with zero expectation. In section 6, as we investigate some applications, we will see why it is natural to allow $w$ to be stochastic.

We define $\Omega_T$ as the set of all stochastic processes on $[0,T]$ such that

$$ E \|w\|_{2,T}^2 := E \int_0^T w'(t)w(t)dt $$

is well-defined and bounded. Then it is possible to define a finite horizon cost function in terms of an induced norm:

$$ J(\Sigma, T) = \sup_{w, \beta} \frac{E \|z_{w, \beta}\|_{2,T}^2}{E \|w\|_{2,T}^2 + \beta T} $$

(2)

where $w \in \Omega_T$ is constrained to be such that $w(t)$ is $\mathcal{F}_v^t$-measurable. Here $\mathcal{F}_v^t$ is the $\sigma$-algebra generated by $\{v(s), 0 \leq s \leq t\}$. The latter says nothing more than the only stochastics contained in $w(t)$ is a dependence on past values of $v$. Note that the term $\beta T$ is a little surprising. As we already stated we have one input $d$ which is white noise with $Ed(t)d'(t) = \beta I$ for all $t$. Hence $E\|d\|_{2,T}^2 = \beta T$, and we get a cost function

$$ J(\Sigma, T) = \sup_{w, \beta} \frac{E \|z_{w, \beta}\|_{2,T}^2}{E \|w\|_{2,T}^2 + \beta T} $$

In discrete time this would be the formal definition. It is in some way the operator norm of $\Sigma$ on $[0,T]$ induced by the norm $E\cdot \|2,T$, except that, but for the magnitude, the distribution of the random signal $d$ is fixed. However in continuous time white noise is not well-defined. This forces us to work with $v$ instead of $d$. But the above reasoning motivates the definition of $J(\Sigma, T)$. We will derive a test, whether $J(\Sigma, T)$ is smaller than a given bound, in terms of a Riccati differential equation.

We would like to extend this definition to the infinite horizon case. The definition of the $H_2$ norm is in terms of the steady state covariance of the output, i.e.

$$ \lim_{T \to \infty} \frac{1}{T} E \|z\|_{2,T}^2 $$

(3)

It seems appropriate to define $J(\Sigma)$ in terms of the steady state behaviour of the signals. However, we would like to work with input and output spaces that are linear vector spaces. The class of stochastic processes for which the limit in (3) exists is easily seen not to be linear. Furthermore we could not show that for $w$ in this class and $\beta > 0$ the output $z$ is in this class. Hence it is better to take input and output functions in $\Omega$ which is defined as the set of stochastic processes $w$ for which

$$ \|w\|_\Omega^2 = \lim_{T \to \infty} \sup_{T} \frac{1}{T} E \|z\|_{2,T}^2 $$

is well-defined and finite. It is easily checked that $\Omega$ is a linear space. On the other hand note that $\|\cdot\|_\Omega$ is not a norm, since it equals zero for all signals in $L_2$ (it is however a semi-norm). With this class we are able to extend $J(\Sigma, T)$ to the infinite horizon case:

$$ J(\Sigma) = \sup_{w, \beta} \frac{\|z_{w, \beta}\|_\Omega^2}{\|w\|_\Omega^2 + \beta} $$

(4)

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where $w \in \Omega$ is constrained to be such that $w(t)$ is $\mathcal{F}_t$-measurable. It will be shown that this cost function is well-defined and finite for stable systems. In the appendix we show that $J(\Sigma)$ equals the square of the $H_\infty$ norm if $\beta = 0$. It is easily seen that $J(\Sigma)$ equals the square of the $H_2$ norm for $w = 0$, which indicates the natural mixture of these norms. We will derive a test, whether $J(\Sigma)$ is smaller than a given bound. This test will be in terms of an algebraic Riccati equation. However we will need quite a bit of work to derive this test. In section 6 we will show that $J(\Sigma)$ has nice interpretations, e.g. for robust performance and for tracking problems.

3 The finite horizon case

First note that for the finite horizon problem, we do not have to deal with stability properties. So we can drop the assumption that $A$ is stable. For $T > 0$ we define the cost function $J(\Sigma, T)$ by (2) where the supremum is taken over all $w$ such that $E \|w\|_{H_2}^2$ is well-defined and finite, $\beta \geq 0$ and $(w, \beta) \neq (0, 0)$. If we only consider deterministic signals $w$ (so $w$ does not depend on $v$), it is easy to compute $J(\Sigma, T)$:

Lemma 3.1: Let $\gamma_1(T)$ be the finite horizon $H_\infty$ norm from $w$ to $z$, and $\gamma_2(T)$ the finite horizon $H_2$ norm from $v$ to $z$:

$$\gamma_1(T) = \sup_{w \in L_2_{loc}} \frac{\|z_{w,0}\|_{2,T}}{\|w\|_{2,T}}$$

and

$$\gamma_2(T) = \sqrt{E \|z_{0,1}\|_{2,T}}$$

If we require in definition (2) that $w$ is a deterministic function and hence independent of $v$, then $J(\Sigma, T) = \max\{\gamma_1^2(T), \gamma_2^2(T)\}$. □

Proof: Note that $w$ is deterministic and hence it is not difficult to see that $E < z_{w,0}, z_{0,0} > T = 0$.

Therefore, we get

$$\frac{E \|z_{w,0}\|_{2,T}^2}{\|w\|_{2,T}^2 + \beta T} = \frac{\|z_{w,0}\|_{2,T}^2 + E \|z_{0,0}\|_{2,T}^2}{\|w\|_{2,T}^2 + \beta T} \leq \frac{\gamma_1^2(T) \|w\|_{2,T}^2 + \gamma_2^2(T) \beta T}{\|w\|_{2,T}^2 + \beta T} \leq \max\{\gamma_1^2(T), \gamma_2^2(T)\}$$

If $\gamma_2(T) \geq \gamma_1(T)$, then this upper bound is attained by $w = 0$. This is seen by noting that $E \|z_{0,0}\|_{2,T} = \beta E \|z_{0,1}\|_{2,T}$.

On the other hand, if $\gamma_1(T) > \gamma_2(T)$, then $J(\Sigma, T) = \gamma_1^2$ which is attained for $\beta = 0$. ■

Now consider the general case, where $w$ is allowed to depend on $v$, i.e. $w$ is stochastic. It is easy to find an upper and lower bound for $J(\Sigma, T)$:
Lemma 3.2: Let \( \gamma_1 \) and \( \gamma_2 \) be as defined in lemma 3.1. For \( J(\Sigma, T) \), defined by (2), we have the following bounds

\[
\max \{ \gamma_1^2(T), \gamma_2^2(T) \} \leq J(\Sigma, T) \leq 2 \max \{ \gamma_1^2(T), \gamma_2^2(T) \}
\]

Proof: By taking \( w = 0 \) we see that \( J(\Sigma, T) \geq \gamma_2^2(T) \), and for \( \beta = 0 \) we get \( J(\Sigma, T) \geq \gamma_1^2(T) \). This proves the first inequality. The second one follows by writing

\[
E\|w,0\|_2^2 \leq E\|w,0 + z_0,\beta\|_2^2 \leq 2E\|z_0,\beta\|_2^2 + \max \{ \gamma_1^2(T), \gamma_2^2(T) \}(E\|w\|_2^2 + \beta T)
\]

To evaluate this cost function, we relate it to another problem. Just as in \( H_\infty \) theory it is useful to work with a quadratic cost criterion. For \( \alpha > 0 \) we define

\[
J(T, \alpha) = \sup_{w,\beta} E(\|z_{w,\beta}\|^2_2 - \alpha\|w\|^2_2 - \alpha \beta T)
\]

Obviously \( J(\Sigma, T) \leq \alpha \) if and only if \( J(T, \alpha) \leq 0 \). We have seen that \( J(\Sigma, T) \geq \gamma_1^2(T) \). We know that for \( \alpha > \gamma_1^2(T) \) the Riccati differential equation

\[
\dot{P}(t) = A'P(t) + P(t)A + \frac{1}{\alpha}P(t)GG'P(t) + C'C , \quad P(0) = 0
\]

has a solution \( P_\alpha \) with \( P_\alpha(t) \geq 0 \) for \( t \geq 0 \). We can prove the following (the proof is given in appendix A):

Theorem 3.3: Let \( \gamma_1(T) \) be the finite horizon \( H_\infty \) norm from \( w \) to \( z \). Then \( J(T, \alpha) < \infty \) if and only if \( \alpha > \gamma_1^2(T) \) and \( \int_0^T \text{Trace} E'P_\alpha(t)Edt \leq \alpha T \). Moreover, in that case, \( J(T, \alpha) = 0 \), which is attained by \( w_*(t) = \alpha^{-1}G'P_\alpha(T-t)x(t) \) and \( \beta^* = 0 \).

Since \( P_\alpha \) decreases if \( \alpha \) increases (while \( \alpha T \) of course increases), it is easy to compute \( J(\Sigma) \). If \( \int_0^T \text{Trace} E'P_\alpha Edt \leq \alpha T \), then \( J(\Sigma, T) \leq \alpha \), and if \( \int_0^T \text{Trace} E'P_\alpha Edt > \alpha T \), then \( J(\Sigma, T) > \alpha \). Hence is possible to give an explicit formula for our cost function:

Corollary 3.4: We have \( J(\Sigma, T) = \inf_{\alpha > \gamma_1^2(T)} \{ \alpha | \int_0^T \text{Trace} E'P_\alpha Edt \leq \alpha T \} \)

and a binary search will lead to \( J(\Sigma, T) \).
4 The infinite horizon case

In the infinite horizon case the cost function will in general be unbounded if \( A \) is unstable. Hence a standing assumption in this section will be that \( A \) is stable. The results found in the previous section can be extended to the infinite horizon case, although the mathematical details become more complicated. Consider the cost function \( J(\Sigma) \), defined by (4), where the supremum is taken over all \( w \in \Omega \) such that \( w(t) \) is \( F_0^\infty \)-measurable and over all \( \beta \geq 0 \) where \( \|w\|_\Omega \) and \( \beta \) are not both zero. We defined the class \( \Omega \) at the end of section 2. Notice that \( \|\cdot\|_\Omega \) is a semi-norm, not a norm. As in the finite horizon case it is possible to give an upper and lower bound for \( J(\Sigma) \):

**Lemma 4.1**: Let \( \gamma_1 \) be the \( H_\infty \) norm from \( w \) to \( z \), and \( \gamma_2 \) be the \( H_2 \) norm from \( v \) to \( z \):

\[
\gamma_1 = \sup_{w \in L_2} \frac{\|z_{w,0}\|_2}{\|w\|_2} \quad \text{and} \quad \gamma_2 = \sqrt{\lim_{T \to \infty} \frac{1}{T} \|z_{0,1}\|_2},
\]

Then we have \( \max \{\gamma_1^2, \gamma_2^2\} \leq J(\Sigma) \leq 2 \max \{\gamma_1^2, \gamma_2^2\} \).

**Proof**: That \( J(\Sigma) \geq \gamma_2^2 \) follows immediately by taking \( w = 0 \). In the appendix we show that \( J(\Sigma) = \gamma_2^2 \) for \( \beta = 0 \), which proves the first inequality. The second one follows by writing

\[
\limsup_{T \to \infty} \frac{1}{T} \|z_{w,\beta}\|_2^2 = \limsup_{T \to \infty} \frac{1}{T} \|z_{w,0} + z_{0,\beta}\|_2^2 + 2 \limsup_{T \to \infty} \frac{1}{T} \|z_{0,\beta}\|_2^2 \leq 2 \gamma_2^2 \limsup_{T \to \infty} \frac{1}{T} \|w\|_2^2 + 2 \gamma_2^2 \beta,
\]

where in the last step we used that \( J(\Sigma) = \gamma_2^2 \) for \( \beta = 0 \).

In the same way as in lemma 3.1, it is easily seen that \( J(\Sigma) = \max \{\gamma_1^2, \gamma_2^2\} \) if we only consider deterministic functions \( w \) which are hence independent of \( v \). The evaluation of \( J(\Sigma) \) goes in an analogous way as for the finite horizon case. For \( \alpha > 0 \) we define

\[
K(\alpha) = \sup_{w,\beta} \limsup_{T \to \infty} \frac{1}{T} \|z_{w,\beta}\|_2^2 - \alpha \limsup_{T \to \infty} \frac{1}{T} \|w\|_2^2 - \alpha \beta.
\]

Obviously \( J(\Sigma) \leq \alpha \) if and only if \( K(\alpha) \leq 0 \). Since \( K(\alpha) \) involves limsup twice, this is not so attractive for computations. We would rather relate it to

\[
J(\alpha) = \sup_{w,\beta} \limsup_{T \to \infty} \frac{1}{T} (\|z_{w,\beta}\|_2^2 - \alpha \|w\|_2^2) - \alpha \beta.
\]

(5)

It is easily seen that \( K(\alpha) \leq J(\alpha) \). To show that they are equal we will first evaluate \( J(\alpha) \). We know that \( J(\Sigma) \geq \gamma_1^2 \), and that for \( \alpha > \gamma_1^2 \) the Riccati equation

\[
A'P + PA + \alpha^{-1} PGG'P + C'C = 0
\]

(6)
has a solution $P_\alpha \geq 0$ such that
\[ A + \alpha^{-1}GG'P_\alpha \]
is stable.

**Theorem 4.2:** Let $\gamma_1$ be the $H_\infty$ norm from $w$ to $z$. Then $J(\alpha) < \infty$ if and only if $\alpha > \gamma_1^2$ and Trace $E'P_\alpha E \leq \alpha$. In that case $J(\alpha) = 0$, which is (not uniquely) attained by $w^* = \alpha^{-1}G'P_\alpha x$ and $\beta^* = 0$. □

The proof is given in appendix B. Using the optimal $w^*$ and $\beta^*$ from this theorem we find the following lemma (also proven in the appendix):

**Lemma 4.3:** For $\alpha > J(\Sigma)$ there holds $J(\alpha) = K(\alpha)$. □

This lemma shows that $J(\Sigma) \leq \alpha$ if and only if $J(\alpha) \leq 0$. In combination with the previous theorems the following is an immediate result:

**Corollary 4.4:** We have $J(\Sigma) = \inf_{\alpha > \gamma_1^2} \{ \alpha \mid \text{Trace } E'P_\alpha E \leq \alpha \}$. □

**Proof:** We have that
\[
J(\Sigma) < \alpha \implies K(\alpha) = J(\alpha)
\]
\[
J(\Sigma) \leq \alpha \implies K(\alpha) \leq 0
\]
Combined this yields
\[
J(\Sigma) < \alpha \implies J(\alpha) \leq 0
\]
On the other hand, we have
\[
J(\alpha) \leq 0 \implies K(\alpha) \leq 0 \implies J(\Sigma) \leq \alpha
\]
The above immediately yields a proof of the corollary. ■

Note that this cost-function equals $\gamma_1^2$ if $E = 0$, and $\gamma_2^2$ if $G = 0$, exactly what is expected. The term Trace $E'P_\alpha E$ is the so called auxiliary cost, which has been used in [5, 8, 11]. In these papers the auxiliary cost is used as an upperbound for the nominal $H_2$ norm, which is minimized under an $H_\infty$ constraint.

Since $P_\alpha$ increases if $\alpha$ decreases, it only requires a binary search to compute $J(\Sigma)$. 7
5 Minimization by state feedback

In the previous section we analysed a cost function $J(\Sigma)$, and we showed that it can be related to a standard $H_\infty$ Riccati equation. Using well-known techniques from $H_\infty$ optimization (see e.g. [9]), the minimization of $J(\Sigma)$ using state feedback is relatively easy. First consider the open-loop system

$$\Sigma : \begin{cases} dx(t) = Ax(t)dt + Bu(t)dt + Gw(t)dt + Edv(t) \\ z(t) = Cx(t) + Du(t) \end{cases}$$

We make the following assumptions:

**Assumption 5.1**

(i) The pair $(A, B)$ is stabilizable,  
(ii) The pair $(C, A)$ is detectable,  
(iii) $D'[G \ D] = [0 \ f]$.  

These assumptions are standard for $H_\infty$ theory. For the third assumption it is sufficient that $D$ is injective. In that case assumption (iii) can be achieved via a preliminary feedback.

For an arbitrary, possibly non-linear, time-varying or dynamic but causal compensator: $u = F(x)$ we denote by $\Sigma \times F$ the closed loop interconnection. The interconnection is well-posed if the state and output signals, for given input and zero initial conditions, are unique and well-defined stochastic processes. We call a compensator stabilizing if the interconnection is well-posed and stable. From lemma 4.1 we know that we can never achieve $J(\Sigma \times F) < \gamma_1^*$, where $\gamma_1^*$ is the infimum over all $\gamma > 0$ for which there exists a stabilizing compensator for which the closed loop $H_\infty$ norm from $w$ to $z$ is less than $\gamma$. We know the following ([9]):

**Corollary 5.2** : Let $\gamma > 0$. There exists an internally stabilizing feedback such that the $H_\infty$ norm from $w$ to $z$ is less than $\gamma$ if and only if there exists a positive semi-definite solution $Q$ of

$$A'Q + QA + \gamma^{-2}QGG'Q - QBB'Q + C'C = 0$$

such that $A + \gamma^{-2}GG'Q - BB'Q$ is stable.  

This enables us to solve the problem of minimizing $J(\Sigma)$ using state feedback:

**Theorem 5.3** : There exists an internally stabilizing feedback $u = F(x)$ such that $J(\Sigma \times F) \leq \alpha$ if and only if there exists a positive semi-definite solution $Q$ of

$$A'Q + QA + \alpha^{-1}QGG'Q - QBB'Q + C'C = 0$$

such that $A + \alpha^{-1}GG'Q - BB'Q$ is stable and such that $\text{Trace } E'Q E \leq \alpha$. In that case the linear, static and time-invariant compensator $u = -B'Qx$ is internally stabilizing and assures $J(\Sigma \times F) \leq \alpha$.  

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Proof: We denote by $R(\Sigma \times F, \alpha)$ the cost function $J(\alpha)$ as defined in 5 for the system $\Sigma \times F$. Obviously (see the previous section) $J(\Sigma \times F) \leq \alpha$ if and only if $R(\Sigma \times F, \alpha) \leq 0$. For any stabilizing feedback $u = F(x)$ we can write using assumption (iii):

$$R(\Sigma \times F, \alpha) = \sup_{w, \beta} \|Z_{w, \beta}\|_2^2 - \alpha \|w\|_2^2 - \alpha \beta$$

$$= \sup_{w, \beta} \limsup_{T \to \infty} \frac{1}{T} \left( \|Cx + Du\|_{2,T}^2 - \alpha \|w\|_{2,T}^2 - \alpha \beta \right)$$

$$= \sup_{w, \beta} \limsup_{T \to \infty} \frac{1}{T} \left( \int_0^T |Cx(t)|^2 + |u(t)|^2 - \alpha |w(t)|^2 + \frac{d}{dt} z'(t)Qx(t)dt - z'(T)Qx(T) \right) - \alpha \beta$$

$$= \sup_{w, \beta} \limsup_{T \to \infty} \frac{1}{T} \left( \int_0^T |Cx(t)|^2 + |u(t)|^2 - \alpha |w(t)|^2 + 2z'(t)QAx(t) + 2u'(t)B'Q x(t) \right)$$

$$+ 2w'(t)G'Qx(t)dt - z'(T)Qx(T)) + \beta(\text{Trace } E'E - \alpha)$$

$$= \sup_{w, \beta} \limsup_{T \to \infty} \frac{1}{T} \left( \int_0^T |u(t) + B'Q x(t)|^2 - \alpha |w(t) - \alpha^{-1}G'Qx(t)|^2 dt \right)$$

$$- z'(T)Qx(T)) + \beta(\text{Trace } E'E - \alpha)$$

Here we used that $\int_0^T z'(t)QEdv(t) = 0$ since $v$ is a Brownian motion process and since $x(t)$ is $\mathcal{F}_t$-measurable. Obviously (for any fixed stabilizing compensator) this supremum is bounded if and only if $\text{Trace } E'E \leq \alpha$, which gives

$$R(\Sigma \times F, \alpha) = \limsup_{T \to \infty} \frac{1}{T} \left( \int_0^T |u(t) + B'Q x(t)|^2 - z'(T)Qx(t)dt \right)$$

which is (not uniquely) attained by $w^* = \alpha^{-1}G'Qx$ and $\beta^* = 0$. By taking $u^* = -B'Q x$, it is easy to check (using that $u^*$ and $w^*$ result in the stable closed-loop matrix $A + \alpha^{-1}GG'Q$) that

$$\lim_{T \to \infty} \frac{1}{T} z'(T)Qx(t) = 0$$

and hence $R(\Sigma \times F, \alpha) = 0$. This completes the proof.

We see that the minimization of $J(\Sigma)$ using state feedback is not so difficult to solve. The minimization using dynamic feedback will be much harder, as in $H_\infty$ theory. This is the subject of future research.

6 Norminterpretation

In this section we will give some intuition why one might be interested in the cost function $J(\Sigma)$ introduced in this paper. One application is related to robust performance while the other is basically the tracking problem.
6.1 Tracking problem

In this subsection we show how a standard tracking problem leads naturally to the performance measure $J(\Sigma)$ as introduced in this paper. Consider the following setup:

Here $r$ is an, a priori unknown, reference signal. The objective is to make the output $z$ resemble this reference signal, i.e. to minimize the tracking error $e$. We can affect this tracking signal by designing a suitable compensator $C$. However, our sensors measuring the output $z$ of the system, will always induce some measurement noise. Clearly, it is natural to assume that the measurement noise is a white noise stochastic process (in case of coloured noise we can easily incorporate this via a shaping filter). However, for the reference signal $r$ it is much better to model this as an unknown signal in the set $\Omega$. After all, the set $\Omega$ contains standard reference signals like steps, sinusoidal functions and bounded piecewise-constant functions. Information with respect to the frequency content of the reference signal can again be incorporated via a suitable weighting.

We would like to minimize the steady state tracking error. We know that if the inputs are in the above classes then the resulting output will be in $\Omega$ and $\| \cdot \|_\Omega$ is indeed a measure for the steady state tracking error.

One might argue that the measurement noise and the reference signal are independent. However, there are two reasons why some dependency can occur. First of all if the reference signal is produced by a human operator, the operator might make the reference signal depending upon observations he personally makes of the closed loop system. Secondly the size of the measurement error at a certain time might depend on the size of the signal to be measured. After all, if the signal to be measured is large, then the measurement error might increase: either because the measurement error is relative to the size of the signal or because of saturation effects.

After the above reasoning it should be clear why the measure $J(\Sigma)$ is a natural measure because it expresses the size of the tracking error relative to the size of the reference signal and the measurement noise.

Concluding it is our feeling that the effect of measurement noise should be measured in an $H_2$ sense while the effect of reference signals on the tracking error is better measured in an $H_\infty$ sense with suitable weighting. The measure introduced in this paper combines these two.

6.2 Robust performance

In this paper we defined a cost function $J(\Sigma)$ which is a natural mixture of the $H_2$ and $H_\infty$ norm of a system with two inputs and one output. We have seen a very attractive equivalent expression for $J(\Sigma)$, which is given in terms of a Riccati equation. In this section we want to give some results which can be derived using this cost function.
We consider the problem of robust performance. Assume that $w = \Delta z$ for some $\Delta$, as is shown in the following picture:

![Diagram](image)

This system described by this interconnection will be denoted by $\Sigma_{\Delta}$. The disturbance system $\Delta$ contains modelling errors, non-linearities, time-delays, and so on. We assume $\Delta$ to be causal, and bounded (say by 1) in the sense of $\Omega$ induced semi-norms (which, for most systems, is equal to being bounded by 1 in $L_2$ induced norm). Moreover, we require $\Delta$ to yield a well-posed interconnection, i.e. given $v$, the disturbance system $\Delta$ and the model $\Sigma$ with zero initial conditions, the signals $w$ and $z$ are unique and well-defined stochastic processes.

This unknown structure of $\Delta$ requires an investigation of the definition of the $H_2$ norm. The stochastic definition of the $H_2$ norm can be extended to non-linear, time-varying systems: we define the $H_2$ norm for a (possibly) non-linear, time-varying system $\Sigma_n$ as:

$$\|\Sigma_n\|_2^2 := \limsup_{T \to \infty} \frac{1}{T} \|z_n\|_{2,T}^2,$$

where $z_n$ is the output of $\Sigma_n$ with a Brownian motion $v_n$ as input where $\mathcal{E}v_n(t)v_n'(s) = I\min(t,s)$. We take $\limsup$ instead of $\lim$, because we can only guarantee that $z_n$ is bounded in this sense, and not that the limit exists (especially for time-varying perturbations). From $\mathcal{J}(\Sigma)$ we can derive an upperbound for the worst-case $H_2$ norm:

**Lemma 6.1:** If $\mathcal{J}(\Sigma) \leq \alpha < 1$, then

$$\sup_{\|\Delta\|_\infty \leq 1} \|\Sigma\|_2^2 \leq \frac{\alpha}{1 - \alpha}.$$  

**Proof:** Notice that $\|\Sigma_{\Delta}\|_2^2 = \|z_{\Delta,z,1}\|_0^2$ ($\beta = 1$). Say $w = \Delta z$ with $\|\Delta\|_\infty \leq 1$. Then

$$\frac{\|z_{\Delta,z,1}\|_0^2}{\|z_{\Delta,z,1}\|_0^2 + 1} \leq \frac{\|z_{w,z,1}\|_0^2}{\|z_{\Delta,z,1}\|_0^2 + 1} = \frac{\|w\|_0^2}{\|w\|_0^2 + 1} \leq \alpha$$

Since $\alpha < 1$, the result follows immediately.

It is not difficult to check that this bound is not tight. For example if $G = 0$ the disturbed system will always have the same $H_2$ norm, namely $\gamma_2$ (notice that $\mathcal{J}(\Sigma) = \gamma_2^2$). And if $E = 0$ it should be zero (here $\mathcal{J}(\Sigma) = \gamma_1^2$). It is possible to give a better bound:

**Lemma 6.2:** If $\gamma_1^2 < 1$, then

$$\sup_{\|\Delta\|_\infty \leq 1} \|\Sigma\|_2^2 \leq \inf_{1 > \alpha > \gamma_1^2} \frac{\text{Trace } E'P_\alpha E}{1 - \alpha}$$

where $P_\alpha$ for $\alpha > \gamma_1^2$ is equal to the unique solution $P \geq 0$ satisfying (6) such that (7) is stable.
Proof: This bound can be derived by a careful reexamination of the sequence of equalities used in the proof of B.3. For $1 > \alpha > \gamma_1^2$ (take $\beta = 1$) we find

$$\sup_w \limsup_{T \to \infty} E^{1/T}(||z_{w,1}||^2_{2,T} - \alpha ||w||^2_{2,T}) - \alpha$$

$$= \sup_w \limsup_{T \to \infty} E^{1/T} \left(-||w - \alpha^{-1}G'P_a x||^2_{2,T} - x'(T)P_a x(T)\right) + \text{Trace } E'P_a E - \alpha$$

which is equal to $\text{Trace } E'P_a E - \alpha$ and this supremum is (not uniquely) attained by $w^* = \alpha^{-1}G'P_a x$. Hence

$$\sup_w \limsup_{T \to \infty} E^{1/T}(||z_{w,1}||^2_{2,T} - \alpha ||w||^2_{2,T}) = \text{Trace } E'P_a E$$

So for $w = \Delta z$ with $||\Delta||_\infty \leq 1$ we get

$$(1 - \alpha) \limsup_{T \to \infty} E^{1/T} ||z_{w,1}||^2_{2,T} \leq \limsup_{T \to \infty} E^{1/T} \left(||z_{\Delta z,1}||^2_{2,T} - \alpha ||\Delta z_{\Delta z,1}||^2_{2,T}\right)$$

$$= \limsup_{T \to \infty} E^{1/T} \left(||z_{w,1}||^2_{2,T} - \alpha ||w||^2_{2,T}\right)$$

$$\leq \text{Trace } E'P_a E$$

From this the result follows immediately.

Note that $P_a$ decreases if $\alpha$ increases ($\alpha < 1$), while $\frac{1}{1-\alpha}$ increases. This makes this bound not so easy to compute. This bound seems to be a tight one, for example if $E = 0$ or $G = 0$ it gives exactly the results given above (notice that $\gamma_1 = 0$ if $G = 0$, and in that case $P_a = L$ where $L$ is the observability grammar).

7 Conclusions

In this paper we investigate a mixed cost function which combines the $H_2$ and the $H_\infty$ norm in a stochastic framework. We consider the case of two inputs and one output. First we chose a suitable class of functions for the input space from which we can define the $H_2$ norm and the $H_\infty$ norm in terms of the same kind of measure on the output. The cost function $J(\Sigma)$ is then defined in a natural way as an induced semi-norm. This paper mainly deals with the analysis of this mixed cost function. Our objective was to show how the $H_2$ and $H_\infty$ norm can be combined in a logical and elegant way. We find a very attractive expression for $J(\Sigma)$ in terms of a standard $H_\infty$ Riccati equation, which is known in literature as the auxiliary cost. In section 5 we show that using this expression the minimization of the cost function for the state feedback case is relative easy. We also give some connections with problems like robust performance and tracking problems, for which we can easily derive results using $J(\Sigma)$.

For the combination of $H_2$ and $H_\infty$ in one mixed cost function there are many open problems which seem rather tractable. We can look at the dual version of the problem stated here, i.e. the case where we have one input and two outputs. Then we can try to consider the general case of two inputs and two outputs. Especially this last set-up is very useful for the problem of robust performance but hard to handle. Furthermore it is interesting to investigate the use of the mixed cost functions for certain applications in more detail.
Appendices

A  Proofs for the finite horizon case

The proof of theorem 3.3 will only be given for \( \alpha = 1 \). The general result can be obtained via scaling.

**Theorem A.1 :** Let \( \gamma_1(T) \) be the finite horizon \( H_\infty \) norm from \( w \) to \( z \). Then \( J(T, 1) < \infty \) if and only if \( \gamma_1^2(T) < 1 \) and \( \int_0^T \text{Trace } E'P(t)Edt \leq T \). In that case \( J(T, 1) = 0 \) and the supremum is attained for \( w^*(t) = G'P(T - t)x(t) \) and \( \beta^* = 0 \). Here \( P(t) \) is defined by

\[
\dot{P}(t) = A'P(t) + P(t)A + P(t)GG'P(t) + C'C ; \quad P(0) = 0
\]

**Proof :** \( J(T, 1) \)

\[
\begin{align*}
J(T, 1) &= \sup_{w, \beta} E(\|x_{w, \beta}\|_{\infty, T}^2 - \|w\|_{\infty, T}^2 - \beta T + \int_0^T \frac{d}{dt} x'(t)P(T - t)x(t)dt) \\
&= \sup_{w, \beta} E \left( \int_0^T [x'(t)C'Cx(t) - w'(t)w(t)] dt - \beta T \\
&\quad + \int_0^T 2x'(t)P(T - t)Edv(t) - \int_0^T x'(t)\dot{P}(T - t)x(t)dt + \int_0^T \beta \text{Trace } E'P(t)Edt \\
&\quad + \int_0^T 2x'(t)P(T - t)[Ax(t) + Gw(t)]dt \right) \\
&= \sup_{w, \beta} E \int_0^T [2x'(t)P(T - t)Gw(t) - w'(t)w(t) - x'(t)P(T - t)GG'P(T - t)x(t)] dt \\
&\quad + \beta(\int_0^T \text{Trace } E'P(t)Edt - T)
\end{align*}
\]

Here we used Ito’s differential rule and the fact that \( E \int_0^T f(t)dv(t) = 0 \) for any \( f(t) \) that is \( \mathcal{F}_t \)-measurable. Hence we get

\[
J(T, 1) = \sup_{w, \beta} \beta(\int_0^T \text{Trace } E'P(t)Edt - T) - E\|w(t) - G'P(T - t)x(t)\|_{\infty, T}^2
\]

which is finite if and only if \( \int_0^T \text{Trace } E'P(t)Edt \leq T \), and in that case \( J(T, 1) = 0 \) and the supremum is attained for \( w^*(t) = G'P(T - t)x(t) \) and \( \beta^* = 0 \).
B Proofs for the infinite horizon case

The results found for the finite horizon case can be extended to the infinite horizon case, although it needs some more extensive calculations. At first we will give an equivalent definition for the $H_\infty$ norm, which is used in lemma 4.1. This result is similar result given in [11]. However, because we work with $\limsup$ instead of $\lim$ and because we do not make assumptions on the input signals to allow a frequency domain analysis, we need to do more work.

Lemma B.1: Let the system (1) be given. We have

$$\sup_{w} \frac{\|z_{w,0}\|_\Omega}{\|w\|_\Omega} = \gamma_1$$

where $\gamma_1$ is the $H_\infty$ norm from $w$ to $z$. \hfill $\square$

Proof: Note that since $\beta = 0$, we only deal with deterministic signals. We first show that the above supremum is larger than or equal to $\gamma_1$. Let $\epsilon > 0$ be arbitrary small and choose $w_* \in L_2$ such that

$$\frac{\|z_{w_*,0}\|_\Omega^2}{\|w_*\|_2^2} > \gamma_1^2 - \frac{\epsilon}{4}$$

This clearly implies that there exists $T_1$ such that for all $t > T_1$:

$$\frac{\|z_{w_*,0}\|_\Omega^2}{\|w_*\|_2^2} > \gamma_1^2 - \frac{\epsilon}{2}$$

Define

$$\kappa = \sqrt{\int_0^\infty \|Ce^t\|^2 dt} \quad \delta = \frac{\|w_*\|_2^2 T_1}{4\kappa \|z_{w_*,0}\|_2^2} \epsilon$$

where $\|\cdot\|$ denotes the largest singular value of a matrix. Since the input $w_*$ is in $L_2$ and $A$ is stable, it is well-known that then

$$x(t) = \int_0^t e^{A(t-\tau)}Ew_*(\tau)d\tau \rightarrow 0$$

as $t \rightarrow \infty$. Using this it can be shown that there exists $T_2 > T_1$ such that for all $t > T_2$:

$$\left| \int_0^t e^{A(t-\tau)}Ew_*(\tau)d\tau \right| < \frac{1}{2}\delta$$

Since $A$ is stable, it is obvious that there exists $s > T_2$ such that for all $t > s$ we have

$$\|e^{At}\| < \frac{1}{2}$$

Next, we will define a input function with period $s$, and compute a lower bound for the corresponding output. Define $w \in \Omega$ by

$$w(ks+t) = w_*(t)$$
for $k \in \mathbb{N}$ and $0 \leq t < s$. Because $w$ is periodic, it is easy to see that this function is indeed in $\Omega$. We will show that $|x(ks)| < \delta$ for all $k \in \mathbb{N}$. Obviously this is true for $k = 0$ ($x(0) = 0$). Suppose it is true for some $k \in \mathbb{N}$. Then

$$|x((k+1)s)| = |e^{As}x(ks) + \int_{ks}^{(k+1)s} e^{A((k+1)s-t)} Ew(t)dt|$$

$$\leq \|e^{As}\| |x(ks)| + \int_{0}^{s} e^{A(s-t)} Ew_s(t)dt$$

$$< \frac{1}{2} \delta + \frac{1}{2} \delta$$

The next step is to give a lower bound for $|z_{w,0}(t,ks)|$ for $k \in \mathbb{N}$:

$$\|z_{w,0}\|_{2,(k+1)s} \leq \|z_{w,0}\|_{2,ks} + \int_{ks}^{(k+1)s} |z_{w,0}(t)|^2 dt$$

$$\leq \|z_{w,0}\|_{2,ks} + \int_{ks}^{(k+1)s} |Ce^{A((k+1)s-t)} x(ks) + z_{w,0}(t-ks)|^2 dt$$

$$\geq \|z_{w,0}\|_{2,ks} + \|z_{w,0}\|_{2,s}^2 - 2k\delta \|z_{w,0}\|_2$$

By applying the same step to this expression, we find in a recursive way

$$\|z_{w,0}\|_{2,ks} \geq k \|z_{w,0}\|_{2,s}^2 - 2k\delta \|z_{w,0}\|_2$$

This enables us to prove the result

$$\limsup_{k \to \infty} \frac{k\|z_{w,0}\|_{2,s}^2 - 2k\delta \|z_{w,0}\|_2}{\|w\|_{2,s}^2} \geq \frac{1}{s} \|w_*\|_{2,s}^2$$

$$\|z_{w,0}\|_{2,s}^2 \geq \|z_{w,0}\|_{2,s}^2 - 2k\delta \|z_{w,0}\|_2$$

We used that $\|w_*\|_{2,T_1}^2 \leq \|w_*\|_{2,s}^2$ since $T_1 < s$. Since $\epsilon > 0$ was arbitrary, this proves that the supremum is larger than or equal to $\gamma_1$.

We will now show that we actually have equality. Suppose there exists $w_*$ such that

$$\frac{\|z_{w,0}\|_{2,s}^2}{\|w_*\|_{2,s}^2} \geq \gamma_1^2 + \epsilon$$

for some $\epsilon > 0$. Let $\delta > 0$ be such that $\delta < \frac{\epsilon}{\epsilon + 2\gamma_1^2}$. By definition of the limsup there exists $T_1$ such that for all $t > T_1$:

$$\frac{1}{t} \|w_*\|_{2,s}^2 \leq (1 + \delta) \|w_*\|_{2,s}^2$$
Moreover, there exists $s > T_1$ such that

$$\frac{1}{s} \|z_{w_0,0}\|_{\frac{2}{s}, s}^2 \geq (1 - \delta) \|z_{w_0,0}\|_1^2$$

Using this we find

$$\sup_{w \in L_2[0,s]} \frac{\|z_{w_0,0}\|_{\frac{2}{s}, s}^2}{\|w\|_{\frac{2}{s}, s}^2} \geq \frac{(1 - \delta) \|z_{w_0,0}\|_1^2}{(1 + \delta) \|w_0\|_1^2} = \frac{(1 - \delta)(\gamma_1^2 + \epsilon)}{(1 + \delta)} > \gamma_1^2$$

Since the infinite horizon $H_\infty$ norm is larger than or equal to the finite horizon $H_\infty$ norm, this is a contradiction. This completes the proof.

To prove that $J(\alpha) = 0$ if $J(\alpha) < \infty$ (theorem 4.2), we need the following lemma:

**Lemma B.2**: If $w \in \Omega$ then $\limsup_{T \to \infty} \frac{1}{T} |x(T)|^2 < \infty$ for all $\beta \geq 0$.

**Proof**: We have

$$\limsup_{T \to \infty} \frac{1}{T} |x(T)|^2 = \limsup_{T \to \infty} \frac{1}{T} \left| \int_0^T e^{A(T-t)}Gw(t)dt + \int_0^Te^{A(T-t)}Edv(t) \right|^2$$

$$\leq 2 \limsup_{T \to \infty} \frac{1}{T} \left| \int_0^T e^{A(T-t)}Gw(t)dt \right|^2 + 2 \limsup_{T \to \infty} \frac{1}{T} \left| \int_0^Te^{A(T-t)}Edv(t) \right|^2$$

The first term can be bounded as follows:

$$\limsup_{T \to \infty} \frac{1}{T} \left| \int_0^T e^{A(T-t)}Gw(t)dt \right|^2 \leq \left( \int_0^\infty \|e^{A't}G\|^2 dt \right) \|w\|_1^2$$

using lemma B.1 and the fact that $A$ is stable. Using standard properties of stochastic integrals we also find

$$\limsup_{T \to \infty} \frac{1}{T} \left| \int_0^Te^{A(T-t)}Edv(t) \right|^2 = \limsup_{T \to \infty} \frac{\beta}{T} \int_0^T \text{Trace} E'e^{A'(T-t)}e^{A(t)}Edt$$

which tends to zero since $A$ is stable.

This lemma enables us to prove theorem 4.2 (for $\alpha = 1$):
Theorem B.3: Let $\gamma_1^2 < 1$, where $\gamma_1$ is the $H_\infty$ norm from $w$ to $z$. Then $J(1) < \infty$ if and only if $\text{Trace } E'PE \leq 1$, and in that case $J(1) = 0$, which is (not uniquely) attained by $w^* = G'Px$ and $\beta^* = 0$. Here $P \geq 0$ satisfies $A'P + PA + PG'P + C'C = 0$ with $A + GG'P$ stable.

Proof: We can write using lemma B.2:

$$
J(1) = \sup_{w, \beta} \lim_{T \to \infty} \mathcal{E} \left( \left\| w_{x,T} \right\|_{2,T}^2 + \left\| w \right\|_{2,T}^2 - \beta + \frac{1}{T} \int_0^T \left[ \frac{d}{dt} x'(t)Px(t) \right] dt - x'(T)Px(T) \right)
$$

$$
= \sup_{w, \beta} \lim_{T \to \infty} \mathcal{E} \left( \left\| w_{x,T} \right\|_{2,T}^2 + \left\| w \right\|_{2,T}^2 - \beta + \frac{1}{T} \int_0^T (2x'(t)PGw(t) - w'(t)w(t) - x'(t)PGG'Px(t)) dt - x'(T)Px(T) \right)
$$

Hence we get

$$
J(1) = \sup_{w, \beta} \beta(\text{Trace } E'PE - 1) + \lim_{T \to \infty} \mathcal{E} \left( -\left\| w - G'Px \right\|_{2,T}^2 - x'(T)Px(T) \right)
$$

which is finite if and only if $\text{Trace } E'PE \leq 1$, and in that case we see that $J(1) = 0$, which is (not uniquely!) attained by $w^*(t) = G'Px(t)$ and $\beta^* = 0$. Note that if we have inputs $w^*$ and $\beta^*$ (i.e. $v = 0$) then it is easy to show that

$$
\lim_{T \to \infty} \mathcal{E} \left( \frac{1}{T} x'(T)Px(T) \right) = 0
$$

Therefore $J(1) = 0$. Any $w$ and $\beta$ which attain this supremum are of the form $\beta^* = 0$ and $w^* = G'Px + w_1$ for some function $w_1$ with $\|w_1\|_\Omega = 0$.

Now we will prove lemma 4.3, which claims that $J(\alpha) = K(\alpha)$. Again we take $\alpha = 1$.

Lemma B.4: If $J(\Sigma) < 1$, then

$$
J(1) = \sup_{w, \beta} \lim_{T \to \infty} \mathcal{E} \left( \left\| w_{x,T} \right\|_{2,T}^2 + \left\| w \right\|_{2,T}^2 - \beta \right)
$$

$$
= \sup_{w, \beta} \left\| w_{x,T} \right\|_{\Omega}^2 + \left\| w \right\|_{\Omega}^2 - \beta = K(1)
$$

Proof: The left-hand side equals $J(1)$, for which theorem B.3 says that this supremum is (not uniquely) attained by a state feedback $w^* = G'Px$. This feedback results in the system

$$
\Sigma^*:\begin{cases}
    dx(t) = (A + GG'P)x(t)dt + Edv(t) \\
    z(t) = Cx(t) \\
    w^*(t) = G'Px(t)
\end{cases}
$$

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where $A + GG'P$ is stable. It is not very hard to show that for every $\beta \geq 0$ the following limit is well-defined and bounded:

$$\lim_{T \to \infty} \frac{1}{T} \|w^*\|_{2,T}^2$$

and hence

$$J(1) = \lim_{T \to \infty} \sup \frac{1}{T} (\|z_{w^*,\beta}\|_{2,T}^2 - \|w^*\|_{2,T}^2) - \beta = \|z_{w^*,\beta}\|_\Omega^2 - \|w^*\|_\Omega^2 - \beta.$$

From this it easily follows that $J(\alpha) \leq K(\alpha)$. That $J(\alpha) \geq K(\alpha)$ is trivial.

References


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<td>92-13</td>
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<td>M.A. Peters, A.A. Stoovogel</td>
<td>Mixed $H_2/H_\infty$ Control in a Stochastic Framework</td>
</tr>
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