Compositionality in the temporal logic of concurrent systems

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Compositionality in the temporal
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by

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COMPUTING SCIENCE NOTES

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Compositionality in the Temporal Logic of Concurrent Systems.

(extended abstract)

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In this paper we develop a new method for describing in temporal logic in a compositional manner sequential composition, its iterated version (loops) and its interaction with (nested) parallelism. The logic we will use will be a linear time logic with as sole temporal operator the until. We will illustrate this method in the construction of a compositional proof system for a CSP-like language. We will prove this system to be sound and (relative) complete.

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INTRODUCTION

Temporal logic formalizes reasoning about structures changing in time ([P]). We can apply temporal logic to program verification by interpreting time as a sequence of states, a state being a representation of the abstract machine by which our programs are run ([Pn]). One of the basic temporal operators is the (binary) until operator, denoted by U . The meaning of a temporal assertion $\psi_1 U \psi_2$ can be described roughly as follows: ψ_2 is guaranteed to hold in the future and ψ_1 holds continuously until then. Given a temporal assertion ψ , program S , we define ψ to be a specification of S , written as $\{S\}\psi$, if ψ holds in every execution sequence of S . In this paper we address the problem of axiomatizing this notion.

In the literature there are given various axiomatizations in temporal logic of various (concurrent) programming languages ([BKP1], [L1], [MP], [NGO], [OG]). But a desirable property of an axiomatization is compositionality, by which we mean that the specification of a composite program is derivable from some specifications of its constituents. How to prove a specification of a program $S_1;S_2$, the execution of which consists of an execution of S_1 followed by S_2 , in a compositional manner, using specifications of S_1 and S_2 , is however not yet answered in a satisfactory way.

There are developed until now essentially two ways to describe sequential composition in temporal logic: one of which consists of introducing location predicates, the other of introducing new temporal operators.

Introducing location predicates consists of adding to the specification language for each program S the predicates $IN(S)$, $AT(S)$, $AFTER(S)$ ([L2]). To evaluate these predicates we have to add a component to each state, a program counter, such that, for example, $IN(S)$ holds in a state if the program counter points to a location in S . Given these predicates we can formulate the following rule for sequential composition.

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$$\frac{\{S_1\}\psi_1, \{S_2\}\psi_2}{\{S_1;S_2\}\psi_1 \wedge (IN(S_1)U^-\psi_2)}$$

The assertion $IN(S_1)U^-\psi_2$ will be true iff $IN(S_1)U\psi_2$ holds or $IN(S_1)$ holds throughout the future (this U^- operator can be easily expressed by the until operator). The assertion $\psi_1 \wedge (IN(S_1)U^-\psi_2)$ describes a sequence in which ψ_1 holds and which can be decomposed in a (possibly infinite) prefix such that in every state of this prefix $IN(S_1)$ holds, that is: control is at some location of S_1 , and ψ_2 holds in the suffix (possibly empty) determined by this prefix.

Note that for this rule to be sound we have to interpret $\{S\}\psi$ as follows: for every computation of S ψ holds in all those sequences which are obtained by appending an arbitrary sequence to this computation. If we interpret $\{S\}\psi$ as: ψ holds in every computation of S , the above rule would be unsound. For example, if the statement S_1 does not change the value of some variable x a valid specification of S_1 , with respect to this interpretation, would be “the value of x does not change”, which can be expressed in temporal logic. But if the statement S_2 does change the value of x then, obviously, this specification does not hold for $S_1;S_2$.

The introduction of location predicates, however, will in general complicate the proof procedure. One needs in addition an axiomatization of these predicates. Furthermore, ideally a specification of a statement should specify only what it does, and not how it accomplishes its task. In a compositional proof system reasoning about the control structure should be carried out only in the proof that a statement meets a particular specification. That is, the control structure should be reflected by the structure of the axioms and rules of the proof system. Location predicates however introduce in the specification language the expressibility of the “flow of control”. In [BKP2], for example, to construct a fully abstract temporal semantics for a particular concurrent programming language these location predicates are dispensed with by restricting sequential composition to a head-tail construction: given an atomic action a and a statement S one can construct the statement $a;S$. Applying our method for describing sequential composition would generalize this fully abstract semantics to the language containing the usual notion of sequential composition.

Another way to describe sequential composition is introduced in [BKP1]: a new temporal operator, called the chop-operator, denoted by C , is defined such that the temporal assertion $\psi_1 C \psi_2$ holds in a particular sequence if this sequence can be decomposed into a prefix in which ψ_1 holds and a suffix determined by this particular prefix in which ψ_2 holds. Now if ψ_1 describes S_1 and ψ_2 describes S_2 then $\psi_1 C \psi_2$ will specify $S_1;S_2$. Unfortunately this C operator can greatly increase the complexity of the decision procedure (of the language at the propositional level), and will in general complicate the proof procedure ([HMM]).

We will show how one can describe sequential composition in a temporal logic with as sole temporal operator the until operator (by which all the other familiar temporal operators as “always”, “sometimes”, are expressible), the propositional level of this logic being decidable in exponential time ([HKP]), in a compositional manner in the context of a CSP-like language ([H]), without making use of location predicates.

Our method can be described roughly as follows. We specify a statement S with respect to some possible continuation of S . In our assertion language (the language for the description of sequences of states) this continuation will be denoted by some particular predicate variable. Let Y be this predicate variable. We define a specification $\{S\}\psi(Y)$ to be true iff for every (infinite) sequence consisting of a prefix which is a computation of S $\psi(Y)$ holds in this sequence when Y is interpreted as the suffix determined by this computation. In this framework it can be shown that if $\{S_1\}\psi_1(Y)$ and $\{S_2\}\psi_2(Y)$ are valid specifications then $\psi_1[\psi_2/Y]$ will specify $S_1;S_2$. Here $\psi_1[\psi_2/Y]$ denotes the result of substituting ψ_2 for the free occurrences of Y in ψ_1 . Furthermore we will show how one can describe iterated sequential composition (loops) and the behaviour resulting from the interaction between sequential and parallel composition in this framework.

This paper is organized as follows. In the first section we describe the programming language the proof theory of which we will study. We will define a semantics for this language in the second section. In the third section we introduce the assertion language used to describe sequences of states and

describe its semantics. Then we present in section four the proof system. In the sections five and six we prove the soundness and completeness of the system. In the last section we discuss how to extend our method for a CSP-like language containing nested parallelism.

1. THE PROGRAMMING LANGUAGE

Given the sets, L , a set of function and predicate symbols, $VAR = \{x, \dots\}$, the set of program variables, $CHAN = \{c, \dots\}$, the set of channel names, the class of statements, denoted by $STAT$, is defined as follows:

$$\begin{aligned}
 S ::= & x := t \\
 & | skip \\
 & | c!t \mid c?y \\
 & | S_1; S_2 \\
 & | \bigoplus_{i=1}^n b_i \rightarrow S_i \\
 & | \star \bigoplus_{i=1}^n b_i \rightarrow S_i
 \end{aligned}$$

Where t denotes a term built up from program variables and function symbols and b denotes a boolean expression built up from terms, predicate symbols, and the usual logical connectives like $\wedge, \vee, \neg, \dots$.

The execution of an assignment statement $x := t$ consists of assigning the value of the term t to the variable x . The execution of a *skip*-statement does not change the state. The statements $c!t$, $c?y$ are called IO commands. The execution of an output statement $c!t$ is synchronized with an execution of an input statement $c?y$, the result of this synchronization being that of the assignment $y := t$. The statement $S_1; S_2$ denotes as usual the sequential composition of S_1 and S_2 . The last two statements are called the alternative command and the iterative command. These commands are generalizations of the *if-then-else-fi* and the *while-do-od* construct ([D]).

The set of programs is defined as follows:

$$P ::= S_1 \parallel \dots \parallel S_n$$

where $VAR(S_i) \cap VAR(S_j) = \emptyset, (1 \leq i \neq j \leq n)$, and every channel occurring in P connects only two processes, one of which is only allowed to read from it, the other only to write to it.

This restriction on channels is introduced in [ZRE] to obtain a compositional Hoare-style proof system for a CSP-like language containing nested parallelism.

$VAR(S)$ denotes the set of program variables occurring in S . $CHAN(S)$ denotes the set of channels occurring in S .

Execution of a program $P \equiv S_1 \parallel \dots \parallel S_n$ consists of the parallel execution of the statements S_i .

2. SEMANTICS

In this section we define a semantics for the programming language defined in the previous section. We fix some set L of function and predicate symbols, and some interpretation $I = (DATA, J)$ for L , where $DATA$ is some domain of values and J some interpretation of the function, predicate symbols of L (the set L and the interpretation I we assume to be fixed throughout the paper). To define the semantics we need the following definitions:

DEFINITION 2.1:

We define Σ , the set of states, with typical element σ , as follows:

$$\Sigma = (VAR \rightarrow DATA) \times (CHAN \rightarrow DATA^*)$$

A state consists of two components. The first component gives values to the program variables. The second component associates with each channel the sequence of data transferred on it, $DATA^*$ denoting the set of finite sequences of elements of $DATA$.

The value of a variable x with respect to a state σ we denote by $\sigma(x)$, and the value of a channel c with respect to a state σ will be denoted by $\sigma(c)$. This will cause no confusion because we assume $CHAN \cap VAR$ to be empty.

DEFINITION 2.2.:

We define the set of contexts CON , with typical element C , as follows:

$$CON = \{(V, W): V \subseteq VAR, W \subseteq CHAN\}$$

With each statement S we associate a $CON(S) \in CON$ such that the first component contains all the program variables of S , the second all the channels occurring in S . More precisely, $CON(S) = (VAR(S), CHAN(S))$, where $VAR(S)$ denotes the set of variables occurring in S and $CHAN(S)$ denotes the set of channels of S .

DEFINITION 2.3.:

Given a context C we define the set of local states $\Sigma(C)$, with typical element s , as follows:

$$\Sigma(C) = (V \rightarrow DATA) \times (W \rightarrow DATA^*), \text{ where } C = (V, W).$$

$\Sigma(C)$ denotes the set of local states specified by the context C . Given $\sigma \in \Sigma$ we define $\sigma(C) \in \Sigma(C)$, the restriction of σ to $\Sigma(C)$ as follows:

$$\sigma(C)(x) = \sigma(x), x \in V$$

$$\sigma(C)(c) = \sigma(c), c \in W$$

The set of finite sequences of local states of $\Sigma(C)$ we denote by $\Sigma(C)^*$, the set of infinite sequences of local states of $\Sigma(C)$ we denote by $\Sigma(C)^\omega$, and the union of these sets, with typical element ρ , is denoted by $\Sigma(C)^\infty$.

DEFINITION 2.4.:

Let $\rho_1, \rho_2 \in \Sigma(C)^\infty$. We define $\rho_1 \otimes \rho_2$, the fusion of ρ_1 and ρ_2 , as follows:

$$\begin{aligned} \rho_1 \otimes \rho_2 &= \rho_1 \text{ if } \rho_1 \in \Sigma(C)^\omega \\ &= \langle s_1, \dots, s_n, s_{n+1}, \dots \rangle \text{ if } \rho_1 = \langle s_1, \dots, s_n \rangle \text{ and } \rho_2 = \langle s_{n+1}, \dots \rangle \\ &\text{undefined, otherwise} \end{aligned}$$

DEFINITION 2.5.:

We next define the operation $last: \Sigma(C)^\infty \rightarrow \Sigma(C)$:

$$\begin{aligned} last(\rho) &= s_n \text{ if } \rho = \langle s_1, \dots, s_n \rangle \\ &= s \text{ if } \rho \in \Sigma(C)^\omega, \end{aligned}$$

In this definition s is some arbitrary fixed local state.

DEFINITION 2.6.:

Let $X \subseteq \Sigma(C)^\infty$, $\rho \in \Sigma(C)$. We define:

$$\begin{aligned} \rho \otimes X &= \{\rho \otimes \rho_1 : \rho_1 \in X\} \text{ if } X \neq \emptyset \text{ or } \rho \in \Sigma(C)^* \\ \rho \otimes X &= \{\rho\} \text{ otherwise.} \end{aligned}$$

$\rho \otimes X$ denotes the set of sequences resulting from fusing ρ with an arbitrary sequence of X .

DEFINITION 2.7.:

Given $s \in \Sigma(C)$, we define:

$$\begin{aligned} s^0 &= \epsilon \text{ (the empty sequence)} \\ s^{n+1} &= s^n \circ \langle s \rangle \end{aligned}$$

The set of (finite and infinite) sequences of global states, with typical element τ , we denote by Σ^∞ .

DEFINITION 2.8.:

Let $\tau = \langle \sigma_n \rangle_n \in \Sigma^\infty$ ($\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$). For $1 \leq n \leq |\tau|$, where $|\tau|$ denotes the length of τ , $\tau(n)$ denotes the n^{th} state of τ . For $1 \leq n < m \leq \omega$ we define $\tau[n, m] = \langle \tau(k+1) \rangle_{n-1 \leq k < m}$: $\tau[n, m]$ denotes the subsequence of τ determined by n and m . Given a context C we define $\tau(C) = \langle \sigma_n(C) \rangle_n$: $\tau(C)$ denotes the restriction of the sequence τ to $\Sigma(C)^\infty$.

DEFINITION 2.9.:

The value of a term t in a state s , where $s \in \Sigma(C)$, with $C = (V, W)$ such that $VAR(t) \subseteq V$, we denote by $\llbracket t \rrbracket(s)$. We define the truth of a boolean expression b , with $VAR(b) \subseteq V$, in a local state $s \in \Sigma(C)$, assuming $C = (V, W)$, in the usual way. Notation: $s \models b$. ($VAR(t)$, $VAR(b)$ denotes the set of program variables occurring in t , b , respectively.)

DEFINITION 2.10.:

Given a local state s , the local state obtained by assigning the value d to a variable x we denote by $s\{d/x\}$, and the local state resulting from assigning the sequence of data d^* to a channel c is denoted by $s\{d^*/c\}$. In general, given a function $f \in A \rightarrow B$, $f\{b/a\}$ is defined as follows:

$$\begin{aligned} f\{b/a\}(c) &= f(c), c \neq a \\ &= b, c = a. \end{aligned}$$

Finally we are ready to define the semantics of the programming language defined above. We define $M(C): STAT(C) \rightarrow \Sigma(C) \rightarrow P(\Sigma(C)^\infty)$, where $STAT(C) = \{S: CON(S) \subseteq C\}$ (let $C_1 = (V_1, W_1)$, $C_2 = (V_2, W_2)$ then $C_1 \subseteq C_2$ iff $V_1 \subseteq V_2$ and $W_1 \subseteq W_2$).

$M(C)(S)(s)$ will deliver as value all the execution sequences of S starting from state s . The context C restricts the semantic description of S to the variables and channels specified by it.

$$\begin{aligned} M(C)(x := t)(s) &= \{s^n \circ \langle s_1 \rangle : 1 \leq n < \omega \wedge s_1 = s\{\llbracket t \rrbracket(s)/x\}\} \\ M(C)(skip)(s) &= \{\langle s^n \rangle : 1 < n < \omega\} \\ M(C)(c!t)(s) &= \{s^n \circ \langle s_1 \rangle : 1 \leq n < \omega \wedge s_1 = s\{s(c) \circ \langle \llbracket t \rrbracket(s) \rangle / c\}\} \\ M(C)(c?y)(s) &= \bigcup_{d \in DATA} \{s^n \circ \langle s_1 \rangle : 1 \leq n < \omega \wedge s_1 = s\{s(c) \circ \langle d \rangle / c\}\{d/y\}\} \\ M(C)(S_1; S_2)(s) &= \bigcup_{\rho \in M(C)(S_1)(s)} \rho \otimes M(C)(S_2)(last(\rho)) \\ M(C)(\square_i b_i \rightarrow S_1)(s) &= \bigcup_{i \in I} M(C)(S_i)(s), I = \{i: s \models b_i\} \\ M(C)(\star \square_i b_i \rightarrow S_i)(s) &= \nu F(s), \end{aligned}$$

where $F: D \rightarrow D$, $D = \Sigma(C) \rightarrow P(\Sigma(C)^\infty)$ is defined such that:

$$\begin{aligned} F(f)(s) &= M(C)(skip)(s) \text{ if } s \models \bigwedge_i \neg b_i \\ &= \bigcup_{i \in I} \bigcup_{\rho \in M(C)(S_i)(s)} \rho \otimes f(last(\rho)) \text{ if } I = \{i: s \models b_i\} \neq \emptyset \end{aligned}$$

Here νF denotes the greatest fixed point of F .

Some comments are in order here. The meaning of the iterative construct is well defined because of the following lemma. This lemma is a weak form of a theorem due to Tarski ([T]), which generalizes an earlier result of Knaster and Tarski.

LEMMA 2.11:

Let $\mathcal{L}$ be a complete lattice, T its upperbound, f a monotonic operator on $\mathcal{L}$.

There exists a greatest fixed point of f , which we denote by νf . This fixed point can be characterized in the following two ways.

$$\nu f = l.u.b. \{x \in \mathcal{L} : x \leq f(x)\} \text{ and}$$

$$\nu f = f^\alpha(T), \alpha \text{ the least ordinal of cardinality greater than that of } \mathcal{L}$$

where

$$f^0(T) = T,$$

$$f^{\beta+1}(T) = f(f^\beta(T)),$$

$$f^\alpha(T) = \bigcap_{\beta < \alpha} f^\beta(T), \alpha \text{ a limit ordinal.}$$

PROOF : see ([T]). $\square$

Now the following ordering turns $D = \Sigma(C) \rightarrow P(\Sigma(C)^\infty)$ into a complete lattice. For $f, g \in D$ we define $f \leq g$ iff for every $s \in \Sigma(C)$ we have $f(s) \subseteq g(s)$, furthermore we define $f \cap g, f \cup g$ as follows: for every $s \in \Sigma(C)$ we have $(f \cap g)(s) = f(s) \cap g(s)$ and $(f \cup g)(s) = f(s) \cup g(s)$. It is easy to check that F is monotonic on this complete lattice D .

We introduced in the meaning of the assignment statement the notion of finite stuttering because in our temporal logic, to be defined in the next section, we can not describe the assignment as taking place in one unit of time.

An output statement $c!t$ is modelled as an assignment $c := c^\circ \langle t \rangle$, the function symbol $\circ$ being interpreted as concatenation. An input statement $c?y$ is modelled as: $y := ?; c := c^\circ \langle y \rangle$. Execution of the statement $y := ?$ consists of assigning some random value to y .

The finite stuttering present in the meaning of the IO statements models synchronization. Note that we force this synchronization to take place, thus abstracting from possible deadlock behaviour. We abstract from deadlock behaviour to focus our attention on the main problem: a compositional proof system. To incorporate deadlock behaviour we have to allow in the semantics of IO statements infinite stuttering. Communications then can be enforced to take place by requiring computation sequences to satisfy some fairness assumptions. To formulate these fairness assumptions we have to add to a state a component which for each channel records if some process is ready to write to or to read from it ([BKP1]).

Note that we also abstract from the failure due to the guards of some alternative construct being false.

In the definition of the parallel composition, see below, we merge only those local computations which are in agreement with respect to the choices made in the evaluation of the input statements.

We lift $M(C)$ to $O(C): STAT \rightarrow P(\Sigma^\infty)$ as follows:

$$O(C)(S) = \emptyset \text{ if } \neg CON(S) \subseteq C.$$

$$O(C)(S) = \{\tau : \tau(C) \in M(C)(S)(\tau(1)(C))\} \text{ otherwise.}$$

So $O(C)(S)$ consists of all those sequences such that the restriction of it to the context C is an element of $M(C)(S)(s)$, where s is the restriction of the initial state to C . Finally we define:

$$O(S_1 \parallel \dots \parallel S_n) = \bigcap_i O(C_i)(S_i; \text{halt})$$

where $C_i = \text{CON}(S_i)$ and halt is an abbreviation of $*\Box \text{true} \rightarrow \text{skip}$. Note that the semantics of programs consists only of infinite sequences.

3. THE ASSERTION LANGUAGE

In this section we will define the language used to describe sequences of states. We assume given the following types: VAL , INT , VAL^* . An element of our domain of values will be of type VAL . A sequence of data will be of type VAL^* . The type of integers we denote by INT . We will use a to range over this set of types. For each type a the set of logical variables of type a we denote by $LVAR^a$. For different types a, a' we have $LVAR^a \cap LVAR^{a'} = \emptyset$. We next define the set of terms t_a of type a .

DEFINITION 3.1.:

We define:

$$\begin{aligned} t_a &::= z_a, z_a \in LVAR^a \\ &| x, x \in VAR, a = VAL \\ &| c, c \in CHAN, a = VAL^* \\ &... \end{aligned}$$

Given the set of predicate variables $PREDV$, with typical element X , the set of assertions is defined as follows:

DEFINITION 3.2.:

We define:

$$\begin{aligned} \psi &::= t_a = t'_a \mid t_a \leq t'_a, a = INT \mid \dots \\ &| \exists z \psi, \psi \text{ a state assertion, } z \text{ a logical variable} \\ &| X, X \in PREDV \mid \neg \psi \mid \psi_1 \wedge \psi_2 \\ &| LET z_a = t_a IN \psi NI \\ &| \psi_1 U \psi_2 \\ &| \nu X \psi, X \text{ occurring positively in } \psi \text{ (in the scope of an even number of negations)} \end{aligned}$$

Note that we allow the usual quantification only over so-called state assertions, i.e., assertions containing no temporal operators. Quantification over an arbitrary assertion is only allowed by means of the *LET*-construct. The assertion $\nu X \psi$ will be interpreted as the greatest fixed point of ψ , ψ viewed as a predicate transformer in X . We introduce the following abbreviations: $\psi_1 \vee \psi_2 \equiv \neg(\neg\psi_1 \wedge \neg\psi_2)$, $\psi_1 \rightarrow \psi_2 \equiv \neg\psi_1 \vee \psi_2$, $\Box \psi \equiv \neg(\text{true} U \neg\psi)$, $\psi_1 U^- \psi_2 \equiv (\psi_1 U \psi_2) \vee \Box \psi$, $LET z_1, \dots, z_n = t_1, \dots, t_n IN \psi NI \equiv LET z_1 = t_1 IN \dots LET z_n = t_n IN \psi NI \dots NI$. The sequence $\neg, \wedge, \vee, \rightarrow, \Box, U$ exhibits the connectives of the logic in decreasing order of binding priority.

We will evaluate an assertion ψ in a logical environment (a function assigning values to logical variables), a predicate environment (a function which interprets predicate variables) and with respect to a pair $\langle \tau, n \rangle$, $\tau \in \Sigma^\omega$, n a natural number. The idea is that the number n in $\langle \tau, n \rangle$ denotes the n^{th} state of τ , which will be considered as the present. Basic assertions, for example, will be evaluated with respect to $\tau(n)$, the n^{th} state of τ .

In the sequel the notation $\langle \tau, n \rangle$ will imply that $\tau \in \Sigma^\omega$, i.e., that τ is an infinite sequence.

DEFINITION 3.3.:

The set of logical environments, with typical element γ , is defined as follows:

$$\prod_a LVAR^a \rightarrow D^a$$

Here $\prod$ denotes the generalized cartesian product. furthermore we define $D^{INT} = \mathbb{N}$, $D^{VAL} = DATA$, $D^{INT^*} = \mathbb{N}^*$ and $D^{VAL^*} = DATA^*$.

DEFINITION 3.4.:

The set of predicate environments, with typical element η , is defined as follows:

$$PREDV \rightarrow P(\Sigma^\omega \times \mathbb{N})$$

Elements of $P(\Sigma^\omega \times \mathbb{N})$ will be denoted by $\Theta, \Theta', \dots$. Singleton sets of $P(\Sigma^\omega \times \mathbb{N})$ including the empty set will be denoted by $\theta, \theta', \dots$.

Next we define $\langle \tau, n \rangle, \gamma, \eta \models \psi$, the truth of the assertion ψ with respect to the environments γ, η and the sequence $\langle \tau, n \rangle$, by induction on the complexity of ψ .

DEFINITION 3.5.:

We treat the following cases:

- $\langle \tau, n \rangle, \gamma, \eta \models X$ iff $\langle \tau, n \rangle \in \eta(X)$
- $\langle \tau, n \rangle, \gamma, \eta \models LET\ z_a = t_a\ IN\ \psi\ NI$ iff $\langle \tau, n \rangle, \gamma\{\llbracket t_a \rrbracket(\gamma)(\tau(n))/z_a\}, \eta \models \psi$

where $\llbracket t_a \rrbracket(\gamma)(\tau(n))$ denotes the result of evaluating t_a in the logical environment γ with respect to the n^{th} state of the sequence τ .

- $\langle \tau, n \rangle, \gamma, \eta \models (\psi_1 U \psi_2)$ iff there exists a $m > n$ such that $\langle \tau, m \rangle, \gamma, \eta \models \psi_2$ and for all $n \leq k < m$ we have $\langle \tau, k \rangle, \gamma, \eta \models \psi_1$
- $\langle \tau, n \rangle, \gamma, \eta \models (\nu X \psi)$ iff $\langle \tau, n \rangle, \gamma, \eta\{\Theta/X\} \models \psi$

where $\Theta = \bigcup \{\Theta' : \Theta' \subseteq \llbracket \psi \rrbracket(\gamma)(\eta\{\Theta'/X\})\}$, with $\llbracket \psi \rrbracket(\gamma)(\eta) = \{\langle \tau, n \rangle : \langle \tau, n \rangle, \gamma, \eta \models \psi\}$.

With respect to the last clause we require that X occurs positively in ψ , that is, only in the scope of an even number of negations. In that case ψ can be viewed as a monotonic operator on the complete lattice $P(\Sigma^\omega \times \mathbb{N})$ (ordered with respect to set inclusion), and the set Θ as its greatest fixed point (see lemma 2.1.: substitute $P(\Sigma^\omega \times \mathbb{N})$ for $\mathcal{L}$ and interpret f as $f(\Theta) = \llbracket \psi \rrbracket(\gamma)(\eta\{\Theta/X\})$).

Next we define the notion of validity of an assertion ψ , written as $\models \psi$. To this end we fix some predicate variable Y . Now we define ψ to be valid iff for every (infinite) sequence and suffix of this sequence ψ holds in this sequence when the predicate variable Y is interpreted as this suffix. More precisely: We define an assertion ψ to be valid if for an arbitrary logical environment γ , predicate environment η , and sequence $\langle \tau, n \rangle$ such that $\eta(Y) = \{\langle \tau, m \rangle\}$, for some $n \leq m \leq \omega$, where we define $\{\langle \tau, \omega \rangle\} = \emptyset$, $\langle \tau, n \rangle, \gamma, \eta \models \psi$ holds. To formalize this definition we need the following:

DEFINITION 3.6.:

Let $X \subseteq \Sigma^\omega$, $\tau \in \Sigma^\omega$. We define:

$$CONT(X, \langle \tau, n \rangle) = \{\{\langle \tau, m \rangle\} : n \leq m \leq \omega \wedge \tau[n, m] \in X\}$$

where we define $\{\langle \tau, \omega \rangle\} = \emptyset$.

$CONT(X, \langle \tau, n \rangle)$ essentially denotes all the suffixes of the sequence $\tau[n, \omega]$ such that the prefix of $\tau[n, \omega]$ determined by such a suffix belongs to this set of sequences X .

In the next definition we define formally the notion of validity as described above.

DEFINITION 3.7.:

We define

$$\models \psi \text{ iff } \forall \gamma, \eta, \langle \tau, n \rangle, \theta \in CONT(\Sigma^\infty, \langle \tau, n \rangle): \langle \tau, n \rangle, \gamma, \eta\{\theta/Y\} \models \psi$$

Note that we have $\models \Box \neg Y \vee (\neg YUY \wedge \Box \neg Y)$. This axiom expresses that the predicate variable Y denotes some unique suffix. In fact the following is the case. Define: $\models \psi$ iff for every $\langle \tau, n \rangle, \gamma, \eta$ we have $\langle \tau, n \rangle, \gamma, \eta \models \psi$, so $\models$ denotes the usual notion of validity. Suppose $\vdash$ completely axiomatizes $\models$.

Then $\vdash$ together with the axiom $\Box \neg Y \vee (\neg YUY \wedge \Box \neg Y)$ completely axiomatizes $\models$ because of the following lemma.

LEMMA 3.8.:

For an arbitrary assertion ψ we have

$$\models \psi \text{ iff } \vdash \Box \neg Y \vee (\neg YUY \wedge \Box \neg Y) \rightarrow \psi.$$

PROOF :

$\Rightarrow$: Let $\gamma, \eta, \langle \tau, n \rangle$ such that

$$\langle \tau, n \rangle, \gamma, \eta \models \Box \neg Y \vee (\neg YUY \wedge \Box \neg Y) \quad (*)$$

We have

$$\langle \tau, n \rangle, \gamma, \eta \models \psi \text{ iff } \langle \tau, n \rangle, \gamma, \eta' \models \psi$$

where $\eta'(Y) = \eta(Y) \cap \{\langle \tau, m \rangle : n \leq m < \omega\}$, and with respect to all other predicate variables η' agrees with η (a proof of this is straightforward but tedious). But (*) then implies that for some $\theta \in CONT(\langle \tau, n \rangle)$ we have $\eta'(Y) = \theta$. Now $\langle \tau, n \rangle, \gamma, \eta\{\theta/Y\} \models \psi$ follows directly from $\models \psi$.

$\Leftarrow$: Evident $\square$

In the sequel we abbreviate $CONT(\Sigma^\infty, \langle \tau, n \rangle)$ to $CONT(\langle \tau, n \rangle)$.

A specification will be a formula $\{S\}_C \psi$ such that $CON(S) \subseteq C = (V, W)$ (pointwise containment), $VAR(\psi) \subseteq V$, $CHAN(\psi) \subseteq W$, and Y occurs positively in ψ . ($VAR(\psi)$ and $CHAN(\psi)$ denote the set of program variables and channels which occur in ψ .) The truth of such a specification is defined as follows:

DEFINITION 3.9.:

We define:

$$\models \{S\}_C \psi \text{ iff } \forall \gamma, \eta, \langle \tau, n \rangle, \theta \in CONT(O(C)(S), \langle \tau, n \rangle): \langle \tau, n \rangle, \gamma, \eta\{\theta/Y\} \models \psi$$

Intuitively: A specification $\{S\}_C \psi$ holds iff ψ holds in every sequence consisting of a prefix which is a computation of S when Y is interpreted as the suffix of this sequence determined by this particular prefix, that is, when Y is interpreted as the continuation of this computation of S .

Programs are specified by formulas of the form $\{P\}\psi$, the semantics of which is given by the following definition:

DEFINITION 3.10.:

We define:

$$\models \{P\}\psi \text{ iff } \forall \gamma, \eta, \langle \tau, n \rangle \text{ such that } \tau[n, \omega] \in O(P) \text{ we have } \langle \tau, n \rangle, \gamma, \eta \models \psi$$

We assume that ψ does not contain free occurrences of the predicate variable Y . This assumption makes sense because at the top level we need not reason about possible continuations.

4. THE PROOF SYSTEM

In this section we present the proof system for specifications as defined in the previous section. The axioms and rules of this system will closely mirror the semantic definitions of section 2.

INVARIANCE

$$\{S\}_C \text{ LET } z=t \text{ IN } z=t \text{ U } z=t \wedge Y \text{ NI}$$

Here S is an atomic statement and if $C=(V,W)$ then $t \in (V \cup W) \setminus \text{change}(S)$,

with $\text{change}(c!t) = \{c\}$,

$$\text{change}(c?t) = \{c\},$$

$$\text{change}(c?y) = \{c, y\},$$

$$\text{change}(x:=t) = \{x\},$$

$$\text{change}(\text{skip}) = \emptyset.$$

This axiom states the invariance of the value of the term t during the computation of the atomic statement S . Note how this *LET*-construct is used to “freeze” the initial value of t . The continuation of this computation is denoted by the variable Y .

ASSIGNMENT

$$\{x:=t\}_C \text{ LET } z=x \text{ IN } z=x \text{ U } x=t[z/x] \wedge Y \text{ NI}$$

This axiom describes the result of the assignment $x:=t$ on the variable x . The initial value of x is stored in the logical variable z . After the execution of $x:=t$ the value of x can then be described by the term $t[z/x]$. The continuation of the execution of the assignment statement is denoted by the predicate variable Y .

OUTPUT

$$\{c!t\}_C \text{ LET } z=c \text{ IN } z=c \text{ U } c=z \circ \langle t \rangle \wedge Y \text{ NI}$$

This axiom describes the effect of the output statement $c!t$ as adding the value of the term t to the sequence of data transferred on the channel c .

INPUT

$$\{c?y\}_C \text{ LET } z_1, z_2=c, y \text{ IN } z_1=c \wedge z_2=y \text{ U } c=z_1 \circ \langle y \rangle \wedge Y \text{ NI}$$

Note that this axiom states nothing about the value of the variable y after the execution of $c?y$. It states only that after the execution of $c?y$ a certain relation must hold between c and y . This axiom describes the effect of the input statement $c?y$ as assigning some arbitrary value to the variable y and adding this value to the sequence of data transferred on the channel c . The logical variables z_1 and z_2 are used to freeze the initial values of the channel c and the variable y .

SEQUENTIAL COMPOSITION

$$\frac{\{S_1\}_C \psi_1, \{S_2\}_C \psi_2}{\{S_1; S_2\}_C \psi_1[\psi_2/Y]}$$

In this rule $\psi_1[\psi_2/Y]$ denotes the substitution of ψ_2 for Y in ψ_1 .

This substitution can be performed only if no free occurrences of predicate and logical variables of ψ_2 get bound. If this is the case we first have to rename some bound predicate and logical variables of ψ_1 . The intuition behind this rule can be described as follows. If $\vdash \{S_i\}_C \psi_i$ then one can view ψ_i as describing some computation of S_i followed by some continuation denoted by the predicate variable Y . But in the context of $S_1; S_2$ occurrences of Y in ψ_1 , denoting some continuation of S_1 , will denote a sequence consisting of a computation of S_2 , i.e., a sequence satisfying ψ_2 when Y in ψ_2 is interpreted as the continuation of $S_1; S_2$. Now we can make occurrences of Y in ψ_1 refer to sequences satisfying ψ_2 by substituting ψ_2 for Y in ψ_1 .

CHOICE

We have the following rule for the alternative command:

$$\frac{\{S_i\}_C \psi_i}{\{\Box_i b_i \rightarrow S_i\}_C \bigvee_i (b_i \wedge \psi_i)}$$

ITERATION

The iterative command is described by the following rule:

$$\frac{\{S_i\}_C \psi_i, \{skip\}_C \psi}{\{\star \Box_i b_i \rightarrow S_i\} \nu X (\bigvee_i (b_i \wedge \psi_i[X/Y]) \vee (\bigwedge_i \neg b_i \wedge \psi))}$$

Here X is a fresh predicate variable.

The idea of this rule can be described as follows. The predicate variable Y in ψ_i denotes some continuation of S_i . But in the context of $\star \Box_i b_i \rightarrow S_i$ the continuation of S_i consists of an execution of $\star \Box_i b_i \rightarrow S_i$. This can be described by applying the fixed point operator to a fresh predicate variable which is substituted for Y in ψ_i . Note that the assertion ψ , which specifies the statement *skip*, is used to describe finite stuttering.

PARALLEL COMPOSITION

We have the following rule for programs:

$$\frac{\{S_i; halt\}_{C_i} \psi_i}{\{S_1 \parallel \dots \parallel S_n\} \wedge_i \psi_i [false/Y]}$$

where $C_i = CON(S_i)$.

We may substitute *false* for Y in ψ_i because at this level there are no continuations. Note that freedom from interference is guaranteed by the syntactic restriction that $VAR(\psi_i) \subseteq VAR(S_i)$ and $CHAN(\psi_i) \subseteq CHAN(S_i)$.

We conclude this presentation of the proof system with the following three rules:

CONSEQUENCE RULES

$$\frac{\{S\}_C \psi_1, \psi_1 \rightarrow \psi_2}{\{S\}_C \psi_2} \quad \frac{\{P\} \psi_1, \psi_1 \rightarrow \psi_2}{\{P\} \psi_2}$$

CONJUNCTION RULE

$$\frac{\{S\}_C \psi_1, \{S\}_C \psi_2}{\{S\}_C \psi_1 \wedge \psi_2}$$

5. SOUNDNESS

The soundness of the system is proved by induction on the length of the derivation. In fact, it suffices to show that the axioms are valid, and that the rules preserve validity, i.e., the validity of the conclusion of a rule follows from that of its premises. In this section we prove the soundness of the rule for sequential composition.

LEMMA 5.1.:

If $\models \{S_1\}_C \psi_1$ and $\models \{S_2\}_C \psi_2$ then $\models \{S_1; S_2\}_C \psi_1[\psi_2/Y]$.

PROOF:

Take a $\gamma, \eta, \langle \tau, n \rangle$ and $\theta \in \text{CONT}(O(C)(S_1; S_2), \langle \tau, n \rangle)$. Furthermore, let $\Theta = \llbracket \psi_2 \rrbracket(\gamma)(\eta\{\theta/Y\})$. (Recall that for an arbitrary ψ, γ, η : $\llbracket \psi \rrbracket(\gamma)(\eta) = \{\langle \tau, n \rangle : \langle \tau, n \rangle, \gamma, \eta \models \psi\}$.) We have

$$\langle \tau, n \rangle, \gamma, \eta\{\theta/Y\} \models \psi_1[\psi_2/Y] \text{ iff } \langle \tau, n \rangle, \gamma, \eta\{\Theta/Y\} \models \psi_1$$

A proof of the above equivalence we omit, it is straightforward but tedious. As $\theta \in \text{CONT}(O(C)(S_1; S_2), \langle \tau, n \rangle)$ there exist n_1 and n_2 such that

$$n \leq n_1 \leq \omega \text{ and } \tau[n, n_1] \in O(C)(S_1),$$

$$n_1 \leq n_2 \leq \omega \text{ and } \theta = \{\langle \tau, n_2 \rangle\}, \text{ and if } n_1 < \omega \text{ then } \tau[n_1, n_2] \in O(C)(S_2).$$

Let $\theta' = \{\langle \tau, n_1 \rangle\}$. So we have $\theta' \in \text{CONT}(O(C)(S_1), \langle \tau, n \rangle)$. From $\models \{S_1\}_C \psi_1$ we then infer $\langle \tau, n \rangle, \gamma, \eta\{\theta'/Y\} \models \psi_1$. Now suppose $n_1 < \omega$: From $\theta \in \text{CONT}(O(C)(S_2), \langle \tau, n_1 \rangle)$ and $\models \{S_2\}_C \psi_2$ we infer $\langle \tau, n_1 \rangle, \gamma, \eta\{\theta/Y\} \models \psi_2$. So we have $\theta' \subseteq \llbracket \psi_2 \rrbracket(\gamma)(\eta\{\theta/Y\})$ (if $n_1 = \omega$ then $\theta' = \emptyset$). From $\langle \tau, n \rangle, \gamma, \eta\{\theta'/Y\} \models \psi_1$ and the monotonicity of ψ_1 in Y we conclude: $\langle \tau, n \rangle, \gamma, \eta\{\Theta/Y\} \models \psi_1$, or, equivalently, $\langle \tau, n \rangle, \gamma, \eta \models \psi_1[\psi_2/Y]$. $\square$

It is not difficult to see that for the rule for sequential composition to be sound it is necessary that Y occurs positively in ψ_1 . Consider the following: Let $S_1, S_2 \equiv x := x$ and $\psi_1 \equiv \neg Y, \psi_2 \equiv \text{true}, C = (\{x\}, \emptyset)$. Then $\models \{S_i\}_C \psi_i$. Note that ψ_1 expresses that the execution of the statement $x := x$ does not terminate immediately. But it is not the case that $\models \{x := x; x := x\}_C \text{false!}$

6. COMPLETENESS

In this section we prove that an arbitrary well-formed specification about a statement or a program is derivable using as additional axioms all the valid assertions (the underlying temporal theory). (Note that we assume the interpretation I to be fixed.) We prove completeness by associating with each statement S and context C such that $\text{CON}(S) \subseteq C$ an assertion ψ such that ψ defines $O(C)(S)$. First we will formalize this notion of an assertion defining the semantics of a statement.

DEFINITION 6.1:

Given the environments γ and η we define the extension of an assertion ψ , which we denote by $\llbracket \psi \rrbracket_{\gamma, \eta}$, as follows:

$$\llbracket \psi \rrbracket_{\gamma, \eta} = \min\{\langle \tau, n \rangle, \theta : \theta \in \text{CONT}(\langle \tau, n \rangle) \wedge \langle \tau, n \rangle, \gamma, \eta\{\theta/Y\} \models \psi\}$$

where the operation *min* takes all the minimal elements with respect to the following ordering:

$$(\langle \tau, n \rangle, \theta) \leq (\langle \tau', m \rangle, \theta') \text{ iff } \tau = \tau' \wedge n = m \wedge \theta \subseteq \theta'.$$

Note that $\theta \subseteq \theta'$, for $\theta, \theta' \in \text{CONT}(\langle \tau, n \rangle)$ iff $\theta = \theta'$ or $\theta = \emptyset$.

We demand ψ to be monotonic in Y , therefore we apply the operation *min* to the set of pairs $(\langle \tau, n \rangle, \theta)$, with $\theta \in \text{CONT}(\langle \tau, n \rangle)$, such that ψ holds in $\langle \tau, n \rangle$ when Y is interpreted as θ .

DEFINITION 6.2:

Given the environments γ and η we define the set of sequences determined by an assertion ψ , which we denote by $O_{\gamma, \eta}(\psi)$, as follows:

$$O_{\gamma, \eta}(\psi) = \{\tau[n, m] : (\langle \tau, n \rangle, \theta) \in [\psi]_{\gamma, \eta}, \text{ with } \theta = \{\langle \tau, m \rangle\}\}$$

Now $O_{\gamma, \eta}(\psi) \subseteq O(C)(S)$ for an arbitrary γ, η formalizes the notion of ψ defining the semantics of S relative to the context C . We next prove that for an arbitrary statement S and context C (with $\text{CON}(S) \subseteq C$) there exists an assertion ψ such that

$$\vdash \{S\}_C \psi \text{ and for every } \gamma, \eta \text{ we have } O_{\gamma, \eta}(\psi) \subseteq O(C)(S).$$

Completeness then follows: Let $\vdash \{S\}_C \phi$, $\{S\}_C \phi$ being a well-formed specification. Furthermore, let ψ such that $\vdash \{S\}_C \psi$ and for every γ, η we have $O_{\gamma, \eta}(\psi) \subseteq O(C)(S)$. As we have $\vdash \{S\}_C \phi$ by an application of the consequence rule it thus suffices to show that $\vdash \psi \rightarrow \phi$. So let $\gamma, \eta, \langle \tau, n \rangle, \theta \in \text{CONT}(\langle \tau, n \rangle)$ such that $\langle \tau, n \rangle, \gamma, \eta \{\theta/Y\} \models \psi$. First suppose that $(\langle \tau, n \rangle, \theta) \in [\psi]_{\gamma, \eta}$, say $\theta = \{\langle \tau, m \rangle\}$: So we have $\tau[n, m] \in O_{\gamma, \eta}(\psi) \subseteq O(C)(S)$. By $\vdash \{S\}_C \phi$ we then infer that $\langle \tau, n \rangle, \gamma, \eta \{\theta/Y\} \models \phi$. If on the other hand $(\langle \tau, n \rangle, \theta) \notin [\psi]_{\gamma, \eta}$ we have $(\langle \tau, n \rangle, \emptyset) \in [\psi]_{\gamma, \eta}$. So we have $\tau[n, \omega] \in O_{\gamma, \eta}(\psi) \subseteq O(C)(S)$. Now by $\vdash \{S\}_C \phi$ we infer that $\langle \tau, n \rangle, \gamma, \eta \{\emptyset/Y\} \models \phi$. But ϕ is monotonic in Y so we conclude $\langle \tau, n \rangle, \gamma, \eta \{\theta/Y\} \models \phi$.

LEMMA 6.1.:

For an arbitrary statement S and context C , with $\text{CON}(S) \subseteq C$, there exists an assertion ψ such that $\vdash \{S\}_C \psi$, $\{S\}_C \psi$ being a well-formed specification, and for every γ, η we have $O_{\gamma, \eta}(\psi) \subseteq O(C)(S)$.

PROOF:

We show that for every statement S and context C , with $\text{CON}(S) \subseteq C$, there exists an assertion ψ such that $\vdash \{S\}_C \psi$ and if $\langle \tau, n \rangle, \gamma, \eta \models \psi$ (γ, η arbitrary) then for some $\theta = \{\langle \tau, m \rangle\} \in \text{CONT}(\langle \tau, n \rangle)$, with $\theta \subseteq \eta(Y)$, we have $\tau[n, m] \in O(C)(S)$. This suffices: Let $(\langle \tau, n \rangle, \theta) \in [\psi]_{\gamma, \eta}$, it then follows that for some $\theta' = \{\langle \tau, m \rangle\} \subseteq \theta$ we have $\tau[n, m] \in O(C)(S)$. By $\vdash \{S\}_C \psi$ (which follows from $\vdash \{S\}_C \psi$ and the soundness of the system) we have $\langle \tau, n \rangle, \gamma, \eta \{\theta'/Y\} \models \psi$. But $(\langle \tau, n \rangle, \theta) \in [\psi]_{\gamma, \eta}$ so we conclude $\theta' = \theta$.

The proof goes by induction on the complexity of S . We treat the cases of S an atomic statement and of $S \equiv S_1; S_2$.

S atomic: We illustrate this case by a simple example. Let $S \equiv x := y$ and $C = (\{x, y\}, \emptyset)$.

Now:

$$\vdash \{x := y\}_C \psi_1$$

by the invariance axiom, where $\psi_1 \equiv \text{LET } z_1 = y \text{ IN } z_1 = y \text{ U } z_1 = y \wedge Y \text{ NI}$. Furthermore we have:

$$\vdash \{x := y\}_C \psi_2$$

by the assignment axiom, where $\psi_2 \equiv \text{LET } z_2 = x \text{ IN } z_2 = x \text{ U } x = y \wedge Y \text{ NI}$. So we infer

$$\vdash \{x := y\}_C \psi_1 \wedge \psi_2$$

by the conjunction rule. Finally we conclude

$$\vdash \{x:=y\}_C \psi$$

by the consequence rule, where $\psi \equiv LET \ z_1, z_2 = y, x \ IN \ z_1 = y \wedge z_2 = x \ U \ z_1 = y \wedge x = y \wedge Y \ NI$ (use the validity of the assertion $\Box \neg Y \vee (\neg Y \ U \ Y \wedge \Box \neg Y)$, which expresses that the variable Y denotes some unique continuation). Now it is straightforward to check that if $\langle \tau, n \rangle, \gamma, \eta \models \psi$ then for some $\theta = \{\langle \tau, m \rangle\} \subseteq \eta(Y)$ we have $\tau[n, m] \in O(C)(x:=y)$.

$S \equiv S_1; S_2$: The induction hypothesis gives us assertions ψ_1 and ψ_2 such that, among others, $\vdash \{S\}_C \psi_1$ and $\vdash \{S_2\}_C \psi_2$. So by the rule for sequential composition we have $\vdash \{S_1; S_2\}_C \psi_1[\psi_2/Y]$. Let $\gamma, \eta, \langle \tau, n \rangle$ such that $\langle \tau, n \rangle, \gamma, \eta \models \psi_1[\psi_2/Y]$. We show that there exists a $n \leq m \leq \omega$ such that $\tau[n, m] \in O(C)(S)$ and $\{\langle \tau, m \rangle\} \subseteq \eta(Y)$: Let $\Theta = [\psi_2](\gamma)(\eta)$. We have

$$\langle \tau, n \rangle, \gamma, \eta \models \psi_1[\psi_2/Y] \text{ iff } \langle \tau, n \rangle, \gamma, \eta \models \Theta/Y \models \psi_1.$$

By the induction hypothesis we have for some $n \leq k \leq \omega$

$$\{\langle \tau, k \rangle\} \subseteq \Theta \text{ and } \tau[n, k] \in O(S_1)(C).$$

If $k = \omega$ then $\tau[n, \omega] \in O(C)(S)$ and we are done ($\{\langle \tau, \omega \rangle\} = \emptyset$). So suppose $k < \omega$: By the induction hypothesis ($\langle \tau, k \rangle, \gamma, \eta \models \psi_2$) we have for some $k \leq m \leq \omega$ that

$$\{\langle \tau, m \rangle\} \subseteq \eta(Y) \text{ and } \tau[k, m] \in O(C)(S_2).$$

By the definition of the semantics of statements it then follows that $\tau[n, m] \in O(C)(S_1; S_2)$, which concludes the proof. $\square$

Now let $P \equiv S_1 \parallel \dots \parallel S_n$. We prove that an arbitrary valid specification of P is derivable. It suffices to show that there exists an assertion ψ such that for every $\gamma, \eta, \langle \tau, n \rangle$ if $\langle \tau, n \rangle, \gamma, \eta \models \psi$ then $\tau[n, \omega] \in O(P)$ and $\vdash \{P\} \psi$. From this completeness easily follows: $\vdash \{P\} \phi$ implies $\vdash \psi \rightarrow \phi$ (note that as Y does not occur in ψ and ϕ we have $\vdash \psi \rightarrow \phi$ iff for every $\gamma, \eta, \langle \tau, n \rangle$ if $\langle \tau, n \rangle, \gamma, \eta \models \psi$ then $\langle \tau, n \rangle, \gamma, \eta \models \phi$). Now by the previous lemma there exist assertions ψ_i such that for every γ, η we have $O_{\gamma, \eta}(\psi_i) \subseteq O(C_i)(S_i; halt)$, with $C_i = CON(S_i)$, and $\vdash \{S_i; halt\}_C \psi_i$. Let $\langle \tau, n \rangle, \gamma, \eta \models \bigwedge_i \psi_i[false/Y]$, with $\langle \tau, n \rangle, \gamma, \eta$ arbitrary. It then follows that $\langle \tau, n \rangle, \gamma, \eta \models \{\emptyset/Y\} \models \psi_i$, or $(\langle \tau, n \rangle, \emptyset) \in [\psi_i]_{\gamma, \eta}$. As $O_{\gamma, \eta}(\psi_i) \subseteq O(C_i)(S_i; halt)$ we have $\tau[n, \omega] \in O(C_i)(S_i; halt)$, so we infer $\tau[n, \omega] \in O(P)$. Thus we can put $\psi \equiv \bigwedge_i \psi_i[false/Y]$.

7. NESTED PARALLELISM

In this last section we show how to handle nested parallelism. One of the main problems in specifying nested parallelism, in a compositional manner, in temporal logic is how to describe the behaviour arising from executions of parallel statements in which one component terminates while some other components are still executing. (To define $S_1 \parallel S_2$ in terms of $S_i; halt$, as we did in section 2, does not work because this generates only infinite computations so that we can not describe the semantics of statements like $(S_1 \parallel S_2); S_3$.) To be able to describe such behaviour we extend our temporal language with quantification over propositional variables ($[W]$). Let $PROP$ be a set of propositional variables with typical element p . States are extended with a component $PROP \rightarrow BOOL$ which gives meaning to the propositional variables, $BOOL$ being the class of boolean values. The class of assertions is extended by the following rule.

$$\psi ::= \exists p \psi$$

An assertion $\exists p \psi$ is interpreted as follows.

DEFINITION 7.1.:

We define:

$\langle \tau, n \rangle, \gamma, \eta \models \exists p \psi$ iff for some τ' differing from τ only with respect to p we have $\langle \tau', n \rangle, \gamma, \eta \models \psi$.

Here two sequences τ and τ' are said to differ with respect to a propositional variable p iff every two states $\tau(n)$ and $\tau'(n)$ agree with respect to the sets VAR , $CHAN$ and $PROP \setminus \{p\}$. Next we extend our class of statements by the following rule:

$$S ::= S_1 \parallel S_2$$

$$\text{where } IN(S_1) \cap IN(S_2) = OUT(S_1) \cap OUT(S_2) = VAR(S_1) \cap VAR(S_2) = \emptyset$$

$IN(S)$ ($OUT(S)$) denotes the set of input (output) channels occurring in S . This restriction on channels generalizes the restriction as described in the first section. A program is now just a statement. To describe the semantics of statements of the form $S_1 \parallel S_2$ we will use these propositional variables to signal the termination of the subprocesses executing the statements S_1 and S_2 . We will associate with S_i a variable p_i such that p_i is false during the execution of S_i and becomes true when the execution of S_i has terminated. To model this compositionally we extend our notion of a context. We define a context C to be a triple (V_1, V_2, V_3) , where $V_1 \subseteq VAR$, $V_2 \subseteq CHAN$, and $V_3 \subseteq PROP$. When we associate a context $C = (V_1, V_2, V_3)$ with a statement S , as will be the case in a specification $\{S\}_C \psi$, then the variables of V_3 are supposed to signal the termination of those processes executing a statement of which S is a substatement. This implies that during the execution of S all the variables of V_3 must be false. This is formalized by the following axiom:

$$\bullet \{S\}_C \neg p \ U \ Y$$

where, assuming $C = (V_1, V_2, V_3)$, $p \in V_3$. This axiom states that while S has not terminated the process, the termination of which is signaled by p , has not terminated either. Note that Y marks the termination of S .

To describe the termination protocol we introduce the auxiliary statements $p?$ and $p := ?$ the semantics of which is given by the following specifications:

$$\bullet \{p?\}_C p \wedge Y.$$

So $p?$ tests if p holds and terminates, otherwise it aborts.

$$\bullet \{p := ?\}_C \neg p \ U \ Y.$$

This specification describes the effect of assigning an arbitrary value to p , assuming p to be true initially. Let p_1 signal the termination of S_1 and p_2 that of S_2 , the termination protocol then is described by the parallel execution of the statements $p_i?; \text{WHILE } \neg p_j \text{ DO } p_j := ? \text{ OD}$, $i \neq j$.

Now we can formulate the following proof rule for nested parallelism:

$$\bullet \frac{\{S_i\}_{C_i} \psi_i, \{p_i?; \text{WHILE } \neg p_j \text{ DO } p_j := ? \text{ OD}\}_{C_i} \phi_i}{\{S_1 \parallel S_2\}_C \exists p_1, p_2 (\psi_1[\phi_1 / Y] \wedge \psi_2[\phi_2 / Y])}$$

where, assuming $C = (V_1, V_2, V_3)$, we have $C_i = (V_1 \setminus VAR(S_j), V_2 \cap CHAN(S_i), V_3 \cup \{p_i\})$, with p_1, p_2 fresh propositional variables (neither occurring in ψ_i , free or bound, nor in V_3) and $i \neq j \in \{1, 2\}$. Furthermore, $C'_i = (V_1 \setminus VAR(S_j), V_2 \cap CHAN(S_i), V_3)$.

First note that the first two components of C_i guarantee freedom from interference: ψ_i will neither be allowed to mention variables of S_j nor those channels of S_j which do not connect S_j with S_i . Furthermore by adding the variable p_i to V_3 we know by the above axiom that during the execution of S_i p_i will be false. Note that as p_j is not a member of V_3 nothing can be said about this variable by ψ_i .

We abstract from the control information encoded by the variables p_1 and p_2 by existentially quantifying them in the conclusion of the rule.

A proof of the soundness and completeness of this proof system for nested parallelism goes along similar lines as the one given. In these proofs we use a slightly different notion of a continuation, that is, we had to modify the interpretation of the variable Y to make it consistent with the quantification over propositional variables. This different interpretation of the notion of a continuation can be axiomatized in a similar way as the old one.

CONCLUSION

We have developed a new method for specifying in a compositional manner, in a temporal logic with as sole temporal operator the until, without making use of location predicates, sequential composition, its iterated version, and its interaction with (nested) parallelism, in the context of a CSP-like language. We did so by specifying statements with respect to some possible continuation, denoted by some predicate variable. The reasoning process about these continuations can be easily formalized in the logic. We proved this method to be sound and complete with respect to a denotational semantics which associates with each program a set of sequences of states. The rules and axioms of the proof system closely mirror the semantical operations describing the different ways programs can be constructed.

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