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On the stability of stationary shock waves in nozzle flows with homogeneous condensation

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The stability limit of stationary normal shock waves in supercritical nozzle flows with homogeneous condensation is investigated by the singularity theory of the quasi-one-dimensional steady-state differential equations of motion. In this case it is shown that a catastrophic change in the phase portraits of the flow variables, where the spiral point turns into a nodal point, occurs when the initial relative humidity exceeds a critical value, resulting in the alteration of the quasi-one-dimensional steady-state condensation structures. In particular, a criterion for the limit of stability, based on the break-up of structural stability of the steady equations of motion in the quasi-one-dimensional approximation, is established by variational analysis and a correlation for the critical initial relative humidity is derived for fixed nozzle geometry keeping appropriate reservoir conditions fixed in the same approximation. A comparison of the values of the critical initial relative humidity, calculated by this correlation, shows excellent agreement with those of experiments and/or numerical simulations for moist air expansions in various slender nozzles under different reservoir conditions.

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I. INTRODUCTION

High speed flows with homogeneous condensation are particularly important in steam turbine technology. Indeed, supersonic flow is found in the outer parts of large blades and unsteady condensation there implies erosion of turbine blades, loss of efficiency, and unwanted vibration problems. The investigation of unsteady condensation is also useful in phase separation. High speed flows with condensation are also of interest for the investigation of nucleation and droplet growth phenomena. Surveys of such flows are well documented in the literature.\(^1\)-\(^4\)

Various investigations\(^5\)-\(^11\) have been devoted to nonequilibrium flows with homogeneous condensation in converging–diverging nozzles. In this case the flow in the condensation zones remains steady and supersonic as long as the latent heat release to the flow is below a critical value for a given nozzle geometry and reservoir conditions (subcritical flows). If this critical value is exceeded, the flow becomes thermally choked (e.g., see Delale et al.\(^12\)) and stationary shock waves followed by local subsonic heat addition zones, where the flow is accelerated back to supersonic speeds, occur (supercritical flows). Detailed structure of the condensation zones for both subcritical and supercritical flows can be found in Delale et al.\(^10,11\) using an asymptotic predictive method given by Blythe and Shih\(^7\) and Clarke and Delale.\(^3\) Two-dimensional numerical simulations of such flows were given by Schnerr\(^9\) and Schnerr and Dohrmann.\(^13\) If the heat addition to the flow is further increased (e.g., by increasing the initial relative humidity keeping appropriate reservoir conditions fixed), the flow becomes unsteady and different patterns of periodic flow structures with moving shock waves occur (e.g., see Adam and Schnerr\(^14\)). Unsteady flow phenomena with moving shocks were first discovered by Schmidt\(^15\) in moist air expansions. Subsequently, they were observed in wet-steam flows. The first numerical calculations of such flows were presented by Saltanov and Tkaleko\(^16\) for moist air and by Guha and Young\(^17\) for wet-steam. Two-dimensional numerical simulations have recently been carried out by White and Young\(^18\) for wet-steam and by Adam and Schnerr\(^14\) for moist air.

The aim of this paper is to investigate the stability limit of stationary normal shock waves in the quasi-one-dimensional approximation. For this reason we consider the differential equations of motion for quasi-one-dimensional nozzle flows and identify the singularities. From the well-known global solutions of supercritical flows with stationary normal shock waves, for fixed nozzle configuration we increase the initial relative humidity keeping appropriate reservoir conditions fixed. We then determine the variations in the nature of singularities due to arbitrarily small variations in the initial relative humidity. Consequently, we show that the spiral singularity in the phase portraits of the global solutions (responsible for the existence of stationary normal shock waves) turns into a nodal point singularity as the initial relative humidity is increased beyond some limit, which we identify as the limit of stability of stationary shock waves for condensing nozzle flows in the quasi-one-dimensional approximation. By the impulse-momentum theorem of quasi-one-dimensional steady nozzle flows, we determine the
variation of the shock position due to the variation in the initial relative humidity and we find the break-up of structural stability of the equations of motion as the limit where nucleation has been quenched by orders of magnitude so that no sufficient latent heat is available to accelerate the flow in the subsonic heating zones back to supersonic speeds. This suggests that the flow behind the shock will remain subsonic everywhere corresponding to a global solution with a nodal point singularity. Consequently, a criterion is established for the critical Mach number, which, in turn, is related to the critical initial relative humidity at the limit of stability in the quasi-one-dimensional approximation. Comparison of the results of this critical initial relative humidity with those of numerical simulations and experiments for different nozzle configurations under different reservoir conditions shows excellent agreement.

II. DIFFERENTIAL EQUATIONS OF MOTION AND THEIR SINGULARITIES

We consider the quasi-one-dimensional steady-state compressible flow of a mixture of a condensable vapor with or without a carrier gas through a converging–diverging nozzle, as shown in Fig. 1, with reservoir temperature $T_0$, reservoir specific humidity $w_0$ (or reservoir mixture pressure $p_0$), and reservoir relative humidity $\varphi_0$. The differential system of equations governing the motion can then be written in normalized form as (for normalization, notation and other details see Delale et al.10,12)

$$\frac{1}{M} \frac{dM}{dx} = \frac{1}{(M^2-1)} \left[ \frac{1}{A} \frac{dA}{dx} + \frac{\mu_m/\mu_v}{g} \frac{dg}{dx} \right]$$

$$+ \frac{(\mu_m/\mu_v)[2-\Gamma+(2-\Gamma)M^2]}{2[1-(\mu_m/\mu_v)g][1+(\Gamma-1)M^2/2]} \frac{dg}{dx}$$

$$- \frac{(1+\Gamma M^2)}{2[1+q/c_{pm}]} \frac{d}{dx} \left( \frac{q}{c_{pm}} \right) ,$$

(1)

$$\frac{1}{\rho} \frac{dp}{dx} = - \frac{M^2}{(M^2-1)} \left[ \frac{1}{A} \frac{dA}{dx} + \frac{\mu_m/\mu_v}{g} \frac{dg}{dx} \right]$$

$$- \frac{[1+(\Gamma-1)M^2/2]}{M^2[1+q/c_{pm}]} \frac{d}{dx} \left( \frac{q}{c_{pm}} \right) ,$$

(2)

$$\frac{1}{\rho} \frac{dp}{dx} = - \frac{\Gamma M^2}{(M^2-1)} \left[ \frac{1}{A} \frac{dA}{dx} + \frac{\mu_m/\mu_v}{g} \frac{dg}{dx} \right]$$

$$- \frac{[1+(\Gamma-1)M^2/2]}{[1+q/c_{pm}]} \frac{d}{dx} \left( \frac{q}{c_{pm}} \right) ,$$

(3)

$$\frac{1}{\bar{T}} \frac{dT}{dx} = - \frac{(\Gamma-1)M^2}{(M^2-1)} \left[ \frac{1}{A} \frac{dA}{dx} + \frac{\mu_m/\mu_v}{g} \frac{dg}{dx} \right]$$

$$- \frac{(\Gamma M^2-1)[1+(\Gamma-1)M^2/2]}{(\Gamma-1)M^2[1+q/c_{pm}]} \frac{d}{dx} \left( \frac{q}{c_{pm}} \right) ,$$

(4)

where $q$ is the heat added to the flow (in normalized form) and is given by

$$q = \frac{\mu_m}{\mu_v} g L ,$$

(5)

$M$ is the flow Mach number, based on the local frozen speed of sound, and is given by

$$M^2 = \frac{u^2}{a_f^2} = \frac{[1+(\mu_m/\mu_v)(\gamma-1)g]}{\gamma[1-(\mu_m/\mu_v)g]} \frac{u^2}{T}$$

(6)

and

$$\Gamma = \frac{\gamma}{1+(\mu_m/\mu_v)(\gamma-1)g} .$$

(7)

In Eqs. (1)–(7), $x$ is the normalized axial coordinate, $A$ is the normalized cross-section area (normalized with respect to the

![FIG. 1. Geometric configurations of nozzles employed. (a) 1D circular arc nozzle S2 with throat height $2\gamma_{t} = 30$ mm and the radius of curvature at the throat $R_{t} = 400$ mm [see Schnerr (Ref. 9)]. (b) 2D circular arc nozzle S1 with throat height $2\gamma_{t} = 120$ mm and the radius of curvature at the throat $R_{t} = 100$ mm [see Schnerr (Ref. 9)]. (c) 1D nozzle G1 with throat height $2\gamma_{t} = 20$ mm and the radius of curvature at the throat $R_{t} = 4000$ mm [see Lamanna (Ref. 24)].](image)
cross-section area at the throat), \( p, \rho, u, \) and \( T \) are, respectively, the normalized mixture pressure, the normalized mixture density, the normalized flow speed and the normalized mixture temperature, \( g \) is the condensate mass fraction, \( L \) is the normalized latent heat of condensation, \( \mu_m \) and \( \mu_v \) are, respectively, the molecular weight of the mixture at the reservoir and the molecular weight of the vapor and \( \gamma \) is the isentropic exponent of the mixture at the reservoir. It is essential to notice that the second order quantity \( g dL/dT \) is neglected compared to unity in Eq. (6) although the dependence of the latent heat on temperature is retained elsewhere in the above system of equations. It is also worthwhile to mention that the flow Mach number, given by Eq. (6), is different from the Mach number used by Delale et al.\(^{12} \) and that the normalization is carried in a similar manner, but using reservoir conditions instead of the conditions used at the saturation point in Delale et al.\(^{10} \).

The differential system of equations (1)–(7) together with the nonequilibrium condensation rate equation for \( g \) constitute the basic equations of nozzle flows with nonequilibrium condensation for given cross section area of the nozzle. It is well known that the flow passes through the throat almost isentropically so that the classical saddle point singularity of the system (1)–(7) occurs at the throat with \( M=1 \). Nonequilibrium condensation thus occurs in the supersonic portion of the nozzle (for details of the condensation zones, see Clarke and Delale\(^8 \) and Delale et al.\(^{10,11} \)). As a result, a considerable amount of heat is released in the heat addition zones. As long as the amount of heat released is below a critical value, the flow field remains continuous throughout the nozzle (subcritical flows). When the amount of heat exceeds this critical value (say, by increasing the initial relative humidity), the flow becomes thermally choked (see Delale et al.\(^{12} \)). In this case flows with embedded normal shock waves occur (supercritical flows). Further increase in initial relative humidity can lead to the loss of stability of the stationary shock resulting in unsteady flow oscillations, as observed in experiments and unsteady two-dimensional numerical simulations (e.g., see Adam and Schnerr\(^{14} \)). In what follows we give a complete description of quasi-one-dimensional steady-state flow patterns for condensing nozzle flows by studying the singularities of the differential system of equations (1)–(4).

### A. Classification of singularities and flow patterns

The singularities of the differential system (1)–(4) for given heat addition distributions were considered by Möhring\(^{19} \) (also discussed in detail in Zierep\(^{20} \) and Delale et al.,\(^{12} \) and Delale and van Dongen\(^{21} \)). Figure 2 shows a global continuous solution with one singularity at \( x_1 = 0 \) (the classical saddle point singularity at the throat without heat addition). Figures 3(a)–3(c) show solutions with three singularities in the \( M-x \) phase plane where \( x_1 = 0 \) is the classical saddle point at the throat, \( x_2 \) is a nodal point with heat addition, and \( x_3 \) is a saddle point with heat addition. Figures 4(a)–4(c), on the other hand, show possible \( M-x \) phase portraits with three singularities: The classical saddle point at \( x_1 = 0 \) (at the throat), the spiral singularity at \( x_2 \), and the saddle point with heat addition at \( x_3 \). In cases where the flow Mach number \( M \) reaches unity due to considerable latent heat addition from condensation, it can be shown that the

\[
\psi = \left( \frac{dM}{dx} \right)^* \tag{9}
\]

\[
h(x) = r(x) - s(x) \tag{10}
\]

with

\[
r(x) = \frac{1}{A} \frac{dA}{dx} \tag{11}
\]

and

\[
s(x) = \left( \frac{1 + \Gamma}{2(1 + q/c_{pm})} \right) \frac{d}{dx}\left( \frac{q}{c_{pm}} \right) - \frac{\mu_m/\mu_v}{1 - (\mu_m/\mu_v)g} \frac{dg}{dx} \tag{12}
\]

and where the superscript * denotes evaluation at \( M=1 \). In particular,

\[
h^* = h(x^*) = r^* - s^* = 0. \tag{13}
\]

Now, if we define \( G \) and \( D \) by

\[
G = (1 + \Gamma^*) \left( \frac{dh}{dx} \right)^* \tag{14}
\]

and

\[
D = \left[ \Gamma^* \left( \frac{1}{A} \frac{dA}{dx} \right)^* \right]^{12} + 4G, \tag{15}
\]

we obtain the solution of Eq. (8) for \( \psi \) as

\[
\psi = \left( \frac{dM}{dx} \right)^* - \frac{\Gamma^* (1/AdA/dx)^* + D^{1/2}}{4}. \tag{16}
\]

The classification of the singularities at \( M=1 \) and \( h^*=0 \) of the differential system (1)–(4) then follows: The singular point is

- (i) a turning point without heat addition if \( G=D=0 \),
- (ii) a saddle point with or without heat addition if \( G\geq 0 \) and \( D>0 \),
- (iii) a nodal point with heat addition if \( G<0 \) and \( D\geq 0 \),
- (iv) a spiral point with heat addition if \( G<0 \) and \( D<0 \), and
- (v) a vortex point without heat addition if \( G<0 \) and \( D<0 \).

Possible global solutions of the system of equations (1)–(4) possessing one or three singularities in the \( M-x \) phase plane are well-known (e.g., see Möhring,\(^{19} \) Zierep,\(^{20} \) Delale et al.,\(^{12} \) and Delale and van Dongen\(^{21} \)).
amount of heat added to the flow, given by Eq. (5), becomes equal to the critical value \( q^* \) defined by (for details see Delale et al.\textsuperscript{12} and Delale and van Dongen\textsuperscript{21})

\[
q^* = \left(\frac{[(p + pu^2)/\rho u]^2}{4\Theta(g)}\right) - 1,
\]

(17)

where \( \Theta(g) \) is given by

\[
\Theta(g) = \frac{(\Gamma + 1)}{2\Gamma} \left[ 1 - \frac{\mu_m}{\mu_v} g \right].
\]

We can now classify the possible steady-state flow patterns, that can be realized in condensing nozzle experiments, in connection with the above-mentioned global solutions.

1. **Subcritical flows**

For these flows the amount of heat added to the flow remains everywhere less than the critical amount given by Eq. (17). Consequently, the curve \( s(x) \) lies completely below the curve \( r(x) \) [see Fig. 5(a)], and the flow field remains everywhere continuous and supersonic in the diverging section of the nozzle. Such flows may assume any of the global solutions presented in Figs. 2 (with one singularity), 3(a), and 4(a) (with three singularities). All of these three types of global solutions, with different topological phase portraits, are possible and were calculated by Younis\textsuperscript{22} for specified heat addition.

2. **Supercritical flows**

These flows are defined as those flows for which the amount of latent heat released by nonequilibrium condensation exceeds the critical amount of heat given by Eq. (17). They correspond to global solutions with three singularities: the classical saddle at the throat, a nodal or spiral singularity, and a saddle point with heat addition (although they have been observed only with a spiral singularity). Therefore, continuous global solutions as well as discontinuous global solutions with normal shock waves are possible, in principle.

The typical variations of the curves \( r(x) \) and \( s(x) \) for the continuous and discontinuous cases (in the latter with a normal shock wave at \( x = z \)) are shown, respectively, in Figs. 5(b) and 5(c). The underlying characteristics of supercritical flows is the existence of a localized subsonic region in the heat addition zones whether the transition occurs continuously or with a normal shock wave (until now, the former has never been realized). A typical global solution with a continuous transition from supersonic to subsonic speeds through a nodal point and back to supersonic speeds through a saddle point in the heat addition zones is shown in Fig. 3(b). Discontinuous global solutions with normal shock waves at \( x = z \) [corresponding, respectively, to the phase portraits presented in Figs. 4(b) and 4(c) and typical for the variations of the curves \( r(x) \) and \( s(x) \) of Fig. 5(c)] are shown in the \( M-x \) phase planes in Figs. 6(a) and 6(b). Younis\textsuperscript{22} has calculated flows with given heat addition cor-
responding to those in the phase portraits of Figs. 6(a) and 6(b). No global supercritical solutions have ever been observed or calculated corresponding to cases presented in the $M-x$ phase planes of Fig. 3(b) or 3(c) with a nodal point singularity. [Actually, the phase portrait of Fig. 3(c) requires anomalous heat addition with very steep gradients in the local subsonic region, achieved continuously through a nodal point in this case, which does not seem to be possible for supercritical flows with nonequilibrium condensation investigated here and, thus, can be discarded on physical grounds for such flows]. Consequently, supercritical flows are believed to be realized with shock waves associated with a spiral singularity. (Such flows are said to be thermally choked. For a detailed discussion of thermal choking, see Delale et al. 12 and Delale and van Dongen. 21)

**B. Stability limit of supercritical flows with stationary normal shock waves in the quasi-one-dimensional approximation**

The steady-state flow patterns of nozzle flows with nonequilibrium condensation presented in the preceding section can experimentally be realized by increasing the initial relative humidity $\varphi_0$ for given nozzle cross-section area. This can be achieved in several different ways:

(a) By keeping the initial specific humidity $\omega_0$ and the temperature $T'_0$ at the reservoir fixed;

(b) by keeping the initial specific humidity $\omega_0$ and the total pressure $p'_0$ at the reservoir fixed;

(c) by keeping the total pressure $p'_0$ and the temperature $T'_0$ at the reservoir fixed (this latter case is not possible for pure vapor).
For sufficiently low initial humidity, the steady-state flow pattern corresponding to the global solution with only one singularity exhibited in Fig. 2 is realized. When the initial relative humidity is further increased in one of the several ways listed above, depending on nozzle geometry, reservoir conditions, and the condensation process (nucleation and droplet growth laws), the heat addition to the flow may be distributed in such a way that subcritical flow patterns corresponding to global solutions exhibited in Fig. 3(a) or 4(a) may occur (although the amount of heat added to the flow lies below the critical amount in both cases, it is believed that heat addition is more intense and localized in the latter case). Further increase in the relative humidity $\varphi_0$ for the same geometry and fixed reservoir conditions leads to supercritical flow patterns discussed above where the amount of heat added to the flow exceeds the critical amount given by Eq. (17). Although the supercritical flow pattern with continuous transition to subsonic speeds corresponding to the global solution presented in Fig. 3(b) is, in principle, possible, it has never been observed or calculated (e.g., see Younis\textsuperscript{22}). Therefore, supercritical flows are almost always realized with a normal shock wave corresponding to the global solutions with spiral singularities presented in Figs. 6(a) and 6(b).

We now discuss the stability limit of such stationary normal shock waves in the quasi-one-dimensional approximation as the initial relative humidity $\varphi_0$ is increased further for the same nozzle geometry keeping the appropriate reservoir conditions fixed. For this reason we first investigate the sign of the variation of $G$ at the spiral point $x=x_2$ given by

$$\delta G_2 = \delta r \left( \frac{dh}{dx} \right)_2 + (1 + \Gamma_2) \delta \left( \frac{dh}{dx} \right)_2$$

(19)

using Eq. (14), where subscript 2 is used to denote evaluation at the spiral singularity $x=x_2$, as the initial relative humidity is increased from $\varphi_0$ to $\varphi_0 + \delta \varphi_0$. As confirmed by experiments and by the global analysis of the next section, the normal shock wave moves towards the throat as $\varphi_0$ is increased (in one of the several different ways mentioned above) so that the variations $\delta x_2$ and $\delta x_3$ are all negative. This implies weakening of the normal shock as $\varphi_0$ is increased. Consequently, a less and less amount of heat is required to accelerate the flow back to supersonic speeds resulting in broadening of the function $s(x)$ between the spiral singularity at $x=x_2$ and the saddle point at $x=x_3$ over approximately the same distance as $\varphi_0$ is increased. Since the function $r(x)$ is kept fixed, the function $s(x)$ meets $r(x)$ (see Fig. 7) with a smaller slope at $x=x_2$ as $\varphi_0$ is increased so that we have

$$\delta \left( \frac{ds}{dx} \right)_2 < 0. \quad (20)$$

On the other hand, we have

$$\delta \left( \frac{dr}{dx} \right)_2 = \frac{d^2 r}{dx^2} \delta x_2. \quad (21)$$

We first show that $(d^2 r/dx^2)_2 < 0$ for slender nozzles. Near the throat, this can easily be achieved by using the parabolic approximation for the normalized area as

$$A = 1 + \alpha x^2 \quad (22)$$

where $\alpha$ is the slenderness parameter ($\ll 1$) of the nozzle defined by

$$\alpha = \frac{2 y_i'}{R_i'} \quad (23)$$

with $2 y_i'$ and $R_i'$ denoting, respectively, the nozzle throat height and the radius of wall curvature at the throat. At sufficiently larger distances away from the throat, the function $r(x)$, as exhibited in Fig. 7, can be shown to be concave downwards so that the assertion $(d^2 r/dx^2)_2 < 0$ is satisfied.
for all \( x_2 > 0. \) Since \( \delta x_2 < 0. \) we have \( \delta (d r/dx) z > 0. \) Now by differentiating the defining relation \((10)\) and then taking its variation at \( x = x_2, \) we obtain
\[
\frac{\delta (d h)}{dx} = \frac{\delta (d r)}{dx} - \frac{\delta (d s)}{dx} > 0. \tag{24}
\]
For the investigation of the sign of \( \delta G_2, \) given by Eq. \((19),\)
we also need to consider the sign of \( \delta \Gamma_2 \) defined by
\[
\delta \Gamma_2 = \frac{(\mu_m - \mu_v) \gamma}{[1 - (\mu_m/\mu_v) g]^2} \delta g_2 + \frac{\delta \gamma}{[1 - (\mu_m/\mu_v) g]^2} \delta g_2 + \frac{\delta \gamma}{[1 - (\mu_m/\mu_v) g]^2} \delta g_2,
\]
where the variations \( \delta \gamma \) and \( \delta \mu_m \) are nonvanishing only for the case where \( \phi_0 \) is increased, taking the reservoir pressure \( p_0 \) and reservoir temperature \( T_0 \) fixed. We first note that
\[
\delta g_2 = \left( \frac{d g_2}{dx} \right)_2 \delta x_2 < 0. \tag{26}
\]
From the variational analysis carried out in the appendix, we also have \( \delta \gamma \leq 0 \) whereas \( \delta \mu_m \) can be positive, zero or negative as \( \phi_0 \) is increased. Nevertheless, in Eq. \((25),\) \( \delta \mu_m \) multiples \( g_2 \) \((\approx 1)\) whereas the variations \( \delta g_2 \) and \( \delta \gamma \) multiply \( O(1) \) quantities. Consequently, even when \( \delta \mu_m \neq 0, \) the last term on the right-hand side \( (\text{rhs}) \) of Eq. \((25)\) can be neglected with respect to the first and second terms so that we have
\[
\delta \Gamma_2 < 0. \tag{27}
\]
We also note that
\[
\frac{dh}{dx} < 0 \tag{28}
\]
since the singularity at \( x = x_2 \) is a spiral point with \( G_2 < 0. \) By Eqs. \((24), \) \((27), \) and \((28),\) we thus have shown that
\[
\delta G_2 > 0. \tag{29}
\]
for slender nozzles \((\alpha \approx 1)\) as the initial relative humidity is increased from \( \phi_0 \) to \( \phi_0 + \delta \phi \) keeping the reservoir conditions stated above fixed. To obtain the stability limit of stationary normal shock waves, we now discuss the sign of the variation of \( \Delta \) at the spiral singularity \( x = x_2 \) given by
\[
\delta \Delta = \frac{1}{A} \left[ \frac{d A}{dx} \right]_2 \delta \Gamma_2 + \frac{1}{A} \left[ \frac{d A}{dx} \right]_2 \delta \gamma + 4 \delta G_2. \tag{30}
\]
It can be shown that, in the parabolic approximation of the normalized area given by Eq. \((22),\) the terms \( (1/A d A/dx)_2 \) and \( (d/dx (1/A d A/dx))_2 \) are of \( O(\alpha) \) with \( \alpha \approx 1 \) whereas \( \delta \Gamma_2, \delta \gamma, \) and \( \delta G_2 \) are all of the same order of magnitude. Therefore, the first and second terms on the rhs of Eq. \((30)\) can be neglected with respect to the term \( 4 \delta G_2. \) Utilizing Eq. \((29),\) we obtain the remarkable result that
\[
\delta D_2 > 0 \tag{31}
\]
for slender nozzles \((\alpha \approx 1)\) as the initial relative humidity is increased from \( \phi_0 \) to \( \phi_0 + \delta \phi \) keeping the appropriate reservoir conditions fixed. Now that we have shown that both \( \delta D_2 \) and \( \delta G_2 \) are positive for supercritical flow patterns with stationary normal shock waves corresponding to the discontinuous global solutions presented in Figs. \((a) \) and \((b),\) both \( G_2 \) and \( D_2 \) increase toward zero as \( \phi_0 \) is increased resulting in broadening of the heat addition function \( s(x), \) thereby of the flow variables \( (\text{see Fig. } 7) \) behind the normal shock position. It is obvious from the defining relation \((15)\) that \( D_2 \) tends to zero for \( G_2 < 0. \) When this happens, the spiral singularity at \( x = x_2 \) disappears and, instead, a nodal point singularity appears resulting in a catastrophic change in the \( M - x \) phase plane. The normal shock wave still persists at \( x = z \) so that a continuous global solution through a nodal point, such as those presented in Figs. \((a) \) and \((b),\) is not possible in this case. Consequently, a discontinuous global solution with a nodal point at \( x_2 \) in the topological phase portraits of Fig. \( 3 \) should emerge. On the other hand, the broadening of the heat function in this case results in a heat addition diagram where \( s(x) \) completely lies under the curve \( r(x) \) behind the normal shock \( [\text{i.e., } s(x) < r(x) \text{ for all } x > z]. \) Therefore, only a discontinuous global solution with a normal shock at \( x = z, \) for which the flow field behind the shock remains completely subsonic, can be possible. Consequently, the limit
\[
D_2 \rightarrow 0 \quad \text{as} \quad \phi_0 - (\phi_0)_{cr} \tag{32}
\]
can be taken as the stability limit beyond which quasi-one-dimensional steady-state supercritical flow patterns seem not possible. Increasing the initial relative humidity \( \phi_0 \) beyond this limit may result in the movement of the shock with bifurcation to unsteady flow patterns, as confirmed by experiments.

### III. BREAK-UP OF STRUCTURAL STABILITY OF SUPERCRITICAL FLOWS WITH STATIONARY NORMAL SHOCK WAVES

In this section, by using variational analysis and physical arguments, we show how the catastrophic change associated with the global solution of the differential system \((1) - (4)\) results in the break-up of structural stability of quasi-one-dimensional steady-state supercritical flow. To achieve this, we relate the variation \( \delta x_2 \) of the shock position to the variations \( \delta x_2 \) and \( \delta x_3 \) of the spiral singularity at \( x = x_2 \) and of the saddle point at \( x = x_3, \) both influenced by excessive heat addition due to condensation, and to the variation in the pressure distribution as the initial relative humidity is increased from \( \phi_0 \) to \( \phi_0 + \delta \phi \) keeping the appropriate reservoir conditions fixed.

#### A. Variation of the shock position due to changes in reservoir conditions

Let us define a new variable \( \beta \) and relate it to the flow Mach number by


\[
\beta = \frac{q - q_*}{c_{pm} + q} = \frac{(M^2 - 1)^2}{2(\Gamma + 1)M^2[1 + (\Gamma - 1)M^2/2]}. 
\]

Using this definition, Eq. (17) can be written as

\[
\frac{p + p\mu^2}{\rho a} = \left[ 4\Theta(g) \left( 1 + \frac{q}{c_{pm}} \right) \left( 1 + \frac{1}{\beta} \right) \right]^{1/2}. 
\]

Now by utilizing the integral momentum theorem together with Eqs. (33) and (34), we obtain in normalized form the expression

\[
\int_0^{x_s} p \left( \frac{dA}{dx} \right) dx = 2 \left[ \frac{2}{\gamma + 1} \right]^{1/(\gamma - 1)} \left[ 1 - \frac{2\gamma}{(\gamma + 1)} \Theta(g) \left( 1 + \frac{q_s}{c_{pm}} \right) \right]^{1/2} \times \left[ 1 + \frac{q}{c_{pm}} \right]^{1/2}. 
\]

In particular, at the saddle point singularity \( x = x_3 \), we have

\[
J = \int_0^{x_3} p \left( \frac{dA}{dx} \right) dx + \int_{x_3}^z \left( \frac{dA}{dx} \right) dx. 
\]

Therefore, the variation of \( J \) becomes

\[
\delta J = \int_0^{x_3} \delta \left( \frac{dA}{dx} \right) dx + \left[ p_+ \left( \frac{dA}{dx} \right) \right] \delta x_3 + \left[ p_- \left( \frac{dA}{dx} \right) \right] \delta x_3, 
\]

where \( \delta x_3 \) and \( \delta x_3 \), respectively, denote the variation in the shock position and in the position of the saddle point with heat addition and \( p_+ \) and \( p_- \) are, respectively, the pressure just downstream and upstream of the shock. Since the flow is nearly frozen up to the shock location (e.g., see Clarke and Delale\(^8\) and Delale et al.\(^9\)), \( \delta \rho \) vanishes in the interval \( 0 < x < z \), except for the onset zone, with thickness of \( O(K) \) where \( K \ll 1 \) denotes the nucleation parameter. Thus the contribution to the integral

\[
\int_0^z \delta \rho \left( \frac{dA}{dx} \right) dx = O(K \delta \rho) 
\]

becomes of second order and can be neglected. By Eqs. (38) and (39), we obtain the remarkable result

\[
\delta z = - \left[ \frac{\delta J - \int_0^{x_3} \delta \rho (dA/dx) dx + \int_{x_3}^z \delta \left( p_+ (dA/dx) \right) \delta x_3}{p_+ \Pi_z (dA/dx)_{x_3}} \right], 
\]

where \( \Pi_z \) is the shock strength defined by

\[
\Pi_z = \frac{p_+ - p_1}{p_+}. 
\]

Equation (40) relates the variation \( \delta z \) in the shock position to the shock strength \( \Pi_z \), to the rate of area change at the shock location, to the variation \( \delta J \) in the impulse function and to the variation in the impulse arising from the variation \( \delta \rho \) of the pressure distribution. As for the signs of different variations entering Eq. (40), we first note that the saturation point \( x_s \) (defined as the point where the supersaturation ratio of the condensable vapor is unity) in the converging section of the nozzle shifts closer to the reservoir so that

\[
\delta x_s > 0, 
\]

as the initial relative humidity is increased by an amount \( \delta \varphi_0 \) keeping appropriate reservoir conditions fixed. This would lead to achieving higher supersaturation ratios at the same location up to the shock location resulting in the shift of the onset zone in the diverging section of the nozzle towards the throat so that

\[
\delta z < 0, 
\]

as confirmed by experiments. Consequently the shock strength decreases so that

\[
\int_0^{x_3} \delta \rho \left( \frac{dA}{dx} \right) dx < 0. 
\]

Weakening of the shock also results in less amount of heat necessary to accelerate the flow back to supersonic speeds downstream of the shock so that

\[
\delta x_3 > 0. 
\]

One can also show that (details are given in the Appendix)

\[
\delta J > 0 
\]

demonstrating consistency.

B. Break-up of structural stability

We now reconsider Eq. (40) and write it in the form

\[
\frac{\delta z}{\delta \varphi_0} = - \left[ \frac{\delta J - \int_0^{x_3} \delta \rho (dA/dx) dx + \int_{x_3}^z \delta \left( p_+ (dA/dx) \right) \delta x_3}{p_+ \Pi_z (dA/dx)_{x_3}} \right] / \left[ p_+ \Pi_z (dA/dx)_{x_3} \right]. 
\]

using variational derivatives. When the initial relative humidity is increased in the supercritical flow regime, the shock moves closer to the throat so that the denominator of Eq. (48) decreases considerably mainly due to the decrease in the shock strength \( \Pi_z \) and, secondarily due to the reduction in \( (dA/dx)_{x_3} \). Consequently, the magnitude of the abso-
with the initial relative humidity \( \varphi_0 \) for fixed nozzle geometry and appropriate reservoir conditions \([\varphi_0] \), corresponds to the hypothetical limit where \( \delta z / \delta \varphi_0 \to -\infty \) at the throat and \((\varphi_0)_e\) corresponds to the limit of stability.

The absolute value of the variational derivative \( \delta z / \delta \varphi_0 \) assumes higher and higher values as \( \varphi_0 \) is further increased. In particular, we have

\[
\frac{\delta z}{\delta \varphi_0} \to -\infty \quad \text{as} \quad z \to 0^+
\]

since, in this case, both \( \Pi_z \) and \( (dA/dx)_z \) would vanish, say, for a hypothetical value \((\varphi_0)_e\) of the initial relative humidity.

A typical variation of the normal shock location \( z \) with respect to the initial relative humidity \( \varphi_0 \) is shown in Fig. 8. It is now obvious that Eq. (48), derived from the steady-state impulse law, will be violated at some \((\varphi_0)_e < (\varphi_0)_0\), where

\[
\left| \frac{\delta z}{\delta \varphi_0} \right| \gg 1,
\]

where subscript \( cr \) is used to denote such critical quantities.

In order to see how the condition stated in Eq. (50) leads to the break-up of structural stability of the differential system (1)-(4), we consider the asymptotically structured condensation zones for supercritical flows of regime I of Delale et al. Figure 9 shows the various condensation zones, whose nondimensional thicknesses are measured with respect to the nucleation parameter \( K \ll 1 \) (details are given in Delale et al.). Since the onset zone, which appears just upstream of the shock, has nondimensional thickness of \( O(K) \), the variation \( \delta z \) in the nondimensional shock position cannot exceed \( O(K) \) for arbitrarily small increase \( \delta \varphi_0 \) in the initial relative humidity. In particular, condition (50) implies that \( \delta z \) can become of \( O(1) \) on measure \( K \) for sufficiently small variation \( \delta \varphi_0 \) of \( O(K) \), which may be taken as a condition for the violation of quasi-one-dimensional steady-state condensation structures. Therefore, a sufficient condition for the break-up of structural stability of the differential system (1)-(4) can be taken as

\[
\frac{\delta z}{\delta \varphi_0} \bigg|_{cr} = O(K^{-1}).
\] (51)

When this condition is satisfied, nucleation rates will be by orders of magnitude lower than those necessary to release sufficient latent heat downstream of the shock in order to accelerate the flow back to supersonic speeds. Thus the flow downstream of the shock, in this case, would become fully subsonic corresponding to the discontinuous global solutions with a nodal point at \( x = x_2 \) \((D_2 > 0)\) mentioned before, where the limit given by Eq. (32) is reached or exceeded.

C. A criterion for the alteration of quasi-one-dimensional steady-state condensation structures

Assuming that the numerator of the right-hand side (rhs) of Eq. (48) does not exceed \( O(1) \) on measure \( K \), Eq. (51) implies

\[
(\Pi_z)_{cr} \left( \frac{dA}{dx} \right)_z = cK,
\]

where \( c \) is a parameter, of \( O(1) \) in magnitude on measure \( K \), describing a measure of the numerator of the rhs of Eq. (48) due to heat addition downstream of the shock. In particular, the shock strength \( \Pi_z \) is related to the shock Mach number \( M_z \), which for stationary shock waves is the flow Mach number at \( x = z \), by

\[
\Pi_z = \frac{2 \gamma}{(\gamma + 1)} \left( M_z^2 - 1 \right).
\]

Then Eqs. (22), (52), and (53) lead to

\[
1 + \frac{c^2 K^2(\gamma + 1)^2}{16 \alpha \gamma^2 (M_z^2 - 1)^2} = \frac{1}{M_z} \left[ \frac{2}{(\gamma + 1)} + \frac{(\gamma - 1)}{(\gamma + 1)} M_z^2 \right]^{(\gamma + 1)/[2(\gamma - 1)]},
\]

provided that the classical isentropic Mach number-area relation is granted in the nearly frozen zones upstream of the shock location \( x = z \). In Eq. (54), the nozzle slenderness parameter \( \alpha \) is to be evaluated by Eq. (23) and the isentropic exponent of the mixture \( \gamma \) is to be evaluated at the reservoir. The nucleation parameter \( K \) depends on the particular nucleation rate equation employed and can be written in terms of the reservoir conditions and properties of the constituents of the two-phase mixture (e.g., see Blythe and Shih and Delale et al.). The unknown parameter \( c \) in Eq. (54) needs a separate consideration. Of course, the value of \( c \) can be calculated from the numerator of the rhs of Eq. (48), which requires complete solution of the problem (such a closed-form
asymptotic solution is already given by Delale et al.\(^\text{11}\)). This computation is tedious and often not practical; therefore, we here seek an estimate of \(c\) only. In particular, the quantity

\[
\frac{\mu_m L g_3}{\mu_0 c_{pm}} \left( \frac{1 + \mu_m L g_3}{\mu_0 c_{pm}} \right) \quad (55)
\]

seems to indicate a measure of \(\delta \dot{R} / \partial \varphi_0\) (for details see the Appendix) in Eq. (48), which is also a measure for the magnitude of the numerator of the rhs of Eq. (48). Thus, an estimate of \(c\) can be obtained by assuming a power-law correlation of the form

\[
c = E \left( \frac{q_T}{c_{pm}} \right) \quad (56)
\]

where \(q_T = (\mu_m / \mu_0) L g_T\) is the total heat that can be added to the fluid in complete relaxation to equilibrium with \(g_T\) denoting the final equilibrium condensate mass fraction (for a mixture of condensable vapor and a carrier gas, \(g_T = \omega_0\)) and \(f = q_3 / q_T\) with \(q_3\) denoting the amount of latent heat released in the local subsonic region downstream of the shock. The constants \(E\) and \(F\) can then be determined from experiments over a wide range of reservoir conditions, as will be demonstrated in the next section. Equation (54) together with Eq. (56) then yield the critical shock Mach number \(M_\text{cr}\) for which the quasi-one-dimensional steady-state supercritical condensation structures are no longer possible. This critical shock Mach number, which can be taken as the flow Mach number at the onset of condensation, can then be related to the critical initial relative humidity \(\varphi_0\) for the given nozzle using transonic similarity rules (e.g., see Zierep and Lin\(^\text{33}\) and Schnerr\(^\text{9}\)).

### IV. RESULTS

In this section we investigate the stability limit of stationary normal shock waves caused by latent heat addition from moist air expansion, in connection with the critical initial relative humidity of the preceding section, in two slender nozzles, designated as S2 and G1 and used in the experiments of Schnerr\(^\text{9}\) and Lamanna\(^\text{24}\) with geometric configurations shown in Fig. 1. A condensation model used by Schnerr and Dohrmann\(^\text{13}\) in steady two-dimensional numerical simulations, by Adam and Schnerr\(^\text{14}\) in unsteady two-dimensional numerical simulations and by Delale et al.\(^\text{10,11}\) in the asymptotic solution of quasi-one-dimensional nozzle flows, which employs the classical nucleation equation and the Hertz-Knudsen droplet growth law with the surface tension and the accommodation coefficient fitted to the experiments of Peters and Paikert,\(^\text{23}\) has already been successful in achieving excellent agreement with the static pressure measurements and with the observed shock locations in supercritical flows. The same condensation model will be used to examine the stability limit here. Figure 10 shows typical distributions, obtained by the quasi-one-dimensional asymptotic solution of Delale et al.\(^\text{11}\) for the pressure and Mach number under atmospheric supply conditions in the circular arc nozzle S2 as the initial relative humidity \(\varphi_0\) is varied in the range 0.70 < \(\varphi_0\) < 0.80 keeping the initial specific humidity \(\omega_0\) and the reservoir temperature \(T_0\) fixed. Broadening of the distributions is observed downstream of the shock as the initial relative humidity is increased. In this case the results of numerical simulations suggest a value of \(\varphi_0\) = 0.8 for the critical initial relative humidity. Figure 11 shows similar
distributions by the quasi-one-dimensional asymptotic solution of Delale et al.\textsuperscript{11} in G1 nozzle under similar supply conditions by varying the initial relative humidity in the range $0.67 \leq \varphi_0 < 0.65$ now keeping the reservoir temperature $T_0'$ and the reservoir pressure $p_0'$ fixed. Similar results obtained by Lamanna\textsuperscript{24} using unsteady two-dimensional numerical simulations in G1 nozzle are also shown in Fig. 12 (in nozzle G1 the boundary layer correction to the area is also taken into account). The asymptotic and numerically simulated predictions agree within a few percent (the differences can be attributed, mainly, to those arising from two-dimensional effects and to those between the shock fitting computations in the asymptotic solution on one hand and the shock capturing computations in the numerical simulation on the other hand). Once again broadening of the distributions downstream of the shock, a characteristic indicating an approach to the limit of stability of stationary shock waves, is observed as the initial relative humidity is increased. Experiments and results of numerical simulations suggest a value of $(\varphi_0)_{cr}$ = 0.67 for this case. By correlating the results of all of these experiments for the nozzles S2 and G1 by Eq. (56) and by utilizing the shock Mach number-initial relative humidity correlation of Schnerr\textsuperscript{9} together with the condensation model of Schnerr and Dohrmann,\textsuperscript{13} we find, to a good approximation, the result

$$c = 6.0 \times 10^3 \left[ \frac{L' \omega_0 / (\epsilon_{pm}' T_0')} {1 + 0.5 L' \omega_0 / (\epsilon_{pm}' T_0')} \right]^{3.511}$$

(57)

in the range of reservoir temperatures 280 K $< T_0' <$ 310 K and reservoir pressures 0.856 bar $\leq p_0' \leq$ 1 bar.

With the above correlation for an estimate of $c$ in the range of reservoir conditions stated, we are now in a position to get an estimate of the critical initial relative humidity $(\varphi_0)_{cr}$ (beyond which the quasi-one-dimensional steady-state condensation structures are altered) for fixed reservoir pressure and temperature using the criterion established in Sec. III C. The procedure is as follows.

(i) We first guess a value for $\varphi_0$, say $\varphi_0'$.

(ii) We solve for the corresponding value of the initial specific humidity $\bar{\omega}_0$ iteratively from the defining relation

$$\bar{\omega}_0 = \frac{\bar{\omega}_0 p_{cr}'(T_0')}{p_0'(\mu_m / \mu_v)}$$

(58)

where $p_{cr}'$ is the saturation pressure of the vapor at $T_0'$ (note that $\mu_m$ also depends on $\omega_0$).

(iii) We evaluate $c$ by Eq. (57).

(iv) We use an estimate for the nucleation parameter $K$ given by Delale et al.\textsuperscript{10} corresponding to the classical nucleation equation and the surface tension employed.

(v) We solve for the critical Mach number from Eq. (54) where it is understood that $\gamma$ will be evaluated using the value of $\omega_0$ given by Eq. (58) and $\alpha$ will be evaluated by Eq. (23).

(vi) We use the transonic similarity rule criterion for the onset of condensation by Schnerr\textsuperscript{9} to relate the critical shock Mach number $M_{cr}$ (now taken as the onset Mach number) to the initial relative humidity to arrive at an estimate of the critical initial relative humidity $(\varphi_0)_{cr}$ as

$$(\varphi_0)_{cr} = \left[ \frac{\gamma + 1} {2 + (\gamma - 1) M_{cr}^2} \right]^{1/b}$$

(59)

where $b$ is given by

$$b = 0.0498 \left( \frac{dT'}{dx'} \right)_{t_{cr}}^{0.3010}$$

(60)

with the cooling rate at the throat $-(dT'/dx')_t$ (measured in K/cm) given by

$$\left( \frac{dT'}{dx'} \right)_{t_{cr}} = 2 \frac{(\gamma - 1)} {((\gamma + 1) T_0')} \frac{1} {((\gamma + 1)(\gamma - 1) R_0')}^{1/2}$$

(61)

(vii) The procedure is repeated iteratively by assigning the value obtained by Eq. (59) to the initial guess $\varphi_0'$. An estimate of $(\varphi_0)_{cr}$ is obtained when the initial guess and the value obtained by Eq. (59) agree within any desired accuracy.

We now compare the results for $(\varphi_0)_{cr}$ of the quasi-one-dimensional approximation obtained by the above algorithm for different nozzle configurations and different reservoir conditions with those of experiments and/or of unsteady two-dimensional numerical simulations at the limit of stability, where bifurcation to unsteady flow with shock oscillations occurs (e.g., see Adam and Schnerr\textsuperscript{14} and Lamanna\textsuperscript{24}). Table I shows such a comparison. Excellent agreement is achieved between the values of $(\varphi_0)_{cr}$ given above in the quasi-one-dimensional approximation and the values of the initial rela-

FIG. 12. Variations in the normalized pressure $p = p'/p_0'$ and Mach number $M$ distributions, obtained by the numerical simulation of Lamanna (Ref. 24), along the axis of G1 nozzle with normalized axial coordinate $x = x'/(2x_1')$ for fixed reservoir temperature $T_0' = 282.6$ K and reservoir pressure $p_0' = 0.878$ bar as the initial relative humidity is increased from $\varphi_0 = 0.6$ (solid lines in the figure with $z' = 28$ mm, $x_1' = 36.5$ mm, and $x_1' = 50.4$ mm) to $\varphi_0 + \delta \varphi_0 = 0.65$ (dashed lines in the figure with $z' = 22$ mm, $x_1' = 33.5$ mm, and $x_1' = 49.1$ mm).
tive humidity at the limit of stability observed in experiments and unsteady two-dimensional numerical simulations for the slender nozzles S2 and G1 under different reservoir conditions. Although boundary layer effects along the nozzle wall may become important at the limit of stability, these effects can simply be corrected for by modifying the cross-section area, as was done in the computations for the G1 nozzle. One may argue that two-dimensional effects can also become important at the limit of stability; however, the stationary shock waves observed in condensing flows through slender nozzles are normal shock waves and they occur close to the throat so that the quasi-one-dimensional approximation seems to be justified. In addition, most of the unsteady periodic flow shock oscillations in the experiments of Barschdorff and of Adam and Schnerr through slender nozzles show almost unsteady quasi-one-dimensional flow behavior. This is why such a good agreement for the critical initial relative humidity is achieved between the values obtained above in the quasi-one-dimensional approximation and those observed in experiments and/or numerical simulations in slender nozzles. In two-dimensional nozzles, such as the S1 nozzle of Fig. 1, differences between \((\varphi_0)_{cr}\) of the above algorithm and the values of the initial relative humidity at the limit of stability observed in experiments and numerical simulations can clearly be identified, as expected. Even in this case, the difference is less than 10% so that a rough estimate for the critical initial relative humidity at the limit of stability of stationary shock waves can still be obtained by the above algorithm in the quasi-one-dimensional approximation.

The above criterion for the limit of stability depends on the following:

(a) The nozzle geometry, in particular, the slenderness parameter \(a\) of the nozzle.

(b) The dimensional cooling rate \(- (dT'/dx')_s\). [This is because the correlation by Schnerr used above relates the critical Mach number to the critical initial relative humidity via the dimensional cooling rate given by Eq. (61) in evaluating \(b\) by Eq. (60). A correlation relating \(b\) to the nondimensional cooling rate \(- (dT/dx)\), and, possibly, other nondimensional parameters seems more convenient from dimensional analysis viewpoint.]

(c) The nondimensional heat \(L' g_3 / (c'_pm T'_0)\) added to the flow in the local subsonic region just downstream of the shock.

(d) The initial specific humidity \(\omega_0\).

(e) The adiabatic exponent \(\gamma\).

(f) The nucleation parameter \(K\). Although correlations relating the critical initial relative humidity \((\varphi_0)_{cr}\) corresponding to the limit of stability are not available in the literature, it is instructive to note that various investigations on shock oscillations in the unsteady flow domain by dimensional analysis suggest that the nondimensional frequency of shock oscillations depends on similar nondimensional parameters listed above (e.g., see Zierep and Lin, Mosnier, and Wegener and Mosnier).

V. CONCLUSIONS

The stability of stationary shock waves in supercritical nozzle flows with homogeneous condensation is investigated by examining the variational changes in the nature of the singularities of the differential equations of motion due to the increase in the initial relative humidity keeping appropriate reservoir conditions fixed. It is shown that a catastrophic change is observed in the phase portraits, where the spiral singularity responsible for the discontinuous global solution with stationary shock waves turns into a nodal point singularity as the initial relative humidity is increased beyond some limit. Then, a quasi-one-dimensional steady-state flow solution is no longer possible. A criterion, based on the break-up of structural stability of the quasi-one-dimensional steady-state differential equations of motion, is established yielding an equation for the critical shock Mach number, and this critical shock Mach number is, in turn, related to the critical initial relative humidity using a correlation based on

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### TABLE I. Comparison of the values of the critical initial relative humidity obtained by the proposed criterion with those of experiments and/or numerical simulations.

<table>
<thead>
<tr>
<th>Nozzle</th>
<th>Dimension and geometric parameters</th>
<th>Fixed reservoir conditions</th>
<th>Estimate of (c) by Eq. (57)</th>
<th>Expt. and/or numerical simulations [Adam and Schnerr (Ref. 14) and Lamanna (Ref. 24)]</th>
<th>Proposed criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>S2</td>
<td>(1 - D), (y'_1 = 15) mm, (R'_1 = 400) mm</td>
<td>(T'_0 = 295) K, (P'_0 = 1) bar</td>
<td>1.29</td>
<td>0.68</td>
<td>0.691</td>
</tr>
<tr>
<td>S2</td>
<td>(1 - D), (y'_1 = 15) mm, (R'_1 = 400) mm</td>
<td>(T'_0 = 305) K, (P'_0 = 1) bar</td>
<td>4.18</td>
<td>0.58</td>
<td>0.565</td>
</tr>
<tr>
<td>G1</td>
<td>(1 - D), (y'_1 = 10) mm, (R'_1 = 4000) mm</td>
<td>(T'_0 = 282.6) K, (P'_0 = 0.878) bar</td>
<td>0.14</td>
<td>0.67</td>
<td>0.676</td>
</tr>
<tr>
<td>S1</td>
<td>(2 - D), (y'_1 = 60) mm, (R'_1 = 100) mm</td>
<td>(T'_0 = 295) K, (P'_0 = 1) bar</td>
<td>2.07</td>
<td>0.74</td>
<td>0.793</td>
</tr>
</tbody>
</table>
transonic similarity rule.9 The criterion is then checked against results of experiments and of unsteady two-dimensional numerical simulations of Adam and Schnerr14 and Lamanna24 for the expansion of moist air in distinct slender nozzles for supercritical flows under different reservoir conditions, and excellent agreement for the values of the critical initial relative humidity is demonstrated for these nozzles. Even for two-dimensional nozzles, values for the critical initial relative humidity obtained by the criterion seem to agree within a few percent with those of experiments and numerical simulations. Application of the criterion for the stability limit of stationary shock waves in pure steam condensation seems also possible and is reserved for future work.

\[
\delta J = \left( \frac{2}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}} \left[ \frac{2}{\gamma + 1} \ln \left( \frac{\gamma + 1}{2} \right) - 1 \right] + \left[ \frac{2}{\gamma + 1} \Theta (g_3) \right] \left( 1 + \frac{q_3}{c_{pm}} \right) \left[ \frac{1}{2} \left( \frac{2}{\gamma + 1} \ln \left( \frac{\gamma + 1}{2} \right) - 1 \right) \right] \frac{\delta \gamma}{\gamma (1 - 1)} - \left( \frac{2}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}} \delta T_0 \left( \frac{T_0}{T_0} + \frac{c_{pm}}{c_{pm}} \right),
\]

(A1)

where \( T_0 \) is the mixture reservoir temperature, \( c'_{pm} \) is the mixture specific heat at constant pressure defined by

\[
c'_{pm} = \omega_0 c'_{pv} + (1 - \omega_0) c'_{pi},
\]

(A2)

with \( c'_{pv} \) and \( c'_{pi} \) denoting, respectively, the specific heats at constant pressure of the condensable vapor and of the carrier gas, \( \mu_m \) is the mixture molecular weight defined by

\[
\frac{1}{\mu_m} = \frac{\omega_0}{\mu_v} + \frac{(1 - \omega_0)}{\mu_i}.
\]

(A3)

with \( \mu_v \) and \( \mu_i \) denoting, respectively, the molecular weights of the condensable vapor and of the carrier gas, and \( c_{pm} = \frac{\mu_m c'_{pm}}{\mu} \).

We now calculate the variations entering Eq. (A1) as the initial relative humidity is increased by an amount \( \delta \varphi_0 > 0 \) for any of the three different possible cases given below.

(i) The initial specific humidity \( \omega_0 \) and the mixture reservoir temperature \( T_0 \) are kept fixed.

In this case we have

\[
\delta \gamma = \delta c'_{pm} = \delta T_0 = 0.
\]

(A4)

Therefore, Eq. (A1) for this case becomes

\[
\delta J = - \left( \frac{2}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}} \left[ \frac{2}{\gamma + 1} \Theta (g_3) \right] \left( 1 + \frac{q_3}{c_{pm}} \right) \left[ \frac{1}{2} \left( \frac{2}{\gamma + 1} \ln \left( \frac{\gamma + 1}{2} \right) - 1 \right) \right] \frac{\delta \gamma}{\gamma (1 - 1)} - \left( \frac{2}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}} \frac{\delta T_0}{T_0} \left( \frac{T_0}{T_0} + \frac{c_{pm}}{c_{pm}} \right).
\]

(A5)

Since \( \delta g_3 = (dg/dx) \delta x_3 < 0 \), it follows that \( \delta J > 0 \) for this case. We also have \( \delta g_3 / g_3 = O(\delta \varphi_0) \) so that \( \delta J = O(\delta \varphi_0) \).

(ii) The initial specific humidity \( \omega_0 \) and the mixture reservoir pressure \( p_0 \) are kept fixed.

In this case we have

\[
\delta \gamma = \delta c'_{pm} = 0 \quad \text{but} \quad \delta T_0 \neq 0.
\]

(A6)

Therefore, Eq. (A1) for this case reduces to

\[
\delta J = - \left( \frac{2}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}} \left[ \frac{2}{\gamma + 1} \Theta (g_3) \right] \left( 1 + \frac{q_3}{c_{pm}} \right) \left[ \frac{1}{2} \left( \frac{2}{\gamma + 1} \ln \left( \frac{\gamma + 1}{2} \right) - 1 \right) \right] \frac{\delta \gamma}{\gamma (1 - 1)} - \left( \frac{2}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}} \frac{\delta T_0}{T_0} \left( \frac{T_0}{T_0} + \frac{c_{pm}}{c_{pm}} \right).
\]

(A7)

It can further be shown that

\[
\frac{\delta T_0}{T_0} = - \frac{\delta \varphi_0}{L \varphi_0^2} \leq 0,
\]

(A8)

where \( L = L' \mu_v / (\partial T_0) \) is the normalized latent heat with \( L' \) and \( \partial T_0 \) denoting, respectively, the actual latent heat and the universal gas constant. It can be shown that \( \delta g_3 / g_3 = O(\delta \varphi_0) \) whereas \( \delta T_0 / T_0 = O(\delta \varphi_0) \) (since \( L > 1 \)) so that \( \delta J = O(\delta \varphi_0) \) and is positive for this case as well.

(iii) The mixture reservoir temperature \( T_0 \) and the mixture reservoir pressure \( p_0 \) are kept fixed (not applicable to the condensation of pure vapor).

In this case we have

\[
\delta c'_{pm} = 0 \quad \text{and} \quad \delta \gamma = \delta T_0.
\]
\[ \delta T'_{0}= 0 \quad \text{but} \quad \delta \gamma \neq 0 \quad \text{and} \quad \delta c'_{pm} \neq 0. \]  
(A9)

It can be shown after some manipulations that

\[ \frac{\delta c'_{pm}}{c'_{pm}} = \frac{c'_{pm} - c'_{pl}}{c'_{pm}} \delta \omega_{0} = -\omega_{0} \left[ 1 - \left( \frac{1}{\mu_{i}} - \frac{1}{\mu_{v}} \right) \frac{c'_{pm} - c'_{pl}}{c'_{pl}} \right] \frac{\delta \varphi_{0}}{\varphi_{0}} \]  
(A10)

and

\[ \frac{\delta \gamma}{\gamma(\gamma-1)} = -\omega_{0} \left[ 1 - \left( \mu_{i} \mu_{v} \right) \frac{c'_{pm} - c'_{pl}}{c'_{pl}} \right] \frac{\delta \varphi_{0}}{\varphi_{0}}. \]  
(A11)

Therefore, both \( \delta \gamma[\gamma(\gamma-1)] \) and \( \delta c'_{pm}/c'_{pm} \) are of \( O(\omega_{0}\delta \varphi_{0}) \) and can be neglected with respect to the term \( \delta \varphi_{0}/\varphi_{0} \), which is of \( O(\delta \varphi_{0}) \), since \( \omega_{0}=1 \). It then follows that Eq. (A1) for \( \delta J \) approximately reduces to the expression given by Eq. (A4) for this case. Thus, \( \delta J = O(\delta \varphi_{0}) \) and is positive for this case as well.