The finite difference based fast adaptive composite grid method

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The Finite Difference Based Fast Adaptive Composite Grid Method

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Abstract

The fast adaptive composite grid (FAC) method is an iterative method for solving discrete boundary value problems on composite grids. McCormick introduced the method in [10] and considered the convergence behaviour for discrete problems resulting from finite volume element discretization on composite grids. In this paper we consider discrete problems resulting from finite difference discretization on composite grids. We distinguish between two obvious discretization approaches at the grid points on the interfaces between fine and coarse subgrids. The FAC method for solving such discrete problems is described. In the FAC method several intergrid transfer operators appear. We study how the convergence behaviour depends on these intergrid transfer operators. Based on theoretical insights, (quasi-)optimal intergrid transfer operators are derived. Numerical results illustrate the fast convergence of the FAC method using these intergrid transfer operators.

A.M.S. Classifications: 65N22, 65N50

Keywords : finite difference discretization, composite grid, iterative method, fast adaptive composite grid method, convergence rate

1 Introduction

Many practical boundary value problems produce solutions which contain several *high activity regions*. In these regions the solution varies much more rapidly than in the remaining part of the domain. This behaviour of the solution may be caused by the differential operator itself, by the forcing term in the differential equation, by the boundary conditions or by an irregular boundary (e.g. a re-entrant corner).

Due to the large variations of the solution in the high activity regions, a relatively small mesh size of the discretization grid is required in these regions to obtain a sufficiently accurate approximation of the solution. Outside the high activity regions the behaviour of the solution is much more smooth and there a (much) larger mesh size of the discretization grid is sufficient. So, it seems reasonable to approximate the solution using *several uniform grids with various mesh sizes covering different parts of the domain* (see e.g. [1],[4],[9]). At least one grid should cover the entire domain. The mesh size of this *global coarse grid* is chosen in agreement with the smooth behaviour of the solution outside the high activity regions. Further, several *local grids* are used which are also uniform. Each of them covers only a (small) part of the domain and contains a high activity region. The grid size of each of these grids is chosen in agreement with the behaviour of the solution in the corresponding high activity region. In this way every part of the domain is covered by a uniform grid whose mesh size is in agreement with the behaviour of the continuous solution in that part of the domain. This refinement strategy is known as *local uniform grid refinement*. The solution is approximated on the *composite grid* which is the union of the uniform subgrids.

The *fast adaptive composite grid (FAC) method* [8, 9, 10] is an iterative method for solving a given discrete problem on a composite grid. This discrete problem may result from finite difference discretization, finite element discretization or finite volume discretization on the composite grid. In each iteration step of the FAC method a residual is computed on the composite grid, corrections are computed on the uniform subgrids and the composite grid approximation is corrected. In this way all actual computing, i.e solving (linear) systems of equations, takes place on the uniform subgrids. In [10] theory is presented for the FAC method in a variational setting (Variational FAC) and extended to the nonvariational case of Finite Volume Element based FAC.

In this paper we consider the FAC method for solving discrete problems resulting from *finite difference discretizations* on a composite grid. When discretizing on a composite grid one has to distinguish between locally uniform and locally nonuniform grid points. At locally uniform grid points (the greater number of grid points in a composite grid) standard finite difference stencils can be used. At locally nonuniform grid points (i.e. grid points on the *interfaces* between the uniform subgrids) one can distinguish between two obvious discretization approaches. One approach is to treat these grid points as grid points of the (uniform) coarsest grid which is related to the interface (see e.g. [3]). We refer to this approach as the *uniform interface discretization*. The second approach is to consider the composite grid as a really nonuniform grid near the interface and to use nonuniform finite difference stencils at the interface grid points (see e.g. [11]). We refer to this approach as *nonuniform interface discretization*.

In the FAC method intergrid transfer operators between the composite grid and the uniform subgrids are used. We shall show that the convergence behaviour of the FAC method depends mainly on *the way in which the composite grid residuals are restricted on the interfaces between the uniform subgrids*. An *optimality criterion* for the restriction operators is

defined. It is shown that, for a composite grid problem resulting from the uniform interface discretization, *each restriction operator which equals the trivial injection on the interfaces satisfies this optimality criterion*. With such a restriction operator, the FAC method for the one-dimensional Poisson problem becomes a direct solver. For more general problems the fast convergence of this FAC method is illustrated by numerical results. For composite grid problems resulting from nonuniform interface discretization one should not use trivial injection on the interface. For the one-dimensional Poisson problem a *weighted restriction operator which satisfies the optimality criterion* is derived. The FAC method with this restriction operator is a direct solver for this problem. For more general problems it is not clear (to us) how a (linear) weighted restriction operator satisfying the optimality criterion can be derived. Using the optimality criterion a *nonlinear FAC method* is developed. The convergence rate of this nonlinear FAC method for composite grid problems resulting from nonuniform interface discretization is comparable with the high convergence rate of the FAC method for composite grid problems resulting from the uniform interface discretization.

We note that there are other methods for approximating the continuous solution using a composite grid. In particular we mention the *local defect correction method*. For this method we refer to [4],[2],[3].

The remainder of this paper is organized as follows. In Subsection 2.1 we introduce a model composite grid which is composed of a uniform global coarse grid and a uniform local fine grid. In Subsection 2.2 we consider finite difference discretization on this model composite grid. In Subsection 2.3 the FAC iteration is introduced. In Subsection 2.4 we analyse the error propagation in the FAC method. The dependence of the convergence rate on the restriction operators is considered and an optimality criterion for the restriction operators is defined. In Section 3 we analyse the convergence behaviour of the FAC method for the one-dimensional Poisson problem. In Section 4 we consider the convergence behaviour of the FAC method for two-dimensional elliptic boundary value problems. A nonlinear FAC method for solving composite grid problems resulting from nonuniform interface discretization is developed.

2 The Fast Adaptive Composite Grid (FAC) Method

2.1 Notation

The *fast adaptive composite grid method* is an iterative solver for discrete problems resulting after discretizing a boundary value problem on a composite grid.

As a *model boundary value problem* we use

$$\begin{aligned} \mathcal{L}U &= f & \text{in } \Omega, \\ U &= g & \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

with $\Omega = (0, 1) \times (0, 1)$ and $\mathcal{L}$ a linear elliptic second-order differential operator.

As a *model composite grid* we use a nonuniform grid composed of a *uniform global coarse grid*, Ω^H , and a *uniform local fine grid*, Ω_l^h . The grid Ω^H is a uniform grid with size H covering the domain Ω . The grid Ω_l^h is a uniform grid with size $h < H$ covering a local region $\Omega_l \subset \Omega$. The composite grid is denoted by $\Omega^{H,h}$ for which we have $\Omega^{H,h} = \Omega^H \cup \Omega_l^h$. In the remainder we also use a *uniform local coarse grid*, Ω_l^H . This is a uniform grid with size H covering Ω_l . Examples of Ω^H , Ω_l^h , Ω_l^H and $\Omega^{H,h}$ are shown in Figure 1.

The *refinement factor* σ is defined by $\sigma := H/h$. We assume that $\sigma \in \mathbb{N}$ and that the *interface* Γ between Ω_l and the remainder of Ω coincides with grid lines of Ω^H . We introduce

Figure 1: Examples of Ω^H , Ω_l^h , $\Omega^{H,h}$, Γ^H , Γ^h and Ω_l^H respectively.

the *coarse interface grid* $\Gamma^H = \Gamma \cap \Omega^H$ and the *fine interface grid* $\Gamma^h = \Gamma \cap \Omega^h$, where Ω^h is the uniform grid with size h covering Ω . Examples of coarse and fine interface grids are shown in Figure 1.

To each grid a *set of grid functions* is related. For example, the set of grid functions related to Γ^H is denoted by $\mathcal{F}(\Gamma^H)$. For a grid function $v_\Gamma^H \in \mathcal{F}(\Gamma^H)$ we have $v_\Gamma^H = (v_\Gamma^H(\mathbf{x}))_{\mathbf{x} \in \Gamma^H}$. Given an ordering of the grid points, each grid function can be represented as a vector.

In the remainder we use the following *partitioning* of the composite grid:

$$\Omega^{H,h} = \Omega_l^h \cup \Gamma^H \cup \Omega_C^H, \quad (2.2)$$

where Ω_C^H consists of all grid points of Ω^H that lie neither inside Ω_l nor on Γ (see Figure 2).

2.2 Finite difference discretization on composite grids

The composite grid is composed of uniform subgrids. Thus, in the greater part of the domain the composite grid is *locally uniform*. In this part finite difference stencils related to a uniform grid can be used. So, first we consider finite difference discretization on the *uniform* grids Ω^H , Ω_l^h and Ω_l^H . When discretizing on uniform grids standard finite difference stencils can be used (see the stencils in Section 4).

The discrete problem resulting after finite difference discretization of boundary value problem (2.1) on the *uniform global coarse grid* Ω^H is denoted by

$$L^H u^H = f^H, \quad (2.3)$$

Figure 2: Partitioning of the composite grid. $\times$: grid points of Ω_C^H ; $\bullet$: grid points of Ω_l^h ; $\circ$: grid points of Γ^H .

Figure 3: Slave points ($\circ$) and composite grid points ($\bullet$).

with $u^H, f^H \in \mathcal{F}(\Omega^H)$ and $L^H : \mathcal{F}(\Omega^H) \rightarrow \mathcal{F}(\Omega^H)$. Here L^H is a finite difference operator related to the uniform discretization grid Ω^H .

The discrete problem which results after finite difference discretization of the boundary value problem

$$\begin{aligned} \mathcal{L}U &= f & \text{in } \Omega_l, \\ U &= g & \text{on } \partial\Omega \cap \partial\Omega_l, \\ U &= \tilde{g} & \text{on } \Gamma, \end{aligned} \quad (2.4)$$

on the *uniform local fine grid* Ω_l^h is denoted by

$$L_l^h u_l^h + L_\Gamma^h(\tilde{g}|_{\Gamma^h}) = f_l^h. \quad (2.5)$$

Here $u_l^h, f_l^h \in \mathcal{F}(\Omega_l^h)$ and $L_l^h : \mathcal{F}(\Omega_l^h) \rightarrow \mathcal{F}(\Omega_l^h)$, $L_\Gamma^h : \mathcal{F}(\Gamma^h) \rightarrow \mathcal{F}(\Omega_l^h)$. The right hand sides f and g in (2.4) are incorporated in f_l^h . The term $L_\Gamma^h(\tilde{g}|_{\Gamma^h})$ deals with the Dirichlet boundary condition on the interface Γ .

In the following we will also use the discrete problem which results after finite difference discretization of boundary value problem (2.4) on the uniform local coarse grid Ω_l^H :

$$L_l^H u_l^H + L_\Gamma^H(\tilde{g}|_{\Gamma^H}) = f_l^H, \quad (2.6)$$

with $u_l^H, f_l^H \in \mathcal{F}(\Omega_l^H)$, $L_l^H : \mathcal{F}(\Omega_l^H) \rightarrow \mathcal{F}(\Omega_l^H)$ and $L_\Gamma^H : \mathcal{F}(\Gamma^H) \rightarrow \mathcal{F}(\Omega_l^H)$. In the discretization process on the local coarse grid the same finite difference stencils are used as in the discretization process on the global coarse grid. So we have:

$$(L_l^H(w^H|_{\Omega_l^H}))(\mathbf{x}) + (L_\Gamma^H(w^H|_{\Gamma^H}))(\mathbf{x}) = (L^H w^H)(\mathbf{x}), \quad \mathbf{x} \in \Omega_l^H. \quad (2.7)$$

We assume that in the discretization processes on the uniform grids maximum 9-point stencils are used.

Now we come to finite difference discretization on the *composite grid* $\Omega^{H,h}$. At all grid points $\mathbf{x} \in \Omega^{H,h}$ the difference operator in (2.1) is replaced by some finite difference approximation. We denote the resulting *composite grid problem* by

$$L^{H,h} u^{H,h} = f^{H,h}, \quad (2.8)$$

with $u^{H,h}, f^{H,h} \in \mathcal{F}(\Omega^{H,h})$ and $L^{H,h} : \mathcal{F}(\Omega^{H,h}) \rightarrow \mathcal{F}(\Omega^{H,h})$.

In the discretization process on the composite grid the finite difference stencils from (2.3) are used at grid points $\mathbf{x} \in \Omega_C^H$:

$$(L^{H,h} w^{H,h})(\mathbf{x}) = (L^H(w^{H,h}|_{\Omega^H}))(\mathbf{x}) \quad \mathbf{x} \in \Omega_C^H. \quad (2.9)$$

At the grid points $\mathbf{x} \in \Omega_l^h$ the uniform finite difference stencils from (2.5) are used. At some grid points of Ω_l^h near the interface, these finite difference stencils involve unknowns at so called *slave points* $\mathbf{x} \in \Gamma^h \setminus \Gamma^H$ (see Figure 3). For the elimination of the unknowns at the slave points (using the unknowns at the grid points $\mathbf{x} \in \Gamma^H$) we introduce an interpolation operator $p_\Gamma : \mathcal{F}(\Gamma^H) \rightarrow \mathcal{F}(\Gamma^h)$. When interpolating between a point of $\partial\Omega$ and a grid point of Γ^H , a zero value at the point of $\partial\Omega$ is used. The given boundary value at this grid point appears in the right hand side vector of the composite grid problem. We will use piecewise linear or piecewise quadratic interpolation. We have:

$$(L^{H,h}w^{H,h})(\mathbf{x}) = (L_l^h(w^{H,h}|_{\Omega_l^h}))(\mathbf{x}) + (L_\Gamma^h p_\Gamma(w^{H,h}|_{\Gamma^H}))(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^h. \quad (2.10)$$

At grid points $\mathbf{x} \in \Gamma^H$ the composite grid is locally nonuniform. These grid points can be treated in several ways in the discretization process. One way is to treat these points as though they were grid points of the uniform global coarse grid. In this case the finite difference stencil at a grid point $\mathbf{x} \in \Gamma^H$ involves grid points of $\Omega^{H,h} \cap \Omega^H$ only and we have:

$$(L^{H,h}w^{H,h})(\mathbf{x}) = (L^H(w^{H,h}|_{\Omega^H}))(\mathbf{x}), \quad \mathbf{x} \in \Gamma^H \quad (2.11)$$

(see [3]). We will refer to this discretization process as *the uniform interface discretization*. Another way is to treat these points as grid points of an arbitrary nonuniform grid (see e.g. [11]). Then we do not use the underlying global coarse grid in the discretization process. We refer to such a discretization process as a *nonuniform interface discretization*. Examples of uniform and nonuniform discretizations on composite grids are given in Section 3 and Section 4. In the remainder we assume that all finite difference operators $L^{H,h}$, L^H , L_l^h and L_Γ^h are nonsingular.

2.3 Description of the FAC method

We describe the FAC method for solving the *composite grid problem* (2.8). Approximations of $u^{H,h}$ in (2.8) are computed in an iterative way. At each step two discrete problems on *uniform* grids are solved exactly: one discrete problem defined on Ω^H and one defined on Ω_l^h . The resulting solutions are used to improve the current approximation.

Let $\tilde{u}^{H,h}$ be an approximation of $u^{H,h}$ and consider the *composite grid defect* $d^{H,h} := f^{H,h} - L^{H,h}\tilde{u}^{H,h}$. The exact correction $v^{H,h} := u^{H,h} - \tilde{u}^{H,h}$ satisfies $L^{H,h}v^{H,h} = d^{H,h}$.

The composite grid defect $d^{H,h}$ is restricted to the global coarse grid and to the local fine grid. For the latter restriction we use the trivial injection: $d_l^h := d^{H,h}|_{\Omega_l^h}$.

For the former restriction we introduce the restriction operator $\tilde{r} : \mathcal{F}(\Omega^{H,h}) \rightarrow \mathcal{F}(\Omega^H)$ and we define $d^H := \tilde{r}d^{H,h}$. Outside Ω_l and away from Γ the trivial injection is used:

$$(\tilde{r}d^{H,h})|_{\Omega_C^H} := d^{H,h}|_{\Omega_C^H}. \quad (2.12)$$

At this point we do not specify $\tilde{r}$ inside Ω_l and on Γ . However, we assume that $\tilde{r}$ inside Ω_l does not involve grid points outside Ω_l .

An approximation $v^H \in \mathcal{F}(\Omega^H)$ of $v^{H,h}$ is computed by solving the global coarse grid problem

$$L^H v^H = d^H, \quad (2.13)$$

with L^H from (2.3). Next an approximation $v_l^h \in \mathcal{F}(\Omega_l^h)$ of $v^{H,h}$ is computed. The approximation v^H resulting from (2.13) is used to define Dirichlet boundary conditions on the interface. Values at grid points $\mathbf{x} \in \Gamma^h \setminus \Gamma^H$ are obtained by interpolation. We use the interpolation operator p_Γ from (2.10). We recall that when interpolating between a point on $\partial\Omega$ and a grid point on Γ^H , a zero value at the point on $\partial\Omega$ is used. The following local fine grid problem results (cf. (2.5), (2.10))

$$L_l^h v_l^h = d_l^h - L_\Gamma^h p_\Gamma(v^H|_{\Gamma^H}), \quad (2.14)$$

with L_l^h and L_Γ^h as in (2.10).

The approximations v^H from (2.13) and v_l^h from (2.14) are used to correct the approximation $\tilde{u}^{H,h}$ of $u^{H,h}$:

$$\bar{u}^{H,h}(\mathbf{x}) := \begin{cases} \tilde{u}^{H,h}(\mathbf{x}) + v_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ \tilde{u}^{H,h}(\mathbf{x}) + v^H(\mathbf{x}) & \mathbf{x} \in \Omega^{H,h} \setminus \Omega_l^h \end{cases}.$$

The fast adaptive composite grid method is an iterative process which combines local and global discrete problems in the way described above. At each iteration step an approximation of $u^{H,h}$ is computed.

FAC algorithm

Start: Initial composite grid approximation $u_0^{H,h}$ given.

$i = 1, 2, \dots$

a. Initialization

Computation of the composite grid defect

$$d^{H,h} := f^{H,h} - L^{H,h} u_{i-1}^{H,h} \quad (2.15a)$$

Restriction of the composite grid defect to the global coarse grid

$$d^H := \tilde{r} d^{H,h} \quad (2.15b)$$

Restriction of the composite grid defect to the local fine grid

$$d_l^h := d^{H,h}|_{\Omega_l^h} \quad (2.15c)$$

b. Exact solution of the global problem

$$L^H v^H = d^H \quad \text{on } \Omega^H \quad (2.15d)$$

c. Exact solution of the local problem

$$L_l^h v_l^h = d_l^h - L_\Gamma^h p_\Gamma(v^H|_{\Gamma^H}) \quad \text{on } \Omega_l^h \quad (2.15e)$$

d. Correction of the composite grid approximation

$$u_i^{H,h}(\mathbf{x}) := u_{i-1}^{H,h}(\mathbf{x}) + v_l^h(\mathbf{x}) \quad \mathbf{x} \in \Omega_l^h \quad (2.15f)$$

$$u_i^{H,h}(\mathbf{x}) := u_{i-1}^{H,h}(\mathbf{x}) + v^H(\mathbf{x}) \quad \mathbf{x} \in \Omega^{H,h} \setminus \Omega_l^h \quad (2.15g)$$

Iteration (2.15) was introduced by McCormick in [10] as the *fast adaptive composite grid method in delayed correction form*. Note that at this point we are still free to choose the restriction operator $\tilde{r}$ inside Ω_l and on Γ .

In each iteration step in (2.15) a defect on the composite grid is computed. After the first FAC step, this composite grid defect is equal to zero at all grid points which do not lie on the interface.

Lemma 2.1 *The FAC iterates $u_i^{H,h}$, $i \geq 1$, satisfy*

$$(f^{H,h} - L^{H,h}u_i^{H,h})(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_l^h \cup \Omega_C^H. \quad (2.16)$$

Proof. Let $i \geq 1$. Define

$$v^{H,h}(\mathbf{x}) := \begin{cases} v_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ v^H(\mathbf{x}) & \mathbf{x} \in \Omega^{H,h} \setminus \Omega_l^h \end{cases},$$

with v^H from (2.15d) and v_l^h from (2.15e). Then

$$f^{H,h} - L^{H,h}u_i^{H,h} = f^{H,h} - L^{H,h}u_{i-1}^{H,h} - L^{H,h}v^{H,h}.$$

At $\mathbf{x} \in \Omega_C^H$ we have:

$$\begin{aligned} (f^{H,h} - L^{H,h}u_{i-1}^{H,h})(\mathbf{x}) - (L^{H,h}v^{H,h})(\mathbf{x}) &\stackrel{(2.15a,2.9)}{=} d^{H,h}(\mathbf{x}) - (L^H v^H)(\mathbf{x}) \\ &\stackrel{(2.12,2.15b)}{=} (d^H - L^H v^H)(\mathbf{x}) \\ &\stackrel{(2.15d)}{=} 0. \end{aligned}$$

At $\mathbf{x} \in \Omega_l^h$ we have:

$$\begin{aligned} (f^{H,h} - L^{H,h}u_{i-1}^{H,h})(\mathbf{x}) - (L^{H,h}v^{H,h})(\mathbf{x}) &\stackrel{(2.15a,2.10)}{=} d^{H,h}(\mathbf{x}) - (L_l^h v_l^h)(\mathbf{x}) - (L_\Gamma^h p_\Gamma(v^H|_{\Gamma^H}))(\mathbf{x}) \\ &\stackrel{(2.15c)}{=} (d_l^h - L_l^h v_l^h - L_\Gamma^h p_\Gamma(v^H|_{\Gamma^H}))(\mathbf{x}) \\ &\stackrel{(2.15e)}{=} 0. \end{aligned} \quad \square$$

By Lemma 2.1 we find that, except for the first FAC step, we have to compute the composite grid defect on the interface only. After the first FAC step, the restriction of the composite grid defect to the local fine grid and to the grid points of the global coarse grid which are not part of the interface can be omitted. Thus, it is clear that the convergence rate of the FAC method only depends on the choice for the restriction operator $\tilde{r}$ on the interface. In the following subsection we analyse the error propagation in the FAC iteration.

2.4 Error propagation

The FAC iteration (2.15) is of the form:

$$u_i^{H,h} = M_c u_{i-1}^{H,h} + N_c f^{H,h}. \quad (2.17)$$

Since $u^{H,h}$ is a fixed point of the FAC iteration, we have

$$u^{H,h} = M_c u^{H,h} + N_c f^{H,h}. \quad (2.18)$$

In order to derive an expression for the iteration matrix we define the characteristic functions $\chi : \mathcal{F}(\Omega^H) \rightarrow \mathcal{F}(\Omega^{H,h})$ and $\chi_l : \mathcal{F}(\Omega_l^h) \rightarrow \mathcal{F}(\Omega^{H,h})$ by

$$(\chi w^H)(\mathbf{x}) := \begin{cases} w^H(\mathbf{x}) & \mathbf{x} \in \Omega_C^H \cup \Gamma^H \\ 0 & \mathbf{x} \in \Omega_l^h \end{cases}, \quad (\chi_l w_l^h)(\mathbf{x}) := \begin{cases} 0 & \mathbf{x} \in \Omega_C^H \cup \Gamma^H \\ w_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \end{cases}.$$

Further, we introduce the trivial injections $r : \mathcal{F}(\Omega^{H,h}) \rightarrow \mathcal{F}(\Omega^H)$ and $r_\Gamma : \mathcal{F}(\Omega^H) \rightarrow \mathcal{F}(\Gamma^H)$:

$$r w^{H,h} := w^{H,h}|_{\Omega^H}, \quad (2.19)$$

$$r_\Gamma w^H := w^H|_{\Gamma^H}. \quad (2.20)$$

Theorem 2.2 *The iteration matrix of the FAC iteration (2.15) satisfies*

$$M_c = \chi(L^H)^{-1} \{L^H r - \tilde{r} L^{H,h}\} - \chi_l(L_l^h)^{-1} L_\Gamma^h p_\Gamma r_\Gamma (L^H)^{-1} \{L^H r - \tilde{r} L^{H,h}\}. \quad (2.21)$$

Proof. First, we introduce the trivial injection $r_l : \mathcal{F}(\Omega^{H,h}) \rightarrow \mathcal{F}(\Omega_l^h)$ satisfying

$$r_l w^{H,h} := w^{H,h}|_{\Omega_l^h}.$$

Using (2.15), the definitions of χ , χ_l , r , r_l and r_Γ , and with $I^{H,h} : \mathcal{F}(\Omega^{H,h}) \rightarrow \mathcal{F}(\Omega^{H,h})$ denoting the identity operator on $\Omega^{H,h}$ we get

$$\begin{aligned} u_i^{H,h} &= u_{i-1}^{H,h} + \chi v^H + \chi_l v_l^h \\ &= u_{i-1}^{H,h} + \chi(L^H)^{-1} \tilde{r}(f^{H,h} - L^{H,h} u_{i-1}^{H,h}) \\ &\quad + \chi_l(L_l^h)^{-1} r_l(f^{H,h} - L^{H,h} u_{i-1}^{H,h}) - \chi_l(L_l^h)^{-1} L_\Gamma^h p_\Gamma r_\Gamma (L^H)^{-1} \tilde{r}(f^{H,h} - L^{H,h} u_{i-1}^{H,h}) \\ &= \{I^{H,h} - \chi(L^H)^{-1} \tilde{r} L^{H,h} - \chi_l(L_l^h)^{-1} r_l L^{H,h} + \chi_l(L_l^h)^{-1} L_\Gamma^h p_\Gamma r_\Gamma (L^H)^{-1} \tilde{r} L^{H,h}\} u_{i-1}^{H,h} \\ &\quad + \{\chi(L^H)^{-1} \tilde{r} + \chi_l(L_l^h)^{-1} r_l - \chi_l(L_l^h)^{-1} L_\Gamma^h p_\Gamma r_\Gamma (L^H)^{-1} \tilde{r}\} f^{H,h}. \end{aligned}$$

So, $M_c = I^{H,h} - \chi(L^H)^{-1} \tilde{r} L^{H,h} - \chi_l(L_l^h)^{-1} r_l L^{H,h} + \chi_l(L_l^h)^{-1} L_\Gamma^h p_\Gamma r_\Gamma (L^H)^{-1} \tilde{r} L^{H,h}$.

By (2.10) we have $r_l L^{H,h} = L_l^h r_l + L_\Gamma^h p_\Gamma r_\Gamma r$.

Using this we get

$$M_c = I^{H,h} - \chi_l r_l - \chi(L^H)^{-1} \tilde{r} L^{H,h} + \chi_l(L_l^h)^{-1} L_\Gamma^h p_\Gamma r_\Gamma (L^H)^{-1} \{\tilde{r} L^{H,h} - L^H r\}.$$

Note that $I^{H,h} - \chi_l r_l = \chi r$. Then (2.21) follows immediately. $\square$

The error of the i -th iterate is defined by $v_i^{H,h} := u^{H,h} - u_i^{H,h}$. From (2.17) and (2.18) we obtain

$$v_i^{H,h} = M_c v_{i-1}^{H,h}, \quad i \geq 1. \quad (2.22)$$

At the right hand side of (2.21) the term $\{L^H r - \tilde{r} L^{H,h}\}$ appears.

We define global coarse grid functions

$$\tilde{f}_i^H := L^H r v_{i-1}^{H,h} - \tilde{r} L^{H,h} v_{i-1}^{H,h}, \quad i \geq 1. \quad (2.23)$$

These functions *govern the error propagation*. In order to make this clear, we define the global coarse grid problems

$$L^H v_i^H = \tilde{f}_i^H, \quad i \geq 1. \quad (2.24)$$

and the related local fine grid problems

$$L_l^h v_{l,i}^h + L_\Gamma^h p_\Gamma r_\Gamma v_i^H = 0, \quad i \geq 1. \quad (2.25)$$

Lemma 2.3 *The error of the i -th FAC iterate satisfies*

$$v_i^{H,h} = \chi v_i^H + \chi_l v_{l,i}^h, \quad i \geq 1, \quad (2.26)$$

with v_i^H and $v_{l,i}^h$ from (2.24) and (2.25) respectively.

Proof. Substitute (2.21), (2.23)-(2.25) in (2.22). $\square$

By Lemma 2.3 we have that the size of the error in the i -th FAC iterate is determined by the size of the global coarse grid function $\tilde{f}_i^H$. At the right hand side of (2.23) the restriction operator $\tilde{r}$ appears. We shall analyse the influence of this restriction operator on the error propagation in the FAC method. In agreement with the partitioning of the composite grid in (2.2) we distinguish between three parts of the restriction operator.

The first part is related to Ω_C^H . We have assumed that $\tilde{r}$ is the trivial injection on Ω_C^H (cf. (2.12)). Since the composite grid operator and the global coarse grid operator are identical on Ω_C^H the right hand side in (2.23) is equal to zero at all grid points $\mathbf{x} \in \Omega_C^H$.

Lemma 2.4 *We have*

$$\tilde{f}_i^H(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega_C^H, \quad i \geq 1. \quad (2.27)$$

The proof of Lemma 2.4 follows immediately from the definition of $\tilde{f}_i^H$, (2.9) and (2.12).

The second part of $\tilde{r}$ is related to Ω_l^h . In the following lemma we show that $\tilde{f}_i^H$ does not depend on this part of $\tilde{r}$. We use the *trivial injection* $r_l : \mathcal{F}(\Omega_l^h) \rightarrow \mathcal{F}(\Omega_l^H)$ given by:

$$r_l w_l^h := w_l^h|_{\Omega_l^H}. \quad (2.28)$$

Lemma 2.5 *We have*

$$\tilde{f}_i^H(\mathbf{x}) = (L_l^H r_l v_{l,i-1}^h)(\mathbf{x}) + (L_\Gamma^H r_\Gamma v_{i-1}^H)(\mathbf{x}), \quad \mathbf{x} \in \Omega_l^H, \quad i \geq 2, \quad (2.29)$$

with v_i^H as in (2.24) and $v_{l,i}^h$ as in (2.25).

Proof. At $\mathbf{x} \in \Omega_l^h$ we have $(L^{H,h} v_i^{H,h})(\mathbf{x}) = (L_l^h v_{l,i}^h)(\mathbf{x}) + (L_\Gamma^h p_\Gamma r_\Gamma v_i^H)(\mathbf{x}) = 0, \quad i \geq 1.$

Since we have assumed that $\tilde{r}$ inside Ω_l does not involve grid points outside Ω_l^h , we have $(\tilde{r} L^{H,h} v_{c,i})(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_l^H, \quad i \geq 1.$

Thus, at $\mathbf{x} \in \Omega_l^H$ and for $i \geq 2$, we have

$$\begin{aligned} \tilde{f}_i^H(\mathbf{x}) &= (L^H r v_{i-1}^{H,h})(\mathbf{x}) \\ &\stackrel{(2.7)}{=} (L_l^H r_l v_{l,i-1}^h)(\mathbf{x}) + (L_\Gamma^H r_\Gamma v_{i-1}^H)(\mathbf{x}). \end{aligned} \quad \square$$

The restriction operator $\tilde{r}$ does not appear on the right hand side of (2.29). So the rate of convergence of the FAC method is independent of the restriction operator $\tilde{r}$ *inside* Ω_l .

Expression (2.29) does not hold for $i = 1$; the error reduction in the first FAC step does depend on the restriction operator $\tilde{r}$ inside Ω_l . These results are in agreement with Lemma 2.1.

The third part of $\tilde{r}$ is related to the interface. We recall that, for a given $i \geq 1$, (cf. (2.24),(2.25)),

$$\begin{aligned} L^H v_i^H &= \tilde{f}_i^H, \\ L_l^h v_{l,i}^h &= -L_\Gamma^h p_{\Gamma\Gamma} v_i^H. \end{aligned}$$

A bad choice of $\tilde{r}$ on the interface may result in relatively large components $|\tilde{f}_i^H(\mathbf{x})|$, $\mathbf{x} \in \Gamma^H$ (compared to $|\tilde{f}_i^H(\mathbf{x})|$, $\mathbf{x} \in \Omega_l^H$) and hence in relatively large errors $|v_i^H(\mathbf{x})|$, $\mathbf{x} \in \Omega^H$. For $i \geq 2$ only the components $\tilde{f}_i^H(\mathbf{x})$, $\mathbf{x} \in \Gamma^H$ depend on $\tilde{r}$ (cf. Lemma 2.4 and Lemma 2.5). Clearly $|\tilde{f}_i^H(\mathbf{x})|$, $\mathbf{x} \in \Gamma^H$ is minimal if $\tilde{f}_i^H(\mathbf{x}) = 0$ at all $\mathbf{x} \in \Gamma^H$. Therefore, we define the following *optimality criterion*:

A restriction operator $\tilde{r} : \mathcal{F}(\Omega^{H,h}) \rightarrow \mathcal{F}(\Omega^H)$ satisfies the optimality criterion for some given composite grid problem $L^{H,h} u^{H,h} = f^{H,h}$ (cf. (2.8)), if and only if

$$\tilde{r} L^{H,h} w^{H,h}(\mathbf{x}) = L^H r w^{H,h}(\mathbf{x}), \quad \mathbf{x} \in \Gamma^H,$$

for all $w^{H,h} \in \mathcal{F}(\Omega^{H,h})$.

Remark 2.6 If $\tilde{r}$ satisfies the optimality criterion, then $\tilde{f}_i^H(\mathbf{x}) = 0$, $\mathbf{x} \in \Gamma^H$, $i \geq 1$ (cf (2.23)).

Theorem 2.7 *Consider a composite grid problem resulting from the uniform interface discretization approach. Let the restriction operator $\tilde{r} : \mathcal{F}(\Omega^{H,h}) \rightarrow \mathcal{F}(\Omega^H)$ be equal to the trivial injection on the interface:*

$$(\tilde{r} w^{H,h})|_{\Gamma^H} := w^{H,h}|_{\Gamma^H}. \quad (2.30)$$

Then $\tilde{r}$ satisfies the optimality criterion.

Proof. Follows immediately from (2.11), (2.19) and (2.30). $\square$

We note that the result in Theorem 2.7 does not depend on the boundary value problem nor on the composite grid. As we shall see in the following sections, such a general statement as in Theorem 2.7 can not be made for a nonuniform interface discretization.

Remark 2.8 Consider a composite grid problem resulting from the uniform interface discretization. In [2] it is shown that there is a close correspondence between the FAC method with $\tilde{r}$ the trivial injection (i.e. $\tilde{r} = r$, r from (2.19)) and the local defect correction method introduced in [4].

3 The FAC Method for the One-Dimensional Poisson Problem

In this section we consider the one-dimensional Poisson problem

$$\begin{aligned} -U_{xx} &= f(x), & 0 < x < 1 \\ U(0) &= g_0, \quad U(1) = g_1. \end{aligned} \quad (3.1)$$

For this problem a complete analysis of the convergence behaviour of the FAC method can be given, both for the uniform interface discretization and the nonuniform interface discretization. Furthermore, for the nonuniform interface discretization a restriction operator which satisfies the optimality criterion can be derived.

The composite grid is composed of a coarse grid with size H covering $(0, 1)$ and a fine grid with size $h < H$ covering the local region $\Omega_l = (x_l, x_r)$ with $0 < x_l = lH < x_r = rH < 1$, $l, m \in \mathbb{N}$. The boundary value problem (3.1) is discretized on this composite grid.

At grid points $\mathbf{x} \in \Omega_C^H$ and $\mathbf{x} \in \Omega_l^h$ we use the standard second order accurate central difference schemes, described by the stars

$$\frac{1}{H^2}[-1 \ 2 \ -1], \quad \frac{1}{h^2}[-1 \ 2 \ -1], \quad (3.2)$$

respectively. Note that in the one-dimensional case we do not have to interpolate on the interface.

For the *uniform interface discretization*, the interface grid points are treated as global coarse grid points. The stencils at these grid points are:

$$\frac{1}{H^2}[-1 \ 2 \ \underbrace{0 \ \dots \ 0}_{\sigma-1} \ -1] \quad \text{at } x_l, \quad \frac{1}{H^2}[-1 \ \underbrace{0 \ \dots \ 0}_{\sigma-1} \ 2 \ -1] \quad \text{at } x_r. \quad (3.3)$$

We recall that $\sigma = H/h$. The finite difference stencils in (3.3) involve the two grid points at distance H of x_l and x_r . We use the FAC method for solving the composite grid problem (3.2),(3.3). Each restriction operator $\tilde{r}$ satisfying $(\tilde{r}w^{H,h})(x_l) = w^{H,h}(x_l)$, $(\tilde{r}w^{H,h})(x_r) = w^{H,h}(x_r)$ satisfies the optimality criterion (cf. Theorem 2.7). For the one-dimensional Poisson problem, the FAC method with $\tilde{r}$ the trivial injection (i.e. $\tilde{r} = r$, r from (2.19)) is a *direct solver*.

Theorem 3.1 *The FAC method with $\tilde{r}$ the trivial injection yields the exact solution of the composite grid problem (3.2)-(3.3) after two steps.*

Proof. We have to prove that $u^{H,h} - u_2^{H,h} \equiv 0$.

By Lemma 2.5 we have for $x \in \Omega_l^H$

$$\begin{aligned} \tilde{f}_2^H(x) &= (L_l^H r_l v_{l,1}^h)(x) + (L_\Gamma^H r_\Gamma v_1^H)(x) \\ &= (L_l^H r_l v_{l,1}^h)(x) - (L_l^H v_{l,1}^H)(x) \end{aligned}$$

with

$$L_l^H v_{l,1}^H + L_\Gamma^H r_\Gamma v_1^H = 0$$

and

$$L_l^h v_{l,1}^h + L_\Gamma^h r_\Gamma v_1^H = 0.$$

Thus, $v_{l,1}^h$ and $v_{l,1}^H$ are the exact solutions of the discrete problems resulting from central difference discretization of

$$\begin{aligned} V_{xx} &= 0, & x_l < x < x_r, \\ V(x_l) &= v_1^H(x_l), \\ V(x_r) &= v_1^H(x_r), \end{aligned}$$

on the uniform grids Ω_l^h and Ω_l^H , respectively. Since the solution of this two point boundary value problem is a linear function, we have that

$$v_{l,1}^H = V|_{\Omega_l^H} \quad \text{and} \quad v_{l,1}^h = V|_{\Omega_l^h}.$$

Since r_l is the trivial injection we have that $\tilde{f}_2^H(x) = 0$ at all $x \in \Omega_l^H$. By Lemma 2.4 and Theorem 2.7 we have that $\tilde{f}_2^H(x) = 0$ at all $x \in \Omega_C^H \cup \Gamma^H$. Thus, $\tilde{f}_2^H \equiv 0$ and from (2.24) - (2.26) it follows that $v_2^{H,h} \equiv 0$. $\square$

Next we consider a *nonuniform interface discretization* on the composite grid. At the interface grid points the following finite difference stencils are used:

$$\frac{1}{H^2} \left[-\frac{2\sigma}{\sigma+1} \quad 2\sigma \quad -\frac{2\sigma^2}{\sigma+1} \right] \quad \text{at } x_l, \quad \frac{1}{H^2} \left[-\frac{2\sigma^2}{\sigma+1} \quad 2\sigma \quad -\frac{2\sigma}{\sigma+1} \right] \quad \text{at } x_r. \quad (3.4)$$

These stencils correspond to a first order accurate finite difference approximation of $-\frac{d^2}{dx^2}$ at the interface grid points. The stencils involve the interface grid point and its two direct neighbours (one at distance H and one at distance h).

We use the FAC method for solving the composite grid problem (3.2),(3.4). In the FAC method we use the restriction operator $\tilde{r}$ defined by:

$$\begin{aligned} (\tilde{r}w^{H,h})(x) &:= w^{H,h}(x), & x \in \Omega_C^H \cup \Omega_l^H, \\ (\tilde{r}w^{H,h})(x) &:= \lambda w^{H,h}(x) & x \in \Gamma^H = \{x_l, x_r\}. \end{aligned} \quad (3.5)$$

So, $\tilde{r}$ is the trivial injection at grid points which do not lie on the interface. At the interface grid points $\tilde{r}$ is a weighted restriction with weighting factor λ .

Theorem 3.2 *Consider the nonuniform interface discretization (3.2),(3.4) and the FAC method (2.15) with $\tilde{r}$ as in (3.5). Then $\tilde{r}$ satisfies the optimality criterion if and only if $\lambda = \frac{\sigma+1}{2\sigma}$. Furthermore, if $u_1^{H,h} \neq u^{H,h}$, then*

$$\frac{\|u^{H,h} - u_{i+1}^{H,h}\|_\infty}{\|u^{H,h} - u_i^{H,h}\|_\infty} = \left| 1 - \lambda \frac{2\sigma}{\sigma+1} \right|, \quad i \geq 1. \quad (3.6)$$

Proof. Let $i \geq 1$. Suppose that $v_i^{H,h}(x_l) = \alpha$ and $v_i^{H,h}(x_r) = \beta$. Using (2.24)-(2.26) and the fact that $\tilde{f}_i^H(x) = 0$, $x \in \Omega_C^H$, $i \geq 1$ (cf. Lemma 2.4), it is easy to show that:

$$\begin{aligned} v_i^{H,h}(x) &= \alpha \frac{x}{x_l} & x \in \Omega^H \cap (0, x_l], \\ v_i^{H,h}(x) &= \alpha + (\beta - \alpha) \frac{x - x_l}{x_r - x_l} & x \in \Omega_l^h, \\ v_i^{H,h}(x) &= \beta \frac{1 - x}{1 - x_r} & x \in \Omega^H \cap [x_r, 1). \end{aligned}$$

We assume $\alpha \neq 0$ or $\beta \neq 0$, since otherwise $v_i^{H,h} \equiv 0$. A straightforward calculation yields

$$\begin{aligned}
\tilde{f}_{i+1}^H(x_l) &= L^H r v_i^{H,h}(x_l) - \tilde{r} L^{H,h} v_i^{H,h}(x_l) \\
&= \frac{1}{H^2} \{ 2v_i^{H,h}(x_l) - v_i^{H,h}(x_l - H) - v_i^{H,h}(x_l + H) \\
&\quad - 2\sigma \lambda v_i^{H,h}(x_l) + \frac{2\sigma\lambda}{\sigma+1} v_i^{H,h}(x_l - H) + \frac{2\sigma^2\lambda}{\sigma+1} v_i^{H,h}(x_l + h) \} \\
&= \frac{1}{H} \frac{\sigma+1-2\lambda\sigma}{\sigma+1} \frac{\alpha x_r - \beta x_l}{x_l(x_r - x_l)}, \\
\tilde{f}_{i+1}^H(x_r) &= L^H r v_i^{H,h}(x_r) - \tilde{r} L^{H,h} v_i^{H,h}(x_r) \\
&= \frac{1}{H^2} \{ 2v_i^{H,h}(x_r) - v_i^{H,h}(x_r - H) - v_i^{H,h}(x_r + H) \\
&\quad - 2\sigma \lambda v_i^{H,h}(x_r) + \frac{2\sigma^2\lambda}{\sigma+1} v_i^{H,h}(x_r - h) + \frac{2\sigma\lambda}{\sigma+1} v_i^{H,h}(x_r + H) \} \\
&= \frac{1}{H} \frac{\sigma+1-2\lambda\sigma}{\sigma+1} \frac{\beta - \alpha - \beta x_l + \alpha x_r}{(1-x_r)(x_r - x_l)}.
\end{aligned}$$

It is immediately clear that $\tilde{f}_{i+1}^H(x_l) = \tilde{f}_{i+1}^H(x_r) = 0$ if and only if $\lambda = \frac{\sigma+1}{2\sigma}$. Hence, $\tilde{r}$ from (3.5) satisfies the optimality criterion if and only if $\lambda = \frac{\sigma+1}{2\sigma}$.

Similarly as in the proof of Theorem 3.1 it can be shown that $\tilde{f}_i^H(x) = 0$, $x \in \Omega_l^H$, $i \geq 2$. By Lemma 2.4 we have $\tilde{f}_i^H(x) = 0$, $x \in \Omega_C^H$, $i \geq 1$. Thus, $\tilde{f}_{i+1}^H$ has only two non-zero components: $\tilde{f}_{i+1}^H(x_l)$ and $\tilde{f}_{i+1}^H(x_r)$.

In this special case the inverse $(L^H)^{-1}$ is known and $v_{i+1}^{H,h}(x_l)$ and $v_{i+1}^{H,h}(x_r)$ can be computed easily:

$$\begin{aligned}
v_{i+1}^{H,h}(x_l) &= ((L^H)^{-1} \tilde{f}_{i+1}^H)(x_l) \\
&= H x_l (1 - x_l) \tilde{f}_{i+1}^H(x_l) + H x_l (1 - x_r) \tilde{f}_{i+1}^H(x_r) \\
&= (1 - \lambda \frac{2\sigma}{\sigma+1}) \alpha, \\
v_{i+1}^{H,h}(x_r) &= ((L^H)^{-1} \tilde{f}_{i+1}^H)(x_r) \\
&= H x_l (1 - x_r) \tilde{f}_{i+1}^H(x_l) + H x_r (1 - x_r) \tilde{f}_{i+1}^H(x_r) \\
&= (1 - \lambda \frac{2\sigma}{\sigma+1}) \beta.
\end{aligned}$$

Using (2.24)-(2.26) and Lemma 2.4 it is easy to show that $v_{i+1}^{H,h}(x) = (1 - \lambda \frac{2\sigma}{\sigma+1}) v_i^{H,h}(x)$, $x \in \Omega^{H,h}$. Now (3.6) follows immediately. $\square$

Theorem 3.2 clearly shows how the convergence behaviour of the FAC iterates depends on the weighting factor λ in (3.5). For $\lambda > 1 + \frac{1}{\sigma}$ the FAC iterates diverge. If $\lambda = 1$ then the error reduction factor equals $\frac{\sigma-1}{\sigma+1}$. So, if $\tilde{r}$ is the trivial injection, then the convergence rate of the FAC method deteriorates for $\sigma \rightarrow \infty$.

From the proof of Theorem 3.3 it is clear that a restriction operator satisfying the optimality criterion can be derived, because the error $v_i^{H,h}$ can be given explicitly as function of the values $v_i^{H,h}(\mathbf{x})$, $\mathbf{x} \in \Gamma^H$. Obviously, this is not possible for more general problems (e.g. problems with variable coefficients, convection-diffusion problems, two-dimensional problems). For these problems it is not clear how to derive a restriction operator satisfying the optimality criterion for a nonuniform interface discretization.

4 The FAC Method for the Two-Dimensional Poisson Problem

By means of numerical results the fast convergence of the FAC method with a restriction operator satisfying the optimality criterion will be illustrated for the uniform interface discretization. Further, the optimality criterion is used to derive a fastly converging nonlinear variant of the FAC method for a nonuniform interface discretization.

We consider the two-dimensional Poisson problem with Dirichlet boundary conditions:

$$\begin{aligned} -\Delta U &= f & \text{in } \Omega = (0, 1) \times (0, 1), \\ U &= g & \text{on } \partial\Omega. \end{aligned} \quad (4.1)$$

The composite grid is composed of a coarse grid with size H covering $(0, 1) \times (0, 1)$ and a fine grid with size $h < H$ covering the local region $\Omega_l = (0, x_\Gamma) \times (0, y_\Gamma)$ with $0 < x_\Gamma = mH < 1$, $0 < y_\Gamma = nH < 1$, $m, n \in \mathbb{N}$.

The boundary value problem (4.1) is discretized on this composite grid. Like in the previous section we consider two types of discretizations on the composite grid: *uniform interface discretization* and *nonuniform interface discretization*. We introduce the following partitioning of the coarse interface grid Γ^H :

$$\Gamma^H = \Gamma_{vert}^H \cup \Gamma_{hor}^H \cup \{(x_\Gamma, y_\Gamma)\} \quad (4.2a)$$

$$\Gamma_{vert}^H = \{(x, y) \in \Gamma^H \mid x = x_\Gamma, 0 < y < y_\Gamma\}, \quad (4.2b)$$

$$\Gamma_{hor}^H = \{(x, y) \in \Gamma^H \mid y = y_\Gamma, 0 < x < x_\Gamma\}. \quad (4.2c)$$

In the discretization process the following finite difference stencils are used.

At grid points $\mathbf{x} \in \Omega_C^H \cup \{(x_\Gamma, y_\Gamma)\}$ and $\mathbf{x} \in \Omega_l^h$ we use the standard second order accurate 5-point stencils

$$\frac{1}{H^2} \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}, \quad \frac{1}{h^2} \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}, \quad (4.3)$$

respectively. If in the latter scheme an unknown at a *slave point* is involved (see Figure 3), then the coefficient corresponding with this slave point is replaced by a *linear combination* of the two nearest grid points of Γ^H (i.e. p_Γ from Section 2.2 is the linear interpolation operator).

For the *uniform interface discretization* the interface grid points of $\Gamma_{hor}^H \cup \Gamma_{vert}^H$ are treated as global coarse grid points. Then, e.g. at a grid point $\mathbf{x} \in \Gamma_{hor}^H$, the following stencil is used:

$$\frac{1}{H^2} \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & 0 & \\ & \vdots & \\ & 0 & \\ & -1 & \end{bmatrix} \left. \vphantom{\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & 0 & \\ & \vdots & \\ & 0 & \\ & -1 & \end{bmatrix}} \right\} \sigma - 1. \quad (4.4)$$

This stencil involves the four grid points at a distance H of $\mathbf{x}$.

For the *nonuniform interface discretization*, a finite difference stencil involving the interface

grid point and its four direct neighbours is used at each interface grid point. For example, at a interface grid point $\mathbf{x} \in \Gamma_{hor}^H$ the following finite difference stencil is used:

$$\frac{1}{H^2} \begin{bmatrix} & -\frac{2\sigma}{\sigma+1} & \\ -1 & 2+2\sigma & -1 \\ & -\frac{2\sigma^2}{\sigma+1} & \end{bmatrix}. \quad (4.5)$$

This stencil corresponds to a first order accurate finite difference approximation of U_{yy} . The stencil involves the interface grid point and its four direct neighbours; three at distance H and one at distance h . A similar stencil is used at the interface grid points $\mathbf{x} \in \Gamma_{vert}^H$. This nonuniform interface discretization approach corresponds to the five-point finite volume approach in [7].

In (4.3)-(4.5) the usual modifications are used at grid points close to the boundary $\partial\Omega$.

The discretization error for the uniform interface discretization described above is studied in [3]. Properties of the global discretization error, both for the uniform interface discretization (cf. (4.4)) and the nonuniform interface discretization (cf. (4.5)) are illustrated in the following example.

Example 4.1 We take f, g in (4.1) such that $U(x, y) = (\tanh(25(x + y - \frac{1}{8})) + 1)/2$. This solution contains a high activity region near the line segment $x + y = 1/8$ inside the unit square. We take $\Omega_l = \{(x, y) \in \Omega \mid 0 < x, y < 1/4\}$. In Table 1 we show values of the global discretization error $\|u^{H,h} - U\|_{\Omega_{H,h}}^\infty$ for several values of H and $\sigma = H/h$. The values in the upper part of the table are related to the uniform interface discretization. The values in the lower part of the table are related to the nonuniform interface discretization. We see that the quality of the two discretizations is comparable. Further we see that if we take H fixed, then decreasing h (i.e. increasing σ) results in h^2 convergence until a certain threshold value σ_m is reached. For $H = 1/8$ we see a threshold value $\sigma_m = 16$ for both discretizations. Also note that in Table 1 there is only little variation in the values if we take h fixed and vary σ . For example, along the diagonal from $(H, \sigma) = (1/28, 1)$ to $(H, \sigma) = (1/8, 16)$ all values are of comparable size. This means that the global discretization errors corresponding to the composite grid problems with $H = 1/8$, $h = 1/128$ are approximately of the same size as the global discretization error related to the standard discrete problem on the global uniform grid with $h = 1/128$.

As stated in Theorem 2.7, the trivial injection, $\tilde{r} = r$, satisfies the optimality criterion for a composite grid problem resulting from the uniform interface discretization. The following example shows that the convergence rate of the FAC method with $\tilde{r} = r$ is very high for the uniform interface discretization (4.3),(4.4). For the composite grid problem resulting from the nonuniform interface discretization (4.3),(4.5) the convergence rate of the FAC method with $\tilde{r} = r$ is very low.

Example 4.2 We take the same problem as in Example 4.1. For Ω_l we take $\Omega_l = (0, 1/2) \times (0, 1/2)$. In Table 2 the error reduction factors

$$\rho_i := \frac{\|u_i^{H,h} - u^{H,h}\|_\infty}{\|u_{i-1}^{H,h} - u^{H,h}\|_\infty}$$

for $i = 1, 2, \dots, 5$ are presented for several values of $\sigma = H/h$ and $H = 1/8$ (similar results are obtained for other values of H). We see that for the uniform interface discretization the

H	1	2	4	8	16	32	σ
1/8	$2.55e-1$	$5.96e-2$	$2.29e-2$	$5.39e-3$	$1.49e-3$	$1.55e-3$	
1/16	$6.08e-2$	$2.29e-2$	$5.53e-3$	$1.34e-3$	$8.86e-4$		
1/32	$2.29e-2$	$5.61e-3$	$1.41e-3$	$3.33e-4$			
1/64	$5.63e-3$	$1.43e-3$	$3.51e-4$				
1/128	$1.44e-3$	$3.55e-4$					
1/256	$3.57e-4$						
1/8	$2.55e-1$	$5.94e-2$	$2.29e-2$	$5.48e-3$	$1.29e-3$	$1.25e-3$	
1/16	$6.08e-2$	$2.29e-2$	$5.58e-3$	$1.38e-3$	$7.35e-4$		
1/32	$2.29e-2$	$5.62e-3$	$1.42e-3$	$3.37e-4$			
1/64	$5.63e-3$	$1.43e-3$	$3.20e-4$				
1/128	$1.44e-3$	$3.56e-4$					
1/256	$3.57e-4$						

Table 1: Global discretization errors in Example 4.1.

i	uniform interface discr.			nonuniform interface discr.		
	$\sigma = 2$	$\sigma = 4$	$\sigma = 8$	$\sigma = 2$	$\sigma = 4$	$\sigma = 8$
1	$4.86e-2$	$4.86e-2$	$4.86e-2$	$4.86e-2$	$4.86e-2$	$4.86e-2$
2	$2.05e-2$	$2.63e-2$	$2.77e-2$	$3.44e-1$	$6.67e-1$	$9.38e-1$
3	$2.07e-2$	$2.66e-2$	$2.80e-2$	$3.43e-1$	$6.67e-1$	$9.38e-1$
4	$2.11e-2$	$2.70e-2$	$2.85e-2$	$3.42e-1$	$6.66e-1$	$9.38e-1$
5	$2.17e-2$	$2.77e-2$	$2.92e-2$	$3.41e-1$	$6.66e-1$	$9.38e-1$

Table 2: Error reduction factors ρ_i in Example 4.2.

FAC iterates converge very fast. Further, the error reduction factors only slightly depend on the refinement factor σ . For the nonuniform interface discretization the error reduction is much worse. Now a strong dependence of the error reduction factors on the refinement factor σ can be observed. A similar dependence has been observed for the one-dimensional Poisson problem (cf. Theorem 3.2). We note that for $\sigma = 16$ the FAC iterates diverge for the nonuniform interface discretization.

The numerical results in Table 2 show that one should not use the trivial injection on the interface in the FAC method for solving a composite grid problem resulting from a nonuniform interface discretization. For the one-dimensional Poisson problem a simple weighted restriction has been derived which yields an optimal convergence behaviour. Below we shall derive a suitable *nonlinear restriction operator on the interface* for the two-dimensional Poisson problem. We use theoretical insights from Section 2.4.

Suppose that the restriction operator $\tilde{r}$ is defined by

$$\begin{aligned}
(\tilde{r}w^{H,h})(\mathbf{x}) &:= w^{H,h}(\mathbf{x}) & \mathbf{x} \in \Omega_C^H \cup \Omega_I^h \cup \{(x_\Gamma, y_\Gamma)\}, \\
(\tilde{r}w^{H,h})(\mathbf{x}) &:= \lambda(\mathbf{x})w^{H,h}(\mathbf{x}) & \mathbf{x} \in \Gamma_{\text{vert}}^H \cup \Gamma_{\text{hor}}^H.
\end{aligned} \tag{4.6}$$

The global coarse grid function $\tilde{f}_i^H$ from (2.23) satisfies

$$\tilde{f}_i^H(\mathbf{x}) = (L^H r v_{i-1}^{H,h})(\mathbf{x}) - \lambda(\mathbf{x})(L^{H,h} v_{i-1}^{H,h})(\mathbf{x}) \quad \mathbf{x} \in \Gamma_{\text{hor}}^H \cup \Gamma_{\text{vert}}^H, \quad i \geq 1. \tag{4.7}$$

We recall that if $\tilde{r}$ satisfies the optimality criterion, then $\tilde{f}_i^H(\mathbf{x}) = 0$ for $i \geq 1$ and $\mathbf{x} \in \Gamma^H$. The results in Table 2 for the uniform interface discretization show that then the FAC iterates converge very fast.

So, in the i -th iteration step we want to use weighting factors $\lambda(\mathbf{x})$ such that $\tilde{f}_i^H \approx 0$ at all $\mathbf{x} \in \Gamma_{hor}^H \cup \Gamma_{vert}^H$. At the right hand side of (4.7) components of the unknown error vector $v_{i-1}^{H,h} = u^{H,h} - u_{i-1}^{H,h}$ appear. In the FAC method (2.15), for a *given* restriction operator $\tilde{r}$ (e.g. the trivial injection), approximations of $v_{i-1}^{H,h}$ are computed both on the uniform global coarse grid and on the uniform local fine grid. Combining these approximations (cf. (2.15f,g)) yields an approximation $\tilde{v}_{i-1}^{H,h}$ of $v_{i-1}^{H,h}$. This approximation is used to eliminate the exact error $v_{i-1}^{H,h}$ at the right hand side of (4.7). The resulting term is put equal to zero, which yields:

$$\lambda(\mathbf{x}) = (L^H \tilde{r} \tilde{v}_{i-1}^{H,h})(\mathbf{x}) / (L^{H,h} \tilde{v}_{i-1}^{H,h})(\mathbf{x}) \quad \mathbf{x} \in \Gamma_{hor}^H \cup \Gamma_{vert}^H. \quad (4.8)$$

Thus, the approximation $\tilde{v}_{i-1}^{H,h}$ of $v_{i-1}^{H,h}$ is used to compute suitable weighting factors on the interface. It is not used to update the approximation $u_{i-1}^{H,h}$. The defect of $u_{i-1}^{H,h}$ is restricted to the global coarse grid using the above computed weighting factors on the interface. Using this restricted defect, new approximations of $v_{i-1}^{H,h}$ are computed on the global coarse grid and on the local fine grid (cf. (2.15d,e)). These approximations are used to compute a new iterate $u_i^{H,h}$.

Remark 4.3 The term $L^{H,h} v_{i-1}^{H,h}$ at the right hand side of (4.7) equals the composite defect related to the approximation $u_{i-1}^{H,h}$: $L^{H,h} v_{i-1}^{H,h} = L^{H,h} u^{H,h} - L^{H,h} u_{i-1}^{H,h} = f^{H,h} - L^{H,h} u_{i-1}^{H,h}$.

So, at each grid point $\mathbf{x} \in \Gamma_{hor}^H \cup \Gamma_{vert}^H$ the value $L^{H,h} v_{i-1}^{H,h}(\mathbf{x})$ is known. These values can be used for computing the weighting factors $\lambda(\mathbf{x})$ (cf. (4.8)):

$$\lambda(\mathbf{x}) = (L^H \tilde{r} \tilde{v}_{i-1}^{H,h})(\mathbf{x}) / (L^{H,h} \tilde{v}_{i-1}^{H,h})(\mathbf{x}) \quad \mathbf{x} \in \Gamma_{hor}^H \cup \Gamma_{vert}^H. \quad (4.9)$$

With these weighting factors error reduction factors comparable with the values in Table 2 for the nonuniform interface discretization are obtained. Hence, one should not use the weighted restriction in (4.6) with the weighting factors from (4.9).

Since a *nonlinear* restriction operator $\tilde{r}(u_{c,i-1})$ is used, the resulting iteration is also nonlinear. Below, we describe one step of this *nonlinear FAC* (NFAC) iteration for a composite grid problem $L^{H,h} u^{H,h} = f^{H,h}$ resulting from a *nonuniform interface discretization*. Notation is similar as in (2.15).

NFAC iteration step

a. Initialization

Computation of the composite grid defect

$$d^{H,h} := f^{H,h} - L^{H,h} u_{i-1}^{H,h}$$

Restriction of the composite grid defect to the uniform subgrids

$$d^H := d^{H,h}|_{\Omega^H}, \quad d_l^h := d^{H,h}|_{\Omega_l^h}$$

b. Exact solution of the discrete problems

$$\begin{aligned} L^H v^H &= d^H && \text{on } \Omega^H \\ L_l^h v_l^h &= d_l^h - L_\Gamma^h p_\Gamma(v^H|_{\Gamma^H}) && \text{on } \Omega_l^h \end{aligned}$$

c. Computation of the weighting factors

$$\begin{aligned} \lambda(\mathbf{x}) &:= (L^H r v^{H,h})(\mathbf{x}) / (L^{H,h} v^{H,h})(\mathbf{x}) \quad \mathbf{x} \in \Gamma_{hor}^H \cup \Gamma_{vert}^H \\ \text{where } v^{H,h}(\mathbf{x}) &= \begin{cases} v_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ v^H(\mathbf{x}) & \mathbf{x} \in \Omega^{H,h} \setminus \Omega_l^h \end{cases} \end{aligned}$$

d. Correction of the restricted defect on the interface

$$d^H(\mathbf{x}) := \lambda(\mathbf{x}) d^{H,h}(\mathbf{x}) \quad \mathbf{x} \in \Gamma_{hor}^H \cup \Gamma_{vert}^H$$

e. Exact solution of the discrete problems

$$\begin{aligned} L^H \bar{v}^H &= d^H && \text{on } \Omega^H \\ L_l^h \bar{v}_l^h &= d_l^h - L_\Gamma^h p_\Gamma(\bar{v}^H|_{\Gamma^H}) && \text{on } \Omega_l^h \end{aligned}$$

f. Correction of the composite grid approximation

$$u_i^{H,h}(\mathbf{x}) := \begin{cases} u_{i-1}^{H,h}(\mathbf{x}) + \bar{v}_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ u_{i-1}^{H,h}(\mathbf{x}) + \bar{v}^H(\mathbf{x}) & \mathbf{x} \in \Omega^{H,h} \setminus \Omega_l^h \end{cases}$$

We note that in one NFAC step, two discrete problems on the uniform global coarse grid and two discrete problems on the uniform local fine grid have to be solved. In (2.15) only one discrete problem on the uniform global coarse grid and one discrete problem on the uniform local fine grid have to be solved in each iteration step. So, one NFAC step is approximately twice as expensive as one FAC step.

The values in Table 2 show that the error reduction in the first FAC step with trivial injection on the interface is the same for the nonuniform interface discretization as for the uniform interface discretization. Hence, given a composite grid problem resulting from a nonuniform interface discretization approach and some initial approximation $u_0^{H,h}$ first one FAC step with trivial injection operator is performed and then a number of NFAC steps (with nonlinear restriction on the interface) are performed.

In our final example we show that the convergence behaviour of NFAC for a composite grid problem resulting from a nonuniform interface discretization is comparable with the convergence behaviour of FAC with trivial injection on the interface for a composite grid problem resulting from the uniform interface discretization.

Example 4.4 We consider the same problem as in Example 4.2 and the nonuniform interface discretization (4.3),(4.5). In Table 3 the error reduction factors ρ_i for the nonlinear FAC method described above are presented for several values of σ and $H = 1/8$. The error reduction factors are approximately the same as those in Table 2 for the FAC method for the uniform interface discretization (4.3),(4.4). In this FAC method trivial injection on the interface is used which satisfies the optimality criterion.

i	$\sigma = 2$	$\sigma = 4$	$\sigma = 8$
1	$4.86e - 2$	$4.86e - 2$	$4.86e - 2$
2	$2.11e - 2$	$2.65e - 2$	$2.78e - 2$
3	$2.12e - 2$	$2.67e - 2$	$2.82e - 2$
4	$2.14e - 2$	$2.69e - 2$	$2.89e - 2$
5	$2.19e - 2$	$2.73e - 2$	$2.99e - 2$

Table 3: Error reduction factors ρ_i in Example 4.4.

Remark 4.5 The NFAC algorithm is not restricted to the two-dimensional Poisson problem. Results very similar to those in Table 3 have been found for elliptic boundary value problems with smoothly varying coefficient functions.

We end with some concluding remarks. The FAC method is very suitable for solving composite grid problems resulting from the uniform interface discretization. For such problems the trivial injection can be used for restricting composite grid defects to the global coarse grid. The trivial injection satisfies the optimality criterion. The convergence rate of the method is very high.

For composite grid problems resulting from a nonuniform interface discretization a restriction operator satisfying the optimality criterion can not be given in general. The convergence rate of FAC depends strongly on the way composite grid defects are restricted on the interface. Using theoretical insights a nonlinear restriction operator is derived. The convergence rate of the related nonlinear FAC method is high. The error reduction per iteration step is comparable with the error reduction per step in the FAC method with trivial injection for the uniform interface discretization. One step of this nonlinear FAC method is approximately twice as expensive as one FAC step.

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References

- [1] Brandt, A., Bai, D.: *Local mesh refinement multilevel techniques*, SIAM J. Sci. Statist. Comput., 8 (1987), pp 109-134.
- [2] Ferket, P.J.J., Reusken A.A.: *Further analysis of the local defect correction method*, RANA 94-25, Department of Mathematics and Computing Science, Eindhoven University of Technology, 1994 (to appear in Computing).
- [3] Ferket, P.J.J., Reusken, A.A.: *A finite difference discretization method for elliptic problems on composite grids*, RANA 95-01, Department of Mathematics and Computing Science, Eindhoven University of Technology, 1995 (to appear in Computing).
- [4] Hackbusch, W.: *Local defect correction method and domain decomposition techniques*, Computing, Suppl 5 (1984), pp 89-113.

- [5] Hackbusch, W.: *Multigrid Methods and Applications*, Springer, Berlin, 1985.
- [6] Hackbusch, W.: *Elliptic Differential Equations. Theory and Numerical Treatment*, Springer, Berlin, 1992.
- [7] Heroux, M., McCormick, S., McKay, S., Thomas, J.W.: *Applications of the fast adaptive composite grid method*, in: Proceedings fo the Third Copper Mountain Connference on Multigrid Methods, 1987.
- [8] McCormick, S.: *Fast adaptive composite grid (FAC) methods: Theory for the variational case*, Computing, Suppl. 5 (1984), pp 115-121.
- [9] McCormick, S., Thomas, J.: *The fast adaptive composite grid (FAC) method for elliptic equations*, Math. Comp., 46 (1986), pp 439-456.
- [10] McCormick, S.F.: *Multilevel Adaptive Methods for Partial Differential Equations*, SIAM, Philadelphia, 1989.
- [11] Mitchell, A.R., Griffiths, D.F.: *The Finite Difference Method in Partial Differential Equations*, John Wiley & Sons, New York, 1980.
- [12] Varga, R.S.: *Matrix Iterative Analysis*, Prentice Hall, New York, 1962.