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A note on a fluid queue
driven by an $M/M/1$ queue

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A NOTE ON A FLUID QUEUE DRIVEN BY AN M/M/1 QUEUE
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Abstract. In this note we consider the fluid queue driven by an M/M/1 queue as analysed by Virtamo and Norros [5]. We show that the stationary buffer content in this model can be easily analysed by looking at embedded time points. This approach gives the stationary buffer content distribution in terms of the modified Bessel function of the first kind of order one. By using a suitable integral representation for this Bessel function we show that our results coincide with the ones of [5].

1. A fluid queue driven by an alternating renewal process. Consider a fluid queue with a constant leak rate $c_1$. The input of the fluid queue is governed by an alternating renewal process $X_1, Y_1, X_2, Y_2, \ldots$. The sequences $X_1, X_2, \ldots$, and $Y_1, Y_2, \ldots$, are i.i.d. sequences with distribution function $F_X(\cdot)$ and $F_Y(\cdot)$, respectively. During the alternating periods of lengths $X_i$, resp. $Y_i$, the input rate of the fluid queue is equal to $c_2 (> c_1)$, resp. $c_0 (< c_1)$. Although the results of this section can be easily extended to general values of $c_0, c_1$ and $c_2$, we assume in the sequel for convenience that $c_0 = 0$, $c_1 = 1$ and $c_2 = 2$. Clearly as stability condition for the queue we have $EX < EY$. Our goal is to find the stationary buffer content distribution.

Define $Z_i$ as the buffer content at the beginning of the $i$-th period $X_i$. Then the $Z_i$'s satisfy the recurrence relation (with $Z_1 := 0$)

\begin{equation}
Z_{i+1} = \max(Z_i + X_i - Y_i, 0).
\end{equation}

This relation is equal to the one for the waiting time in a $G/G/1$ queue with interarrival time distribution $F_Y(\cdot)$ and service time distribution $F_X(\cdot)$. Hence we conclude that the distribution of the stationary buffer content, $Z$, at the beginning of an $X$-period is equal to the stationary waiting time distribution in a $G/G/1$ queue.

If we introduce the random variable $T$ as the stationary buffer content at an arbitrary point in time, then standard renewal theory arguments show that (see also Kella and Whitt [3])

\begin{equation}
T \overset{d}{=} \begin{cases} Z + \tilde{X}, & \text{w.p. } EX/(EX + EY), \\ \max(Z + X - \tilde{Y}, 0), & \text{w.p. } EY/(EX + EY), \end{cases}
\end{equation}

where all random variables involved are independent and $\tilde{X}$ (resp. $\tilde{Y}$) denotes the residual life time of the random variable $X$ (resp. $Y$).

In the special case that the $Y_i$'s are exponentially distributed with parameter $\lambda$ we have $\tilde{Y} \overset{d}{=} Y$ and so we obtain from (1) and (2) that

\begin{equation}
T \overset{d}{=} Z + \tilde{X} \cdot 1[U=1]
\end{equation}

where

\begin{equation}
U = \begin{cases} 1, & \text{w.p. } EX/(EX + EY), \\ 0, & \text{w.p. } EY/(EX + EY). \end{cases}
\end{equation}

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1
Hence, denoting the Laplace-Stieltjes transforms of $X$, $Z$ and $T$, resp., by $X^*(\cdot)$, $Z^*(\cdot)$ and $T^*(\cdot)$, we have, because $Z$ is now the waiting time in the $M/G/1$ queue,

$$Z^*(s) = \frac{s(1 - \lambda EX)}{s - \lambda + \lambda X^*(s)},$$

and

$$T^*(s) = Z^*(s) \left[ \frac{EY}{EX + EY} + \frac{EX}{EX + EY} \frac{1 - X^*(s)}{sEX} \right] = \frac{1 - \lambda EX}{1 + \lambda EX} \cdot \frac{s + \lambda - \lambda X^*(s)}{s - \lambda + \lambda X^*(s)}.

Eq. (3) coincides with eq. (6.27) of Gaver and Miller [2]. However, in [2] it is derived by analysing a three-dimensional Markov process (buffer content, input rate and elapsed time of $X$-period).

2. A fluid queue driven by an $M/M/1$ queue. Let us now restrict our attention to the model analysed in [5]. In this case the successive $X$ and $Y$ periods in the alternating renewal process are the successive busy and idle periods in the $M/M/1$ queue. So, $f_X(\cdot) = F'_X(\cdot)$ is the density of the busy period in an $M/M/1$ queue with arrival intensity $\lambda$ and service intensity $\mu$, i.e. (see eq. (5.145) of [4]),

$$f_X(x) = \frac{1}{x \sqrt{\rho}} e^{-\left(\lambda + \mu\right)x} I_1(2x \sqrt{\lambda \mu}),$$

where $\rho := \lambda/\mu$ and $I_1(\cdot)$ denotes the modified Bessel function of the first kind of order one. The Laplace-Stieltjes transform of this busy period is given by (see eq. (5.144) of [4])

$$X^*(s) = \frac{\mu + \lambda + s - ((\mu + \lambda + s)^2 - 4\mu \lambda)^{1/2}}{2\lambda}.$$  

Since the idle period in the $M/M/1$, and thus $Y$, is exponentially distributed with parameter $\lambda$, the transform $T^*(\cdot)$ satisfies eq. (3). Substitution of (4) in (3) yields after some algebra

$$T^*(s) = (1 - 2\rho) \cdot \frac{s + \lambda + \sqrt{(s + \lambda + \mu)^2 - 4\mu \lambda}}{2s - 2\lambda + \mu} = 1 - 2\rho + 4\rho \cdot \frac{\mu/2 - \lambda}{s + \mu/2 - \lambda} - 2\rho \cdot \frac{\mu/2 - \lambda}{s + \mu/2 - \lambda} \cdot X^*(s).$$

Hence, we can conclude that

$$(5) \quad P(T > t) = 4\rho e^{-\left(\mu/2 - \lambda\right)t} - 2\rho \cdot \left[ \int_0^t e^{-\left(\mu/2 - \lambda\right)(t-x)} f_X(x)dx \right] + 1 - \int_0^t f_X(x)dx.]$$

To show that this expression for the buffer content distribution is equivalent with eq. (6.4) of [5], we insert in (5) the following integral representation for $I_1(\cdot)$ (see eq. (9.6.18) in [1])

$$I_1(z) = \frac{z/2}{\sqrt{\pi} \Gamma(\frac{3}{2})} \int_{-1}^1 \sqrt{1 - x^2} e^{zx} dx.$$
Then after changing order of integration and using the fact that, for $b \geq 1$,

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{x-b} dx = -\pi(b - \sqrt{b^2 - 1})$$

we get that

$$P(T > t) = 1_{[c\geq1/4]} \cdot (4\rho - 1) \cdot e^{-(\mu/2-\lambda)t}$$

$$+ (2 - 4\rho) \int_{-1}^{1} \frac{2}{\pi} \frac{\sqrt{1-x^2}}{8x^2 - (12\sqrt{\rho} + 6/\sqrt{\rho})x + 4\rho + 5 + 1/\rho} e^{[2\sqrt{\lambda}x-(\lambda+\mu)]t} dx.$$

To compare this equation with eq. (6.4) of [5] one should substitute in the latter $c = 1/2$ and replace $y$ by $\mu t$. Then it is seen that both equations coincide, after removal of the erroneous minus-sign before the indicator function in the equation for $w_1$ in [5].

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