A general equilibrium model of international trade with exhaustible natural resource commodities
Geldrop, van, J.H.; Withagen, C.A.A.M.

Published: 01/01/1987

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):
Memorandum COSOR 87-10

A general equilibrium model of international trade with exhaustible natural resource commodities.

by

Jan van Geldrop and Cees Witbagen

January 1987
Jan van Geldrop
Department of Mathematics and Computing Science

Cees Witbagen
Department of Philosophy and Social Sciences

Eindhoven University of Technology
PO Box 513
5600 MB Eindhoven
The Netherlands
1. Introduction.

In the last decade economic theory has been enriched by an abundant literature on natural exhaustible resources. It is commonly agreed upon that the origins of this (since Hotelling (1931)) renewed attention are rooted in the 1973 oil crisis and Forrester's (1971) book on World Dynamics and the subsequent work of the Club of Rome. The latter type of work concentrates on global resource problems, whereas the oil crisis revealed the vulnerability of some parts of the world through international trade problems. Economic theory has been developed on both aspects. We refer to Peterson and Fisher (1977) and Withagen (1981) for surveys and to Dasgupta and Heal (1979) for a standard introduction. Here we shall be dealing with trade in exhaustible resource commodities.

The existing literature can be divided into two broad categories: one branch follows a partial equilibrium approach, the other is of a general equilibrium nature. In the partial equilibrium literature one studies the optimal exploitation of an exhaustible natural resource in an open economy where (world) demand conditions for the raw material are given for the optimizing economy. In this area interesting contributions were made by a.o. Vousden (1974), Kemp and Suzuki (1975), Aarrestad (1978) and Kemp and Long (1979, 1980 a,b) for the competitive case, by Dasgupta, Eastwood and Heal (1978), who deal with monopoly, by Lewis and Schmalensee (1980 a,b) for oligopolistic markets, whereas, finally, Newbery (1981), Ulph and Folie (1980) and Ulph (1982) study a cartel-versus-fringe market structure. Relatively minor attention has been paid to general equilibrium models of international trade in raw materials from exhaustible resources. A, to our knowledge, exhaustive survey can be found in Withagen (1985). Kemp and Long (1980 c) present a two economy model. One of the economies is resource-rich, the other is resource-poor but has the disposal of a technology to convert the raw material into a consumer good. Each economy then aims at the maximization of its social welfare function under the condition that equilibrium on its current account prevails. Kemp and Long analyse general equilibrium under several assumptions with respect to the market behaviour of each participant in trade. Chiarella (1980) extends the analysis into two directions: first, he introduces labour and capital as factors of non-resource production (which takes place according to a Cobb-Douglas function) and, second, he allows also for lending and borrowing between the countries involved. Elbers and Withagen (1984) drop the dichotomy between the economies by postulating each country to possess an exhaustible resource. The withdrawal from the resource is costly. They address the problem of existence of a general equilibrium and give a characterization.

The purpose of the present study is to generalize and to add several new aspects to this general equilibrium approach. It is straightforward to see that there is a fair number of good reasons to do so. First of all it goes without saying that a general equilibrium analysis of trade should be preferred to a partial equilibrium treatment, if only from a methodological point of view. Furthermore, it may enable us to explain the time path of a crucial variable in the theory of exhaustible resources, namely the internationally ruling interest rate. In the partial equilibrium approach this variable is always assumed a fixed constant, which will turn out below to be justified only in a very special case. Second, the concise summing up of the presently known models shows that the theory can (and should) be extended in a number of non-trivial ways. The assumption of unilateral ownership of a natural resource can be dropped and the assumption of a Cobb-Douglas technology, describing non-resource production...
The plan of the paper is as follows. Section 2 describes the model. The central question is: do there exist prices which generate a general competitive equilibrium and, if the answer is in the affirmative, how can these prices and the corresponding commodity allocation over time for the economies participating in trade be characterized? It turns out that the literature on existence of equilibrium when an infinity of (dated) commodities is involved is not readily capable of providing the answer to the former question. Therefore we turn to an alternative approach which may also shed light on the latter question.

In Section 3 we study a system whose solution is shown to be a Pareto-efficient (PE) allocation. There also properties of PE allocations are derived. Section 4 addresses the existence of PE allocations for arbitrary weighting factors (one for each economy). Section 5 deals with some comparative dynamics. Finally Section 6 summarizes and concludes. The formal proofs are given in Appendix A, B and C.

2. The model.

We consider two economies which can be described as follows. Economy $i$'s ($i=1,2$) social welfare functional is given by

$$J^*(C_i) = \int_0^{\infty} e^{-\rho_i t} U_i(C_i(t)) \, dt,$$

(2.1)

where $t$ denotes time, $\rho_i$ is the rate of time preference ($\rho_i > 0$), $C_i(t)$ is the rate of consumption at time $t$ and $U_i$ is the instantaneous utility function. It is assumed that $U_i$ is increasing, strictly concave and provides a strong incentive to consume. Furthermore the elasticity of marginal utility is bounded.

(P.1) $U'_i(C_i) > 0, U''_i(C_i) < 0$ for all $C_i > 0, U'_i(0) = \infty$

$$\eta_i(C_i) := \frac{U'_i C_i}{U_i} = \text{bounded.}$$

Each economy has the disposal of an exhaustible resource of which the initial (at $t = 0$) size is denoted by $S_0$.

The resources are not replenishable. Let $E_i(t)$ be the rate of exploitation of resource $i$. Then it is required that

$$\int_0^{\infty} E_i(t) \, dt \leq S_0, \quad i=1,2, \tag{2.2}$$

$$E_i(t) \geq 0, \quad i=1,2. \tag{2.3}$$

Exploitation is not costless. It is assumed that, in order to exploit, one has to use capital (which is perfectly malleable with the consumer good) as an input. Following Heal (1978), Kay and Mirrlees (1975) and Kemp and Long (1980 b) we postulate an extraction technology of the fixed proportions type:

$$K_i^*(t) = a_i E_i(t), \tag{2.4}$$
where \( a_i \) is a positive constant and \( K'_i(t) \) is the amount of capital used at \( t \) by economy \( i \). Economy 1 is cheaper in exploitation than economy 2.

(P.2) \( a_1 < a_2 \).

Capital can also, together with the (homogeneous) resource commodity, be allocated to non-resource production. Let \( R_i(t), K'_i(t), Y_i(t) \) and \( F_i \) denote the rate of use of the resource good, the use of capital, the rate of non-resource production and the technology, respectively.

Then

\[
Y_i(t) = F_i (K'_i(t), R_i(t)).
\]

About \( F_i \) the following assumptions, customary in models of international trade, are made. \( F_i \) exhibits constant returns to scale (CRS), is increasing, concave and differentiable for positive arguments. It is furthermore assumed that each input is necessary for production. Finally, the elasticity of substitution (\( \mu_i \)) is bounded. In the sequel this set of assumptions will be referred to as \( P3 \). Apart from CRS they are rather innocuous. CRS however may seem unrealistic when labour is a factor of production. This is so but for the time being we shall stick to it in order to keep the model manageable.

It is customary in models of international trade to make assumptions with regard to the relation between technologies, for example in terms of capital-intensity. We shall make such assumptions as well by using the concept of factor price-frontier (fpf). This concept will be intuitively clear but a rigorous statement can be found in appendix A.

(P4) The fpf's have, in the strictly positive orthant, a finite number of points in common.

This assumption merely states that the non-resource technologies essentially differ among economies.

Define

\[
Z_i(t) = \{ C_i(t), Y_i(t), K'_i(t), R_i(t), K_i(t), E_i(t) \}.
\]

Let \( K_{i0} \) be economy \( i \)'s given initial non-resource wealth and \( K_i(t) \) its wealth at time \( t \). A general competitive equilibrium is defined as follows.

\[
\{ K_1(t), K_2(t), Z_1(t), Z_2(t), p(t), r(t) \}
\]

with \( p(t) \geq 0 \) and \( r(t) \geq 0 \) constitutes a general competitive equilibrium if

1) for \( i = 1,2 \) : \( Z_i(t) \) maximizes \( J^i(C_i) \) subject to (2.2) - (2.5) and

\[
\int_0^\infty \pi(t) \{ p(t) R_i(t) + r(t) (K'_i(t) + K_i(t)) + C_i(t) \} \, dt \leq
\int_0^\infty \pi(t) \{ p(t) E_i(t) + Y_i(t) \} \, dt + K_{i0},
\]

where

...
2) no excess demand:

\[ R_1(t) + R_2(t) \leq E_1(t) + E_2(t), \quad (2.7) \]

\[ K_1(t) + K_2^s(t) + K_3^s(t) + K_2^2(t) \leq K_1(t) + K_2(t), \quad (2.8) \]

\[ C_1(t) + C_2(t) + \dot{K}_1(t) + \dot{K}_2(t) \leq Y_1(t) + Y_2(t). \quad (2.9) \]

3) \( p(t) = 0 \) when (2.7) holds with strict inequality; \( r(t) = 0 \) when (2.8) holds with strict inequality.

Condition (2.6) needs some clarification. It is assumed that there exists a perfect world market for resource commodities which have spot price \( p(t) \) and a perfect world market for capital services with spot price \( r(t) \). Since capital is also a store of value this implies the existence of a perfect world market for "financial" capital where the rate of interest is \( r(t) \). Hence condition (2.6) requires each economy to make plans such that the discounted value of total sales exceeds the discounted value of total expenditures.

3 Pareto efficiency

It will turn out to be convenient to consider the set of Pareto-optima. In this section we shall define some properties of this set, which carry over to a general equilibrium. Without proof we state the first law of classical welfare economies.

**Theorem 1**

Let \( \{K_1(t), K_2(t), Z_1(t), Z_2(t), p(t), r(t)\} \) be a general equilibrium. Then \( \{K_1(t), K_2(t), Z_1(t), Z_2(t)\} \) is PE.

Next, consider the problem of maximizing

\[ J(C_1, C_2) = \int_0^\infty \left[ \alpha e^{-\rho t} U_1(C_1(t)) + \beta e^{-\rho t} U_2(C_2(t)) \right] dt \quad (3.1) \]

subject to

\[ \int_0^\infty E_i(t) dt \leq S_{i0}, \quad i = 1, 2, \quad (3.2) \]

\[ E_i(t) \geq 0, \quad i = 1, 2, \quad (3.3) \]

\[ K_i^s(t) = a_i E_i(t), \quad i = 1, 2, \quad (3.4) \]

\[ Y_i(t) = F_i(K_i(t), R_i(t)), \quad i = 1, 2, \quad (3.5) \]
\[
\begin{align*}
\dot{K}(t) &= Y_1(t) + Y_2(t) - C_1(t) - C_2(t), \\
R_1(t) + R_2(t) &\leq E_1(t) + E_2(t), \\
K(t) &\geq K_1(t) + K_2(t) + K_3(t) + K_4(t),
\end{align*}
\]
where \( K(0) \) is given and equals \( K_{10} + K_{20} \) and \((P 1), (P 2)\) and \((P 3)\) are satisfied. Clearly, an allocation is Pareto-efficient (PE) if, for some positive \( \alpha \) and \( \beta \), it solves the above problem. Furthermore, if an allocation is a solution of the problem, it is PE. Remark that \( C_1 = 0 \) if \( \alpha = 0 \) and \( C_2 = 0 \) if \( \beta = 0 \). So it will always be assumed that \( \alpha > 0 \) or \( \beta > 0 \).

Attention will be restricted to continuous \( K \) (the "state variable") and piece-wise continuous instruments \( E_i, K_i \) etc. These seem to be the only classes of functions which bear economic relevance in this context. Discontinuity in the stock of capital would mean an infinite rate of investment or disinvestment, which cannot be given a meaningful interpretation here, whereas discontinuities of the second kind in the instruments should be ruled out for the same reason.

This restriction enables us to invoke the Pontryagin maximum principle.
Define the Hamiltonian.
\[
H(C_1, C_2, K_1, K_2, R_1, R_2, E_1, E_2) = \alpha e^{-\rho t} U_1(C_1) + \beta e^{-\rho t} U_2(C_2) + \phi (F_1(K_1, R_1) + F_2(K_2, R_2) - C_1 - C_2) + \theta_1(-E_1) + \theta_2(-E_2)
\]
and the Lagrangean
\[
L(C_1, C_2, K_1, K_2, R_1, R_2, E_1, E_2, K) = H(\cdot) + \sigma_1 E_1 + \sigma_2 E_2 + \phi p(E_1 + E_2 - R_1 - R_2) + \phi r(K - K_1 - K_2 - a_1 E_1 - a_2 E_2).
\]
Remark that along an optimal trajectory \( K(t) > 0 \) for all \( t \) because otherwise consumption would equal zero which is ruled out by \( U_1'(0) = \infty \). Now suppose that \( \{K(t), u(t)\} : = \{K(t), C_1(t), C_2(t), K_1(t), K_2(t), R_1(t), R_2(t), E_1(t), E_2(t)\} \) with \( K(t) > 0 \) and \( u(t) \geq 0 \) for all \( t \) solves the problem posed above. Then there exist
- non-negative constants \( \theta_1 \) and \( \theta_2 \)
- continuous \( \phi(t) \)
- \( v(t) := (r(t), p(t), \sigma_1(t), \sigma_2(t)) \geq 0 \) which is continuous except possible at points of discontinuity of \( u(t) \) such that for all \( t \geq 0 \)
\[
\begin{align*}
\alpha e^{-\rho t} U_1'(C_1(t)) &= \phi(t), \\
\beta e^{-\rho t} U_2'(C_2(t)) &= \phi(t), \\
\phi(t)(p(t) - a_i r(t)) + \sigma_i(t) &= \theta_i, \quad i = 1, 2, \\
-\phi(t) \psi(t) &= r(t),
\end{align*}
\]
(3.9) (3.10) (3.11) (3.12)
\[ F_i(K_i^j(t), R_i(t)) - r(t) K_i^j(t) - p(t) R_i(t) \geq 0 \]
\[ F_i(K_i^j, \bar{R}_i) - r(t) \bar{K}_i^j - p(t) \bar{R}_i \quad \text{for all} \quad (\bar{K}_i^j, \bar{R}_i) \geq 0, \quad i = 1, 2. \tag{3.13} \]
\[ \theta_i(S_{0i} - \int_{0}^{\infty} E_i(t) dt) = 0, \quad i = 1, 2. \tag{3.14} \]
\[ \sigma_i(t) E_i(t) = 0, \quad i = 1, 2. \tag{3.15} \]
\[ r(t) (K(t) - K_1(t) - K_2(t) - K_1(t) - K_2(t)) = 0. \tag{3.16} \]
\[ p(t) (E_1(t) + E_2(t) - R_1(t) - R_2(t)) = 0. \tag{3.17} \]

These conditions are all straightforward (see Takayama (1974)) except for (3.13) which may need some clarification. It follows from the necessary conditions that the Hamiltonian is maximized with respect to the instruments subject to (3.3) - (3.5) and (3.7) - (3.8). The conditions stated above are not only the necessary conditions for an optimal trajectory. It will be shown in Theorem 12 that they are sufficient as well. It is furthermore easily seen that the integral (3.1) is bounded from above (the proof is given in Appendix B).

The equations can be given a nice economic interpretation when \( \phi, \theta, p, \) and \( r \) are thought of as the shadow-prices of non-resource production, a non-exploited unit of resource \( i \), resource input in non-resource production and capital input respectively. \( r \) and \( p \) will hereafter be referred to as the (real) shadow-price of capital and the (real) shadow-price of exploited commodities.

Much of the subsequent analysis can conveniently be illustrated graphically in \((r, p)\)-space. See figure 3.1 below.

![Figure 3.1](image-url)
In view of the homogeneity of $F_i$, the left hand side of (3.13) equals zero for both $i$. Let $i$ be fixed for the moment. The set of real factor shadow-prices for which (3.13) holds is convex and closed. In order for economy $i$ to produce the non-resource commodities it is necessary that the real shadow-prices belong to the boundary of the set just mentioned. This boundary will be called factor price frontier (fpr). It is negatively sloped and hits (one of) the axes and/or has (one of) the axes as an asymptote. There cannot be positive asymptotes because of the necessity of both inputs. Examples of fpr's are the curves 1 - 1 and 2 - 2 in figure 3.1. See Appendix A for the derivation of these results. With the interpretation of $r$ and $p$ as shadow-prices, the line $r = p/a_i$ is the locus where shadow-profits of exploitation from resource $i$ are nil.

Let us now list some properties of the solution of the system given by (3.2) - (3.17). To each of the following theorems we add a description of the line of proof or merely the intuition on which the proof is based. This might be misleading for its simplicity but those readers who are interested in the formal proofs are invited to go carefully through the appendices.

Theorem 2.
The real shadow-prices move along that fpr which, given $r$, has maximal $p$.

This is clear from the fact that otherwise maximal shadow-profits of one of the economies would not be zero.

Theorem 3.
$p \ (t) > 0$ for all $t \geq 0$.

If the theorem would not hold, there would be no exploitation ($\sigma_i > 0$ from (3.11)), nor non-resource production. But then capital is useless, its price zero and (3.13) is violated.

Theorem 4.
$r \ (t) > 0$ for all $t \geq 0$.

Theorem 5.
$p \ (t)$ and $r \ (t)$ are continuous for all $t \geq 0$.

The intuition behind Theorem 5 is simple. Inspection of equation (3.11) learns that it should hold along intervals of time where $\sigma_i$ is zero for some $i$ for then a jump of one of the shadow-prices is accompanied by a jump of the other in the same direction which contradicts Theorem 2. As long as the shadow-price of capital services is positive there is supply of such services and hence non-resource production takes place, which necessarily requires exploitation. So then the condition with respect to $\sigma_i$ holds. What one wants to exclude therefore is the possibility of $r \ (t)$ becoming zero. This can be done by showing that in order for such a phenomenon to arise the capital-resource input ratio in one of the economies becomes infinity within finite time, which is not possible with bounded elasticities of substitution. This type of argument has also been followed by Dasgupta and Heal (1974).
Let \((\bar{r}, \bar{p})\) be defined as the point of intersection of the line \(r = p/a_1\) and the fpf for which in that point \(r\) is maximal. See figure 3.1. Evidently real factor shadow-prices should allow for non-negative shadow profits from exploitation of the cheapest resource. Hence \(r(t) \leq \bar{r}\).

**Theorem 6.**
For the real shadow-price trajectories there are two possibilities.

1. \((r(t), p(t)) = (\bar{r}, \bar{p})\) for all \(t \geq 0\);
2. \(r(t) < \bar{r}, \ p(t) > \bar{p}\) for all \(t \geq 0\).

If possibility 2 occurs then \(r(t)\) monotonically decreases towards zero and \(p(t)\) monotonically increases to infinity or a given constant, namely where a fpf is tangent to the \(p\)-axis.

If the real shadow-prices equal \((\bar{r}, \bar{p})\) for an instant of time, then it follows from (3.11) that they will have these values forever. In this case the second economy will never exploit (otherwise \(\theta_2 < 0\)). The first part of the second statement then immediately follows. The time-path of the real shadow-prices follows from (3.11), (3.12) and the fact that \(\theta_1\)'s are constants. The result is in fact a kind of Hotelling rule.

**Theorem 7.**
If the real shadow-prices are \((\bar{r}, \bar{p})\) then \(\min(\rho_1, \rho_2) > \bar{r}\).

Since the capital-resource input ratio is constant in this case and the stock of the resource is finite
\[
\int_0^\infty K(t)\, dt < \infty.
\]
This implies \(K(t) \to 0\ as \ t \to \infty\). It follows from (3.9), (3.10) and (3.12) that \(C_i\) is decreasing only if \(\rho_i > \bar{r}\). This is necessary to prevent \(K(t)\) from becoming negative (see (3.6)).

We now turn to the description of commodity trajectories. Some of these immediately follow from the shadow-price paths.

**Theorem 8.**
Suppose that the fpf's do not coincide for \(r = p/a_1\) and that if they intersect for positive real shadow-prices, the number of points of intersection is finite. Then non-resource production is always specialized.

This follows from Theorem 2.

**Theorem 9.**
Exploitation is always specialized. Moreover, the second resource is only taken into exploitation after exhaustion of the first resource.

The first part of the theorem follows from the fact that if there were simultaneous exploitation \(r\phi\) and \(p\phi\) would be constants (see 3.11) and real shadow-prices would both be increasing, contradicting Theorem 6. The second part is less straightforward but rests on the idea that cheaper resources should be exhausted first (see also Solow and Wan (1977)). The order of non-resource production can easily be traced by looking at figure 3.1 as an example. One could start at point 1 where economy 1 is producing.
Over time point $S$ is reached. Afterwards economy 2 takes over ad infinitum. The resource of economy 1 gets exhausted. This happens after point $U$ has been passed. Hence, eventually the second economy carries out both production and exploitation.

**Theorem 10.**
Consumption in each economy will decrease eventually, towards zero. If $r(t)\rightarrow 0$ the share in total consumption will move in favour of the economy with the smaller ratio of the rate of time preference and elasticity of marginal utility, at zero consumption. 

The theorem follows from (3.9) and (3.10), where it should be recalled that $-\hat{\phi}\phi \rightarrow 0$ as $t \rightarrow \infty$.

**Theorem 11.**
Production gets more capital-intensive over time. 

This is a consequence of the decreasing real shadow-price of capital services.

Finally there is

**Theorem 12.**

\[
\{K(t), u(t)\} \text{ is PE.}
\]

The proof of this theorem concentrates on asserting that $\phi(t)K(t) \rightarrow 0$ as $t \rightarrow \infty$: the shadow-value of the stock of capital goes to zero. This being done, the rest of the proof is straightforward, using concavity of the functions involved.

4 General equilibrium

This section addresses the relation between PE-allocations as discussed in the preceding section and general equilibrium.

Let $\{K_1(\tau), K_2(\tau), Z_1(\tau), Z_2(\tau), p(\tau), r(\tau)\}$ constitute a general equilibrium. Then, according to Theorem 1, $\{K_1(\tau), K_2(\tau), Z_1(\tau), Z_2(\tau)\}$ is Pareto-efficient. Since conditions (3.2) - (3.17) are necessary and sufficient for Pareto-efficiency it follows that all the results of the previous section, which characterize PE allocations, hold. This is summarized in

**Theorem 13.**

Let $\{K_1(\tau), K_2(\tau), Z_1(\tau), Z_2(\tau), p(\tau), r(\tau)\}$ be a general equilibrium. Then Theorems 2 - 11 hold.

This result needs no further comment except possibly for the following. In partial equilibrium models involving exhausible resources it is customary to use constant interest rates. In view of Theorem 7 (and Theorem 15 of the next section) it can be doubted if this is in general justified.
5 Existence of general equilibrium

Hereto nothing has been said about the existence of a general equilibrium. This is the aim of the present section. In view of the two qualitatively different possible price-trajectories it seems useful to distinguish between them. We shall therefore first examine under which initial states of the economies the equilibrium interest rate is constant \( F \). Subsequently we shall go into the existence problem when the initial conditions do not allow for a constant interest rate.

It should be stressed here that we are only interested in equilibria which display a continuous stock of capital and piece-wise continuous rates of consumption, extraction etc. This implies that the results on existence of equilibria with infinitely many commodities, obtained by Bewley (1970 and 1972), Hart e.a. (1974), Jones (1983) and others cannot be invoked in this case in general. The class of functions they allow for is much larger than the one we wish to employ and there is no guarantee whatsoever that their equilibrium has the desired continuity properties. Models of economic growth where the infinity of the horizon is explicit, have been studied by a.o. Radner (1961), McKenzie (1968) and Gale (1967). However these models do not take into account the exhaustibility of resources. Moreover, and this seems crucial, they work in discrete time, which, as is well-known, is a more tractable concept when dealing with existence problems. Finally, Mitra (1978 and 1980) considers a model that is closely related to ours, although there is no international trade aspect in it. He gives an elegant existence proof. However, he works in discrete time and assumes away time preference. One is therefore tempted to conclude that the present model cannot be set in a format which makes an application of known results possible.

The analysis will be conducted along the following lines. If there exists a general equilibrium it is Pareto-efficient. This implies that there exist \( \alpha \) and \( \beta \) \((\beta = 1- \alpha)\) such that the equilibrium commodity trajectories and the equilibrium price trajectories solve \((3.2) - (3.17)\), where the additional variables \( \sigma_i \) and \( \theta_i \) are implicitly defined. We then use our knowledge of PE allocations to obtain the existence results.

If \( r^*(I) = \bar{r} \), the shadow-price of the resources, \( \theta_j \), equals zero. This implies that the balance of payments condition \((2.6)\) can be written as

\[
\int_0^\infty e^{rw} C_i(t) dt \leq K_{i0}, \quad i = 1, 2.
\]

In equilibrium an equality will prevail since the instantaneous utility functions are increasing. These observations give rise to the following.

Calculate \( C_1 \) and \( C_2 \) from

\[
U_1^*(C_1) = \frac{\Phi(0)}{\alpha} e^{(\phi_1 - \bar{r})}, \quad (5.1)
\]

\[
U_2^*(C_2) = \frac{\Phi(0)}{1-\alpha} e^{(\phi_2 - \bar{r})}, \quad (5.2)
\]

where \( \Phi(0) \) and \( \alpha \) are fixed positive constants for the moment and \( 0 < \alpha < 1 \). If \( \rho_1 > \bar{r} \) and \( \rho_2 > \bar{r} \) there clearly exist \( \hat{\Phi}(0) > 0 \) and \( 0 < \hat{\alpha} < 1 \) such that
\[ \int_0^t e^{-\tau} C_1(\hat{\Phi}(0), \hat{\alpha}, t) \, dt = K_{10} \tag{5.3} \]
\[ \int_0^t e^{-\tau} C_2(\hat{\Phi}(0), \hat{\alpha}, t) \, dt = K_{20} \tag{5.4} \]

where \( C_1 \) and \( C_2 \) are the solutions of (5.1) and (5.2). \( \hat{\Phi}(0) \) and \( \hat{\alpha} \) are unique. Let \( K(t) \) be the solution of

\[ \dot{K}(t) = rK(t) - C_1(\hat{\Phi}(0), \hat{\alpha}, t) - C_2(\hat{\Phi}(0), \hat{\alpha}, t) \]
\[ K(0) = K_{10} + K_{20} \]

In view of (5.3) and (5.4) \( K(t) > 0 \).

Let, without loss of generality, the first economy have the more efficient non-resource technology at \( \mathcal{F} \).

Define \( X \) by \( \mathcal{F} = F_{1K}(X, 1) \). Consider

\[ S_{10} - S_1(t) := \int_0^t \frac{K(s)}{\mathcal{X} + a_1} \, ds \tag{5.5} \]

The right hand side of (5.5) is total extraction of the first economy's resource before \( t \), along the proposed program. For

\[ K = K_1 + K'_1 = K_1 + a_1 R_1 \]

and

\[ \frac{K}{E_1} = \mathcal{X} + a_1 \]

Now if \( S_1(t) \leq S_{10} \) for all \( t \), all the conditions for a general equilibrium are satisfied. The desired condition is therefore that \( S_{10} \) is sufficiently large relative to \( K_{10} + K_{20} \). Hence we state

**Theorem 14.**

Suppose \( p_1 > \mathcal{F} \) and \( p_2 > \mathcal{F} \). If \( S_{10} \) is large relative to \( K_{10} + K_{20} \) there exists a unique general equilibrium with \((r(t), p(t)) = (\mathcal{F}, \mathcal{F})\). \[ \square \]

Matters become seriously more complicated when the conditions of Theorem 14 do not hold. But we know that in this case the interest rate will monotonically fall and the resource price will monotonically increase. Furthermore the first resource will be exhausted first and the second will be exhausted in infinity.

Suppose that we fix \( \alpha \) (and \( \beta = 1 - \alpha \), \( r(0), \theta_1 \) and that we let exploitation of the second resource start when the rate of interest reaches \( r^* \). Obviously we must take \( \alpha \) on the unit interval, \( r(0) < \mathcal{F}, \theta_1 \) positive and \( r^* < r(0) \). In addition, exploitation of the second resource must be profitable eventually: \( p^* - a_2 r^* \geq 0 \), where \( p^* \) is the resource price which, together with \( r^* \), yields zero profits in non-resource production. Appendix C shows how for any vector of such parameters initial values \( K_{10}, K_{20}, S_{10}, S_{20} \) can be calculated which would induce a general equilibrium. More formally, define \( \nu = (r_0, r^*, \alpha, \theta_1) \).
Theorem 15.
Suppose

\[ S_{10} = S_{10}(\nu), S_{20} = S_{20}(\nu), K_{10} = K_{10}(\nu), K_{20} = K_{20}(\nu). \]

then there exists a general equilibrium characterized by \( r(0) = r_0 \), \( \dot{r}(t) < 0, E_1(t) > 0 \) as long as \( r(t) > r^* \), \( E_2(t) > 0 \) for all \( t \) such that \( r(t) < r^* \).

This theorem does not solve the existence problem. The functions \( S_{10}(\nu) \) and \( K_{10}(\nu) \) are very hard to treat analytically. In fact we want them to be sufficiently surjective, which is difficult to verify. Two advantages of this approach should be mentioned. First, it is constructive. Second, it can easily be generalized for an arbitrary number of economies participating in trade.

6 Conclusions.

In this paper we have presented a general equilibrium model of trade in natural exhaustible resource commodities. General equilibrium has been characterized for quite general utility functionals and non-resource production functions. The model furthermore takes exploitation costs into account. In these respects considerable progress has been made compared with other general equilibrium approaches. Existence of an equilibrium has been examined, departing from the characterization of Pareto-efficient allocations.

However some weaknesses of the model should be mentioned, thereby pointing out where further research could focus on. First there is the assumption of CRS in non-resource production. Second, the simplicity of the description of the extraction technology. Third, one could introduce (manufactured) substitutes for the resources. Finally there is the assumption of perfect mobility of capital goods which allows for instantaneous switches in productive activities from one economy to the other. Nonetheless some positive conclusions can be drawn.

Our analysis does not justify to apply partial equilibrium models with constant interest rates to markets for exhaustible resource commodities, except for the rather special case where one of the resources is economically speaking abundant. The results can easily be generalized into the direction of more resources with different exploitation costs and more non-resource production possibilities and hence the model offers a good starting point to study for example the world oil market which seems to become more and more competitive.

Appendix A

In this appendix some duality results are derived which are frequently used in the main text and appendix B. The following notation will be adopted. For a vector \( y = (y_1, y_2) \) we write

\[ y \geq 0 \quad \text{if} \quad y_i \geq 0 \quad \text{for all} \quad i, \]
Consider a concave and homogeneous production function $F(K, R)$ with the following properties:

- $F(0, R) = F(K, 0) = 0$,
- $F_{K} = \frac{\partial F}{\partial K} > 0$ for all $(K, R) > 0$,
- $F_{R} = \frac{\partial F}{\partial R} > 0$ for all $(K, R) > 0$.

Define

$$V = \{(r, p) \mid F(K, R) - rK - pR \leq 0 \text{ for all } (K, R) \geq 0\}.$$

Clearly $V$ is convex and closed. Furthermore $(r, p) \in V$ implies $(r, p) \geq 0$. Let $\delta V$ be the boundary of $V$.

**Lemma A1.**

$$\delta V = \{(r, p) \geq 0 \mid (r, p) \in V \text{ and there exists } (\bar{K}, \bar{R}) \geq 0 \text{ such that } F(\bar{K}, \bar{R}) - r\bar{K} - p\bar{R} = 0\}.$$

**Proof.**

$(r, p) \in \delta V \iff (r, p) \in V$ and there exists a sequence $(r_n, p_n) \notin V$ with $(r_n, p_n) \to (r, p)$.

$(r_n, p_n) \notin V \iff$ there exists $(K_n, R_n)$ such that

$$F(K_n, R_n) - r_nK_n - p_nR_n > 0.$$

$(K_n, R_n)$ can be chosen on the unit circle, converging to $(\bar{K}, \bar{R}) \geq 0$ with

$$F(\bar{K}, \bar{R}) - r\bar{K} - p\bar{R} \geq 0.$$

Since $(r, p) \in V$ this expression equals zero. Conversely, suppose $(r, p) \in \text{int } V$ and there exists $(\bar{K}, \bar{R}) \geq 0$ such that

$$F(\bar{K}, \bar{R}) - r\bar{K} - p\bar{R} = 0.$$

But then there exists $(\bar{r}, \bar{p}) \in V$, close to $(r, p)$ such that

$$F(\bar{K}, \bar{R}) - r\bar{K} - p\bar{R} > 0,$$

a contradiction.

**Lemma A2.**

Suppose $(r_1, p_1) \in \delta V$, $(r_2, p_2) \in \delta V$ and $p_2 > p_1$. Then...
i) \( r_2 \leq r_1 \),

ii) if \( r_2 = r_1 \) then \( r_1 = 0 \) and \( p_1 > 0 \).

**Proof.** The proof follows immediately from the previous lemma.

**Lemma A3.**

Suppose \((r_1, p_1) \in \delta V\), \((r_2, p_2) \in \delta V\) and \( r_2 > r_1 \). Then

i) \( p_2 \leq p_1 \)

ii) if \( p_2 = p_1 \) then \( p_1 = 0 \) and \( r_1 > 0 \).

**Proof.** This follows directly from Lemma 1.

**Lemma A4.**

\((r, p) \in \delta V \land (r, p) > (\bar{r}, \bar{p}) \Rightarrow (\bar{r}, \bar{p}) \in V\).

**Proof.** This is clear from the definition of \( V \) and Lemma A1.

Consequently \( V \) and \( \delta V \) can take the shapes as depicted below.

![Graphs](image)

**Remarks.**

1. It cannot be that the curves displayed have an \( \bar{r} > 0 \) or a \( \bar{p} > 0 \) as an asymptote in view of the necessity of both inputs.

2. The above analysis obviously bears similarity with duality approaches in production theory. However, in for example Diepert (1982) nothing is said about one of the input prices being zero. In resource theory this seems inevitable. But for positive prices the results are the same.

Next, attention is paid to the case of two production functions \( F_1 \) and \( F_2 \) with respective inputs \( K \) and \( R \) indexed by 1 and 2. \( V_1 \) and \( V_2 \) are defined analogously to \( V \). Define furthermore
$W := V_1 \cap V_2.$

**Lemma A5.**

$\delta W = \{(r, p) \geq 0 \mid (r, p) \in W \text{ and there exists } (K, \overline{K}) \geq 0 \text{ such that } F_1(K, \overline{K}) - rK = 0 \text{ for some } i\}.$

**Proof.** This is evident. □

Without proof we state

**Lemma A6.**

Lemmata A2, A3 and A4 hold with $V$ replaced by $W$. □

Now suppose there exists $(r, p), (K_1, R_1)$ and $(K_2, R_2)$ such that

$$F_1(K_1, R_1) - rK_1 - pR_1 \geq F_1(K, \overline{K}) - r\overline{K} - p\overline{K} \text{ for all } (K, \overline{K}) \geq 0,$$

$$F_2(K_2, R_2) - rK_2 - pR_2 \geq F_2(K, \overline{K}) - r\overline{K} - p\overline{K} \text{ for all } (K, \overline{K}) \geq 0.$$

This is equivalent to (3.13).

**Lemma A7.**

(3.13) implies that $(r, p) \in W$. □

**Lemma A8.**

$(K_i, R_i) > 0$ for some $i$ implies $(r, p) \in \delta W$. □

Clearly Lemma A7 proves theorem 2. Lemma A8 will turn out useful in Appendix B.

**Appendix B**

In this appendix the theorems of Section 3 are proved. For conveniency the assumptions of the model are restated here.

(P1) $U_i' > 0$, $U_i'' < 0$, $U_i'(0) = \infty$, $\eta_i(C_i) = \frac{U_i'C_i}{U_i}$ is bounded,

(P2) $a_1 < a_2$. 
(P3) $F_i(K_i^R, R_i)$ is concave and homogeneous and satisfies

$F_i(0, R_i) = F_i(K_i^R, 0) = 0.$

$F_{ik} = \partial F_i/\partial K_i^R > 0$ for all $(K_i^R, R_i) > 0,$

$F_{ik} = \partial F_i/\partial R_i > 0$ for all $(K_i^R, R_i) > 0.$

The elasticity of substitution $\phi_i$ is bounded.

(P4) The set $\{(r, p) \mid (r, p) > 0$ and $(r, p) \in \delta V_1 \cap \delta V_2\}$ is finite.

Assumption (P4) says that essentially the $F_i$'s differ. In the sequel the argument $t$ is omitted when there is no danger of confusion. The following notation will be used.

$x_i = K_i^R/R_i, \quad f_i(x_i) = F_i(x_i, 1).$

It is easily seen that

$F_{ik} = f_i', \quad F_{ik} = f_i - x_i f_i', \quad f_i < 0, \quad f_i(0) = 0.$

$\mu_i = -f_i'(f_i - x_i f_i) x_i f_i.$

Finally

$y(t+) = \lim_{A \uparrow 0} y(t + h), \quad y(t-) = \lim_{A \uparrow 0} y(t + h).$

Theorem 2 has been proved in Appendix A.

We first present a lemma that is frequently used in the sequel.

**Lemma B1.**

$r(t) > 0 \Rightarrow Y_1(t) > 0 \lor Y_2(t) > 0.$

**Proof.** Suppose that there exists $t_1 \geq 0$ such that $r(t_1) > 0$ and $Y_1(t_1) = Y_2(t_1) = 0.$ Then $K_1^r(t_1) = K_2^r(t_1) = 0,$ otherwise $(r, p) \in W$ and Lemma A7 is violated. If $p(t_1) = 0$ then $\sigma_i(t_1) > 0$ for both $i$ (since $\theta_i = \phi(t) (p(t) - a_i r(t)) + \sigma_i(t) \geq 0$) and $E_1(t_1) = E_2(t_1) = R_1(t_1) = R_2(t_1) = 0$ (3.11, 3.15 and 3.7). If $p(t_1) > 0$ then also $R_1(t_1) = R_2(t_1) = 0$ since $(r, p) \in W.$ Therefore in both cases $K_i^r(t_1) = 0, \quad i = 1, 2.$ But $K(t_1) > 0.$ It follows from 3.17 that $r(t_1) = 0,$ a contradiction.

$\square$

**Theorem 3.**

$p(t) > 0$ for all $t \geq 0.$

**Proof.** Suppose there exists $t_1$ such that $p(t_1) = 0.$ Since $(0, 0) \in V_i, \quad i = 1, 2, \quad r(t_1) > 0.$ $\theta_i \geq 0$ for both $i,$ hence $\sigma_i(t_1) > 0$ for both $i$ (from (3.11)) implying $E_1(t_1) = E_2(t_1) = R_1(t_1) = R_2(t_1) = 0.$ Therefore $Y_i(t_1) = 0, \quad i = 1, 2$ and Lemma B1 is violated.

$\square$

**Theorem 4.**

$r(t) > 0$ for all $t \geq 0.$
Proof. The proof will be given in several steps.

1) Suppose that there exist \( t_1 \geq 0 \) and \( i \) such that \( r(t_1) = 0 \) and \( R_i(t_1) > 0 \). Then \( K_i'(t_1) > 0 \), otherwise \( r(t_1), p(t_1) \in V_i \) in view of the fact that \( p(t_1) > 0 \). Hence \( r(t_1), p(t_1) \in \delta V_i \) (Lemma A1). But \( F_{ik} > 0 \) and this implies \( (r(t_1), p(t_1)) \notin \delta V_i \). Hence, \( r(t_1) = 0 \) implies \( R_i(t_1) = 0 \), \( i = 1, 2 \).

2) Suppose that there exists \( t_1 > 0 \) such that \( r(t_1) > 0 \), whereas \( r(0) = 0 \). Then \( \phi(t_1) \leq \phi(0) \) in view of (3.12). Since \( r(t_1) > 0 \), \( Y_i(t_1) > 0 \) for some \( i \) and therefore \( R_i(t_1) > 0 \) for this \( i \). This implies that there exists \( j \) such that \( E_j(t_1) > 0 \) and

\[
\theta_j = \phi(t_1)(p(t_1) - a_j r(t_1)) = \phi(0)(p(0) - a_j r(0)) + \sigma_j(0).
\]

So \( p(t_1) > p(0) \) and \( (r(t_1), p(t_1)) > (r(0), p(0)) \). But then \( (r(0), p(0)) \notin W \), contradicting Lemma A7. Hence \( r(0) = 0 \) implies \( r(t) = 0 \) for all \( t \geq 0 \).

3) \( r(0) > 0 \). This is so since, if \( r(0) = 0 \), \( r(t) = 0 \) for all \( t \) (2) implying \( R_i(t) = 0 \) for all \( t \) and both \( i \); hence \( E_i(t) = 0 \) for all \( t \) and both \( i \) (3.17) contradicting (3.14).

4) Suppose there exists \( t_1 > 0 \) such that \( r(t_1) = 0 \) and \( r(t) > 0 \) for all \( 0 < t < t_1 \).

a) Suppose that \( r(t_1-) > 0 \).

In view of the piece-wise continuity of \( r \) there exists \( \tilde{L} < t_1 \), close enough to \( t_1 \), such that \( r(\tilde{L}) > 0 \). Hence \( Y_i(\tilde{L}) > 0 \) for some \( i \) (Lemma B1). Then also \( p(\tilde{L}) \leq p(t_1) \) otherwise \( (r(t_1), p(t_1)) \notin V_i \) (Lemma A4). Since \( \phi(t) \) is continuous by assumption we then have

\[
\sigma_j(\tilde{L}) > \sigma_j(t_1) \geq 0 \quad (j = 1, 2).
\]

Therefore \( E_1(\tilde{L}) = E_2(\tilde{L}) = R_1(\tilde{L}) = R_2(\tilde{L}) = 0 \), contradicting \( Y_i(\tilde{L}) > 0 \) for some \( i \). So

b) \( r(t_1-) = 0 \).

Assumption P4 implies that there exists an interval \((\tau, t_1)\) such that \( Y_i(t) > 0 \) for all \( t \in (\tau, t_1) \) and just one \( i \). Since \( E_i(t) \) is piece-wise continuous for both \( i \) the interval \((\tau, t_1)\) can be partitioned in a finite number of subintervals such that \( E_i(t) (i = 1, 2) \) is continuous along each subinterval. Observe first that there is no subinterval with \( E_1 > 0 \) and \( E_2 > 0 \) along a non-degenerated subinterval of it. For suppose that there exist \( \tilde{L} \) and \( \tilde{T} \) with \( \tau \leq L < \tilde{L} < \tilde{T} < t_1 \) such that \( E_1(t) > 0 \) and \( E_2(t) > 0 \) for all \( t \in [\tilde{L}, \tilde{T}] \). This implies that \( \phi(p - a_j r) \) and \( \phi(p - a_j r) \) are constants for \( \tilde{L} \leq t \leq \tilde{T} \). But \( \phi(t) < 0 \) for \( \tilde{L} \leq t \leq \tilde{T} \). Since \( a_1 \neq a_2 \) \( \phi \) and \( \phi \) are constants, implying that \( (r(\tilde{L}), p(\tilde{L})) > (r(\tilde{T}), p(\tilde{T})) \). Therefore \( (r(\tilde{L}), p(\tilde{L})) \notin W \), which is not allowed by Lemma A7. So for all partitions \( E_1(t) = 0 \iff E_2(t) > 0 \). Hence there exists \( \tau < t_1 \) such that along the interval \((\tau, t_1)\) \( E_1(t) = 0 \) and \( E_2(t) > 0 \), or \( E_1(t) > 0 \) and \( E_2(t) = 0 \). Assume, without loss of generality, that \( Y_1(t) > 0 \) and \( E_1(t) > 0 \) for all \( t \in (\tau, t_1) \). Straightforward calculations yield
\[
\dot{x}_1/x_1 = [-a_1 (r - \dot{r})^2 f_1'' (x_1) x_1^2] + \mu_1 f_1 (x_1)x_1, \quad t \in (\overline{t}, t_1),
\]

where it should be recalled that \( \mu_1 \) is the elasticity of substitution. If \( \dot{r} \geq 0 \) for some \( t \in (\overline{t}, t_1) \) then \( \dot{x}_1 (t) \leq 0 \) because \( f' (x_1) < 0 \). Therefore

\[
\dot{x}_1/x_1 \leq \mu_1 f_1 (x_1)x_1.
\]

Since \( r (t) \rightarrow 0 \), \( x_1 \) must become arbitrarily large. But

\[
\lim_{x_1 \rightarrow \infty} f_1 (x_1)x_1 = 0
\]

so that \( x_1 \) is bounded on any finite interval.

Theorem 5.

i) \( r (t^+) = r (t^-) \).

ii) \( p (t^+) = p (t^-) \).

Proof. \( \theta_i \) is constant. Hence

\[
\sigma_i (t^-) - \sigma_i (t^+) = \phi (t) \left[ a_i (r (t^-) - r (t^+)) - (p (t^-) - p (t^+)) \right], \quad i = 1, 2.
\]

a) Suppose that \( r (t_1^-) > r (t_1^+) \) for some \( t_1 > 0 \). Then there exist \( \overline{t} \) and \( \overline{t} \), close enough to \( t_1 \), with \( \overline{t} < t_1 < \overline{t} \) such that \( r (\overline{t}) > r (\overline{t}) \). Since \( r (\overline{t}) > 0 \), \( Y_i (\overline{t}) > 0 \) for some \( i \) (Lemma B1). Then also \( p (\overline{t}) \leq p (\overline{t}) \) otherwise \( (r (\overline{t}), p (\overline{t})) \notin V_i \) (Lemma A4). Therefore

\[
\sigma_j (\overline{t}) > \sigma_j (\overline{t}) \geq 0 \quad (i = 1, 2) \quad \text{and} \quad E_1 (\overline{t}) = E_2 (\overline{t}) = R_1 (\overline{t}) = R_2 (\overline{t}) = 0 \quad \text{contradicting that}
\]

\( Y_i (\overline{t}) > 0 \) for some \( i \).

b) Suppose that \( r (t_1^-) < r (t_1^+) \) for some \( t_1 > 0 \). In this case the proof is analogous to the proof under a. This shows the validity of part i) of the theorem. The proof of part ii) is similar and will not be given here.

Let \( (\overline{r}, \overline{p}) \) be defined by \( \overline{p} = a_1 \overline{r} \) and \( (\overline{r}, \overline{p}) \in V \). See figure B1.
In view of the assumptions made $(\bar{r}, \bar{p})$ exists. In fact $\bar{r}$ is the maximal feasible $r$.

**Theorem 6.**

i) $(r(t_1), p(t_1)) = (\bar{r}, \bar{p})$ for some $t_1 \geq 0 \Rightarrow (r(t), p(t)) = (\bar{r}, \bar{p})$ for all $t \geq 0$.

ii) Suppose $r(0) < \bar{r}$. Then
   
   a) $r(t) > r(t_1)$ and $p(t) < p(t_1)$ for all $t > t_1 \geq 0$.
   
   b) $\lim_{t \to \infty} r(t) = 0$.

**Proof.**

**Ad i).**

1) $r(t) \leq \bar{r}$ for all $t$. For suppose that for some $t_1 \geq 0$ $r(t_1) > \bar{r}$. Then $Y_i(t_1) > 0$ for some $i$ (Lemma B1). $p(t_1) \leq \bar{p}$ otherwise $(\bar{r}, \bar{p}) \in \delta W$ (Lemmata A6 and A4). Since $\theta_j \geq 0$ for both $j$ it follows that $\sigma_j(t_1) > 0$ for both $j$, which implies that $E_j(t_1) = R_j(t_1) = 0$ for both $j$ contradicting $Y_i(t_1) > 0$ for some $i$.

2) Suppose $(r(t_1), p(t_1)) = (\bar{r}, \bar{p})$ for some $t_1 \geq 0$. Then $E_2(t_1) = 0$ because $\bar{p} - a_2 \bar{r} < 0$ and $\theta_2 \geq 0$. But, since $r(t_1) > 0, R_1(t_1) + R_2(t_1) > 0$ (Lemma B1), implying that $E_1(t_1) > 0$. Therefore $\theta_1 = 0$ and $p(t) - a_1 r(t) \leq 0$ for all $t$. If, for some $t_2 \geq 0, p(t_2) - a_1 r(t_2) < 0$ then $E_1(t_2) = 0$ (Lemma B1). A fortiori $E_2(t_2) = 0$. Hence $Y_i(t_2) = 0$ for both $i$, contradicting Lemma B1. So $p(t) = a_1 r(t)$ for all $t \geq 0$. This proves the first part of the theorem.
Ad ii).

1) Suppose \( r(t_1) \leq r(t_2). \) \( r(t) > 0 \) for all \( t \) (Theorem 4), hence \( Y_i(t_1) > 0 \) for some \( i \) and \( Y_i(t_2) > 0 \) for some \( i \) (Lemma B1). Therefore \( (r(t_1), p(t_1)) \in \delta V_i \) for some \( i \) and \( (r(t_2), p(t_2)) \in \delta V_i \) for some \( i \). So \( (r(t_1), p(t_1)) \in \delta W \) and \( (r(t_2), p(t_2)) \in \delta W \). (Lemma A5). Using Lemmata A6 and A2 we find \( p(t_1) \geq p(t_2) \). By the same argument \( r(t_1) = r(t_2) \) if and only if \( p(t_1) = p(t_2) \).

2) Now suppose \( r(t_1) = r(t_2) \).

   \[ \theta_j = \phi(t)(p(t) - a_j r(t)) + \sigma_j(t) \text{ for all } t \text{ and both } j. \]

   Because \( \phi \) is continuously decreasing and \( p(t_1) = p(t_2) \), \( \sigma_j(t) > 0 \) for both \( j \). So \( R_1(t_2) + R_2(t_2) = 0 \) and \( Y_j(t_2) = 0 \), contradicting Lemma B1. We can therefore restrict ourselves to the following case.

3) Suppose \( r(t_1) < r(t_2) \). Then \( p(t_1) > p(t_2) \) in view of 1). Then a fortiori \( Y_1(t_2) = Y_2(t_2) = 0 \), contradicting Lemma B1. This proves ii) a).

4) Suppose

   \[ \lim_{t \to \infty} r(t) = \bar{r} > 0. \]

   Then \( \dot{\phi}(t) \phi(t) \to -\bar{r} \) as \( t \to \infty \) (3.12). So \( \phi(t) \to 0 \) as \( t \to \infty \). Furthermore \( p(t) \to \beta > 0 \) as \( t \to \infty \). In the case at hand \( \theta_j > 0 \) for both \( j \). Hence \( \theta_j/\phi(t) \to \infty \) as \( t \to \infty \), implying that \( \sigma_j(t) > 0 \) for \( t \) large enough. Then \( E_j(t) = 0 \) for both \( j \) and \( t \) large enough, contradicting \( Y_i(t) > 0 \) for all \( t \) and at least one \( i \).

\[ \square \]

**Theorem 7.**

\( (r(t), p(t)) = (\bar{r}, \beta) \Leftrightarrow p_1 > \bar{r}, \ p_2 > \bar{r}. \)

**Proof.** \( 0 = p(t) - a_1 r(t) > p(t) - a_2 r(t) \). Hence \( E_2(t) = 0 \) for all \( t \geq 0 \). Since \( \dot{\phi}\phi = -\bar{r} \) it follows that \( \phi(t) = \phi(0) e^{-\bar{r}t} \). Hence

\[ U_1'(C_1) = \frac{\phi(0)}{\alpha} e^{(\alpha - \beta)t}; U_2'(C_2) = \frac{\phi(0)}{\beta} e^{(\alpha - \beta)t}. \]

Furthermore

\[ K = K_1^* + K_2^* + a_1 (R_1 + R_2) \]

\[ K/R = K_1^*/R + K_2^*/R + a_1, \]

where \( R := R_1 + R_2. \)

If \( Y_i(t_1) > 0 \) then \( K_1^*(t_1)/R_1(t_1) \) is constant. If \( Y_i(t_1) = 0 \) then \( K_1^*(t_1)/R_1(t_1) = 0. \) So there are constants \( b_1 \) and \( b_2 \), such that

\[ K/R \leq b_1 + b_2 + a_1. \]

Hence
\[ \int_0^\infty K(t)\,dt < \infty. \]

In view of the homogeneity of \( F \)
\[ F_i(K_i^2, R_i) = K_i^2 + FR_i. \]

Therefore
\[ \dot{K} = F K - C_1 - C_2 \]
and
\[ K(t) = K_0 \varepsilon^{\frac{t}{K}} - \int_0^t \varepsilon^{\int_0^t C(s)\,ds}, \]
where \( C = C_1 + C_2 \). It follows from \( \dot{K} \leq FK, K \geq 0 \) and \( \int_0^\infty K(t)\,dt < \infty \) that \( K(t) \to 0 \). To see this remark that
\[ \int_0^T (K - F K)\,(dt) = K(T) - K(0) - F \int_0^T K\,dt \]
So
\[ F \int_0^T K\,dt + K(0) = K(t) + \int_0^T (F K - \dot{K})\,dt \]
The left hand side of this expression is bounded. The second term of the right hand side is monotonically increasing (since \( \dot{K} \leq FK \)). Therefore \( K(t) \to 0 \) for otherwise \( \int_0^\infty K\,dt \) would diverge.

Now, if \( \rho_1 \leq F \) or \( \rho_2 \leq F \) then \( C(t) \geq C > 0 \) for some \( C \) and for \( t \) large enough \( K(t) \) becomes negative, which is not allowed.

As a corollary we mention

**Corollary B1.**
\( (r(t), p(t)) = (F, \dot{p}) \Rightarrow \phi(t)K(t) \to 0 \) as \( t \to \infty. \)

**Theorem 8.**
Suppose \( (r, a_1r) \in \partial W \). Take \( t_2 \geq t_1 \). Suppose \( Y_1(t) > 0 \) and \( Y_2(t) > 0 \) for all \( t \in [t_1,t_2] \). Then \( t_1 = t_2 \).

**Proof.** Suppose there exist \( t_1 \) and \( t_2 \) with \( t_2 > t_1 \) such that \( Y_1(t) > 0 \) and \( Y_2(t) > 0 \) for all \( t \in [t_1,t_2] \). If \( (r(t), p(t)) = (F, \dot{p}) \) for all \( t \geq 0 \) then \( (r, a_1r) \in \partial W \). If \( (r(0), p(0)) \neq (F, \dot{p}) \) then \( r \) is monotonically
decreasing. Hence P4 is violated.

Theorem 9.

i) Take \( t_2 \geq t_1 \). Suppose \( E_1(t) > 0 \) and \( E_2(t) > 0 \) for all \( t \in [t_1, t_2] \). Then \( t_1 = t_2 \).

ii) \( E_2(t) > 0 \Rightarrow \int_0^t E_1(s) ds = S_{10} \).

Ad i).

The argument has already been given in the proof of theorem 4 but will be repeated here for convenience. Suppose there exist \( t_1 \) and \( t_2 \) with \( t_2 > t_1 \) such that \( E_1(t) > 0 \) and \( E_2(t) > 0 \) for all \( t \in [t_1, t_2] \). Then, from (3.11) and (3.15)

\[
\phi(p - a_j r) = \theta_j, \quad j = 1, 2, \quad t \in [t_1, t_2].
\]

Since \( a_1 \neq a_2 \), \( \phi \) and \( \phi r \) are constants along \([t_1, t_2]\). But \( \phi(t_2) < \phi(t_1) \) and therefore \( (r(t_2), \phi(t_2)) > (r(t_1), \phi(t_1)) \). So \( (r(t_1), \phi(t_1)) \notin W \) (Lemma A4), which is not allowed (Lemma A7).

Ad ii).

Suppose there exists \( t_1 \) such that \( E_2(t_1) > 0 \) and

\[
\int_0^{t_1} E_1(s) ds < S_{10}.
\]

a) \( \int_0^{t_1} E_1(s) ds < S_{10} \).

In this case \( \theta_1 = 0 \), implying \( (r(t), \phi(t)) = (\mathcal{F}, \mathcal{P}) \) for all \( t \geq 0 \). Therefore \( \sigma_2(t) > 0 \) for all \( t \) as well as \( E_2(t) = 0 \) for all \( t \), contradicting \( E_2(t_1) > 0 \).

b) \( \int_0^{t_1} E_1(s) ds = S_{10} \).

There exists an interval \([t_1, t_2]\), \( t_1 \leq t_1 < t_2 \), with \( E_1(t) > 0 \) and continuous, whereas, along the interval, \( E_2(t) = 0 \). Take \( t_1 < t_2 < t_2 \).

\[
E_1(t_1) \geq 0, \quad E_2(t_1) > 0, \quad \sigma_2(t_1) = 0. \quad \text{Hence}
\]

\[
\theta_1 \geq \phi(t_1) (p(t_1) - a_1 r(t_1)), \quad \sigma_2(t_1) = 0.
\]

\[
\sigma_2(t_1) = \phi(t_1) (p(t_1) - a_2 r(t_1)).
\]

\[
E_1(t_2) > 0, \quad E_2(t_2) = 0, \quad \sigma_1(t_2) = 0. \quad \text{Hence}
\]
\[ \phi(t_2) (p(t_2) - a_1 r(t_2)) \geq \phi(t_1) (p(t_1) - a_1 r(t_1)), \]
\[ \phi(t_1) (p(t_1) - a_2 r(t_1)) \geq \phi(t_2) (p(t_2) - a_2 r(t_2)). \]

Multiplication of the left and right hand sides of the first inequality by \( a_2 \) and of the second inequality by \( a_1 \) and adding yields
\[ (a_2 - a_1) (p(t_2) - p(t_1)) \geq 0, \]
implying \( p(t_2) \geq p(t_1). \)

Just addition of the inequalities yield
\[ (a_2 - a_1) (r(t_2) - r(t_1)) \geq 0, \]
implying \( r(t_2) \geq r(t_1). \)

Therefore \((r(t_2), p(t_2)) > (r(t_1), p(t_1)) \) and \((r(t_1), p(t_1)) \) belong to \( W \), contradicting Lemma A7.

**Theorem 10.**

i) There exists \( T \geq 0 \) such that, \( C d_{oi}(t) < 0, i = 1, 2, \) for all, \( t > T \)

ii) \( C_i(t) \to 0 \) as \( t \to \infty \) and \( C_i(C_1 + C_2) \to 0 \) as \( t \to \infty \) if and only if
\[ (p_{2-r(\infty)))n_2(0) > (p_1) n_1(0). \]

**Proof.**

Ad i).

It follows from (3.9) and (3.10) that
\[ \frac{\dot{C}_i}{C_i} = (p_1 - r) n_i(C_i). \]
\( n_i(C_i) < 0. \) If \( r(i) = \overline{r} \) then \( p_i > \overline{r} \) for both \( i. \) If \( r(0) < \overline{r} \) then \( r(t) \to 0 \) as \( t \to \infty. \)

Ad ii).

The first part of ii) follows immediately from i) since \( n_i(C_i) \) is bounded. The asymptotic growth rate of \( C_i \) is \((p_1 - r(\infty)) n_i(0). \) This proves the second part.

**Theorem 11.**

Suppose \((r(0), p(0)) \neq (\overline{r}, \overline{p}). \) \( Y_i(t) > 0 \) implies \( \dot{x}_i(t) > 0. \)

**Proof.** This is immediate from the fact that \( r \) decreases and \( f_i'' < 0. \)

It will turn out to be crucial in the proof of Theorem 12 that \( \phi(t) K(t) \to 0 \) as \( t \to \infty. \) This holds if \((r(0), p(0)) = (\overline{r}, \overline{p}) \) (Corollary B1). For \( r(0) < \overline{r} \) it is proved in the following lemma.
Lemma B2.

\( (r(0), p(0)) \neq (\vec{F}, \vec{p}) \Rightarrow \lim_{t \to \infty} \phi(t) K(t) = 0. \)

Proof. For \( t \) large enough exploitation and production are specialized. Therefore indices \( i \) are omitted here.

Define \( Z \) by

\[
K = \frac{p-ar}{r} Z.
\]

\[
\dot{K} = \frac{(\dot{p} - ar)x - r(p-ar)}{r^2} Z + \frac{p-ar}{r} \dot{Z}.
\]

\[
= \frac{f(x)}{a+x} K - (C_1 + C_2) \quad \text{(from (3.6)).}
\]

It follows from (3.11) and (3.12) that

\[
(\dot{p} - ar) = r(p-ar).
\]

Furthermore \( f(x) = xr + p \) (from the homogeneity of \( F \)) and

\[
\dot{p} = \frac{r(p-ar)x}{x+a},
\]

using \( \dot{r} = f'' \dot{x} \), \( \dot{p} = -xxf'' \). Then it is easily shown that

\[
\dot{Z} = -\frac{r}{p-ar} (C_1 + C_2).
\]

So \( Z \) is monotonically decreasing. Denote by \( r^* \) the instant of time after which there is complete specialization and \( r (r^*) \) by \( r^* \). Then

\[
\int_{r^*}^{\infty} E(t) \, dt = \int_{\rho}^{\infty} \frac{K}{a+x} \, dt
\]

must converge. Hence

\[
\int_{r^*}^{\infty} \frac{K}{a+x} \, dt = \int_{\rho}^{0} \frac{p-ar}{r} \frac{Z}{a+x} \, dt \frac{dp}{dr} \frac{dr}{dr}
\]

\[
= \int_{\rho}^{0} \frac{p-ar}{r} \frac{Z}{a+x} \frac{-a-x}{r(p-ar)} \, dr
\]

\[
= \int_{r^*}^{\infty} \frac{Z}{r^2} \, dr < \infty.
\]

We also have
\[
d\!Z/d\!r = \frac{d\!Z}{d\!t} \frac{d\!t}{d\!p} \frac{d\!p}{d\!r} = -\frac{r}{p-ar} (C_1 + C_2) \frac{1-a \cdot d\!r}{r(p-ar)} \frac{d\!p}{d\!r} = \frac{C_1 + C_2}{(p-ar)^2} (a+x) > 0.
\]

Take \( \epsilon < r^* \) and consider
\[
\int_\epsilon^{r^*} \frac{Z}{r^2} d\!r = -\frac{1}{r} \int_\epsilon^{r^*} \frac{d\!Z}{d\!r} d\!r = \frac{Z(r^*)}{r^*} + \frac{Z(\epsilon)}{\epsilon} + \int_\epsilon^{r^*} \frac{(C_1 + C_2) (a+x)}{r(p-ar)^2} d\!r.
\]

It follows that
\[
\int_0^{r^*} \frac{(C_1 + C_2) (a+x)}{r(p-ar)^2} d\!r
\]
converges and \( \lim Z(r)/r \) exists. It equals zero if \( Z/r \to A > 0 \) for some \( A \) then \( \int_0^{r^*} (Z/r)^2 d\!r \) would diverge. Finally
\[
\phi K = \frac{6K}{p-ar} = \theta Z/r.
\]

Hence \( \phi(t)K(t) \to 0 \) as \( t \to \infty \).

**Theorem 12.**

\( \{K(t), u(t)\} \) is PE.

**Proof.** Consider a program that is feasible, i.e. fulfills (3.2) - (3.8). Denote it by upper bars.
\[
\int_0^T \left( \alpha e^{-\gamma t} U_1(C_1) + \beta e^{-\gamma t} U_2(C_2) - \alpha e^{-\gamma t} U_1(C_1) - \beta e^{-\gamma t} U_2(C_2) \right) d\!t
\]
\[
\geq \int_0^T \left\{ \alpha e^{-\gamma t} U_1(C_1) (C_1-C_1) + \beta e^{-\gamma t} U_2(C_2) (C_2-C_2) \right\} d\!t
\]
\[
= \int_0^T \phi(C_1 + C_2 - C_1 - C_2) d\!t
\]
\[
= \int_0^T \phi(Y_1 + Y_2 - \bar{Y}_1 - \bar{Y}_2) d\!t - \int_0^T \phi(K - \bar{K}) d\!t
\]
(if \( K_I > 0 \) then \( F_{IK} = r \) and \( F_{IK} = p \). Therefore we continue)
Here we prove one additional theorem.

**Theorem B.1.**
Let \((\alpha, \beta)\) given with \(\alpha + \beta = 1\). For all \(K_0\) there exists \(M (K_0)\) such that

\[
\int_0^\infty \left\{ \alpha e^{-pt} U_1 (C_1) + \beta e^{-pt} U_2 (C_2) \right\} dt \leq M (K_0) .
\]

**Proof.** Take \((r, p) \in \text{int } V_1 \cap \text{int } V_2\). Then

\[ F_i (K_i, R_i) \leq r K_i + p R_i, i = 1, 2. \]

Hence

\[
\bar{K} \leq r (K_1 + K_2) + p (R_1 + R_2)
\]

\[
\leq r K + p E - C,
\]

where \(E = E_1 + E_2, C = C_1 + C_2\). Therefore

\[
\int_0^1 e^{-st} C (s) ds \leq p (S_10 + S_20) + K_0.
\]

There are positive \(A_1, A_2, B_1,\) and \(B_2\) such that

\[
U_1 (C_1) \leq A_1 + B_1 C_1 ; U_2 (C_2) \leq A_2 + B_2 C_2
\]

So
\[ \alpha e^{-\rho t} U_1(C_1) + \beta e^{-\rho t} U_2(C_2) \leq e^{-\rho t} (\alpha A_1 + \alpha B_1 C_1 + \beta A_2 + \beta B_2 C_2) \leq e^{-\rho t} (\alpha A_1 + \beta A_2 + (\alpha B_1 + \beta B_2)C), \]

where \( \rho = \min(\rho_1, \rho_2) \).

Hence

\[ \alpha e^{-\rho t} U_1(C_1) + \beta e^{-\rho t} U_2(C_2) \leq e^{-\rho t} (A + B C) \leq e^{-\rho t} (A + B C) \text{ for } 0 < r < \rho. \]

It follows that

\[ \int_0^T (\alpha e^{-\rho t} U_1(C_1) + \beta e^{-\rho t} U_2(C_2)) dt \leq \frac{A}{r} + B p(S_{10} + S_{20}) + B K_0. \]

Appendix C

This appendix derives \((S_{10}, S_{20}, K_{10}, K_{20})(v)\), used in Section 5.

Consider the quadruple \( v = (r^*, r_0, \alpha, \theta) \) with \( 0 < r^* < \ell, 0 < r^* < r_0 < \ell, 0 < \alpha < 1 \) and \( \theta_1 > 0 \) where \( \ell \) is defined by \( r = a_2p, (r, p) \in \delta W \). Define

\[ \theta_2 = \frac{p^* - a_2 r^*}{p^* - a_1 r^*} \theta_1, \]

where \( p^* = p(r^*) \) such that \((r^*, p^*) \in \delta W\).

Let \( r(t, v) \) be the solution of

\[ \dot{r} = -\frac{p(r) - a_1 r}{a_1 + x(r)} r, \quad r(0) = r_0, \quad r^* < r \leq r_0, \]

\[ \dot{r} = -\frac{p(r) - a_2 r}{a_2 + x(r)} r, \quad r \leq r^*, \]

where \( p(r) \) is such that \((r, p) \in \delta W \) and \( x = \frac{dp}{dr} \) on \( \delta W \). Remark that these differential equations follow from (3.11) - (3.13) and the constancy of the \( \theta_i \)’s.

Let \( t(r, v) \) be the inverse function of \( r(t, v) \). Hence

\[ t(r, v) = \int_r^{r_0} \frac{a_1 + x(s)}{s (p(s) - a_1 s)} ds, \quad r^* \leq r \leq r_0. \]
\[ t(r, v) = t(r^*, v) + \int_{r}^{r^*} \frac{a_2 + x(s)}{s(p(s) - a_2 s)} \, ds, \quad 0 < r \leq r^*. \]

Define \( C_1(r, v) \) and \( C_2(r, v) \) by

\[
U'_1(C_1(r, v)) = \frac{1}{\alpha} e^{\rho_1 t(r, v)} \frac{\theta_1}{p - a_1 r}, \quad r^* \leq r \leq r_0
\]

\[
U'_2(C_1(r, v)) = \frac{1}{\alpha} e^{\rho_1 t(r, v)} \frac{\theta_2}{p - a_1 r}, \quad 0 < r \leq r^*
\]

\[
U'_1(C_2(r, v)) = \frac{1}{\alpha} e^{\rho_2 t(r, v)} \frac{\theta_1}{p - a_2 r}, \quad r^* \leq r \leq r_0
\]

\[
U'_2(C_2(r, v)) = \frac{1}{\alpha} e^{\rho_2 t(r, v)} \frac{\theta_2}{p - a_2 r}, \quad 0 < r \leq r^*
\]

For given \( r(t) \) and \( v \) the amounts extracted from the resources can be calculated as follows (see also the proof of Lemma B2):

\[
S_{10}(v) = \int_{r^*}^{r_0} \frac{Z^+(r, v)}{r^2} \, dr,
\]

\[
S_{20}(v) = \int_{0}^{r^*} \frac{Z^-(r, v)}{r^2} \, dr,
\]

where

\[
Z^+(r, v) = \frac{\theta_2}{\theta_1} \int_{0}^{r^*} \frac{a_2 + x}{(p - a_2 s)^2} C(s, v) \, ds - \int_{r}^{r^*} \frac{a_1 + x}{(p - a_1 s)^2} C(s, v) \, ds,
\]

\[
Z^-(r, v) = \int_{0}^{r} \frac{a_2 + x}{(p - a_2 s)^2} C(s, v) \, ds,
\]

with \( C(s, v) = C_1(s, v) + C_2(s, v) \).

Finally define

\[
K_{10}(v) = (p_0 - a_1 r_0) \left\{ \int_{r^*}^{r_0} \frac{a_1 + x}{s(p - a_1 s)^2} C_1(s, v) \, ds \right. \\
+ \frac{\theta_2}{\theta_1} \int_{0}^{r^*} \frac{a_2 + x}{s(p - a_2 s)^2} C_1(s, v) \, ds - S_{10}(v) \right\}
\]
Straightforward calculations yield

$$K_0(v) = K_{10}(v) + K_{20}(v) = \frac{p_0 - a_1 r_0}{r_0} Z^*(r_0, v).$$

We conclude that if there exists $v$ such that

$$S_{10}(v) = S_{10}, \quad S_{20}(v) = S_{20}
K_{10}(v) = K_{10}, \quad K_{20}(v) = K_{20}$$

there exists a general equilibrium characterized by

$$\dot{r} = - \frac{p - a_1 r}{a_1 + x} r, \quad r(0) = r_0, \quad 0 \leq t < t(r^*, v)$$

$$\dot{r} = - \frac{p - a_2 r}{a_2 + x} r, \text{ with } r \text{ continuous in } t(r^*, v), t \geq t(r^*, v).$$

Hence the remaining problem is whether or not such a $v$ exists. The functions $(S_{10}, S_{20}, K_{10}, K_{20})$ $(v)$ are continuous in $v$ and continuously differentiable in $v$ unless of course $r_0$ and $r^*$ are located in points where $\delta W$ is not differentiable. These properties however are not sufficient to have a solution. Unfortunately the formulae derived above do not allow for much analytical work.
References


