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van Berkel, C.A.M.

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A study of Gel'fand-Shilov spaces
of functions of two real variables
by
C.A.M. van Berkel
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Abstract
This report contains a study of Schwartz spaces and Gel'fand-Shilov spaces of functions on $\mathbb{R}$ and $\mathbb{R}^2$. Some classical results valid for these spaces are summarized and some new results are added. Besides the usual descriptions in terms of Cartesian coordinates which can be found in the classical literature on the subject, an approach with polar coordinates is discussed. Thus, applying the Hecke-Bochner identities, characterizations of the two-variable function spaces are deduced in terms of Hankel transformations. The problem is considered whether the factored function $(r, \phi) \mapsto g(r) h(\phi), r \geq 0, -\pi < \phi \leq \pi,$ belongs to one of the above mentioned spaces.
In this report we present several descriptions of the Schwartz spaces and of their subspaces, the Gel’fand-Shilov spaces. These spaces are said to be of type $S$. We start with the function spaces $S(\mathbb{R})$ and $S_0^0(\mathbb{R})$ and next we consider $S_{C_0^m}(\mathbb{R})$ and $S_{C_0}^n(\mathbb{R})$. Descriptions of the last mentioned spaces in terms of Cartesian coordinates are often based on inductive arguments. However, besides Cartesian coordinates also polar coordinates can be chosen to describe the two variable function spaces. Now the correspondence between $S(\mathbb{R})$ and $S(\mathbb{R}^2)$ and also between $S_0^0(\mathbb{R})$ and $S_0^0(\mathbb{R}^2)$ is less obvious. The results for $S(\mathbb{R}^2)$ and $S_0^0(\mathbb{R}^2)$ admit generalizations to $S(\mathbb{R}^q)$ and $S_0^0(\mathbb{R}^q)$, $q \geq 3$, when spherical coordinates are used. In a subsequent report we describe these generalizations.

When treating problems in which the Fourier transformation plays an important role the spaces of type $S$ arise in a natural way. In fact recently it is shown that spaces of type $S$ can be characterized in terms of growth behaviour at infinity of the functions themselves as well as of their Fourier transforms. The Fourier transformation on $L^2(\mathbb{R}^2)$ can be decomposed into certain Hankel transformations. Since the Hankel transformations are defined for functions on $\mathbb{R}^+$ we have studied these transformations acting on the subspaces of even functions in the $S$-type spaces (cf. [EB]). We mention the characterizations of the even $S$-type spaces in terms of decay at infinity of the functions themselves and of their Hankel transforms.

Suppose a function can be expanded in a Fourier series $\sum a_n(r) e^{i n \phi}$, where $(r, \phi)$ are polar coordinates. We wonder if there are necessary and sufficient conditions on the $a_n$, defined on $\mathbb{R}^+$, such that the above Fourier series determines an $S(\mathbb{R}^2)$-type function. In this paper we show that such conditions on the $a_n$ indeed exist and it may be of no surprise to the reader that the Hankel transformations are an essential tool in our analysis. In fact $a_n(r)$ has to belong to $r^{1/2} S_{a, even}^n(\mathbb{R})$, $n \in \mathbb{Z}$, in order that the above Fourier series becomes a candidate for an $S_0^0(\mathbb{R}^2)$-function. Of course in this respect we also have to deal with convergence problems.

Finally we pose the following polar coordinates factorization problem. Considering factored functions of the form $f(r \cos \phi, r \sin \phi) = g(r) h(\phi)$, which combinations $g(r) h(\phi)$ determine $S$-type functions?

Applications of our results are sought amongst others when solving differential equations with the aid of the classical method of separation of variables.

In the remaining part of this introduction we give a more detailed summary of Chapter I which deals with the Schwartz spaces $S(\mathbb{R})$ and $S(\mathbb{R}^2)$. We do not present such a summary of Chapter II, which discusses the Gel’fand-Shilov spaces $S_0^0(\mathbb{R})$ and $S_0^0(\mathbb{R}^2)$, since except for technical details results in the second chapter are deduced and specialized from the first one.

We first introduce the Hankel transformations $H_{v, \mu}$ and we emphasize well-known relations between these transformations and the Fourier transformation: The Fourier transform of a spherical symmetric function is spherical symmetric too and can be expressed as a Hankel transform. In general the Fourier transform is expressed as a series of Hankel transforms, applying the Hecke-Bochner identities (cf. Section 3.4). Then we introduce the Schwartz space $S(\mathbb{R})$ and we deal with linear operations such as multiplication, differentiation, Fourier transformation, dilatation etc. The space $S(\mathbb{R})$ is characterized in terms of domains of unbounded operators and thus
topologized by means of several equivalent systems of seminorms. Special attention is devoted to sequential convergence. We present the description of $S(\mathbb{R})$ in terms of the Hermite expansion coefficients of its elements.

Next we investigate $S_{\text{even}}(\mathbb{R})$, the subspace of even functions in $S(\mathbb{R})$. Also therein we consider linear operations, such as $IF$, $BH_{n,\mu}$ and $x^{-1} \frac{d}{dx}$. The characterization of $S_{\text{even}}(\mathbb{R})$ is presented in terms of Laguerre expansion coefficients of its elements.

In Section 3 we introduce the Schwartz space $S(\mathbb{R}^2)$ and we repeat the above program for the present case of two independent variables. Now we can also utilize different coordinate systems, e.g. polar coordinates. Let $f$ be a function with Fourier expansion

$$f(r \cos \phi, r \sin \phi) = \sum a_n(r) e^{in\phi}.$$ 

For the Fourier transform $IF$ we then have

$$(IF f)(r \cos \phi, r \sin \phi) = \sum (-i)^{|n|} (BH_{|n|,1} a_n) (r) e^{in\phi}.$$ 

We deduce necessary and sufficient conditions on the $a_n$ and $BH_{|n|,1} a_n$ in order that $f$ belongs to $S(\mathbb{R}^2)$. This is one of the main theorems of this report. In particular we obtain a description of radial symmetric functions in $S(\mathbb{R}^2)$.

Finally we prove a polar coordinates factorization problem: The factored function $(r,\phi) \mapsto g(r) h(\phi)$ determines an $S(\mathbb{R}^2)$-function if and only if $h$ extends to an infinitely differentiable $2\pi$-periodic function on $\mathbb{R}$ and $g$ extends to a function in $r^{ln1} S_{\text{even}}(\mathbb{R})$ for all $n \in \mathbb{Z}$ for which

$$\int_{-\pi}^{\pi} h(\phi) e^{-in\phi} d\phi \neq 0.$$ 

This problem will be treated also in the Gel'fand-Shilov spaces $S^\beta_\alpha(\mathbb{R}^2)$. However we are not yet able to give the solution for $1 < \beta < \alpha$. 
Chapter I Rapidly decreasing functions

1. The Hankel transformations $\mathcal{H}_{\nu, \mu}$

Formally we define the classical Hankel transformations $\mathcal{H}_{\nu, \mu}$ by

$$
(\mathcal{H}_{\nu, \mu} f)(x) = \int_0^\infty(xy)^{-\frac{\nu+1}{2}} J_\nu(xy) y^\mu \, dy, \quad x > 0,
$$

(1.1)

where $J_\nu$ is the Bessel function of the first kind and of order $\nu$. In the introduction of the Hankel transformations we apply the generalized Laguerre polynomials. First we present a short summary of properties of these polynomials. For a more complete list of properties see [MOS, Section 5.5].

The generalized Laguerre polynomials $L_n^{(\nu)}$, $n \in \mathbb{N}_0$, $\nu > -1$, are the orthogonal polynomials associated with the interval $(0, \infty)$ and the weight function $w(x) = x^\nu e^{-x}$. In [MOS] they are normalized by the condition that the coefficient of $x^n$ in $L_n^{(\nu)}(x)$ equals $(-1)^n / n!$.

Formulas involving the $L_n^{(\nu)}$:

Explicit expression

$$
L_n^{(\nu)}(x) = \sum_{m=0}^{n} (-1)^m \frac{n + \nu}{n - m} \frac{x^m}{m!} \binom{n + \nu}{n} {}_1F_1(-n; \nu + 1; x).
$$

(1.2)

Rodrigues' formula

$$
L_n^{(\nu)}(x) = x^{-\nu} e^{\frac{x}{2}} \frac{\sqrt{\pi}}{n!} \frac{d^n}{dx^n} \left[ e^{-\frac{x^2}{2}} x^{n+\nu} \right].
$$

(1.3)

Orthogonality relation

$$
\int_0^\infty e^{-x} x^\nu L_n^{(\nu)}(x) L_m^{(\nu)}(x) \, dx = \frac{\Gamma(\nu+n+1)}{\Gamma(n+1)} \delta_{mn}.
$$

(1.4)

Integral relation

$$
L_n^{(\nu)}(x) = \frac{(-1)^n}{2^\nu} \frac{\sqrt{\pi}}{\Gamma(\nu+1)} \left[ e^{-\frac{r^2}{2}} \int_0^\infty t^{\nu} L_n^{(\nu)}(t) J_\nu(\sqrt{xt}) \, dt \right].
$$

(1.5)

Differential equation

$$
\frac{d^2}{dx^2} \left[ L_n^{(\nu)}(x) \right] + (\nu + 1 - x) \frac{d}{dx} \left[ L_n^{(\nu)}(x) \right] + n L_n^{(\nu)}(x) = 0.
$$

(1.6)

In the sequel we consider $\mu \geq 0$, $\nu > -1$.

From (1.4) we derive the orthogonality relation
\[
\int_0^{\infty} \left[ x^{-\frac{1}{2}+\mu+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{(v)} (x^2) \right] \cdot \left[ x^{-\frac{1}{2}+\mu+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_{m}^{(v)} (x^2) \right] x^\mu \, dx = \\
= \frac{\Gamma(v+n+1)}{2\Gamma(n+1)} \delta_{nm}
\]

(1.7)

and from (1.5) the integral expression
\[
x^{-\frac{1}{2}+\mu+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{(v)} (x^2) = \\
= (-1)^n \int_0^{\infty} (xy)^{-\frac{1}{2}+\mu+\frac{1}{2}} J_v(xy) y^{-\frac{1}{2}+\mu+\frac{1}{2}} e^{-\frac{1}{2}y^2} L_n^{(v)} (y^2) y^\mu \, dy.
\]

(1.8)

These relations inspire us to define the generalized Laguerre functions of order \( (v,\mu) \) by
\[
L_n^{(v,\mu)} (x) := \left[ \frac{2\Gamma(n+1)}{\Gamma(v+n+1)} \right]^{\frac{1}{2}} x^{-\frac{1}{2}+\mu+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{(v)} (x^2), \quad x > 0, \, n \in \mathbb{N}_0.
\]

(1.9)

Now relation (1.7) says that the \( L_n^{(v,\mu)} \) establish an orthonormal system in the Hilbert space \( L_2((0,\infty), x^\mu \, dx) \). This Hilbert space shall be denoted by \( X_\mu \) in the sequel. In fact the set \( \{ L_n^{(v,\mu)} \mid n \in \mathbb{N}_0 \} \) is an orthonormal basis in \( X_\mu \).

The differential equation (1.6) tells us that the functions \( L_n^{(v,\mu)} \) are eigenfunctions of the differential operator
\[
-\frac{d^2}{dx^2} - \frac{\mu}{x} \frac{d}{dx} + \frac{v^2 - \frac{1}{2}(\mu-1)^2}{x^2} + x^2
\]

(1.10)

and their respective eigenvalues are \( 4n + 2v + 2 \) \( (n \in \mathbb{N}_0) \). Associated to this operator we introduce the self-adjoint operator \( A_{v,\mu} \) in \( X_\mu \) defined by
\[
A_{v,\mu} f = \sum_{n=0}^{\infty} (4n + 2v + 2) (f, L_n^{(v,\mu)})_{X_\mu} L_n^{(v,\mu)}
\]

(1.11)

for
\[
f \in D(A_{v,\mu}) = \{ g \in X_\mu \mid \sum_{n=0}^{\infty} n^2 \mid (g, L_n^{(v,\mu)})_{X_\mu} \mid^2 < \infty \}.
\]

(1.12)

Here \( (\cdot, \cdot)_{X_\mu} \) denotes the inner product of \( X_\mu \). If we take \( f \) in the dense linear span of the \( L_n^{(v,\mu)} \), so \( f \in \langle L_n^{(v,\mu)} \mid n \in \mathbb{N}_0 \rangle \) then
\[
A_{v,\mu} f = \left\{ -\frac{d^2}{dx^2} - \frac{\mu}{x} \frac{d}{dx} + \frac{v^2 - \frac{1}{2}(\mu-1)^2}{x^2} + x^2 \right\} f.
\]

(1.13)

In fact \( D(A_{v,\mu}) \) consists of \( C^1((0,\infty)) \)-functions with second generalized derivative. More details can be found in [Ma].

From the integral expression (1.8) we derive the integral relation
\[-8-\]

\[-It\]

\[-L\]

\[-v\]

\[-L\]

\[-(x)\]

\[-=\]

\[-J\]

\[-(xy)\]

\[-l1\]

\[-+\]

\[-J\]

\[-(y)\]

\[-yl1\]

\[-dy.\]

\[-o\]

\[-(1.14)\]

The integral on the right hand side of this equality is just the classical Hankel transform \(\mathcal{H}_{v,\mu} L_n^{(v,\mu)}\) from definition (1.1). So relation (1.14) can be written as

\[-(-1)^n L_n^{(v,\mu)} = \mathcal{H}_{v,\mu} L_n^{(v,\mu)}.\]

These relations inspire us to define the Hankel transformation \(\mathcal{H}_{v,\mu}\) on \(X_\mu\) in the following way.

**Definition 1.1.**

\[\mathcal{H}_{v,\mu} f := \sum_{n=0}^{\infty} (-1)^n (f, L_n^{(v,\mu)})_{X_\mu} L_n^{(v,\mu)}, f \in X_\mu.\]

Clearly, \(\mathcal{H}_{v,\mu}\) is a unitary self-adjoint operator on \(X_\mu\) for which

\[\mathcal{H}_{v,\mu} \mathcal{H}_{v,\mu} = I\]

Here \(I\) is the identity. If \(f \in \{L_n^{(v,\mu)} \mid n \in \mathbb{N}_0\}\) then

\[\left(\mathcal{H}_{v,\mu} f\right)(x) = \int_0^{\infty} (xy)^{-\frac{1}{2} + \frac{\mu}{2}} J_v(xy) f(y) y^{\mu} \, dy, x > 0.\]  

(1.17)

The latter relation is a consequence of formula (1.14) and Definition 1.1. Here we want to prove that the classical Hankel integral transformation, defined in the beginning of this section, is closely allied to the Hankel transformation \(\mathcal{H}_{v,\mu}\) that we defined on \(X_\mu\).

If the function \(y \mapsto (y^{\nu + \frac{1}{2}} + 1)y^{\frac{\nu}{2}} f(y)\) is absolutely integrable over \(\mathbb{R}^+\), then the integral in (1.17) converges absolutely for all \(x > 0\). To see this observe that \(J_v(xy) = O(y^\nu)\) \((y \downarrow 0)\) and \(J_v(xy) = O(y^{-\nu/2})\) \((y \to \infty)\).

**Theorem 1.2.**

Let \(f \in X_\mu\) such that \(y \mapsto (y^{\nu + \frac{1}{2}} + 1)y^{\frac{\nu}{2}} f(y)\) is absolutely integrable over \(\mathbb{R}^+\). Then the integral

\[F(x) := \int_0^{\infty} (xy)^{-\frac{1}{2} + \frac{\mu}{2}} J_v(xy) f(y) y^{\mu} \, dy\]

converges absolutely and uniformly on compacta in \((0, \infty)\). Moreover \(F\) is a continuous representant of \(\mathcal{H}_{v,\mu} f\).

**Proof.**

The proof is inspired on the proof of Theorem (1.10) in \([E1]\). By standard techniques it can be shown that the integral \(F\) converges absolutely and uniformly on compacta in \((0, \infty)\). Hence \(F\) is
continuous. Yet we have to show that $F$ is a representant of $IH_{v, \mu} f$.
Since the span $\langle L_{v(\alpha)} \mid \alpha \in \mathbb{N}_0 \rangle$ is dense in $X_{\mu}$ there exists a sequence $(\phi_n)$ in this span such that $\phi_n \to f$ in $X_{\mu}$.
Let $\Phi_n = \phi_n - f$ and $\Psi_n = IH_{v, \mu} \phi_n - F$. Then we have the following pointwise relation

$$\Psi_n(x) = \int_0^\infty (xy)^{\frac{1}{2} \mu + 1} J_{v}(xy) \Phi_n(y) y^\mu \ dy.$$  

Let $\delta > 0$. Consider the following estimation

$$\int_0^\infty e^{-\delta y^2} |\Psi_n(y)|^2 y^\mu \ dy =$$

$$= \int_0^\infty e^{-\delta y^2} y^\mu \ dy \cdot \left( \int (yv)^{\frac{1}{2} \mu + 1} J_{v}(yv) \Phi_n(w) w^\mu \ dw \right) \times$$

$$\int_0^\infty \left( (yv)^{\frac{1}{2} \mu + 1} J_{v}(yv) \Phi_n(u) u^\mu \ du \right) =$$

$$= \int_0^\infty \left( \int (uw)^{\frac{1}{2} \mu + 1} \Phi_n(u) \Phi_n(w) \left( \int y e^{-\delta y^2} J_{v}(yu) J_{v}(yw) \ dy \right) dw \right) du dv \leq$$

$$\leq \frac{1}{2\delta} \int_0^\infty \left( \int (uw)^\mu \ |\Phi_n(u)|^2 \ |\Phi_n(w)|^2 \ dw \right)^{1/2} du dv \leq$$

$$\leq \frac{1}{2\delta} \left( \int \left( \frac{1}{2} \ |\Phi_n(u)|^2 \ |\Phi_n(w)|^2 \ dw \right)^{1/2} du \right)^2 =$$

$$= \frac{3}{4} \left( \sqrt{2} + 1 \right)^{\mu \gamma} \int_0^\infty \left( \int \Phi_n(u) \left|\Phi_n(u)\right|^2 u^\mu \ du \right)^{1/2}.$$  

Note that, for the equalities marked with (*) and (**), we used respectively one or both of the improper integrals

$$\int_0^\infty J_{v}(at) J_{v}(bt) e^{-\delta t^2} \ t \ dt = \frac{1}{2\gamma} e^{-\left(\alpha^2 + \beta^2\right)^{\gamma/(2\gamma)}} J_{v}\left( \frac{1}{2} \alpha \beta / \gamma \right)$$

$$\int_0^\infty J_{v}(at) J_{v}(bt) e^{-\delta t^2} \ t \ dt = \frac{1}{2\gamma} e^{-\left(\alpha^2 + \beta^2\right)^{\gamma/(2\gamma)}} J_{v}\left( \frac{1}{2} \alpha \beta / \gamma \right)$$

where Re $\nu > -1$ and Re $\gamma > 0$ (see [MOS, p. 93]).

Now we have proved the following inequality. For each $\delta > 0$,
\[ \int_0^\infty e^{-by^2} \left| \Psi_n(y) \right|^2 y^\mu dy \leq 2 \frac{3}{4} (\sqrt{2} + 1)^{\nu/2} \int_0^\infty \left| \Phi_n(u) \right|^2 u^\mu du. \]

So, by Fatou's lemma
\[ \Phi_n \to 0 \text{ in } X_\mu \text{ implies } \Psi_n \to 0 \text{ in } X_\mu. \]

We see that \( F \) is a representant of \( BH_{v,\mu} \), and the proof is complete.

A consequence of the previous theorem is the following.

**Theorem 1.3.**

Let \( f \in X_\mu \). Then for all \( x > 0 \)
\[ (BH_{v,\mu} f) (x) = \text{l.i.m.}_{R \to \infty} \int_0^R (xy)^{\nu+\frac{1}{2}} J_v(xy) f(y) y^\mu dy. \]

Here "l.i.m. = limit in mean" stands for the limit in \( X_\mu \), i.e.
\[ \int_0^\infty \|BH_{v,\mu} f - f\|_{X_\mu}^2 dx \to 0 \text{ as } R \to \infty. \]

**Proof.**

Let \( (R_n) \) be any increasing sequence in \((0,\infty)\) with \( R_n \uparrow \infty \). We take \( f_n \in X_\mu \) as follows
\[ f_n(x) = \begin{cases} 0 & \text{if } x > R_n \\ f(x) & \text{if } 0 < x \leq R_n \end{cases}, \quad n \in \mathbb{N}. \]

Then
\[ \int_0^\infty (x^{\nu+\frac{1}{2}} + 1) x^\mu f_n(x) dx \leq \int_0^R (x^{\nu+\frac{1}{2}} + 1)^2 dx \cdot \left( \int_0^\infty f(x) x^\mu dx \right)^{\frac{1}{2}}. \]

So the function \( x \mapsto (x^{\nu+\frac{1}{2}} + 1) x^\mu f_n(x) \) is absolutely integrable for all \( n \in \mathbb{N} \) \((\nu > -1)\).

The previous theorem yields
\[ (BH_{v,\mu} f) (x) = \int_0^R (xy)^{\nu+\frac{1}{2}} J_v(xy) f(y) y^\mu dy \]

for almost every \( x > 0 \). Furthermore, we have \( f_n \to f \) in \( X_\mu \), hence
\[ \|BH_{v,\mu} f - BH_{v,\mu} f_n\|_{L^2}^2 \to 0 \quad (n \to \infty) \]

or equivalently
\[
\int_0^\infty \left( I_{\nu,\mu}(x) f(x) - \int_0^x (xy)^{-\frac{\mu+\nu}{2}} J_\nu(xy) f(y) y^\mu \, dy \right) \, x^\mu \, dx \rightarrow 0 \quad (n \rightarrow \infty)
\]
and the result follows.

For each \( f \in \{ L_n^{(\nu, \mu)} \mid n \in \mathbb{N}_0 \} \) we have, using Bessel’s differential equation

\[
-x^2 \left( I_{\nu,\mu}(x) f(x) \right) = \int_0^\infty (xy)^{-\frac{\mu+\nu}{2}} \left( - \frac{1}{2} (xy)^2 J_\nu(xy) \right) f(y) y^\mu \, dy =
\]

\[
= \int_0^\infty (xy)^{-\frac{\mu+\nu}{2}} \left( (xy)^2 J_\nu''(xy) + (xy) J_\nu'(xy) - \nu^2 J_\nu(xy) \right) f(y) y^\mu \, dy.
\]

Integration by parts yields

\[
-x^2 \left( I_{\nu,\mu}(x) f(x) \right) = \int_0^\infty (xy)^{-\frac{\mu+\nu}{2}} J_\nu(xy) \left\{ \left( \frac{d}{dy} \left( \frac{y^\mu}{x} \right) - \frac{\nu^2 y^\mu}{4} \right) f(y) \right\} y^\mu \, dy.
\]

From this formula and the equality \( I_{\nu,\mu} I_{\nu,\mu} = I \) we derive the following pointwise relation

\[
(I_{\nu,\mu} x^2 I_{\nu,\mu}) f = \left\{ - \frac{d^2}{dx^2} - \frac{\mu}{x} \frac{d}{dx} + \frac{\nu^2 (\mu-1)^2}{x^2} \right\} f.
\]

In \( X_\mu \) we introduce the operator \( Q \) as follows. By \( Q \) we denote the unbounded operator of multiplication by the identity function in \( X_\mu \) with domain

\[
D(Q ; X_\mu) = \{ f \in X_\mu \mid \int_0^\infty \left| x f(x) \right|^2 x^\mu \, dx < \infty \}.
\]

The operator \( Q \) is self-adjoint. On the basis of relation (1.18) we introduce the operator \( D_{\nu,\mu} \) as follows

\[
D_{\nu,\mu} = I_{\nu,\mu} Q^2 I_{\nu,\mu}
\]

with domain

\[
D(D_{\nu,\mu}) = I_{\nu,\mu} D(Q ; X_\mu).
\]

Being unitarily equivalent with \( Q^2 \), the operator \( D_{\nu,\mu} \) is self-adjoint. We see that the Hankel transformation \( I_{\nu,\mu} \) diagonalizes the differential operator \( D_{\nu,\mu} \).

Note that for all \( f \in \{ L_n^{(\nu, \mu)} \mid n \in \mathbb{N}_0 \} \) we have

\[
D_{\nu,\mu} f = \left\{ - \frac{d^2}{dx^2} - \frac{\mu}{x} \frac{d}{dx} + \frac{\nu^2 (\mu-1)^2}{x^2} \right\} f.
\]

At the end of this section we want to describe a relation between Hankel transformations and Fourier transformations (cf. \([Sn, pp. 62-65]\)).
Hankel transformations find their applications amongst others in the discussion of problems posed in spherical coordinates. The Fourier transform of a spherical symmetric function can be expressed in terms of a Hankel transform. We will explain this in the two-dimensional case.

Let \( f \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2) \) be a spherical symmetric function on \( \mathbb{R}^2 \), i.e. \( f \) is a function of the variable \( r = (x_1^2 + x_2^2)^{1/2} \) only, so

\[
f(x_1, x_2) = f(r, 0).
\]

Consider the Fourier transform \( \mathcal{F} f \). We have, by definition,

\[
(\mathcal{F} f)(\xi_1, \xi_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2) e^{-i(\xi_1 x_1 + \xi_2 x_2)} \, dx_1 \, dx_2.
\]

Introduce polar coordinates \( (r, \theta) \) in the \((x_1, x_2)\)-plane and \((\rho, \phi)\) in the \((\xi_1, \xi_2)\)-plane. Then the integral becomes

\[
(\mathcal{F} f)(\xi_1, \xi_2) = \frac{1}{2\pi} \int_{0}^{2\pi} f(r, 0) r \, dr \int_{0}^{2\pi} e^{-i\rho \cos(\phi - \theta)} \, d\phi.
\]

Because of the periodic nature of the integrand

\[
\int_{0}^{2\pi} e^{-i\rho \cos(\phi - \theta)} \, d\phi = \int_{0}^{\pi} e^{i\rho \cos(\phi - \theta)} \, d\phi = \int_{0}^{\pi} e^{i\rho \cos \phi} \, d\phi
\]

and with the aid of the identity

\[
J_0(z) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{iz \cos \phi} \, d\phi
\]

it follows that the Fourier transform \( \mathcal{F} f \) is a function of the variable \( \rho = (\xi_1^2 + \xi_2^2)^{1/2} \) only, so \( \mathcal{F} f \) is a spherical symmetric function as well.

\[
(\mathcal{F} f)(\xi_1, \xi_2) = \int_{0}^{\infty} f(r, 0) J_0(\rho r) \, dr = (\mathcal{H}_{0,1} \tilde{f}) (\xi)
\]

Here \( \tilde{f}(x) = f(x, 0) \), \( x > 0 \).

The result is readily generalized. Let \( f \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n) \) be a spherical symmetric function on \( \mathbb{R}^n (n \geq 2) \), i.e.

\[
f(x) = f(1 x, 1 e)
\]

for all \( x \in \mathbb{R}^n \) and all \( e \in \mathbb{R}^n \) with length \( 1 e = 1 \). Then its Fourier transform \( \mathcal{F} f \) is spherical symmetric too, and can be expressed as a Hankel transform:
\((IF \ f)(\xi) = \int_0^\infty (r \ |\xi|)^{-\frac{n+1}{2}} J_{\frac{n-1}{2n-1}} (r \ |\xi|) f(r \ \overline{\mathbf{e}}) r^{n-1} \, dr\)

\[= (\mathcal{H}_{\frac{n-1}{n}, n-1} \tilde{f})(|\xi|) \quad (1.25)\]

Here \(\mathbf{e}\) is an arbitrary vector in \(\mathbb{R}^n\) with \(|\mathbf{e}| = 1\) and \(f(x) = f(x \mathbf{e})\), \(x > 0\). For details we refer to [Sn].

Even in the most general case, the Fourier transform of an arbitrary \(L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)\)-function \(f\) can be expressed in terms of Hankel transforms. For that we use an expansion of \(f\) in terms of radial symmetric functions and spherical harmonics on the unit sphere in \(\mathbb{R}^n\).
2. The Schwartz space $S(\mathbb{R})$

2.1. Introduction

Let $S(\mathbb{R})$ denote the subspace of $C^\infty(\mathbb{R})$ which consists of all rapidly decreasing functions. A function $\phi \in C^\infty(\mathbb{R})$ belongs to $S(\mathbb{R})$ if and only if $\sup_{x \in \mathbb{R}} |x^k \phi^{(l)}(x)| < \infty$ for all $k, l \in \mathbb{N}_0$.

In this definition the supremum norm can be replaced by the $L_2(\mathbb{R})$-norm which we denote by $\|\cdot\|_2$. That leads us to the following equivalent definition of the Schwartz space

$$\phi \in S(\mathbb{R}) \text{ if and only if } \forall k \in \mathbb{N}_0 \forall l \in \mathbb{N}_0 : \|x^k \phi^{(l)}(x)\|_2 < \infty.$$ 

The equivalence of the above definitions can be seen with the aid of the inequalities

$$\|f\|_2 \leq \pi^{1/2} (\|f\|_\infty + \|xf\|_\infty) \tag{2.1}$$

and

$$\|f\|_\infty \leq 2^{-1/5} (\|f\|_2 + \|\frac{d}{dx} f\|_2) \tag{2.2}$$

where $\|\cdot\|_\infty$ denotes the supremum norm. The second inequality is known as a Sobolev inequality in mathematical literature.

In fact, the space $S(\mathbb{R})$ consists of all infinitely differentiable functions which, together with their derivatives, approach zero more rapidly than any power of $|x|$ as $|x| \to \infty$. Therefore, it is clear that for all $p \in [1, \infty]$, 

$$S(\mathbb{R}) \subset L_p(\mathbb{R}). \tag{2.3}$$

For instance, the functions

$$x \mapsto p(x) e^{-x^2} \tag{2.4}$$

belong to $S(\mathbb{R})$ for every polynomial $p$.

In Appendix D the Fourier transformation $\mathcal{F}$ on $S(\mathbb{R})$ is introduced. This transformation extends to a unitary transformation on $L_2(\mathbb{R}^2)$ which we also denote by $\mathcal{F}$.

Let us introduce the important operators $Q$ and $P$. By $Q$ we denote the unbounded operator of multiplication by the identity function in $L_2(\mathbb{R})$ with domain

$$D(Q) = \{ f \in L_2(\mathbb{R}) \mid \int_{-\infty}^{\infty} |xf(x)|^2 \, dx < \infty \}. \tag{2.5}$$

The operator $Q$ is self-adjoint. We define the operator $P$ by $P = \mathcal{F} Q \mathcal{F}^*$ with domain $D(P) = \mathcal{F}(D(Q))$. Being unitarily equivalent with $Q$, the operator $P$ is self-adjoint. Since $\mathcal{F}(S(\mathbb{R})) = S(\mathbb{R})$ we have $S(\mathbb{R}) \subset D(P)$. On $S(\mathbb{R})$ the operator $P$ is the differentiation operator,

$$P \phi = i \phi'. \tag{2.6}$$

For a more detailed introduction of the operators $Q$ and $P$ we refer to [We, Chapter 10].
The space $S(\mathbb{R})$ remains invariant under the following operations,

the *multiplication* operation

$$Q : \phi(x) \to x \phi(x),$$

the *differentiation* operation

$$P : \phi(x) \to i \phi'(x),$$

the *multiplier*

$$M_f : \phi(x) \to f(x) \phi(x)$$

here $f$ is a $C^\infty$ function on $\mathbb{R}$ with the property $\forall x \in \mathbb{R} : \sup_{k \in \mathbb{R}} |f^{(k)}(x)| < \infty$,

the *phase-shift* operation

$$\Phi_b : \phi(x) \to e^{ibx} \phi(x), \quad b \in \mathbb{R},$$

the *translation* operation

$$T_h : \phi(x) \to \phi(x-h), \quad h \in \mathbb{R},$$

the *dilatation* operation

$$Z_\lambda : \phi(x) \to \sqrt{\lambda} \phi(\lambda x), \quad \lambda \in \mathbb{R} \setminus \{0\},$$

the *parity* operation

$$\Pi : \phi(x) \to \phi(-x),$$

the pointwise defined *Fourier transformation*

$$\mathcal{F} : \phi(x) \to (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \phi(y) e^{-ixy} dy.$$  

The operators $\Phi_b, T_h, Z_\lambda, \Pi$ and $\mathcal{F}$ are bijections. Note that the operators $Z_\lambda$, $\lambda \neq 0$, are unitary operators on $L^2(\mathbb{R})$. For every $\lambda \neq 0$ the operator $Z_\lambda$ is unitarily equivalent with $Z_{1/\lambda}$, indeed

$$\mathcal{F} Z_\lambda \mathcal{F}^* = Z_{1/\lambda}. \quad \text{(2.8)}$$

Furthermore,

$$Z_1 = I \quad \text{and} \quad Z_\lambda Z_\mu = Z_{\lambda \mu} \quad \text{for all} \quad \lambda, \mu \neq 0. \quad \text{(2.9)}$$

Thus the family $\{Z_\lambda \mid \lambda \in \mathbb{R} \setminus \{0\}\}$ is a one-parameter unitary group and we get a representation of the multiplication group $(\mathbb{R} \setminus \{0\}, \cdot)$ by operators on $L^2(\mathbb{R})$. Similarly the phase-shift operators respectively the translation operators establish a one-parameter unitary group. Each of them is a representation of the additive group $(\mathbb{R}, +)$ by operators on $L^2(\mathbb{R})$. 


2.2. Functional analytic characterizations of the space $S(\mathbb{R})$

We introduce the following notation. Let $A, B$ denote self-adjoint operators in a Hilbert space $X$. Then we write

$$D^\infty(A) := \bigcap_{n=1}^{\infty} D(A^n)$$

$$D^\infty(A, B) := \bigcap_{n \in \mathbb{N}} \bigcap_{k \in \mathbb{N}_0^2} \bigcap_{l \in \mathbb{N}_0^2} D( A^{k_1} B^{l_1} \cdots A^{k_n} B^{l_n})$$

The elements of $D^\infty(A)$ are called the $C^\infty$-vectors of $A$; $D^\infty(A)$ is called the $C^\infty$-domain of $A$. Similarly $D^\infty(A, B)$ is named the joint $C^\infty$-domain of the pair $(A, B)$. (The subspace $D^\infty(A, B)$ will be trivial, in general).

We formulate the so called 'shake lemma'.

**Lemma 2.2.1.**

Let $n \in \mathbb{N}$ and $k, l \in \mathbb{N}_0^n$. Then for each $f \in D^\infty(P, Q)$,

$$p^{k_1} Q^{k_1} \cdots p^{k_n} Q^{k_n} f = \sum_{r \in \mathbb{N}_0^n} c_{k_1}(r) Q^{k_1-r_1} P^{l_1-r_1} f$$

where the coefficients $c_{k_1}(r)$ satisfy

$$c_{k_1}(r) = \begin{cases} \frac{1}{r!} \frac{k!}{(k-r)!} \frac{l!}{l-r!} & \text{if } r \leq l \text{ and } r \leq k \\ 0 & \text{else.} \end{cases}$$

We recall that multi-index notation is used. So

$$1^{r} = r_1 + \cdots + r_n, \ r! = r_1! \cdots r_n!, \ r \leq k \iff \forall j \in \{1, \ldots, n\} : r_j \leq k_j.$$ 

For the proof we refer to [Go, Lemma 6.1].

The Schwartz space $S(\mathbb{R})$ admits the following functional analytic characterizations in terms of the operators $P$ and $Q$.

**Theorem 2.2.2.**

(i) $\phi \in S(\mathbb{R})$ if and only if $\forall k \in \mathbb{N}_0, \forall l \in \mathbb{N}_0 : \phi \in D(Q^k P^l)$

(ii) $S(\mathbb{R}) = D^\infty(P, Q)$

(iii) $S(\mathbb{R}) = D^\infty(P) \cap D^\infty(Q)$

(iv) $S(\mathbb{R}) = D^\infty(P^2 + Q^2)$

For the proofs see [Go, Section 6].
As a corollary we mention the following characterization of $S(\mathbb{R})$ in terms of the decrease at infinity of a function and of its Fourier transform.

Characterization 2.2.3.
A function $\phi \in L_1(\mathbb{R})$ belongs to $S(\mathbb{R})$ if and only if

The functions $x \mapsto x^k \phi(x)$ and $x \mapsto x^k \mathcal{F}\phi(x)$ are bounded (for all $k \in \mathbb{N}_0$).

2.3. The space $S(\mathbb{R})$ as a countably normed space

In this section we define a topology in the space $S(\mathbb{R})$ by means of several countable families of norms. Together with each of these families the space $S(\mathbb{R})$ becomes a countably normed space. The topology does not depend on the choice of the set of norms, because the sets are mutually equivalent.

Furthermore, we present a characterization of $S(\mathbb{R})$ in terms of the Hermite functions. The reader who is not familiar with countably normed spaces is advised to read first Appendix A for a short introduction on this subject.

Corresponding to the functional analytic characterizations of $S(\mathbb{R})$ as presented, we define the following countable sets of norms on $S(\mathbb{R})$.

Definition 2.3.1.

(i) $p_{k,l}(\phi) := \sum_{i=0}^{k} \sum_{j=0}^{l} \|Q^i P^j \phi\|_2$ \hspace{1cm} , $k,l \in \mathbb{N}_0$

(ii') $q_{k,l}(\phi) := \|Q^k P^l \phi\|_2$ \hspace{1cm} , $k,l \in \mathbb{N}_0$

(ii) $r_{n,k,l}(\phi) := \|P^k \cdot Q^k \cdot \cdots P^l \cdot Q^l \phi\|_2$ \hspace{1cm} , $n \in \mathbb{N}$ , $k,l \in \mathbb{N}_0$

(iii) $
\begin{cases}
  s_k(\phi) := \|Q^k \phi\|_2 \\
  t_l(\phi) := \|P^l \phi\|_2
\end{cases}$ \hspace{1cm} , $k,l \in \mathbb{N}_0$

(iv) $u_k(\phi) := \|(P^2 + Q^2)^k \phi\|_2$ \hspace{1cm} , $k \in \mathbb{N}_0$.

It can be proved that the sets of norms in the above definition are equivalent. Observe that the equivalence of the norms $p_{k,l}$ and $r_{n,k,l}$ can be seen with the aid of the shake lemma. By applying the estimation

$$\|Q^k P^l f\|_2^2 = \langle P^l Q^{2k} P^l f, f \rangle_2 =$$
\[
\begin{align*}
\sum_{j=0}^{\min(2k,l)} (i)^j \left[ \begin{array}{c} l \\ j \end{array} \right] \frac{(2k)!}{(2k-j)!} (P^{2l-j} f, Q^{2k-j} f)_{L^2} & \leq \\
\sum_{j=0}^{\min(2k,l)} \left[ \begin{array}{c} l \\ j \end{array} \right] \frac{(2k)!}{(2k-j)!} \| P^{2l-j} f \|_{L^2} \| Q^{2k-j} f \|_{L^2}
\end{align*}
\]
\tag{2.11}
\]

it follows that the norms \( q_{k,l} \) and \( s_{k,l} \) are equivalent. That the norms \( u_k \) are equivalent with one of the other sets of norms follows from recurrence relations for the Hermite functions, which we define below. Clearly the norms \( p_{k,l} \) and \( q_{k,l} \) are equivalent.

If we replace the \( L_2(\mathbb{R}) \)-norms by supremum norms in the above definition we still get equivalent sets of norms. The families \( \{ p_{k,l} \mid k, l \in \mathbb{N}_0 \} \) and \( \{ u_k \mid k \in \mathbb{N}_0 \} \) are directed but the other families are not.

The space \( S(\mathbb{R}) \) endowed with one of the above sets of norms becomes a complete countably normed space. So \( S(\mathbb{R}) \) is a Fréchet space. The topological structure is independent of the particular choice of the set of norms.

Remark 2.3.2.

(1) Convergence of a sequence \( (\phi_n) \subset S(\mathbb{R}) \) in the topology defined by one of the above sets of norms agrees with the convergence defined in \( S(\mathbb{R}) \) in [GS 1, Section 1.1]; in the last mentioned book it is stated that a sequence \( (\phi_n) \subset S(\mathbb{R}) \) converges to \( \phi \) if in every bounded region the derivatives of all orders of the \( \phi_n \) converge uniformly to the corresponding derivatives of \( \phi \) and if

\[
\forall k, l \in \mathbb{N}_0 \exists c_{k,l} > 0 \forall n \in \mathbb{N} : \sup_{x \in \mathbb{R}} | x^k \phi_n^{(l)}(x) | < c_{k,l}
\]

(the constants \( c_{k,l} \) can be chosen independent of \( n \)).

(2) The operators defined in section 2.1 (see (2.7)) are all continuous.

Next we will present a characterization of the space \( S(\mathbb{R}) \) in terms of the Hermite functions \( \psi_n, n \in \mathbb{N}_0 \). These functions are defined by

\[
\psi_n(x) = \frac{e^{-\frac{1}{2} x^2}}{(\pi^{\frac{1}{2}} 2^n n!)^{\frac{1}{2}}} H_n(x), \quad x \in \mathbb{R}.
\]
\tag{2.12}

Here \( H_n \) is the \( n \)-th Hermite polynomial,

\[
H_n(x) = (-1)^n e^{x^2} \left[ \frac{d}{dx} \right] e^{-x^2}, \quad x \in \mathbb{R}.
\]
\tag{2.13}

The \( \psi_n \) constitute an orthonormal basis in \( L_2(\mathbb{R}) \). Since \( S(\mathbb{R}) \subset L_2(\mathbb{R}) \), functions in \( S(\mathbb{R}) \) have
Hermite expansions $\sum_{n=0}^{\infty} a_n \psi_n$.

The functions $\psi_n$ are eigenfunctions of the differential operator

$$- \frac{d^2}{dx^2} + x^2$$

and their respective eigenvalues are $2n + 1$ ($n \in \mathbb{N}_0$).

Associated to this operator we define the self-adjoint operator $T$ by

$$Tf = \sum_{n=0}^{\infty} (2n+1) (f,\psi_n)_2 \psi_n$$

for $f \in D(T) = \{g \in L_2(\mathbb{R}) \mid \sum_{n=0}^{\infty} (2n+1)^2 |(g,\psi_n)_2|^2 < \infty\}$.

Here $(\cdot,\cdot)_2$ denotes the inner product in $L_2(\mathbb{R})$. If we take $f \in \{\psi_n \mid n \in \mathbb{N}_0\}$ then

$$Tf = \left[- \frac{d^2}{dx^2} + x^2\right] f.$$ 

In Appendix F we prove that

$$T = P^2 + Q^2.$$  

The operator $P^2 + Q^2$ is called the Hermite operator. Since $S(\mathbb{R}) = D^\infty(P^2 + Q^2)$ we can reformulate this characterization of $S(\mathbb{R})$ in terms of the Hermite expansion coefficients of its elements.

**Characterization 2.3.2.**

If $f \in S(\mathbb{R})$, then for all $k \in \mathbb{N}_0$ we have

$$\sum_{n=0}^{\infty} (2n+1)^{2k} |(f,\psi_n)_2|^2 < \infty.$$ 

Conversely, if a sequence $(a_n) \in C^{\mathbb{N}_0}$ satisfies $\sum_{n=0}^{\infty} (2n+1)^{2k} |a_n|^2 < \infty$ for all $k \in \mathbb{N}_0$ then $\sum_{n=0}^{\infty} a_n \psi_n$ converges (in the topology of $S(\mathbb{R})$) to a function in $S(\mathbb{R})$.

Put differently,

$$f \in S(\mathbb{R}) \text{ if and only if } \forall k \in \mathbb{N}_0 : (f,\psi_n)_2 = O((2n+1)^{-2k}) \quad (n \to \infty).$$

Note that the norms $u_k$, $k \in \mathbb{N}_0$, can be written as
2.4. The space of even functions in $S(\mathbb{R})$

An interesting subspace of $S(\mathbb{R})$ is the space $S_{\text{even}}(\mathbb{R})$ which consists of all even functions belonging to $S(\mathbb{R})$.

Applying Borel's theorem, cf. [Zu, Chapter 1, Exercise 1] it follows that every $\phi \in S_{\text{even}}(\mathbb{R})$ can be written as

$$\phi(x) = \psi(x^2), \quad x \in \mathbb{R}$$

for some $\psi \in S(\mathbb{R})$. Clearly $\psi$ is not uniquely defined. So the mapping $\phi(x) \rightarrow \phi(\sqrt{x})$, $x > 0$, cannot be extended to a function from $S_{\text{even}}(\mathbb{R})$ onto $S(\mathbb{R})$. The space $S_{\text{even}}(\mathbb{R})$ remains invariant under the operations

$$Q^2 : \phi(x) \mapsto x^2 \phi(x)$$
$$x^{-1}D : \phi(x) \mapsto \frac{1}{x} \phi'(x)$$
$$\text{Re} \Phi_b : \phi(x) \mapsto \cos(bx) \phi(x), \quad b \in \mathbb{R},$$
$$M_f : \phi(x) \mapsto f(x) \phi(x), \quad f \in C^\infty_{\text{even}}(\mathbb{R}) \quad \text{with} \quad \forall k \in \mathbb{N}_0 : \sup_{x \geq 0} |f^{(k)}(x)| < \infty,$$
$$T_h + T_{-h} : \phi(x) \mapsto \phi(x-h) + \phi(x+h), \quad h \in \mathbb{R},$$
$$Z_\lambda : \phi(x) \mapsto \sqrt{1+\lambda^2} \phi(\lambda x), \quad \lambda \in \mathbb{R} \setminus \{0\},$$
$$\Pi : \phi(x) \mapsto \phi(-x)$$
$$IF : \phi(x) \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(y) e^{-ixy} dy = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \phi(y) \cos(xy) dy$$
$$IH_{v,2v+1} : \phi(x) \mapsto \int_0^{\infty} (xy)^{-v} J_v(xy) \phi(y) y^{2v+1} dy, \quad v \geq -\frac{1}{2}.$$
we derive the following operator equality on $S_{\text{even}}(\mathbb{R})$:

$$\left\{ 1 \frac{d}{dz} \right\} [z^{-\mu} J_{\mu}(z)] = -z^{-\mu+1} J_{\mu+1}(z), \quad \mu, z \in \mathbb{C} \quad (2.22)$$

This leads us to the surprising result that the operator $x^{-1} D$ is invertible on $S_{\text{even}}(\mathbb{R})$. Indeed, we can write for $v \geq -\frac{1}{2}$

$$x^{-1} D = -\mathcal{H}_{v+1, 2(v+1)+1} \mathcal{H}_{v, 2v+1} \quad (2.24)$$

and

$$(x^{-1} D)^{-1} = -\mathcal{H}_{v, 2v+1} \mathcal{H}_{v+1, 2(v+1)+1} \quad (2.25)$$

The latter equality yields that

$$(x^{-1} D)^{-1} = -\int_{x}^{\infty} \phi(t) \, dt \quad , \phi \in S_{\text{even}}(\mathbb{R}) \quad (2.26)$$

In [E1, pp. 28,29] Van Eijndhoven calculated the matrix of $x^{-1} D$ with respect to the Laguerre functions $L_{\nu}^{(\frac{1}{2}, 0)}$:

$$((x^{-1} D) L_{k}^{(\frac{1}{2}, 0)}, L_{l}^{(\frac{1}{2}, 0)})_{2} = \begin{cases} 0, & l = k + 1 \\ -1, & l = k, k > l \\ -1 \frac{(k+1)^{\nu+1}}{(k+l+1)} \frac{1}{(l+1)} \cdot (-1)^{k-l}, & 0 \leq l \leq k - 1 \end{cases} \quad (2.27)$$

where $l, k = 0, 1, 2, ...$

In Appendix E we compute the matrix of $(x^{-1} D)^{-1}$ with respect to the same Laguerre functions. It turns out to be the upper triangular matrix

$$((x^{-1} D)^{-1} L_{k}^{(\frac{1}{2}, 0)}, L_{l}^{(\frac{1}{2}, 0)})_{2} = \begin{cases} 0, & l = k + 1 \\ -1, & l = k, k > l \\ -1 \frac{(k+1)^{\nu+1}}{(k+l+1)} \frac{1}{(l+1)} \cdot (-1)^{k-l}, & 0 \leq l \leq k - 1 \end{cases} \quad (2.28)$$

where $l, k = 0, 1, 2, ...$

In [E1, pp. 16-20] Van Eijndhoven proved that an even function $\phi \in C_{\text{even}}(\mathbb{R})$ belongs to $S_{\text{even}}(\mathbb{R})$ if and only if $\sup_{x>0} |x^\nu (x^{-1} D)^l \phi(x)| < \infty$. Furthermore, he showed that the norms $v_{k,l}, k, l \in \mathbb{N}_0$, defined by
\begin{align}
  v_{k,l}(\phi) &= \| Q^k(x^{-1} D)^l \phi \|_2 
  \tag{2.29}
\end{align}

generate the same topology in \( S_{\text{even}}(IR) \) as the topology defined by the norms \( q_{k,l} \), \( k,l \in \mathbb{N}_0 \). We recall that

\[ q_{k,l}(\phi) = \| Q^k P^l \phi \|_2. \]

Again the \( L_2(IR) \)-norm may be replaced by the supremum norm.

We present a characterization of the space \( S_{\text{even}}(IR) \) in terms of the generalized Laguerre functions \( L^{(v,2v+1)}_n \), \( n \in \mathbb{N}_0 \), for fixed \( v \geq -\frac{1}{2} \). We recall that

\[ L^{(v,2v+1)}_n(x) = \left( \frac{2 \Gamma(n + 1)}{\Gamma(v + n + 1)} \right)^{\frac{1}{2}} e^{-\frac{1}{2} x^2} L^{(v)}_n(x^2), \quad x > 0, \ n \in \mathbb{N}_0, \]

where \( L^{(v)}_n \) is the \( n \)-th generalized Laguerre polynomial of type \( v \). From Section 1 we know that the \( L^{(v,2v+1)}_n \) establish an orthonormal basis in \( X_{2v+1} = L_2((0, \infty), x^{2v+1} dx) \). We denote the inner product in \( X_{2v+1} \) by \( \langle \cdot, \cdot \rangle_{X_{2v+1}} \), as in Section 1. Since \( S_{\text{even}}(IR) \subset X_{2v+1} \), functions in \( S_{\text{even}}(IR) \) have Laguerre expansions

\[ \sum_{n=0}^{\infty} a_n L^{(v,2v+1)}_n. \]

The functions \( L^{(v,2v+1)}_n \) are eigenfunctions of the self-adjoint differential operator \( A_{v,2v+1} \) and their respective eigenvalues are \( 4n + 2v + 1 \), see Section 1.

In [E2, Theorem 2.8] Van Eijndhoven proved the following characterization of \( S_{\text{even}}(IR) \) in terms of the Laguerre expansion coefficients of its elements.

**Characterization 2.4.1.**

If \( f \in S_{\text{even}}(IR) \), then for all \( k \in \mathbb{N} \), we have

\[ \sum_{n=0}^{\infty} (4n + 2v + 2)^{2k} | \langle f, L^{(v,2v+1)}_n \rangle_{X_{2v+1}} |^2 < \infty, \]

Conversely, if a sequence \( (a_n) \in C^{\mathbb{N}} \) satisfies

\[ \sum_{n=0}^{\infty} (4n + 2v + 2)^{2k} | a_n |^2 < \infty \]

for all \( k \in \mathbb{N}_0 \) then \( \sum_{n=0}^{\infty} a_n L^{(v,2v+1)}_n \) converges (in the topology of \( S_{\text{even}}(IR) \)) to a function in \( S_{\text{even}}(IR) \).

In other words

\[ D^{\infty}(A_{v,2v+1}) = S_{\text{even}}(IR) \]

as countably normed spaces.

Moreover we mention the useful characterization

\[ D^{\infty}(A_{v,2v+1}) = S_{\text{even}}(IR) \]
We are in a position to determine the $C^\infty$-domain of the operators $A_{v,\mu}$, $v \geq -\frac{1}{2}$, $\mu \geq 0$, introduced in Section 1. For $v > -\frac{1}{2}$, $\mu > 0$ we have the identities

$$L_n^{(v,\mu)}(x) = x^{v+\frac{1}{2}+\mu} L_n^{(v,2v+1)}(x)$$

and

$$(f, L_n^{(v,\mu)})_{X_v} = (x^{-v+\frac{1}{2}+\mu} f, L_n^{(v,2v+1)})_{X_{2v+1}}.$$  

So the following relation between the operators $A_{v,2v+1}$ and $A_{v,\mu}$ is settled herewith

$$x^{-v+\frac{1}{2}+\mu} A_{v,2v+1} x^{-v+\frac{1}{2}+\mu} = A_{v,\mu}.$$  

From this relation and from the above characterization of $S_{\text{even}}(IR)$ we obtain a characterization of $D^\infty(A_{v,\mu})$.

$$D^\infty(A_{v,\mu}) = x^{-v+\frac{1}{2}+\mu} S_{\text{even}}(IR).$$

On $S_{\text{even}}(IR)$ we define the norms $w_k^{(v,\mu)}$, $k \in N_0$, by

$$w_k^{(v,\mu)}(\phi) = \| A_{v,\mu} x^{-v+\frac{1}{2}+\mu} \phi \|_{X_v} = \left\{ \sum_{n=0}^{\infty} (4n + 2v + 2)^{2k} 1 (x^{-v+\frac{1}{2}+\mu} \phi, L_n^{(v,\mu)})_{X_v} \right\}^{\frac{1}{2}}.$$  

These norms are directed and for each choice of $v, \mu$ they generate a topology in $S_{\text{even}}(IR)$ which is the same as the topology defined by the norms $v_{k,l}$, $k,l \in N_0$.

With the aid of the identity

$$x^{-v+\frac{1}{2}+\mu} H_{v,2v+1} x^{-v+\frac{1}{2}+\mu} = H_{v,\mu}$$

and the $H_{v,2v+1}$-invariance of the space $S_{\text{even}}(IR)$,

$$H_{v,2v+1}(S_{\text{even}}(IR)) = S_{\text{even}}(IR),$$

we obtain the $H_{v,\mu}$-invariance of the space $x^{-v+\frac{1}{2}+\mu} S_{\text{even}}(IR)$,

$$H_{v,\mu}(x^{-v+\frac{1}{2}+\mu} S_{\text{even}}(IR)) = x^{-v+\frac{1}{2}+\mu} S_{\text{even}}(IR).$$

At the end of Section 1 we introduced the operator $D_{v,\mu} = H_{v,\mu} Q^2 H_{v,\mu}$. As a consequence of the above formula we derive

$$D_{v,\mu}(x^{-v+\frac{1}{2}+\mu} S_{\text{even}}(IR)) \subset x^{-v+\frac{1}{2}+\mu} S_{\text{even}}(IR).$$

In [EB, Theorem 2.22] we proved the following characterization of $S_{\text{even}}(IR)$ in terms of decrease at infinity of the function and of its Hankel transform.
Characterization 2.4.2.
Let \( v \geq -\frac{1}{2} \). An even function \( \phi \in L(R) \) belongs to \( S_{\text{even}}(R) \) if and only if for all \( k, l \in N_0 \).

\[
\sup_{x \geq 0} |x^k \phi(x)| < \infty \quad \text{and} \quad \sup_{x \geq 0} |x^l (IH_{v,2v+1} \phi)(x)| < \infty
\]

With the same techniques the following can be proved.

Characterization 2.4.3.
Let \( v \geq -\frac{1}{2} \). An even function \( \phi \in L_1(R) \) belongs to \( S_{\text{even}}(R) \) if and only if

\[
\phi \in D^\infty(Q) \quad \text{and} \quad IH_{v,2v+1} \phi \in D^\infty(Q).
\]

From Characterization 2.4.2 and from the above relation (2.37) between the operators \( IH_{v,2v+1} \) and \( IH_{v,\mu} \) we obtain an analogue characterization of the space \( x^{v-\frac{1}{2}+\frac{1}{2}} S_{\text{even}}(R) \).

Characterization 2.4.4.
Let \( v \geq -\frac{1}{2}, \mu \geq 0 \). A function \( \phi \in L_1(R) \) belongs to \( x^{v-\frac{1}{2}+\frac{1}{2}} S_{\text{even}}(R) \) if and only if \( x^{v-\frac{1}{2}+\frac{1}{2}} \phi \) is an even function with the properties

\[
\sup_{x \geq 0} |x^{v-\frac{1}{2}+\frac{1}{2}} \phi(x)| < \infty \quad \text{and} \quad \sup_{x \geq 0} |x^{v-\frac{1}{2}+\frac{1}{2}} (IH_{v,\mu} \phi)(x)| < \infty
\]

for all \( k, l \in N_0 \).

In the next section we deal with the Schwartz space \( S(R^2) \) which consists of \( C^\infty(R^2) \) functions of rapid decrease. Similar results will be stated. In particular we employ the relation between the Fourier transformation in two variables and the Hankel transformation to obtain new characterizations of \( S(R) \).
3. The Schwartz space $S(\mathbb{R}^2)$

3.1. Introduction

In the $n$-dimensional case ($n \geq 2$) we use multi-index notation. A multi-index is an $n$-tuple of non-negative integers $m = (m_1, ..., m_n)$. We adopt the standard notation

\[
\begin{align*}
\quad & m! = m_1! \cdot m_2! \cdot \cdots \cdot m_n! \\
\uparrow & \quad m = m_1 + m_2 + \cdots + m_n \\
\quad & m + 1 = (m_1 + 1, m_2 + 1, \ldots, m_n + 1) \\
\quad & m \leq k \iff m_1 \leq k_1, m_2 \leq k_2, \ldots, m_n \leq k_n \\
\quad & x^m = x_1^{m_1} \cdot x_2^{m_2} \cdot \cdots \cdot x_n^{m_n}, \quad x \in \mathbb{R}^n \\
\quad & D^\alpha = \frac{\partial^{1+\alpha}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.
\end{align*}
\]

(3.1)

In this section we consider the two-dimensional case. So we take $n = 2$.

The Schwartz space $S(\mathbb{R}^2)$ consists of all $C^\infty$-functions $\phi$ on $\mathbb{R}^2$ with the property

\[
\sup_{x \in \mathbb{R}^2} |x^k D^\ell \phi(x)| < \infty
\]

(3.2)

for all $k, \ell \in \mathbb{N}_0^2$ ($\mathbb{N}_0^2$ is the Cartesian product $\mathbb{N}_0 \times \mathbb{N}_0$). As in the one-dimensional case the supremum norm can be replaced by the $L_2(\mathbb{R}^2)$ norm, which we denote by $\|\cdot\|_{L_2(\mathbb{R}^2)}$ or by $\|\cdot\|_2$.

So we can define $S(\mathbb{R}^2)$ as follows

\[
\phi \in S(\mathbb{R}^2) \iff \phi \in C^\infty(\mathbb{R}^2) \quad \text{and} \quad \forall k, \ell \in \mathbb{N}_0^2 \quad \forall x_1, x_2 \in \mathbb{R}^2 : \|x^k D^\ell \phi\|_{L_2(\mathbb{R}^2)} < \infty.
\]

(3.3)

In fact, the space $S(\mathbb{R}^2)$ consists of all infinitely differentiable functions which, together with their derivatives, approach zero more rapidly than any power of $1/|x|$ as $|x| \to \infty$ (here $|x| = \sqrt{x_1^2 + x_2^2}$). Therefore, it is clear that for all $p \in [1, \infty]$,

\[
S(\mathbb{R}^2) \subset L_p(\mathbb{R}^2).
\]

(3.4)

For instance, the functions

\[
x \mapsto p(x) e^{-|x|^2}, \quad x \in \mathbb{R}^2,
\]

(3.5)

belong to $S(\mathbb{R}^2)$ for every polynomial $p$.

In Appendix D the following Fourier transformations are introduced

\[
\begin{align*}
\mathcal{I}F_1 & : \quad \text{The Fourier transformation with respect to the first variable on } S(\mathbb{R}^2). \\
\mathcal{I}F_2 & : \quad \text{The Fourier transformation with respect to the second variable on } S(\mathbb{R}^2). \\
\mathcal{I}F & = \mathcal{I}F_1 \circ \mathcal{I}F_2 & : \quad \text{The Fourier transformation on } S(\mathbb{R}^2).
\end{align*}
\]

The transformations $\mathcal{I}F_1, \mathcal{I}F_2$ and $\mathcal{I}F$ extend to unitary transformations on $L_2(\mathbb{R}^2)$. 
We introduce the important operators $Q_j$ and $P_j$, $j = 1, 2$. By $Q_j$ we denote the unbounded operator of multiplication by the function $(x_1,x_2) \mapsto x_j$ in $L_2(\mathbb{R}^2)$ with domain 

$$D(Q_j) = \{f \in L_2(\mathbb{R}^2) \mid \iint_{\mathbb{R}^2} |x_j f(x_1,x_2)|^2 \, dx_1 \, dx_2 < \infty\}.$$ 

The operator $Q_j$ is self-adjoint. We define the operator $P_j$ by 

$$P_j = \mathcal{I} F Q_j \mathcal{I} F^* = \mathcal{I} F_j Q_j \mathcal{I} F_j^*$$ 

with domain $D(P_j) = \mathcal{I} F(D(Q_j)) = \mathcal{I} F_j(D(Q_j))$. Being unitarily equivalent with $Q_j$, the operator $P_j$ is self-adjoint. Since $\mathcal{I} F(S(\mathbb{R}^2)) = S(\mathbb{R}^2)$ we have $S(\mathbb{R}^2) \subset D(P_j)$. On $S(\mathbb{R}^2)$ the operator $P_j$ is the differentiation operation $i \frac{\partial}{\partial x_j}$.

Combinations of the operators $P_1$ and $P_2$ lead to the well known differential operators grad, div and $\Delta$, defined by

$$\begin{align*}
\text{grad } f &= (-i P_1 f, -i P_2 f) \quad , f \in D(P_1) \cap D(P_2) \\
\text{div } f &= -i P_1 f_1 - i P_2 f_2 \quad , f = (f_1, f_2) \text{ with } f_1 \in D(P_1), f_2 \in D(P_2) \\
\Delta f &= -P_1^2 f - P_2^2 f \quad , f \in D(P_1^2) \cap D(P_2^2).
\end{align*}$$

Grad and div are abbreviations for gradient and divergence respectively; $\Delta$ is called the Laplacian. The following notations are standard,

$$\begin{align*}
\text{grad } f &= \nabla f \\
\text{div } f &= (\nabla, f) \\
\Delta f &= \nabla^2 f = \text{div grad } f.
\end{align*}$$

The space $S(\mathbb{R}^2)$ remains invariant under the following operations, the multiplication operations

$$\begin{align*}
Q_1 : \phi(x) &\rightarrow x_1 \phi(x) \\
Q_2 : \phi(x) &\rightarrow x_2 \phi(x)
\end{align*}$$

the differentiation operations

$$\begin{align*}
P_1 : \phi(x) &\rightarrow i \frac{\partial}{\partial x_1} \phi(x) \\
P_2 : \phi(x) &\rightarrow i \frac{\partial}{\partial x_2} \phi(x)
\end{align*}$$

the pointwise defined Fourier transformations on $S(\mathbb{R}^2)$,

$$\begin{align*}
\mathcal{F}_1 : \phi(x) &\rightarrow (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \phi(y,x_2) e^{-ix_1 y} \, dy
\end{align*}$$
\[ \mathcal{F}_2 : \phi(x) \mapsto (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \phi(x_1, y) e^{-ix_2y} \, dy \]

\[ \mathcal{F} : \phi(x) \mapsto (2\pi)^{-1} \iint_{\mathbb{R}^2} \phi(y_1, y_2) e^{-i(x_1y_1 + x_2y_2)} \, dy_1 \, dy_2. \]

The Fourier transformations are linear homeomorphisms on \( S(\mathbb{R}^2) \).

the multipliers

\[ M_f : \phi(x) \mapsto f(x) \phi(x) \]

here \( f \) is a \( C^\infty \) function on \( \mathbb{R}^2 \) with the property \( \forall k \in \mathbb{N}_0 : \sup_{x \in \mathbb{R}^2} |D^k f(x)| < \infty. \)

the phase-shifts

\[ \Phi_b : \phi(x) \mapsto e^{i(b \cdot x)} \phi(x), \quad b \in \mathbb{R}^2 \]

the translations

\[ T_h : \phi(x) \mapsto \phi(x-h), \quad h \in \mathbb{R}^2. \]

the dilatations

\[ Z_A : \phi(x) \mapsto \sqrt{| \det A |} \phi(x \Lambda) \]

here \( \Lambda \in GL(\mathbb{R}^2) := \{ A \in \mathbb{R}^{2 \times 2} \mid \det A \neq 0 \} \)

the parity operation

\[ \Pi : \phi(x) \mapsto \phi(-x). \]

Moreover,

\[ \text{grad maps } S(\mathbb{R}^2) \text{ into } S(\mathbb{R}^2) \times S(\mathbb{R}^2), \quad \text{grad } \phi = \left[ \frac{\partial}{\partial x_1} \phi, \frac{\partial}{\partial x_2} \phi \right] \]

\[ \text{div maps } S(\mathbb{R}^2) \times S(\mathbb{R}^2) \text{ into } S(\mathbb{R}^2), \quad \text{div } \phi = \frac{\partial}{\partial x_1} \phi_1 + \frac{\partial}{\partial x_2} \phi_2 \]

\[ \Delta \text{ maps } S(\mathbb{R}^2) \text{ into } S(\mathbb{R}^2), \quad \Delta \phi = \frac{\partial^2}{\partial x_1^2} \phi + \frac{\partial^2}{\partial x_2^2} \phi. \]

For the operators \( Z_\Lambda, \Lambda \in GL(\mathbb{R}^2) \), we prove the following.

\textbf{Theorem 3.1.1.}

For each \( \Lambda \in GL(\mathbb{R}^2) \) the operator \( Z_\Lambda \) maps \( S(\mathbb{R}^2) \) bijectively onto \( S(\mathbb{R}^2) \). In particular, for all \( k, l \in \mathbb{N}_0^2 \) there exists \( c_{k,l} > 0 \) such that
Moreover, the operators $Z_A$, $A \in GL(\mathbb{R}^2)$, extend to unitary operators on $L_2(\mathbb{R}^2)$.

**Proof.**

Let $\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \in GL(\mathbb{R}^2)$ and $\phi \in S(\mathbb{R}^2)$. For all $x = (x_1, x_2) \in \mathbb{R}^2$ we define $\xi := x \Lambda = (\lambda_1 x_1 + \lambda_3 x_2, \lambda_2 x_1 + \lambda_4 x_2)$. Also we introduce the normalization $\mu_j := \lambda_j / \det \Lambda$, $j = 1, ..., 4$. Let $R_1 := \max \{ |\mu_j| : 1 \leq j \leq 4 \}$ and $R_2 = R_1 \| \det \Lambda \|$. Then for all $k, l \in \mathbb{N}_0^2$, using Leibnitz's differentiation rule,

$$
\sup_{x \in \mathbb{R}^2} \left| \frac{\partial^{l_1} \partial^{l_2} \phi(x \Lambda)}{\partial x_1^{l_1} \partial x_2^{l_2}} \right| = \sup_{\xi \in \mathbb{R}^2} \left| (\mu_4 \xi_1 - \mu_3 \xi_2)^{k_1} (-\mu_2 \xi_1 + \mu_1 \xi_2)^{k_2} \left[ \frac{\partial}{\partial \xi_1} \right]^{l_1} \left[ \frac{\partial}{\partial \xi_2} \right]^{l_2} \phi(\xi_1, \xi_2) \right|
$$

$$
\leq R_1^{k_1 + k_2} (R_2)^{l_1 + l_2} \sup_{\xi \in \mathbb{R}^2} \left| \left( 1 + i \xi_1 1 + i \xi_2 \right) \left[ \frac{\partial}{\partial \xi_1} \right]^{l_1} \left[ \frac{\partial}{\partial \xi_2} \right]^{l_2} \phi(\xi_1, \xi_2) \right|
$$

$$
\leq \sup_{\xi \in \mathbb{R}^2} \left| \left( k_1 + k_2 \right) \left( m \right) \left[ \frac{\partial}{\partial \xi_1} \right]^{i+j} \left[ \frac{\partial}{\partial \xi_2} \right]^{l_1+l_2-i-j} \phi(\xi_1, \xi_2) \right| < \infty, \text{ since } \phi \in S(\mathbb{R}^2).
$$

Hence $Z_A \in S(\mathbb{R}^2)$, $S(\mathbb{R}^2)$ is the Schwartz space of rapidly decreasing functions. From the equality $Z_A Z_A^{-1} \phi = \phi$ we establish

$$
S(\mathbb{R}^2) = Z_A \cdot Z_A^{-1} (S(\mathbb{R}^2)) = \mathbb{R}^2 \in S(\mathbb{R}^2)
$$

and so $Z_A$ is a bijection with

$$(Z_A)^{-1} = Z_A^{-1}.$$

Now let $f \in S(\mathbb{R}^2)$, then

$$
\iint_{\mathbb{R}^2} \frac{1}{\sqrt{\| \det \Lambda \|}} | f(x \Lambda) |^2 \, dx =
$$
\[ = |\det \Lambda| \int_{\mathbb{R}^4} |f(\xi)|^2 \left| \frac{\partial(x_1, x_2)}{\partial(\xi_1, \xi_2)} \right| d\xi = \int_{\mathbb{R}^4} |f(\xi)|^2 d\xi < \infty, \]

here \( \frac{\partial(x_1, x_2)}{\partial(\xi_1, \xi_2)} \) denotes the Jacobian \( \det \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix} \).

So \( Z_\Lambda \) extends to an operator on \( L_2(\mathbb{R}^2) \) with the properties
\[
Z_\Lambda(L_2(\mathbb{R}^2)) \subset L_2(\mathbb{R}^2) \quad \text{and} \quad \|Z_\Lambda f\|_2 = \|f\|_2.
\]

From the equality \( Z_\Lambda Z_\Lambda^{-1} f = f \) it follows that \( Z_\Lambda \) is a bijection on \( L_2(\mathbb{R}^2) \). Hence \( Z_\Lambda \) is a unitary operator on \( L_2(\mathbb{R}^2) \).

Remark 3.1.2.

(1) If \( \Lambda \in \mathbb{R}^{2 \times 2} \) is a singular matrix and if we define
\[
\tilde{Z}_\Lambda : \phi(x) \to \phi(x \Lambda)
\]
then it can be seen with the same arguments as used in the above proves that \( \tilde{Z}_\Lambda \) maps \( S(\mathbb{R}^2) \) into \( S(\mathbb{R}^2) \).

But \( \tilde{Z}_\Lambda \) is not a bijection. This can be seen as follows. Let \( \Lambda \in \mathbb{R}^{2 \times 2} \) be a singular matrix, \( \Lambda \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \). Then there exists a nonsingular matrix \( P \in \mathbb{R}^{2 \times 2} \) such that
\[
P^{-1} \Lambda P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad P^{-1} \Lambda P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

Let \( B = P^{-1} \Lambda P \). It is obvious that \( \tilde{Z}_B \) maps \( S(\mathbb{R}^2) \) into itself and \( \tilde{Z}_B(S(\mathbb{R}^2)) \neq S(\mathbb{R}^2) \).

Hence \( \tilde{Z}_\Lambda = \tilde{Z}_P \tilde{Z}_B \tilde{Z}_P^{-1} \) maps \( S(\mathbb{R}^2) \) into \( S(\mathbb{R}^2) \), but \( \tilde{Z}_\Lambda \) is not a bijection.

(2) The operators \( \Phi_b, T_k \) and \( \Pi \) are bijections.

Let \( \Lambda \in \mathbb{R}^{2 \times 2} \). By \( \Lambda^T \) we denote the transpose of \( \Lambda \), defined by
\[
(\Lambda^T)_{i,j} = \Lambda_{j,i}, \quad i, j = 1, 2. \tag{3.9}
\]

For each \( \Lambda \in GL(\mathbb{R}^2) \) the operator \( Z_{\Lambda^T} \) is unitarily equivalent with \( Z_{\Lambda^{-1}} \). Indeed we have
Theorem 3.1.3.
Let \( A \in \text{GL}(\mathbb{R}^2) \). Then
\[
\mathcal{F} Z_A \mathcal{F}^* = Z_{A^{-1}}.
\]

Proof:
Let \( \phi \in S(\mathbb{R}^2) \). Then for each \( x \in \mathbb{R}^2 \),
\[
(\mathcal{F} Z_A \phi)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} (Z_A \phi)(y) e^{-i(xy)} dy = 
\]
\[
= \sqrt{|\det A^T|} \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} \phi(y A^T) e^{-i(xy)} dy = \{\xi = y A^T\} = 
\]
\[
= \frac{1}{\sqrt{|\det A|}} \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} \phi(\xi) e^{-i(x\xi^T)} d\xi = (Z_{A^{-1}} \mathcal{F} \phi)(x).
\]
Since \( S(\mathbb{R}^2) \) is dense in \( L^2(\mathbb{R}^2) \) we obtain \( \mathcal{F} Z_A \mathcal{F}^* = Z_{A^{-1}} \mathcal{F} \).

Note that
\[
Z_1 = I \quad \text{and} \quad Z_{A_1} Z_{A_2} = Z_{A_1 A_2} \quad \text{for all} \quad A_1, A_2 \in \text{GL}(\mathbb{R}^2).
\]

Thus the family \( \{ Z_A | A \in \text{GL}(\mathbb{R}^2) \} \) is a unitary group and we get an representation of the non-singular matrix group \( \text{GL}(\mathbb{R}^2), \cdot \) as operators on \( L^2(\mathbb{R}^2) \).

The phase-shifts respectively the translations establish two-parameter unitary groups. Each of
them yields a representation of the additive group \( (\mathbb{R}^2, +) \) as unitary operators on \( L^2(\mathbb{R}^2) \).

3.2. Functional analytic characterizations of the space \( S(\mathbb{R}^2) \)

We introduce the following notation. Let \( A_1, A_2, B_1, B_2 \) denote operators in a Hilbert space \( X \). Then we write
\[
D^\omega(A_1, A_2, B_1, B_2) := \bigcap_{n \in \mathbb{N}} \bigcap_{k_1 \in \mathbb{N}_0^2} \bigcap_{k_2 \in \mathbb{N}_0^2} \bigcap_{l_1 \in \mathbb{N}_0^2} \bigcap_{l_2 \in \mathbb{N}_0^2} D(A_1^{k_1} A_2^{k_2} B_1^{l_1} B_2^{l_2} \cdots A_1^{k_n} A_2^{k_n} B_1^{l_n} B_2^{l_n}).
\]

Before we formulate the 'shake-lemma' in two dimensions we mention the following simple
result.
Let \( n \in \mathbb{N} \) and \( k_1, k_2, l_1, l_2 \in \mathbb{N}_0^2 \). Then for all \( \phi \in D^\omega(P_1, P_2, Q_1, Q_2) \) we have
This result follows from the commutation relations
\[ P_1 Q_2 = Q_2 P_1 \quad \text{and} \quad P_2 Q_1 = Q_1 P_2. \]  

Next we prove the two dimensional 'Shake-lemma'

**Lemma 3.2.1.**

Let \( n \in \mathbb{N} \) and \( k_1, k_2, l_1, l_2 \in \mathbb{N}_0^n \). Then for all \( \phi \in D(\mathbb{R}^n) \) we have
\[
P_1^{l_1} P_2^{l_2} Q_1^{k_1} Q_2^{k_2} \cdots P_1^{l_m} P_2^{l_m} Q_1^{k_m} Q_2^{k_m} \phi = 
\sum_{r_1 \in \mathbb{N}_0^m} \sum_{r_2 \in \mathbb{N}_0^m} c^{(1)}_{k_1,l_1}(r_1) \cdot c^{(2)}_{k_2,l_2}(r_2) \mathcal{Q}_1^{k_1-r_1} \mathcal{Q}_2^{k_2-r_2} \mathcal{P}_1^{l_1-r_1} \mathcal{P}_2^{l_2-r_2} \phi,
\]
where the coefficients \( c^{(j)}_{k_j,l_j}(r_j) \), \( j = 1, 2 \), satisfy
\[
\begin{align*}
c^{(j)}_{k_j,l_j}(r_j) \leq & \frac{k_j!}{r_j! (k_j-r_j)!} \frac{l_j!}{l_j-r_j!} \quad \text{if} \quad r_j \leq l_j \quad \text{and} \quad r_j \leq k_j \\
c^{(j)}_{k_j,l_j}(r_j) = & \quad 0 \quad , \text{else}
\end{align*}
\]

Note that multi-index notation is used.

**Proof:**

Using the above simple result and applying the one dimensional shake-lemma twice, we derive
\[
P_1^{l_1} P_2^{l_2} Q_1^{k_1} Q_2^{k_2} \cdots P_1^{l_m} P_2^{l_m} Q_1^{k_m} Q_2^{k_m} \phi = 
\sum_{r_1 \in \mathbb{N}_0^m} \sum_{r_2 \in \mathbb{N}_0^m} c^{(1)}_{k_1,l_1}(r_1) \mathcal{Q}_1^{k_1-r_1} \mathcal{P}_1^{l_1-r_1} \sum_{r_2 \in \mathbb{N}_0^m} c^{(2)}_{k_2,l_2}(r_2) \mathcal{Q}_2^{k_2-r_2} \mathcal{P}_2^{l_2-r_2} \phi = 
\sum_{r_1 \in \mathbb{N}_0^m} \sum_{r_2 \in \mathbb{N}_0^m} c^{(1)}_{k_1,l_1}(r_1) c^{(2)}_{k_2,l_2}(r_2) \mathcal{Q}_1^{k_1-r_1} \mathcal{Q}_2^{k_2-r_2} \mathcal{P}_1^{l_1-r_1} \mathcal{P}_2^{l_2-r_2} \phi,
\]

where the coefficients \( c^{(j)}_{k_j,l_j}(r_j) \), \( j = 1, 2 \), satisfy the above stated conditions.

We prove the following functional analytic characterizations of the Schwartz space \( S(\mathbb{R}^2) \).
Theorem 3.2.2.

(i) \( \phi \in S(\mathbb{R}^2) \) if and only if \( \forall k \in \mathbb{N}_0, \forall l \in \mathbb{N}_0 : \phi \in D(Q_1^k Q_2^l P_1^k P_2^l) \).

(ii) \( S(\mathbb{R}^2) = D^\omega(P_1, P_2, Q_1, Q_2) \)

(iii) \( S(\mathbb{R}^2) = D^\omega(P_1, P_2) \cap D^\omega(Q_1, Q_2) \)

(iv) \( S(\mathbb{R}^2) = D^\omega(P_1) \cap D^\omega(P_2) \cap D^\omega(Q_1) \cap D^\omega(Q_2) \)

(v) \( S(\mathbb{R}^2) = D^\omega(P_1^2 + P_2^2) \cap D^\omega(Q_1^2 + Q_2^2) \)

(vi) \( S(\mathbb{R}^2) = D^\omega(P_1^2 + Q_1^2) \cap D^\omega(P_2^2 + Q_2^2) \)

(vii) \( S(\mathbb{R}^2) = D^\omega(P_1^2 + Q_2^2) \cap D^\omega(P_2^2 + Q_1^2) \)

(viii) \( S(\mathbb{R}^2) = D^\omega(P_1, Q_1) \cap D^\omega(P_2, Q_2) \)

(ix) \( S(\mathbb{R}^2) = D^\omega(P_1, Q_2) \cap D^\omega(P_2, Q_1) \)

\textbf{Proof:}

First we show that

\[
D^\omega(Q_1, Q_2) = D^\omega(Q_1) \cap D^\omega(Q_2) = D^\omega(Q_1^2 + Q_2^2).
\]

It is not hard to see that

\[
D^\omega(Q_\ell) = \{ f \in L_2(\mathbb{R}^2) \mid \forall k \in \mathbb{N}_0 : x_\ell^k f \in L_2(\mathbb{R}^2) \}, \quad \ell = 1, 2,
\]

\[
D^\omega(Q_1, Q_2) = \{ f \in L_2(\mathbb{R}^2) \mid \forall k, l \in \mathbb{N}_0 : x_1^k x_2^l f \in L_2(\mathbb{R}^2) \},
\]

\[
D^\omega(Q_1^2 + Q_2^2) = \{ f \in L_2(\mathbb{R}^2) \mid \forall k, l \in \mathbb{N}_0 : x_1^k x_2^l f \in L_2(\mathbb{R}^2) \}.
\]

Clearly \( D^\omega(Q_1, Q_2) \subset D^\omega(Q_1) \cap D^\omega(Q_2) \) and \( D^\omega(Q_1^2 + Q_2^2) \subset D^\omega(Q_1) \cap D^\omega(Q_2) \).

For the converse inclusions, let \( \phi \in D^\omega(Q_1) \cap D^\omega(Q_2) \). Then, for each \( k, l \in \mathbb{N}_0 \),

\[
\iint_{\mathbb{R}^2} |x_1^k x_2^l f(x_1, x_2)|^2 \, dx_1 \, dx_2 = \left( \iint_{|x_1| < |x_2|} + \iint_{|x_1| > |x_2|} \right) |x_1^k x_2^l f(x_1, x_2)|^2 \, dx_1 \, dx_2 \leq
\]

\[
\leq \iint_{|x_1| < |x_2|} |x_1^k x_2^l f(x_1, x_2)|^2 \, dx_1 \, dx_2 + \iint_{|x_1| > |x_2|} |x_2^k x_1^l f(x_1, x_2)|^2 \, dx_1 \, dx_2 \leq
\]

\[
\leq \iint_{\mathbb{R}^2} |x_1^k x_2^l f(x_1, x_2)|^2 \, dx_1 \, dx_2 + \iint_{\mathbb{R}^2} |x_2^k x_1^l f(x_1, x_2)|^2 \, dx_1 \, dx_2 < \infty.
\]

Hence

\[
D^\omega(Q_1) \cap D^\omega(Q_2) \subset D^\omega(Q_1, Q_2).
\]

Using the above technique it also follows that
Thus we proved
\[ D^\infty(Q_1, Q_2) = D^\infty(Q_1) \cap D^\infty(Q_2) = D^\infty(Q_1^2 + Q_2^2). \]

Applying the Fourier transformations \( \mathcal{F}, \mathcal{F}_1 \) and \( \mathcal{F}_2 \) respectively, we readily obtain from this
\[
D^\infty(P_1, P_2) = D^\infty(P_1) \cap D^\infty(P_2) = D^\infty(P_1^2 + P_2^2), \\
D^\infty(P_1, Q_2) = D^\infty(P_1) \cap D^\infty(Q_2) = D^\infty(P_1^2 + Q_2^2), \\
D^\infty(Q_1, P_2) = D^\infty(Q_1) \cap D^\infty(P_2) = D^\infty(Q_1^2 + P_2^2).
\]

As in the one-dimensional case we have
\[ D^\infty(P_j^2 + Q_j^2) = D^\infty(P_j, Q_j) = D^\infty(P_j) \cap D^\infty(Q_j), \quad j = 1, 2. \]

The latter equality follows from the fact that \( D^\infty(P_j) \cap D^\infty(Q_j) \) remains invariant under \( P_j \) and \( Q_j \). Analogously \( D^\infty(P_1, P_2) \cap D^\infty(Q_1, Q_2) \) remains invariant under \( P_j \) and \( Q_j \), \( j = 1, 2 \), therefore,
\[ D^\infty(P_1, P_2) \cap D^\infty(Q_1, Q_2) = D^\infty(P_1, P_2, Q_1, Q_2). \]

Now the correctness of the characterizations can be seen as follows. By definition, (i) is true. Applying the shake-lemma, (ii) follows from (i). The above consideration yields the other assertions. \qed

**Remark 3.2.3.**
The equalities \( D^\infty(Q_1, Q_2) = D^\infty(Q_1) \cap D^\infty(Q_2) = D^\infty(Q_1^2 + Q_2^2) \) are a consequence of a more general property in functional analysis. Let \( A_1 \) and \( A_2 \) be strongly commuting self-adjoint operators in a Hilbert space \( X \) (i.e. all their spectral projections commute), then
\[
D^\infty(A_1, A_2) = D^\infty(A_1) \cap D^\infty(A_2) \\
= D^\infty(\{ A_1 \}) \cap D^\infty(\{ A_2 \}) \\
= D^\infty(\{ A_1 \} + \{ A_2 \}) \\
= D^\infty(\{ A_1 \}, \{ A_2 \}),
\]
here \( \{ A_j \} \) denotes the positive square root of \( A_j \), \( \{ A_j \}^2 = (A_j^2)^{1/2}, \quad j = 1, 2. \) \qed

As a consequence of the above theorem we mention the following characterizations of \( S(\mathbb{R}^2) \) in terms of decrease at infinity of a function and of its Fourier transform.

**Corollary 3.2.4.**
Let \( \phi \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2). \) The following assertions are equivalent.
(i) $\phi \in S(\mathbb{R}^2)$
(ii) For each $k, l \in \mathbb{N}_0$ the functions $x \mapsto x^k_1 x^l_2 \phi$ and $x \mapsto x^k_1 x^l_2 \mathcal{I} \phi$ are bounded.
(iii) For each $k \in \mathbb{N}_0$ the functions $x \mapsto x^k_1 \phi$, $x \mapsto x^k_2 \phi$, $x \mapsto x^k_1 \mathcal{I} \phi$, and $x \mapsto x^k_2 \mathcal{I} \phi$ are bounded.
(iv) For each $k \in \mathbb{N}_0$ the functions $x \mapsto (x^k_1 + x^k_2) \phi$ and $x \mapsto (x^k_1 + x^k_2) \mathcal{I} \phi$ are bounded.

\[\]

3.3. The space $S(\mathbb{R}^2)$ as a countably normed space

Comments on countably normed spaces can be found in Section 2.2 and Appendix A.

Corresponding to the functional analytic characterizations of $S(\mathbb{R}^2)$ as presented, we define the following countable sets of norms on $S(\mathbb{R}^2)$.

Definition 3.3.1.

(i) $a_{k,l}(\phi) := \sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2} \sum_{j_1=0}^{l_1} \sum_{j_2=0}^{l_2} \|Q_1^{i_1} Q_2^{i_2} P_1^{j_1} P_2^{j_2} \phi\|_2$, $k, l \in \mathbb{N}_0^2$.

(i') $\tilde{a}_{k,l}(\phi) := \|Q_1^{k_1} Q_2^{k_2} P_1^{l_1} P_2^{l_2} \phi\|_2$, $k, l \in \mathbb{N}_0^2$.

(ii) $b_{n,k,l}(\phi) := \|P_1^{l_1} P_2^{l_2} Q_1^{k_1} Q_2^{k_2} \cdots P_1^{l_n} P_2^{l_2} Q_1^{k_1} Q_2^{k_2} \phi\|_2$, $n \in \mathbb{N}_0$, $l_1, l_2, k_1, k_2 \in \mathbb{N}_0^2$.

(iii) $c_{k,l}(\phi) := \|Q_1^k Q_2^l \phi\|_2$, $k, l \in \mathbb{N}_0^2$.

\[d_l(\phi) := \|P_1^l P_2^l \phi\|_2\]

(iv) $e_{k_1}(\phi) := \|Q_1^{k_1} \phi\|_2$, $f_{k_2}(\phi) := \|Q_2^{k_2} \phi\|_2$.

\[g_{l_1}(\phi) := \|P_1^{l_1} \phi\|_2\], $k_1, k_2, l_1, l_2 \in \mathbb{N}_0$

\[h_{l_2}(\phi) := \|P_2^{l_2} \phi\|_2\]

(v) $p_l(\phi) := \|(Q_1^2 + Q_2^2)^l \phi\|_2$, $k, l \in \mathbb{N}_0$.

\[q_l(\phi) := \|(P_1^2 + P_2^2)^l \phi\|_2\]
\[
\begin{align*}
\text{(vi)} \quad r_k(\phi) & := \| (P^2_\phi + Q^2_\phi)^k \phi \|_2, \quad k, l \in \mathbb{N}_0 \\
\text{(vii)} \quad s_l(\phi) & := \| (P^2_\phi + Q^2_\phi)^l \phi \|_2, \\
\text{(viii)} \quad t_k(\phi) & := \| (P^2_\phi + Q^2_\phi)^k \phi \|_2, \quad k, l \in \mathbb{N}_0 \\
\text{(ix)} \quad u_l(\phi) & := \| (P^2_\phi + Q^2_\phi)^l \phi \|_2
\end{align*}
\]

It can be shown that the sets of norms in the above definition are equivalent and the \( L_2(\mathbb{R}^2) \)-norm may be replaced by the supremum norm to get more equivalent families of norms.

It can be readily checked that the families \( \{a_{k,l} \mid k, l \in \mathbb{N}_0 \} \) and \( \{s_{k,l} \mid k, l \in \mathbb{N}_0 \} \) are directed.

The space \( S(\mathbb{R}^2) \) endowed with each of the above sets of norms becomes a complete countably normed space. So \( S(\mathbb{R}^2) \) is a Fréchet space. The topological structure is independent of the particular choice of the sets of norms.

Remark 3.3.2.

1. Convergence of a sequence \( (\phi_n) \subset S(\mathbb{R}^2) \) in the topology defined by one of the above sets or norms agrees with the convergence defined in \( S(\mathbb{R}^2) \) in [GS 2, Section 1.1]; in the aforementioned book it is stated that a sequence \( (\phi_n) \subset S(\mathbb{R}^2) \) converges to \( \phi \) if in every bounded region the partial derivatives of all orders of the \( \phi_n \) converge uniformly to the corresponding partial derivatives of \( \phi \) and if

\[
\forall k, l \in \mathbb{N}_0 \exists c_{k,l} > 0 \forall n \in \mathbb{N} : \sup_{x \in \mathbb{R}^2} |x^k D^l \phi_n(x)| < c_{k,l}
\]

(the constant \( c_{k,l} \) can be chosen independent of \( n \)).

2. The operators defined in Section 3.1 (see (3.8)) are all continuous.
variables,
\[ \Psi_{n,m}(x) = \psi_n(x_1) \psi_m(x_2), \quad n, m \in \mathbb{N}_0, \quad x \in \mathbb{R}^2, \quad (3.14) \]
here the \( \psi_k, k \in \mathbb{N}_0 \), are the Hermite functions in one variable (see Section 2.2).
The \( \Psi_{n,m} \) constitute an orthonormal basis in \( L_2(\mathbb{R}^2) \). Since \( S(\mathbb{R}^2) \subset L_2(\mathbb{R}^2) \), functions in \( S(\mathbb{R}^2) \) have Hermite expansions
\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,m} \Psi_{n,m} \quad (3.15) \]
with convergence in \( L_2(\mathbb{R}^2) \).
From the one dimensional case we derive that the functions \( \Psi_{n,m} \) are eigenfunctions of the following differential operators
\[ \begin{align*}
-\frac{\partial^2}{\partial x_1^2} + x_1^2, & \quad \text{with respective eigenvalues } 2n + 1, \\
-\frac{\partial^2}{\partial x_2^2} + x_2^2, & \quad \text{with respective eigenvalues } 2m + 1, \\
-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + x_1^2 + x_2^2, & \quad \text{with respective eigenvalues } 2(n + m + 1).
\end{align*} \quad (3.16) \]
Associated to these operators we define the self-adjoint operators \( T_1, T_2 \) and \( T_3 \) by
\[ \begin{align*}
T_1 f &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (2n + 1) \langle f, \Psi_{n,m} \rangle_2 \Psi_{n,m} \\
T_2 f &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (2m + 1) \langle f, \Psi_{n,m} \rangle_2 \Psi_{n,m} \\
T_3 f &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (2n + 2m + 2) \langle f, \Psi_{n,m} \rangle_2 \Psi_{n,m}
\end{align*} \quad (3.17) \]
with
\[ \begin{align*}
D(T_1) &= \{ g \in L_2(\mathbb{R}^2) \mid \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} n^2 \langle g, \Psi_{n,m} \rangle_2^2 < \infty \} \\
D(T_2) &= \{ g \in L_2(\mathbb{R}^2) \mid \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m^2 \langle g, \Psi_{n,m} \rangle_2^2 < \infty \} \\
D(T_3) &= \{ g \in L_2(\mathbb{R}^2) \mid \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (n + m)^2 \langle g, \Psi_{n,m} \rangle_2^2 < \infty \}
\end{align*} \quad (3.18) \]
If we take \( f \in \langle \Psi_{n,m} \mid n, m \in \mathbb{N}_0 \rangle \) then
\[ \begin{align*}
T_1 f &= \left[ -\frac{\partial^2}{\partial x_1^2} + x_1^2 \right] f, \quad T_2 f = \left[ -\frac{\partial^2}{\partial x_2^2} + x_2^2 \right] f \quad \text{and}
\end{align*} \quad (3.19) \]
We show that
\[ D(T_k^1) \cap D(T_k^2) = D(T_k^3) \quad \text{for each } k \in \mathbb{N}. \] (3.20)

Let \( k \in \mathbb{N} \). It is obvious that \( D(T_k^3) \subseteq D(T_k^1) \cap D(T_k^2) \).

For the converse, let \( f \in D(T_k^1) \cap D(T_k^2) \). Then for all \( j \in \{0, \ldots, k\} \),
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (n+m)^{2j} |(f, \Psi_{n,m})_2|^2 \leq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} 4^j (n^{2j} + m^{2j}) |(f, \Psi_{n,m})_2|^2 = 4^j \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} n^{2j} |(f, \Psi_{n,m})_2|^2 + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m^{2j} |(f, \Psi_{n,m})_2|^2 < \infty
\]
(3.21)

hence \( f \in D(T_k^3) \) and therefore, \( D(T_k^1) \cap D(T_k^2) = D(T_k^3) \).

It follows that
\[ T_3 = T_1 + T_2 \quad \text{and} \quad D^\infty(T_3) = D^\infty(T_1) \cap D^\infty(T_2). \] (3.22)

These results enable us to create a new functional analytic characterization of \( S(\mathbb{R}^2) \). As in the one-dimensional case it can be proved that
\[ T_1 = P_1^2 + Q_1^2 \quad \text{and} \quad T_2 = P_2^2 + Q_2^2 \] (3.23)

hence
\[ T_3 = P_1^2 + P_2^2 + Q_1^2 + Q_2^2. \] (3.24)

With the aid of Theorem 3.2.2.(vi) and (3.22) we obtain
\[
S(\mathbb{R}^2) = D^\infty(P_1^2 + Q_1^2) \cap D^\infty(P_2^2 + Q_2^2) = D^\infty(P_1^2 + P_2^2 + Q_1^2 + Q_2^2). \] (3.25)

The operator \( P_1^2 + P_2^2 + Q_1^2 + Q_2^2 \) is called the Hermite operator in two dimensions. Corresponding to the above characterization of \( S(\mathbb{R}^2) \) we define the following directed set of norms on \( S(\mathbb{R}^2) \):
\[
z_k(\phi) := \| (P_1^2 + P_2^2 + Q_1^2 + Q_2^2)^k \phi \|_2 = 2^k \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (n+m+1)^{2k} |(\phi, \Psi_{n,m})_2|^2 \right)^{1/2}, \quad k \in \mathbb{N}_0.
\] (3.26)

This set of norms is equivalent to each of the families of norms defined in the beginning of this section.

We reformulate the above mentioned characterization of \( S(\mathbb{R}^2) \) in terms of the Hermite
expansion coefficients of its elements.

Characterization 3.3.3.

If \( f \in S(\mathbb{R}^2) \), then for all \( k \in \mathbb{N}_0 \) we have

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (n+m)^{2k} \left| \langle f, \Psi_{n,m} \rangle \right|^2 < \infty.
\]

Conversely, if a sequence \( (a_{n,m}) \in C^{\mathbb{N}_0 \times \mathbb{N}_0} \) satisfies \( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (n+m)^{2k} |a_{n,m}|^2 < \infty \) for all \( k \in \mathbb{N}_0 \) then \( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,m} \Psi_{n,m} \) converges (in the topology of \( S(\mathbb{R}^2) \)) to a function in \( S(\mathbb{R}^2) \).

Put differently

\[ f \in S(\mathbb{R}^2) \text{ iff } \forall k \in \mathbb{N}_0 : \langle f, \Psi_{n,m} \rangle = O((n+m)^{-2k}) \ (n,m \to \infty). \]

The Wigner distribution \( W : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2) \) is defined by

\[ W = \mathcal{F}_2 \circ Z_A \text{ with } A = \begin{bmatrix} 1 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}. \]  

(3.26')

Clearly \( W \) is a unitary operator which maps \( S(\mathbb{R}^2) \) onto \( S(\mathbb{R}^2) \). Moreover, for each \( f \in S(\mathbb{R}^2) \) we have the pointwise evaluation

\[ (Wf)(x_1,x_2) = \frac{1}{\sqrt{2}\pi} \int_{-\infty}^{\infty} f(x_1 + \frac{1}{2} s, x_1 - \frac{1}{2} s) e^{-ix_2s} ds , \ (x_1,x_2) \in \mathbb{R}^2. \]  

(3.27)

It is known that (see [Pe])

\[ W \Psi_{n,m} = \Lambda_{n,m} , \ n,m \geq 0 , \]  

(3.28)

where

\[ \Lambda_{n,m}(r \cos \phi, r \sin \phi) = \sqrt{\frac{2}{\pi}} (-1)^n \sqrt{\frac{n!}{m!}} e^{i(m-n)\phi} (r \sqrt{2})^{m-n} e^{-r^2} L_n^{m-n} (2r^2) , \]  

(3.29)

\[ , \ r \geq 0 , \ \phi \in [-\pi,\pi] \ (n,m \geq 0) . \]

Put \( x_1 = r \cos \phi, x_2 = r \sin \phi \) and use the formula

\[ L_n^{(-k)} (x) = (-x)^k \frac{(n-k)!}{n!} L_n^{(k)} (x) , \ 1 \leq k \leq n , \]  

(3.30)

(see [MOS, p. 240]) then we obtain for each \( (x_1,x_2) \in \mathbb{R}^2 \).
\[ \Lambda_{n+k}(x_1, x_2) = \sqrt{\frac{2}{\pi}} (-1)^n \sqrt{\frac{n!}{(n+k)!}} \left( \sqrt{2} \right)^{k+1} \exp\left( -\left( x_1^2 + x_2^2 \right) \right) \frac{L_n^{(k+1)}}{2} (2(x_1^2 + x_2^2)). \]  \hspace{1cm} (3.31)

The \( \Lambda_{n,m} \) constitute an orthonormal basis in \( L_2(\mathbb{R}^2) \):

Let \( f \in L_2(\mathbb{R}^2) \), with Hermite expansion \( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,m} \Psi_{n,m} \).

Then

\[ Wf = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,m} \Lambda_{n,m} = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} a_{n,n+k} \Lambda_{n,n+k}. \]  \hspace{1cm} (3.32)

3.4. A characterization of \( S(\mathbb{R}^2) \) in terms of Fourier series

Definition 3.4.1.

Let \( f \) be a function from \( \mathbb{R}^+ \) into \( \mathbb{C} \). Then \( \mathcal{E}f \) denotes the even extension of \( f \) to the whole of \( \mathbb{R} \).

\[ \mathcal{E}f \in \mathcal{S}(\mathbb{R}^+) \]

The operator \( \mathcal{E} \) enables us to reformulate Characterization 2.4.3 of \( S_{\text{even}}(\mathbb{R}) \).

Characterization 3.4.2.

Let \( f \in X_1 \) and let \( \nu \geq -\frac{1}{2} \). Then equivalent are

(i) \( \mathcal{E}f \in S_{\text{even}}(\mathbb{R}) \)

(ii) \( f \in D^\infty(\mathbb{Q}; X_1) \) and \( H_{v,2v+1} f \in D^\infty(\mathbb{Q}; X_1) \).

We introduce the space \( S_{\text{even}}(\mathbb{R}^+) \), which consists of functions \( f : \mathbb{R}^+ \to \mathbb{C} \) such that \( \mathcal{E}f \in S_{\text{even}}(\mathbb{R}) \).

As a consequence of the above characterization we prove the following

Corollary 3.4.3.

Let \( \nu > -1 \) and \( \mu \geq 0 \). Let \( g \in X_\mu \). The following assertions are equivalent.

(i) \( g \in D^\infty(\mathbb{Q}; X_1) \) and \( H_{v,2v+1} g \in D^\infty(\mathbb{Q}; X_1) \)

(ii) \( r^{-v+\frac{1}{2}} \mu^{-\frac{1}{2}} g \in D^\infty(\mathbb{Q}; X_1) \) and \( r^{-v+\frac{1}{2}} \mu^{-\frac{1}{2}} H_{v,2v+1} g \in D^\infty(\mathbb{Q}; X_1) \)

(iii) \( \mathcal{E}(r^{-v+\frac{1}{2}} \mu^{-\frac{1}{2}} g) \in S_{\text{even}}(\mathbb{R}) \)

(iv) \( g \in r^{-v+\frac{1}{2}} \mu^{-\frac{1}{2}} S_{\text{even}}(\mathbb{R}^+) \)
Proof
Let us first assume that \(-v + \frac{1}{2} \mu - \frac{1}{2} < 0\). We prove that (i) implies (ii). Let \(g \in D^w(Q; X_1)\) and \(\mathcal{H}_{v,\mu} g \in D^w(Q; X_1)\). Define \(h : \mathbb{R}^+ \to \mathbb{C}\) by
\[
    h(r) := \int_0^r (rp)^{-v} J_v(rp) \rho^{1/2 - \mu - 1/2} (\mathcal{H}_{v,\mu} g)(\rho) \rho^\mu d\rho , \quad r \geq 0.
\]
Then \(h\) extends to an even function in \(C^\infty(\mathbb{R})\). Furthermore, for each \(r \geq 0\),
\[
    g(r) = [\mathcal{H}_{v,\mu} (\mathcal{H}_{v,\mu} g)](r) = \int_0^\infty (rp)^{-1/2} J_v(rp) (\mathcal{H}_{v,\mu} g)(\rho) \rho^\mu d\rho = r^{-1/2} h(r).
\]
Since \(h\) is continuous and \(g \in D^w(Q; X_1)\), for all \(k \in \mathbb{N}_0\)
\[
    \int_0^\infty |r^k h(r)|^2 r^2 dr \leq \int_0^1 |r^k h(r)|^2 r^2 dr + \int_1^\infty |r^k g(r)|^2 r^2 dr < \infty.
\]
Hence \(h \in D^w(Q; X_1)\) and so \(r^{-v+\mu-1/2} g \in D^w(Q; X_1)\).
Noting that \(\mathcal{H}_{v,\mu} (\mathcal{H}_{v,\mu} g) = g\) we can replace \(g\) by \(\mathcal{H}_{v,\mu} g\) in the above consideration and we obtain \(r^{-v+\mu-1/2} \mathcal{H}_{v,\mu} g \in D^w(Q; X_1)\). So assertion (ii) holds.
The converse implication, (ii) \(\Rightarrow\) (i), is obvious.
Next assume that \(-v + \frac{1}{2} \mu - \frac{1}{2} \geq 0\). Then the implication (i) \(\Rightarrow\) (ii)' is obvious and the implication (ii) \(\Rightarrow\) (i)' can be proved in the same way as we proved (i) \(\Rightarrow\) (ii)' in the case \(-v + \frac{1}{2} \mu - \frac{1}{2} < 0\). The equivalence of assertions (ii) and (iii) is a consequence of the identity (2.37)
\[
    r^{-v+\mu-1/2} \mathcal{H}_{v,2\mu+1} r^{-v+\mu-1/2} = \mathcal{H}_{v,\mu} ,
\]
and the above Characterization 3.4.2. Obviously, (iii) and (iv) are equivalent. \(\Box\)

Definition 3.4.4.
For each \(n \in \mathbb{Z}\) we define \(e_n : [-\pi, \pi] \to \mathbb{C}\) by
\[
    e_n(\phi) := \frac{e^{in\phi}}{\sqrt{2\pi}} , \quad \phi \in [-\pi, \pi].
\]
The \(e_n\) constitute an orthonormal basis in \(L_2([-\pi, \pi])\), see e.g. [BN, p. 155]. \(\Box\)

Definition 3.4.5.
Let \(g \in X_1\) and \(n \in \mathbb{Z}\). We define
\[ g \otimes e_n := \text{the equivalence class of all functions defined on } \mathbb{R}^+ \times [-\pi, \pi] \text{ which equal } g(r) e_n(\phi) \text{ almost everywhere.} \]

Then \( g \otimes e_n \in L_2(\mathbb{R}^+ \times [-\pi, \pi], r \, drd\phi) \). We have the following orthogonality relations

\[ (f \otimes e_n, g \otimes e_m)_{L_2(\mathbb{R}^+ \times [-\pi, \pi], r \, drd\phi)} = (f, g)_X, \quad \delta_{n,m} \]

for each \( f, g \in X_1 \) and \( n, m \in \mathbb{Z} \).

**Definition 3.4.6.**

For each \( f : \mathbb{R}^2 \rightarrow \mathbb{C} \) we define the polar form \( l^p f : \mathbb{R}^+ \times [-\pi, \pi] \rightarrow \mathbb{C} \) by

\[ (l^p f)(r, \phi) = f(r \cos \phi, r \sin \phi), \quad r \geq 0, \quad \phi \in [-\pi, \pi]. \]

The operator \( l^p \) extends to a unitary operator from \( L_2(\mathbb{R}^2) \) onto \( L_2(\mathbb{R}^+ \times [-\pi, \pi], r \, drd\phi) \).

**Definition 3.4.7.**

By \( C^\infty_{2\pi-\text{per}}(\mathbb{R}) \) we denote the class of all \( 2\pi \)-periodic infinitely differentiable functions \( f : \mathbb{R} \rightarrow \mathbb{C} \).

The next part of this section is based on 'Fourier expansions' in \( S(\mathbb{R}^2) \) and in \( L_2(\mathbb{R}^2) \). To find these expansions we proceed as follows.

Let \( f \in S(\mathbb{R}^2) \). Take a fixed \( r \geq 0 \).

It is clear that the function

\[ \phi \mapsto (l^p f)(r, \phi), \quad -\pi \leq \phi \leq \pi \quad (3.33) \]

extends to a function in \( C^\infty_{2\pi-\text{per}}(\mathbb{R}) \). So this function has a uniformly and absolutely convergent Fourier series,

\[ (l^p f)(r, \phi) = \sum_{n \in \mathbb{Z}} (T_n f)(r) e_n(\phi), \quad -\pi \leq \phi \leq \pi, \quad (3.34) \]

with

\[ (T_n f)(r) = \int_{-\pi}^\pi (l^p f)(r, \phi) e_{-n}(\phi) \, d\phi, \quad n \in \mathbb{Z}. \quad (3.35) \]

Furthermore, we have the Parseval relation
\[
\sum_{n \in \mathbb{Z}} |(T_n f)(r)|^2 = \int_{-\pi}^{\pi} |(\mathcal{P} f)(r, \phi)|^2 d\phi.
\] (3.36)

Applying this equality we deduce
\[
\left\{ \int_{\mathbb{R}^2} |f(x_1, x_2)|^2 \, dx_1 \, dx_2 \right\} = \int_{-\pi}^{\pi} rdr \sum_{n \in \mathbb{Z}} |(T_n f)(r)|^2 \, rdr = \sum_{n \in \mathbb{Z}} \int_{-\pi}^{\pi} |(T_n f)(r)|^2 \, rdr.
\] (3.37)

It follows that \(T_n f \in X_1\), \(n \in \mathbb{Z}\), with
\[
\|f\|_{L^2(\mathbb{R}^2)}^2 = \sum_{n \in \mathbb{Z}} \|T_n f\|_{X_1}^2.
\] (3.38)

For each \(n \in \mathbb{Z}\) we consider \(T_n\) as an operator,
\[
T_n : S(\mathbb{R}^2) \rightarrow X_1.
\] (3.39)

We define the operator \(T : S(\mathbb{R}^2) \rightarrow X_1^\mathbb{Z}\) by
\[
T f = (T_n f)_{n \in \mathbb{Z}}, f \in S(\mathbb{R}^2).
\] (3.40)

As a consequence of relation (3.38) the operator \(T\) is an isometry from \(S(\mathbb{R}^2)\) into the Hilbert space \(\bigoplus_{n \in \mathbb{Z}} X_1\), i.e.
\[
\|T f\|_{\bigoplus_{n \in \mathbb{Z}} X_1} = \|f\|_{L^2(\mathbb{R}^2)} \quad \text{for all} \quad f \in S(\mathbb{R}^2).
\] (3.41)

Since \(S(\mathbb{R}^2)\) is dense in \(L^2(\mathbb{R}^2)\) the mappings \(T_n, n \in \mathbb{Z}\), and \(T\) can be uniquely extended to operators \(T_n, n \in \mathbb{Z}\) and \(T\) on \(L^2(\mathbb{R}^2)\). We prove that \(T\) is a unitary operator from \(L^2(\mathbb{R}^2)\) onto \(\bigoplus_{n \in \mathbb{Z}} X_1\). First we derive some useful lemmas.

**Lemma 3.4.8.**

For each \(n \in \mathbb{Z}\) we have \(D(T_n^*) = X_1\) and
\[
\mathcal{P}(T_n^* g) = g \otimes e_n \quad \text{for all} \quad g \in X_1.
\]

**Proof.**

Let \(f \in S(\mathbb{R}^2)\) and \(g \in X_1\), then
\[
(T_n f, g)_{X_1} = \int_0^\infty \int_{-\pi}^{\pi} (T_n f)(r) \overline{g(r)} \, rdr = \int_{-\pi}^{\pi} (\mathcal{P} f)(r, \phi) \overline{e_{-n}(\phi)} \, d\phi \overline{g(r)} \, rdr
\]
with \( \mathcal{P} g^* = g \otimes e_n \). This yields the wanted result.

\( (*) \) : We observe that the integrals are absolutely convergent, so Fubini's theorem applies.

\[ \int d\phi \int_0^\infty (\mathcal{P} f)(r,\phi) g(r)e_n(\phi) \, r \, dr = (\mathcal{P} f, g \otimes e_n)_{L_2(R^+ \times [-\pi, \pi], \, rdrd\phi)} = \langle f, g^* \rangle_{L_2(R^2)} \]

We introduce the following abbreviation, \( K := L_2(R^+ \times [-\pi, \pi], \, rdrd\phi) \) and state the following result.

**Lemma 3.4.9.**

Let \( (g_n)_{n \in \mathbb{Z}} \in \bigoplus_{n \in \mathbb{Z}} X_1 \). Then \( \sum_{n \in \mathbb{Z}} (g_n \otimes e_n) \in K \) with

\[ \| \sum_{n \in \mathbb{Z}} (g_n \otimes e_n) \|_K^2 = \sum_{n \in \mathbb{Z}} \| g_n \|_{X_1}^2. \]

**Proof.**

From the orthogonality relations, mentioned in Definition 3.4.5, we deduce that for each \( k, l \in \mathbb{Z} \quad (k < l) \),

\[ \| \sum_{n=k}^{l} (g_n \otimes e_n) \|_K^2 = \sum_{n=k}^{l} \| g_n \|_{X_1}^2. \]

So \( \left\{ \sum_{n=-k}^{l} (g_n \otimes e_n) \right\}_{k,l \in \mathbb{N}} \) is a Cauchy sequence in the Hilbert space \( K \) and the result follows.

**Lemma 3.4.10.**

The operator \( T^* \) has domain \( D(T^*) = \bigoplus_{n \in \mathbb{Z}} X_1 \) and for each \( (g_n)_{n \in \mathbb{Z}} \in D(T^*) \) we have

\[ \mathcal{P} T^* (g_n)_{n \in \mathbb{Z}} = \sum_{n \in \mathbb{Z}} (g_n \otimes e_n) \quad \text{and} \quad T^* (g_n)_{n \in \mathbb{Z}} = (g_n)_{n \in \mathbb{Z}}. \]

**Proof.**

Let \( f \in S(R^2) \) and let \( (g_n)_{n \in \mathbb{Z}} \in \bigoplus_{n \in \mathbb{Z}} X_1 \). Then

\[ (T f, (g_n)_{n \in \mathbb{Z}})_{\bigoplus X_1} = \sum_{n \in \mathbb{Z}} (T_n f, g_n)_{X_1} = \sum_{n \in \mathbb{Z}} (f, T_n^* g_n)_{L_2(R^2)} = \]

\[ = \sum_{n \in \mathbb{Z}} \langle \mathcal{P} f, g_n \otimes e_n \rangle_K = \langle \mathcal{P} f, \sum_{n \in \mathbb{Z}} (g_n \otimes e_n) \rangle_K. \]

This proves the first part of the lemma.
For the second part, suppose \( h \in X_1 \) and let \( m \in \mathbb{Z} \). Then
\[
(T_m [T^* (g_n)_{n \in \mathbb{Z}}], \ h)_{X_1} = (T^* (g_n)_{n \in \mathbb{Z}} , \ T_m^* h)_{X_1} =
\]
\[
= \left( \sum_{n \in \mathbb{Z}} (g_n \otimes e_n), \ h \otimes e_m \right)_K = (g_m , \ h)_{X_1}.
\]
Hence \( T_m[T^* (g_n)_{n \in \mathbb{Z}}] = g_m \) and therefore, \( T[T^* (g_n)_{n \in \mathbb{Z}}] = (g_n)_{n \in \mathbb{Z}} \).

Now, as a consequence, we obtain

**Theorem 3.4.11.**

The operator \( T \) is a unitary mapping from \( L_2(\mathbb{R}^2) \) onto \( \bigoplus_{n \in \mathbb{Z}} X_1 \).

**Proof.**

The operator \( T \) is isometric, because \( T \) is isometric. From Lemma 3.4.10 we derive that \( T \) is surjective. Hence \( T \) is unitary (cf. [Co, p. 35]).

**Corollary 3.4.12.**

The Hilbert space \( L_2(\mathbb{R}^2) \) is isometrically isomorphic with the Hilbert space \( \bigoplus_{n \in \mathbb{Z}} X_1 \).

**Corollary 3.4.13.**

If \( f \in L_2(\mathbb{R}^2) \), then \( \mathcal{P} f = \sum_{n \in \mathbb{Z}} (T_n f \otimes e_n) \) where the series converges in \( K \).

**Proof.**

Let \( f \in L_2(\mathbb{R}^2) \), then \( T f \in \bigoplus_{n \in \mathbb{Z}} X_1 \). So by Theorem 3.4.11 and Lemma 3.4.10,
\[
\mathcal{P} f = \mathcal{P}(T^* T f) = \mathcal{P}(T^* (T_n f)_{n \in \mathbb{Z}}) = \sum_{n \in \mathbb{Z}} (T_n f \otimes e_n).
\]

As we have seen, the operators \( \mathcal{P} \) and \( T \) are unitary. Schematically,

\[
\begin{array}{cccc}
\bigoplus_{n \in \mathbb{Z}} X_1 & \xrightarrow{T} & L_2(\mathbb{R}^2) & \xrightarrow{\mathcal{P}} \ \\
& K \end{array}
\] (3.42)

For instance we have
\begin{equation}
LP(L_2(R^2)) = K \quad \text{and} \quad (T \ LP^*) (K) = \bigoplus_{n \in \mathbb{Z}} X_1.
\end{equation}

The latter equality implies that
\begin{equation}
K = \bigoplus_{n \in \mathbb{Z}} (X_1 \otimes e_n).
\end{equation}
Note that the inclusion \( \supset \) follows from Lemma 3.4.9. For the converse, let \( g \in K \). Then, by Corollary 3.4.13,
\begin{equation}
g = LP(IP^* g) = \sum_{n \in \mathbb{Z}} (T_n \ LP^* g \otimes e_n),
\end{equation}
where the series converges in \( K \).

Next we show that the Fourier transform of an arbitrary \( L_2(R^2) \) function can be expressed in terms of Hankel transforms.

**Lemma 3.4.14.**

For each \( n \in \mathbb{Z} \) we have the so called Hecke-Bochner identities
\[ T_n \ LP = (-i)^{\lfloor n \rfloor} \ H_{\lfloor n \rfloor, 1} \ T_n \quad \text{and} \quad T_n \ LP^* = (i)^{\lfloor n \rfloor} \ H_{\lfloor n \rfloor, 1} \ T_n. \]

*Proof.*

Let \( f \in S(R^2) \), then \( \LP f \in S(R^2) \) and
\[ (LP^* \ LP f)(r, \phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\pi} (LP f)(\rho, \theta)e^{-ir\rho\cos(\phi-o)} \rho d\rho. \]

Let \( n \in \mathbb{Z} \), then for all \( r \geq 0 \)
\begin{align*}
(T_n \ LP f)(r) &= \int_{\pi}^{\pi} (LP^* \ LP f)(r, \phi) e^{-i\phi} d\phi = \\
&= \frac{1}{\sqrt{2\pi} \cdot 2\pi} \int_{-\pi}^{\pi} e^{in(\phi-\theta)} d\phi \int_{0}^{\infty} (LP f)(\rho, \theta) \int_{-\pi}^{\pi} e^{i(r\rho\cos(\phi-\theta))} \rho d\rho = \\
&= \frac{1}{\sqrt{2\pi} \cdot 2\pi} \int_{-\pi}^{\pi} e^{in(\phi-\theta)} d\phi \int_{0}^{\infty} (LP f)(\rho, \theta) \cdot 2\pi e^{i\pi n^2/2} f_{\lfloor n \rfloor, 1}(\rho) \rho d\rho = \\
&= (-i)^{\lfloor n \rfloor} (H_{\lfloor n \rfloor, 1} T_n f)(r).
\end{align*}

Hence for all \( f \in S(R^2) \) we have
\[ T_n \ LP f = (-i)^{\lfloor n \rfloor} H_{\lfloor n \rfloor, 1} T_n f. \]

This yields
Note, that at (*), we used the formula
\[ \int_{-\pi}^{\pi} e^{i(\cos t - nt)} dt = 2\pi (i)^{n_1} J_{n_1}(z) \]
cf. [MOS, p. 79].
The second part of this lemma can be shown in a similar way.

Corollary 3.4.15.
If \( f \in L_2(\mathbb{R}^2) \) then
\[ IP(\mathcal{F} f) = \sum_{n \in \mathbb{Z}} (-i)^{n_1} \mathcal{H}_{1n_1,1} T_n f \otimes e_n \]
and
\[ IP(\mathcal{F}^* f) = \sum_{n \in \mathbb{Z}} (i)^{n_1} \mathcal{H}_{1n_1,1} T_n f \otimes e_n. \]

We know that each \( f \in S(\mathbb{R}^2) \) has a "Fourier expansion"
\[ (IP f)(r, \phi) = \sum_{n \in \mathbb{Z}} (T_n f)(r) e_n(\phi). \]

Of course we would like to know what kind of characterizations the \( T_n f \) satisfy. An conversely, which conditions on functions \( g_n, n \in \mathbb{Z} \) ensure that \( \sum_{n \in \mathbb{Z}} g_n(r) e_n(\phi) \) determines a function in \( S(\mathbb{R}^2) \). We proceed stepwise.

Lemma 3.4.16.
Let \( f \in S(\mathbb{R}^2) \). Then for all \( k, l \in N_0 \),
(i) \( \sum_{n \in \mathbb{Z}} \left| n \right|^k \int_{0}^{\infty} (1+r^2)^l |(T_n f)(r)|^2 rdr < \infty \) and
(ii) \( \sum_{n \in \mathbb{Z}} \left| n \right|^l \int_{0}^{\infty} (1+r^2)^l |(T_n f)(r)| rdr < \infty. \)

Proof. Let \( k, l \in N_0 \). Define \( g : \mathbb{R}^2 \to \mathbb{C} \) by
\[ g(x_1, x_2) = (1+x_1^2+x_2^2)^l (x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1})^k f(x_1, x_2), (x_1, x_2) \in \mathbb{R}. \]
Then \( g \in S(\mathbb{R}^2) \) and \( g \) has the polar form

\[
(P g)(r, \phi) = \sum_{n \in \mathbb{Z}} (in)^k (1 + r^2)^j (T_n f)(r) e_n(\phi), \quad r \geq 0, \quad \phi \in [-\pi, \pi].
\]

Hence

\[
\sum_{n \in \mathbb{Z}} |n|^k \int_0^\infty (1 + r^2)^{2j} |(T_n f)(r)|^2 \, rdr \leq \sum_{n \in \mathbb{Z}} n^{2k} \| (1 + r^2)^j T_n f \|_{L^2}^2 = \sum_{n \in \mathbb{Z}} \| T_n g \|_{L^2}^2 = \| g \|_{L^2(\mathbb{R}^2)}^2.
\]

So assertion (i) is true. And application of this assertion yields for all \( k, l \in \mathbb{N}_0 \),

\[
\left\{ \sum_{n \in \mathbb{Z}} |n|^k \int_0^\infty (1 + r^2)^j |(T_n f)(r)| \, rdr \right\}^2 \leq \sum_{n \in \mathbb{Z}} (ln)^k \int_0^\infty (1 + r^2)^j |(T_n f)(r)| \, rdr \leq \sum_{n \in \mathbb{Z}} \left( \frac{1}{n^2 + 1} \right)^{l/k} \int_0^\infty (1 + r^2)^{j+1} |(T_n f)(r)| \, rdr \leq \pi \coth(\pi) \sum_{n \in \mathbb{Z}} \left( \frac{r}{1 + r^2} \right)^{l/k} \int_0^\infty (1 + r^2)^{j+2} \, rdr < \infty
\]

and this shows the validity of assertion (ii).

\[\square\]

Corollary 3.4.17.

Let \( f \in S(\mathbb{R}^2) \), then for all \( n \in \mathbb{Z} \)

\[T_n f \in l^{1,n} S_{\text{even}}(\mathbb{R}^+)\).

Proof.

Let \( n \in \mathbb{Z} \). From Lemma 3.4.16 (i) we obtain \( T_n f \in D^\infty(Q; X_1) \). Since \( f \in S(\mathbb{R}^2) \) we also have \( IF = S(\mathbb{R}^2) \), so by Lemma 3.4.14 and Lemma 3.4.16 (i),

\[IH_{1,n,1} T_n f = l^{1,n} T_n IF f \in D^\infty(Q; X_1)\]

So Corollary 3.4.3 yields
Corollary 3.4.18.

Let \( f \in S(\mathbb{R}^2) \). Then there exist functions \( f_n \in S_{\text{even}}(\mathbb{R}), n \in \mathbb{Z} \) such that

\[
 f(x_1, x_2) = \sum_{n \in \mathbb{Z}} f_n \left( \sqrt{x_1^2 + x_2^2} \right) \left( x_1 + \text{sgn}(n) i x_2 \right)^{|n|}, \quad (x_1, x_2) \in \mathbb{R}^2.
\]

Hence there are also functions \( g_n \in S(\mathbb{R}), n \in \mathbb{Z} \), such that

\[
 f(x_1, x_2) = \sum_{n \in \mathbb{Z}} g_n \left( x_1^2 + x_2^2 \right) \left( x_1 + \text{sgn}(n) i x_2 \right)^{|n|}, \quad (x_1, x_2) \in \mathbb{R}^2.
\]

At this moment we only know that both series are pointwise convergent and convergent in \( L_2(\mathbb{R}^2) \)-sense. For a stronger result cf. Corollary 3.4.22.

Proof.

For each \( n \in \mathbb{Z} \) we define

\[
 f_n := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r^{-|n|} T_n f(r) \, dr.
\]

Then \( f_n \in S_{\text{even}}(\mathbb{R}) \), by Corollary 3.4.17. Hence the polar form \( IP f \) can be written as

\[
 (IP f)(r, \phi) = \sum_{n \in \mathbb{Z}} f_n(r) \cdot r^{-|n|} e^{in\phi}, \quad r \geq 0, \phi \in [-\pi, \pi].
\]

So we obtain, in Cartesian coordinates,

\[
 f(x_1, x_2) = \sum_{n \in \mathbb{Z}} f_n \left( \sqrt{x_1^2 + x_2^2} \right) \left( x_1 + \text{sgn}(n) i x_2 \right)^{|n|}, \quad (x_1, x_2) \in \mathbb{R}^2.
\]

The second characterization follows from the property that each \( \rho \in S_{\text{even}}(\mathbb{R}) \) can be written as

\[
 \rho(x) = \sigma(x^2), \quad x \in \mathbb{R},
\]

for some \( \sigma \in S(\mathbb{R}) \), see Section 2.4. \(\square\)

Remark 3.4.19.

In the above characterization a function in \( S(\mathbb{R}^2) \) is written as an infinite series. The \( n \)-th term of that series is a product of a function in \( S_{\text{even}}(\mathbb{R}) \) and a harmonic homogeneous polynomial of degree \( |n| \) in two variables \( (\rho_n(x_1, x_2) = (x_1 + \text{sgn}(n) i x_2)^{|n|}) \). \(\square\)

We explain how the above corollary applies to Fourier transforms of functions in \( S(\mathbb{R}^2) \).

Let \( f \in S(\mathbb{R}^2) \). By Corollary 3.4.17 we have
\[ f(x_1, x_2) = \sum_{n \in \mathbb{Z}} f_n (\sqrt{x_1^2 + x_2^2}) (x_1 + \text{sgn}(n) i x_2)^{\frac{1}{n+1}}, \quad (x_1, x_2) \in \mathbb{R}^2 \]  

(3.46)

where

\[ f_n = \frac{1}{\sqrt{2\pi}} \mathbb{E}(r^{-\frac{1}{n+1}} T_n f) \in S_{\text{even}}(\mathbb{R}), \quad n \in \mathbb{Z}. \]  

(3.47)

For each \( n \in \mathbb{Z} \) we define

\[ h_n := \frac{(-i)^{n+1}}{\sqrt{2\pi}} \mathbb{E}(f_n, T_n, f). \]  

(3.48)

Then \( h_n \in S_{\text{even}}(\mathbb{R}) \), by the \( \mathcal{H}_{1,2} \)-invariance of \( S_{\text{even}}(\mathbb{R}^+) \). Furthermore, using the identity \( \mathcal{H}_{1,2} \mathcal{H}_{1,1}^{-1} \mathcal{H}_{1,2}^{-1} \mathcal{H}_{1,1} f = \mathcal{H}_{1,1} f \), we derive

\[ (\mathbb{P} \mathbb{F} f)(r, \phi) = \sum_{n \in \mathbb{Z}} (-i)^{n+1} \mathbb{E}(r^{-\frac{1}{n+1}} T_n f)(r, \phi) e_n(\phi) = \sum_{n \in \mathbb{Z}} h_n(r) \cdot r^{-\frac{1}{n+1}} e^{i\phi}, \quad r \geq 0, \phi \in [-\pi, \pi]. \]  

(3.49)

Or, in Cartesian coordinates,

\[ (\mathbb{F} f)(x_1, x_2) = \sum_{n \in \mathbb{Z}} h_n (\sqrt{x_1^2 + x_2^2}) (x_1 + \text{sgn}(n) i x_2)^{\frac{1}{n+1}}, \quad (x_1, x_2) \in \mathbb{R}^2. \]  

(3.50)

Observe that \( h_n \) is related to \( f_n \):

\[ h_n = (-i)^{n+1} \mathbb{E}(r^{-\frac{1}{n+1}} T_n f_n), \]  

(3.51)

where \( f_n \) is the restriction of \( f_n \) to \( \mathbb{R}^+ \) (\( n \in \mathbb{Z} \)).

The next theorem provides necessary and sufficient conditions for functions \( g_n \in X_1, n \in \mathbb{Z} \), which guarantee that

\[ \mathbb{P}^* (\sum_{n \in \mathbb{Z}} g_n \otimes e_n) \in S(\mathbb{R}^2). \]

Before we state this theorem we mention

**Lemma 3.4.20.**

Let \( g \in X_1 \) with the property \( \sup_{r \geq 0} |(1 + r^2) g(r)| < \infty \). Then

\[ \| g \|_{X_1} \leq \frac{1}{2} \sqrt{\pi} \| (1 + r^2) g \|_{\infty}. \]

**Proof.**

The inequality is a consequence of the estimation
Theorem 3.4.21.
Let \( f \in L_2(\mathbb{R}^2) \). The following assertions are equivalent.

(i) \( f \in S(\mathbb{R}^2) \)

(ii) \( \forall k \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^k T_n f \|_{X_1}^2 < \infty \) and \( \forall l \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^l H_{|\mathbb{R}^1,1} T_n f \|_{X_1}^2 < \infty \)

(iii) \( \forall k \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^k T_n f \|_\infty < \infty \) and \( \forall l \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^l H_{|\mathbb{R}^1,1} T_n f \|_\infty < \infty \)

(iv) \( \forall k \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^k T_n f \|_\infty < \infty \) and \( \forall l \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^l H_{|\mathbb{R}^1,1} T_n f \|_\infty < \infty \)

(v) \( \forall k \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^k T_n f \|_{X_1}^2 < \infty \) and \( \forall l \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^l H_{|\mathbb{R}^1,1} T_n f \|_{X_1}^2 < \infty \).

Proof.
We prove the implication diagram

(ii) \( \iff \) (i) \( \iff \) (iii) \( \iff \) (ii),

(iii) \( \iff \) (iv) \( \iff \) (v) \( \iff \) (ii),

(i) \( \iff \) (iv).

The implication (ii) \( \Rightarrow \) (i) is a consequence of Lemma 3.4.16 (i). We prove the converse implication (ii) \( \Rightarrow \) (i). Suppose (ii) holds. We show that \( f \in D^\infty(Q_1^2 + Q_2^2) \) and \( IF f \in D^\infty(Q_1^2 + Q_2^2) \).

For each \( k \in \mathbb{N}_0 \) we have

\[
IP((Q_1^2 + Q_2^2)^kf) = r^{2k} IP f
\]

and

\[
\| r^{2k} IP f \|_K^2 = \sum_{n \in \mathbb{Z}} \| r^{2k} T_n f \|_{X_1}^2 < \infty.
\]

So \( f \in D^\infty(Q_1^2 + Q_2^2) \). Furthermore, for each \( l \in \mathbb{N}_0 \)

\[
IP((Q_1^2 + Q_2^2)^l IF f) = r^{2l} IP(IF f)
\]

and

\[
\| r^{2l} IP(IF f) \|_K^2 = \sum_{n \in \mathbb{Z}} \| r^{2l} H_{|\mathbb{R}^1,1} T_n f \|_{X_1}^2 < \infty.
\]

Therefore, \( IF f \in D^\infty(Q_1^2 + Q_2^2) \). Consequently \( f \in S(\mathbb{R}^2) \).
Clearly (iii) \implies (ii). We prove '(i) \implies (iii)'. Suppose \( f \in S(\mathbb{R}^2) \). Then we obtain from Lemma 3.4.16 (i) that for each \( k \in \mathbb{N}_0 \)

\[
\sum_{n \in \mathbb{Z}} \| r^k T_n f \|_{X_1} = \sum_{n \in \mathbb{Z}} \left( \int_0^\infty \left( T_n f \right)(r) \left( r^2 \right) dr \right)^{\frac{1}{2}} \leq \left( \sum_{n \in \mathbb{Z}} \left( \frac{1}{n^2 + 1} \right) \cdot \sum_{n \in \mathbb{Z}} (n^2 + 1) \int_0^\infty \left( T_n f \right)(r) \left( r^2 \right) dr \right)^{\frac{1}{2}} < \infty.
\]

Since \( \mathcal{H} f \in S(\mathbb{R}^2) \) we similarly derive \( \forall f \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^l \mathcal{H} f \|_{X_1} < \infty. \)

The implication '(iv) \implies (iii)' is a consequence of Lemma 3.4.20. Next we show '(i) \implies (iv)'.

Let \( f \in S(\mathbb{R}^2) \), then for all \( n \in \mathbb{Z} \)

\[
(*) \quad \| T_n f \|_{X_1} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(r \cos \phi, r \sin \phi) e^{-i n \phi} d\phi \leq \sup_{r \geq 0} \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(r \cos \phi, r \sin \phi) e^{-i n \phi} d\phi \leq \sqrt{2\pi} \| f \|_{X_1}.
\]

Let \( k \in \mathbb{N}_0 \). We define \( g_k \in S(\mathbb{R}^2) \) by

\[
g_k(x_1, x_2) = (x_1^2 + x_2^2)^k \left[ 1 - (x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1})^2 \right] f(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2.
\]

The function \( g_k \) has the polar form

\[
(\mathcal{P} g_k)(r, \phi) = \sum_{n \in \mathbb{Z}} r^{2k} (1 + n^2) (T_n f)(r) e_n(\phi), \quad r \geq 0, \quad \phi \in [-\pi, \pi].
\]

Hence, by \((*)\),

\[
\forall n \in \mathbb{Z} : \| (1 + n^2) r^{2k} T_n f \|_{\infty} \leq \sqrt{2\pi} \| g_k \|_{\infty}.
\]

Using these inequalities we deduce that

\[
\sum_{n \in \mathbb{Z}} r^{2k} T_n f \|_{\infty} = \sum_{n \in \mathbb{Z}} \frac{1}{1 + n^2} \int_0^\infty \left( T_n f \right)(r) \left( r^2 \right) dr \leq \sqrt{2\pi} \| g_k \|_{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{1 + n^2} < \infty.
\]

And therefore we also obtain

\[
\sum_{n \in \mathbb{Z}} r^{2k} T_n f \|_{\infty} \leq \frac{1}{2} \sum_{n \in \mathbb{Z}} \| (1 + r^{2k}) T_n f \|_{\infty} < \infty.
\]
Since \( f \in S(\mathbb{R}^2) \) we similarly derive \( \forall \{a_n\} \in \ell_2 : \sum_{n \in \mathbb{Z}} \| r^f H_{|1,1} T_n f \|_\infty < \infty \). It is clear that (iv) \( \Rightarrow \) (v). We prove that (v) \( \Rightarrow \) (ii). Suppose (v) holds, then we obtain from Lemma 3.4.20 and Minkowski’s inequality that for each \( k \in \mathbb{N}_0 \)

\[
\sum_{n \in \mathbb{Z}} \| r^k T_n f \|_1^2 \leq \frac{\pi}{4} \sum_{n \in \mathbb{Z}} (\| r^k T_n f \|_\infty + \| r^{k+2} T_n f \|_\infty)^2 \leq \frac{\pi}{4} \left[ \left( \sum_{n \in \mathbb{Z}} \| r^k T_n f \|_\infty^2 \right)^{1/2} + \left( \sum_{n \in \mathbb{Z}} \| r^{k+2} T_n f \|_\infty^2 \right)^{1/2} \right]^2 < \infty.
\]

Analogously we find

\[
\forall \{a_n\} \in \ell_2 : \sum_{n \in \mathbb{Z}} \| r^f H_{|1,1} T_n f \|_1^2 < \infty.
\]

This completes the proof.

As a consequence we mention the following

**Corollary 3.4.22.**

Let \( f \in S(\mathbb{R}^2) \) and let \( (g_n)_{n \in \mathbb{Z}} \subset S(\mathbb{R}) \) be such that

\[
f(x_1, x_2) = \sum_{n \in \mathbb{Z}} g_n(x_1^2 + x_2^2) (x_1 + sgn(n) i x_2)^{ln^1}, \quad (x_1, x_2) \in \mathbb{R}^2.
\]

The series in the above expansion admits the following kinds of convergence

(i) Absolute convergence with respect to the \( L_2(\mathbb{R}^2) \) norm.

(ii) Absolute convergence with respect to the supremum norm.

(iii) Absolute and uniform convergence on \( \mathbb{R}^2 \).

And most importantly, the series converges in \( S(\mathbb{R}^2) \).

**Proof.**

The assertions (i) and (ii) follow from Theorem 3.4.21 (iii) and (iv). Then assertion (iii) follows from Weierstrass’ test for uniform convergence.

We know that the series

\[
\sum_{n \in \mathbb{Z}} h_n(x_1, x_2) := \sum_{n \in \mathbb{Z}} g_n(x_1^2 + x_2^2) (x_1 + sgn(n) i x_2)^{ln^1}
\]

is pointwise convergent, with sum \( f(x_1, x_2) \). We prove that the series converges to \( f \) in the topology of \( S(\mathbb{R}^2) \), i.e.
(i) \[ \sum_{n=-N}^{M} h_n \to f \quad (N,M \to \infty) \] in the countably normed space \( D^\infty(Q_N^2 + Q_M^2) \)

(ii) \[ \sum_{n=-N}^{M} h_n \to f \quad (N,M \to \infty) \] in the countably normed space \( D^\infty(P_N^2 + P_M^2) \)

(i) Let \( k \in \mathbb{N}_0 \). For each \( N,M \in \mathbb{N} \) let \( V(N,M) = \{ n \in \mathbb{Z} \mid n > M \text{ of } n < -N \} \). Then

\[
\| (x_1^2 + x_2^2)^k \left( \sum_{n=-N}^{M} h_n - f \right)(x_1, x_2) \|_{L_2(\mathbb{R}^2)} = \sum_{n \in V(N,M)} h_n(x_1, x_2) \|_{L_2(\mathbb{R}^2)} \leq \sum_{n \in V(N,M)} \| (x_1^2 + x_2^2)^k \|_{L_2(\mathbb{R}^2)} \to 0 \quad (N,M \to \infty).
\]

(ii) We have

\[
(P^f f_h_n)(r, \phi) = (-i)^{\lceil n/2 \rceil} (PH_{|n|1,1} T_n f)(r) e_n(\phi) \quad \text{and}
\]

\[
(P^f f_h_n)(r, \phi) = \sum_{n \in \mathbb{Z}} (-i)^{\lceil n/2 \rceil} (PH_{|n|1,1} T_n f)(r) e_n(\phi).
\]

Hence for all \( l \in \mathbb{N}_0 \)

\[
\| (x_1^2 + x_2^2)^l \left( \sum_{n=-N}^{M} h_n - f \right)(x_1, x_2) \|_{L_2(\mathbb{R}^2)} \leq \sum_{n \in V(N,M)} \| (x_1^2 + x_2^2)^l \|_{L_2(\mathbb{R}^2)} \to 0 \quad (N,M \to \infty).
\]

So \( \sum_{n \in \mathbb{Z}} h_n \) converges to \( f \) in the topology of \( S(\mathbb{R}^2) \).

Corollary 3.4.23.
Let \( m \in \mathbb{Z} \) and let \( f \in L_2(\mathbb{R}^2) \) with the special polar form

\[
(P^f f)(r, \phi) = g(r) e_m(\phi) \quad , r \geq 0 \quad , \phi \in [-\pi, \pi].
\]

Then

\[ f \in S(\mathbb{R}^2) \quad \text{if and only if} \quad g \in \ell^m \quad \text{even}(\mathbb{R}^+).
\]

Proof.
In this special case we have

\[ T_n f = \delta_{n,m} g \quad , n \in \mathbb{Z}.
\]

So Theorem 3.4.21 yields

\[ f \in S(\mathbb{R}^2) \quad \text{if and only if} \quad \forall k \in \mathbb{N}_0 : \| r^k g \|_{X_1} < \infty \quad \text{and} \quad \| r^k \|_{X_1} < \infty.
\]

Or equivalently,
\[ f \in S(\mathbb{R}^2) \text{ if and only if } g \in D^\infty(Q;X_1) \text{ and } H_{1m,1} g \in D^\infty(Q;X_1). \]

Applying Corollary 3.4.3 we obtain the desired result.

If we take \( m = 0 \) in the above corollary we obtain a result for radial symmetric functions in \( S(\mathbb{R}^2) \). We denote the space of all radial symmetric functions in \( S(\mathbb{R}^2) \) by \( S_{\text{rad sym}}(\mathbb{R}^2) \).

**Corollary 3.4.24.**

Let \( f \in L_2(\mathbb{R}^2) \) be radial symmetric. Define \( g : \mathbb{R}^+ \to \mathcal{C} \) by

\[ g(r) = (\mathcal{P} f)(r, 0), \quad r \geq 0. \]

Then

\[ f \in S_{\text{rad sym}}(\mathbb{R}^2) \text{ if and only if } g \in S_{\text{even}}(\mathbb{R}^+). \]

**Lemma 3.4.25.**

(i) If \( \nu \geq 0 \) then for each \( f \in r^\nu S_{\text{even}}(\mathbb{R}^+) \),

\[ H_{\nu+1,1} r H_{\nu,1} f \in r^{\nu+1} S_{\text{even}}(\mathbb{R}^+) \]

and \( f \) satisfies the pointwise relation

\[ H_{\nu+1,1} r H_{\nu,1} f = \left\{ \frac{\nu}{r} - \frac{d}{dr} \right\} f. \]

(ii) If \( \nu \geq 1 \) then for each \( f \in r^\nu S_{\text{even}}(\mathbb{R}^+) \),

\[ H_{\nu-1,1} r H_{\nu,1} f \in r^{\nu-1} S_{\text{even}}(\mathbb{R}^+) \]

and \( f \) satisfies the pointwise relation

\[ H_{\nu-1,1} r H_{\nu,1} f = \left\{ \frac{\nu}{r} + \frac{d}{dr} \right\} f. \]

**Proof.**

(i) Let \( \nu \geq 0 \) and let \( f \in r^\nu S_{\text{even}}(\mathbb{R}^+) \). From the identity \( r^\mu H_{\mu,1} r^{-\mu} = H_{\mu,2\mu+1} \) and the \( H_{\mu,2\mu+1} \)-invariance of \( S_{\text{even}}(\mathbb{R}^+) \) it follows that

\[ H_{\nu+1,1} r H_{\nu,1} f \in r^{\nu+1} S_{\text{even}}(\mathbb{R}^+) \]

Furthermore, for each \( r > 0 \)
\[
\left[ \frac{v}{r} - \frac{d}{dr} \right] \mathcal{H}^{v}_{v,1} f (r) = \\
= \left[ \frac{v}{r} - \frac{d}{dr} \right] \int_0^\infty J_v(r \rho) \rho f(\rho) \rho d\rho = \int_0^\infty \left( \frac{v}{r \rho} - J_v'(r \rho) \right) \rho f(\rho) \rho d\rho = \\
(\ast) \int_0^\infty J_{v+1}(r \rho) \cdot \rho f(\rho) \rho d\rho = |H^{v+1,1}_{v,1} r f(r)|.
\]

Since \( H^{v,1}_{v,1} H^{v,1}_{v,1} = I \) we obtain from this
\[
H^{v+1,1}_{v,1} r H^{v,1}_{v,1} f = \left[ \frac{v}{r} - \frac{d}{dr} \right] f.
\]

Note that at (\( \ast \)), we used the identity (see [MOS, p. 67])
\[
J_{v+1}(z) = \frac{v}{z} J_v(z) - J_v'(z).
\]

(ii) The proof of this assertion runs similarly to the proof of (i). It makes use of the identity (see [MOS, p. 67])
\[
J_{v-1}(z) = \frac{v}{z} J_v(z) + J_v'(z).
\]

\[\]

Remark 3.4.26.

Let \( v \geq 0 \) and let \( f \in r^v S_{even}(\mathbb{R}^+) \). From relation (1.18) we obtain that \( f \) satisfies the pointwise relation
\[
(\mathcal{H}^{v}_v r^2 \mathcal{H}^{v}_v) f = \left[-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{v^2}{r^2}\right] f.
\]

This relation can easily be deduced from the above lemma,
\[
(\mathcal{H}^{v}_v r^2 \mathcal{H}^{v}_v) f = (\mathcal{H}^{v}_v r \mathcal{H}^{v+1,1}_v) (\mathcal{H}^{v+1,1}_v r \mathcal{H}^{v}_v) f = \\
= \left[ \frac{v+1}{r} + \frac{d}{dr} \right] \left[ \frac{v}{r} - \frac{d}{dr} \right] f = \left[-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{v^2}{r^2}\right] f.
\]

\[\]

Corollary 3.4.27.

Let \( n \in \mathbb{Z} \) and let \( f \in r^{1+n}_{1} S_{even}(\mathbb{R}^+) \). Then
Furthermore, we have the pointwise relations

\[ IH_{n+1,1} f = sgn(n) \left( \frac{n}{r} - \frac{d}{dr} \right) f \quad \text{and} \]

\[ IH_{n-1,1} f = sgn(n-1) \left( \frac{n}{r} + \frac{d}{dr} \right) f. \]

Lemma 3.4.28.

Let \( f \in S(\mathbb{R}^2) \). Then for each \( n \in \mathbb{Z} \),

(i) \( (T_n Q_1 f)(r) = \frac{1}{2} r \left( (T_{n-1} f)(r) + (T_{n+1} f)(r) \right) \), \( r > 0 \)

(ii) \( (T_n Q_2 f)(r) = \frac{-i}{2} r \left( (T_{n-1} f)(r) - (T_{n+1} f)(r) \right) \), \( r > 0 \)

(iii) \( (T_n P_1 f)(r) = \frac{i}{2} \left( (T_{n-1} f)'(r) + (T_{n+1} f)'(r) + \frac{1}{r} [(n+1) (T_{n+1} f)(r) - (n-1) (T_{n-1} f)(r)] \right) \), \( r > 0 \)

(iv) \( (T_n P_2 f)(r) = \frac{1}{2} \left( (T_{n-1} f)'(r) - (T_{n+1} f)'(r) - \frac{1}{r} [(n+1) (T_{n+1} f)(r) + (n-1) (T_{n-1} f)(r)] \right) \), \( r > 0 \).

Proof.

We only prove assertions (i) and (iii), the assertions (ii) and (iv) can be proved in the same way.

So let us fix \( n \in \mathbb{Z} \).

(i) Consider the following calculation

\[
(T_n Q_1 f)(r) = \frac{1}{r} \int_{\pi}^{\pi} r \cos(\phi) (IP f)(r,\phi) e^{-n(\phi)} d\phi = \frac{r}{2} \int_{\pi}^{\pi} (IP f)(r,\phi) (e^{i\phi} + e^{-i\phi}) e^{-n(\phi)} d\phi = \frac{r}{2} \int_{\pi}^{\pi} (IP f)(r,\phi) [e^{-n(\phi)} + e^{-(n+1)(\phi)}] d\phi = \frac{r}{2} \left( (T_{n-1} f)(r) + (T_{n+1} f)(r) \right), \quad r > 0.
\]

(iii) Applying Lemma 3.4.14 we obtain

\[ T_n P_1 f = T_n IF Q_1 IF^* f = (-i)^{|n|} IH_{n+1,1} T_n Q_1 IF^* f. \]

Now we note that \( T_m f \in r^{|m|} S_{even}(\mathbb{R}^+) \), for each \( m \in \mathbb{Z} \) (cf. Corollary 3.4.17) and so by (i), Lemma 3.4.14 and the above corollary we derive

\[ (-i)^{|n|} \left( IH_{n+1,1} T_n Q_1 IF^* f \right)(r) = (-i)^{|n|} \left( IH_{n+1,1} \frac{1}{r} (T_{n-1} IF^* f + T_{n+1} IF^* f) \right)(r) = \]
\[
\begin{align*}
&= -\frac{1}{2} (i^{n+1} (B^l_{1,1} r (i^{n-1} B^l_{n-1,1} T_{n-1} f + i^{n+1} B^l_{n+1,1} T_{n+1} f)) (r) = \\
&= -\frac{i}{2} \text{sgn}(n-1) \cdot \text{sgn}(n-1) \left(\frac{n-1}{r} - \frac{d}{dr} \right) (T_{n-1} f) (r) + \\
&+ \frac{i}{2} \text{sgn}(n) \cdot \text{sgn}(n) \left(\frac{n+1}{r} + \frac{d}{dr} \right) (T_{n+1} f) (r) = \\
&= \frac{i}{2} [(T_{n-1} f)'(r) + (T_{n+1} f)'(r) + \frac{1}{r} [(n+1) (T_{n+1} f) (r) - (n-1) (T_{n-1} f) (r)]] \quad r > 0.
\end{align*}
\]

Remark 3.4.29.

For each \( f \in S(\mathbb{R}^2) \) we have

\[
(IP P_1 f) (r, \phi) = \sum_{n \in \mathbb{Z}} (T_n P_1 f) (r) e_n(\phi);
\]

by relation (iii) of the above lemma and a simple computation

\[
(IP P_1 f) (r, \phi) = \sum_{n \in \mathbb{Z}} i \left( \cos \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi} \right) (T_n f) (r) e_n(\phi).
\]

Furthermore, in general

\[
(IP P_1 f) (r, \phi) = i \left( \cos \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi} \right) (IP f) (r, \phi).
\]

Hence we conclude

\[
(\cos \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi}) \sum_{n \in \mathbb{Z}} (T_n f) (r) e_n(\phi) = \sum_{n \in \mathbb{Z}} (\cos \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi}) (T_n f) (r) e_n(\phi).
\]

Or, in Cartesian coordinates,

\[
\frac{\partial}{\partial x_1} \sum_{n \in \mathbb{Z}} g_n(x_1^2 + x_2^2) (x_1 + \text{sgn}(n) i x_2)^{\epsilon n} = \sum_{n \in \mathbb{Z}} \frac{\partial}{\partial x_1} \left[ g_n(x_1^2 + x_2^2) (x_1 + \text{sgn}(n) i x_2)^{\epsilon n} \right],
\]

where \( g_n \in S(\mathbb{R}) \) such that \( g_n(x^2) = f_n(x), x \geq 0, \) with

\[
f_n = \frac{1}{\sqrt{2\pi}} \mathcal{E}(r^{-\epsilon n} T_n f) \in S_{\text{even}}(\mathbb{R}) \quad (n \in \mathbb{Z})
\]

(cf. Corollary 3.4.18).

Now we can generalize Corollary 3.4.22.

Corollary 3.4.30.

Let \( f \in S(\mathbb{R}^2) \) and let \( (g_n)_{n \in \mathbb{Z}} \subset S(\mathbb{R}) \) be such that
\[
I(n)^2 = \sum_{n \in \mathbb{Z}} g_n(x_1^2 + x_2^2) (x_1 + sgn(n)i x_2)^{i n_1}, \quad (x_1, x_2) \in \mathbb{R}^2. \tag{\ast}
\]

Then for all \(k, l \in \mathbb{N}_0,\)

\[
x_1^k x_2^l \left[ \frac{\partial}{\partial x_1} \right]^l \left[ \frac{\partial}{\partial x_2} \right]^l f(x_1, x_2) =
\]

\[
= \sum_{n \in \mathbb{Z}} x_1^k x_2^l \left[ \frac{\partial}{\partial x_1} \right]^l \left[ \frac{\partial}{\partial x_2} \right]^l g_n(x_1^2 + x_2^2) (x_1 + sgn(n)i x_2)^{i n_1}, \quad (x_1, x_2) \in \mathbb{R}^2.
\]

and this series admits the same types of convergence as the series in (\ast\), (cf. Corollary 3.4.22). 

At the end of this section we present some polar forms of functions in \(S(\mathbb{R}^2).\) Let \(f \in S(\mathbb{R}^2),\)

Then

\[
(\text{IP} f) (r, \phi) = \sum_{n \in \mathbb{Z}} (T_nf) (r) e_n(\phi)
\]

\[
(\text{IP} \mathbb{F} f) (r, \phi) = \sum_{n \in \mathbb{Z}} (-i)^{\frac{|n|}{2}} (\mathbb{H}_{|n|,1,1} T_n f) (r) e_n(\phi)
\]

\[
(\text{IP} \mathbb{F}^* f) (r, \phi) = \sum_{n \in \mathbb{Z}} i^{\frac{|n|}{2}} (\mathbb{H}_{|n|,1,1} T_n f) (r) e_n(\phi)
\]

\[
(\text{IP} Q_1 f) (r, \phi) = \sum_{n \in \mathbb{Z}} \frac{r}{2} (T_{n-1} f + T_{n+1} f) (r) e_n(\phi)
\]

\[
(\text{IP} Q_2 f) (r, \phi) = \sum_{n \in \mathbb{Z}} \frac{r}{2i} (T_{n-1} f - T_{n+1} f) (r) e_n(\phi)
\]

\[
(\text{IP} (Q_1 + Q_2^2) f) (r, \phi) = \sum_{n \in \mathbb{Z}} r^2 (T_n f) (r) e_n(\phi)
\]

\[
(\text{IP} (Q_1 + i Q_2) f) (r, \phi) = \sum_{n \in \mathbb{Z}} r (T_n f) (r) e_{n+1}(\phi)
\]

\[
(\text{IP} (Q_1 - i Q_2) f) (r, \phi) = \sum_{n \in \mathbb{Z}} r (T_n f) (r) e_{n-1}(\phi)
\]

\[
(\text{IP} P_1 f) (r, \phi) = \sum_{n \in \mathbb{Z}} \frac{i}{2} \left\{ (T_{n-1} f)'(r) + (T_{n+1} f)'(r) + \frac{1}{r} \left[ (n+1) (T_{n+1} f) (r) - (n-1) (T_{n-1} f) (r) \right] \right\} e_n(\phi)
\]

\[
(\text{IP} P_2 f) (r, \phi) = \sum_{n \in \mathbb{Z}} \frac{1}{2} \left\{ (T_{n-1} f)'(r) - (T_{n+1} f)'(r) + \frac{1}{r} \left[ (n+1) (T_{n+1} f) (r) + (n-1) (T_{n-1} f) (r) \right] \right\} e_n(\phi)
\]

\[
(\text{IP} (P_1^2 + P_2^2) f) (r, \phi) = \sum_{n \in \mathbb{Z}} (\mathbb{H}_{|n|,1,1} r^2 H_{|n|,1,1} T_n f) (r) e_n(\phi)
\]

\[
= \sum_{n \in \mathbb{Z}} \left[ -(T_n f)'(r) - \frac{1}{r} (T_n f)'(r) + \frac{n^2}{r^2} (T_n f) (r) \right] e_n(\phi)
\]

\[
(\text{IP} (Q_1 P_2 - Q_2 P_1) f) (r, \phi) = \sum_{n \in \mathbb{Z}} -n (T_n f) (r) e_n(\phi)
\]
\[(IP(Q_1 P_1 + Q_2 P_2)f)(r, \phi) = \sum_{n \in \mathbb{Z}} i r (T_n f)'(r) e_n(\phi)\]

\[(IP f Z_\Lambda)(r, \phi) = \sum_{n \in \mathbb{Z}} (T_n f)(r) e_n(\phi + \theta),\]

here \(\Lambda\) represents a rotation over angle \(\theta\).

**Remark 3.4.31.**

Let \(f \in S(\mathbb{R}^2)\). In the beginning of this section we have chosen the orthonormal basis \(\{ \frac{e^{int}}{\sqrt{2\pi}} \}_{n \in \mathbb{Z}}\) in \(L_2([-\pi, \pi])\).

It is also possible to choose the orthonormal basis

\[\left\{ \frac{1}{\sqrt{2\pi}}, \cos(m\phi), \sin(n\phi) \right\} (m, n = 1, 2, ...)\]  

(3.52)

in \(L_2([-\pi, \pi])\). Then the generalized Fourier series corresponding to the function \(IP f\) with respect to this orthonormal set, equals (*)

\[\frac{1}{2\pi} \int_{-\pi}^{\pi} (IP f)(r, \psi) d\psi + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (IP f)(r, \psi) \cos[n(\psi - \phi)] d\psi =\]

\[= \frac{1}{2} (a_0 f)(r) + \sum_{n=1}^{\infty} \{(a_n f)(r) \cos(n\phi) + (b_n f)(r) \sin(n\phi)\}.\]  

(3.53)

Here

\[(a_n f)(r) = \frac{1}{\pi} \int_{-\pi}^{\pi} (IP f)(r, \psi) \cos(n\psi) d\psi \quad (n = 0, 1, 2, ...)\]  

(3.54)

\[(b_n f)(r) = \frac{1}{\pi} \int_{-\pi}^{\pi} (IP f)(r, \psi) \sin(n\psi) d\psi \quad (n = 1, 2, ...).\]

With the aid of this expansion the above theory can be set up in a similar way.

\[
(3.5. A polar coordinates factorization problem.\]

In this section we deal with functions \(f \in L_2(\mathbb{R}^2)\) which have the factored polar form,

\[(IP f)(r, \phi) = g(r) h(\phi)\]

where \(g \in X_1\) and \(h \in L_2([-\pi, \pi])\). We present necessary and sufficient conditions on the functions \(g\) and \(h\) which ensure that \(f \in S(\mathbb{R}^2)\). If we take \(h = 1\) then \(f\) is necessarily a radial

(*) Note that \(C_n(\cos(\psi - \phi)) = \cos[n(\psi - \phi)]\), where \(C_n\) is the Chebyshev polynomial of the first kind and of order \(n\) \((n \in \mathbb{N}_0)\), cf. [MOS p.256].
symmetric function.

**Lemma 3.5.1.**

Let \( g \in X_1 \). Then equivalent are

(i) There is a sequence \((n_k)_{k \in \mathbb{N}} \subset \mathbb{N}_0\) such that \( \lim_{k \to \infty} n_k = \infty \) and \( \forall k \in \mathbb{N} : g \in r^{n_k} S_{\text{even}}(\mathbb{R}^+) \).

(ii) \( \forall l \in \mathbb{N}_0 : g \in r^l S_{\text{even}}(\mathbb{R}^+) \).

(iii) The function \( g \) extends to a function in \( S_{\text{even}}(\mathbb{R}) \). Furthermore, \( \forall l, m \in \mathbb{N}_0 : g^{(m)} \in r^l S_{\text{even}}(\mathbb{R}^+) \).

**Proof.**

We prove the implication diagram

(i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (i).

Clearly (iii) \( \Rightarrow \) (i). First we show that (i) \( \Rightarrow \) (ii).

Let \((n_k)_{k \in \mathbb{N}}\) be a sequence in \( \mathbb{N}_0 \) with \( \lim_{k \to \infty} n_k = \infty \) and suppose

\( \forall k \in \mathbb{N} \exists g_k \in S_{\text{even}}(\mathbb{R}) \forall r > 0 : r^{-n_k} g(r) = g_k(r) \).

We prove the following

**Claim.** \( \forall l \in \mathbb{N}_0 \exists h_l \in S_{\text{even}}(\mathbb{R}) \forall r > 0 : r^{-l} g(r) = h_l(r) \) (i.e. \( g \in r^l S_{\text{even}}(\mathbb{R}^+) \) for each \( l \in \mathbb{N}_0 \)).

Let \( l \in \mathbb{N}_0 \). If \( l = n_k \) for certain \( k \in \mathbb{N} \) then define \( h_l := g_k \). Next suppose such a \( k \) does not exist. Choose \( k \in \mathbb{N} \) such that \( n_k > l \) and let \( n = n_k - l \). Then we have

\( r^{-l} g(r) = r^n g_k(r) , r > 0 \).

Define \( h_l : \mathbb{R} \to \mathcal{C} \) by

\[
 h_l(r) = \begin{cases} 
 r^n g_k(r), & \text{if } r \geq 0 \\
 (-1)^n r^n g_k(r), & \text{if } r < 0.
\end{cases}
\]

Note that the definition of \( h_l \) is independent of the choice of \( n_k \). Clearly \( h_l \) is an even function. If \( n \) is an even number, then it is clear that \( h_l \in S_{\text{even}}(\mathbb{R}) \). Now suppose \( n \) is odd. Because of Lemma 3.5.7 below we only have to prove that all odd derivatives in zero are zero. However it turns out that all derivatives \( h_l^{(j)} \) are zero in \( r = 0 \). The proof is by induction. Obviously \( h_l^{(0)}(0) = h_l(0) = 0 \). Let \( j_0 \in \mathbb{N}_0 \) and suppose that \( h_l^{(j)}(0) = 0 \) for \( 0 \leq j \leq j_0 \). Choose \( k_1 \in \mathbb{N} \) such that \( n_{k_1} - n_k > j_0 \) and let \( p = n_{k_1} - n_k \). It follows that
and so
\[ g_k(r) = \begin{cases} 
  r^p g_k(r) , & \text{if } r \geq 0 \\
  (-1)^p r^p g_k(r) , & \text{if } r < 0 
\end{cases} \]
Applying Leibnitz's differentiation rule we obtain
\[ h_Y^{(p)}(r) = \sum_{j=0}^{n+p} \binom{n+p}{j} (n+p-1) \cdots (n+p-j+1) \cdot \frac{d^j}{d r^j} g_k(r) \cdot \begin{cases} 
  1 , & \text{if } r > 0 \\
  (-1)^{n+p} , & \text{if } r < 0. 
\end{cases} \]
From this formula (viz. \( n + p - j > 1 \)) and the induction hypothesis we obtain
\[ h_Y^{(p)}(r) = h_Y^{(p)}(0) \quad \text{for all } r > 0. \]
This proves the claim.

Next we show that (ii) implies (iii). Suppose \( \forall \ell \in \mathbb{N}_0 : g \in \ell^{l} S_{\text{even}}(\mathbb{R}^+). \) Fix \( m \in \mathbb{N}_0 \) and suppose
\[ \forall \ell \in \mathbb{N}_0 \exists h_{\ell,m} \in S_{\text{even}}(\mathbb{R}) \forall r > 0 : r^{-l} g^{(m)}(r) = h_{l,m}(r). \]
Let \( l \in \mathbb{N}_0 \). Define \( h_{l,m+1} \in S_{\text{even}}(\mathbb{R}) \) by
\[ h_{l,m+1}(r) = l \cdot h_{l+1,m}(r) + \frac{d}{d r} (r \cdot h_{l+1,m}(r)) , \quad r \in \mathbb{R}. \]
Then, for each \( r > 0. \)
\[ r^{-l} g^{(m+1)}(r) = l \cdot r^{-l+1} g^{(m)}(r) + \frac{d}{d r} (r^{-l} g^{(m)}(r)) = h_{l,m+1}(r). \]
The proof of the implication (ii) \( \Rightarrow \) (iii) follows from usual induction arguments. \( \square \)

**Lemma 3.5.2.**

Let \( f \in C^m(\mathbb{R}^2) \) and suppose
\[ (\mathcal{P} f)(r, \phi) = g(r) h(\phi) , \quad r \geq 0, \quad \phi \in [-\pi, \pi]. \]
For each \( l \in \mathbb{N}_0, \) there exist real numbers \( a_{i,k,l}, \quad 0 \leq k \leq l, \quad 0 \leq i \leq 2l - 2k \) such that
The numbers $a_{i,k,l}$ satisfy the recurrence relations

$$a_{j,k,l+1} = a_{j-2,k,l} - 2(2l-j+\frac{1}{2})a_{j-1,k,l} + (2l-j)^2a_{j,k,l} + a_{j,k-1},$$

with initial values

$$a_{0,0,0} = 1, \quad a_{j,k,l} = 0 \text{ if not } (0 \leq k \leq l, 0 \leq i \leq 2l-2k).$$

**Proof.**

Applying induction arguments we obtain the wanted results.

**Theorem 3.5.3.**

Let $f \in L^2(\mathbb{R}^2)$ for which there exist functions $g \in X_1$ and $h \in L^2([-\pi,\pi])$ such that

$$(\mathcal{P} f)(r,\phi) = g(r)h(\phi).$$

The following assertions are equivalent

(i) $f \in S(\mathbb{R}^2)$

(ii) $h$ extends to a function in $C^\infty_{2\pi\text{-per}}(\mathbb{R})$ and

$$\forall n \in \mathbb{Z} \ [(h,e_n)_{L^2([-\pi,\pi])} \neq 0 \Rightarrow g \in r^{1/n} S_{\text{even}}(\mathbb{R}^+)].$$

**Proof:**

For convenience we set $\alpha_n = (h,e_n)_{L^2([-\pi,\pi])}, \ n \in \mathbb{Z}.$

'(i) $\Rightarrow$ (ii)': Suppose $f \in S(\mathbb{R}^2).$ By Lemma 3.4.16, $h$ extends to a function in $C^\infty_{2\pi\text{-per}}(\mathbb{R}).$ For each $n \in \mathbb{Z}$ we have

$$(T_n f)(r) = \alpha_n g(r), \ r \geq 0.$$ 

Applying Theorem 3.4.21 we obtain for all $k,l \in \mathbb{N}_0$

$$\sum_{n \in \mathbb{Z}} |\alpha_n|^2 \|r^k \hat{g}\|_{L^2}^2 < \infty \quad \text{and} \quad \sum_{n \in \mathbb{Z}} |\alpha_n|^2 \|r^l \|_{H^1_{1,1}} \|g\|_{X_1}^2 < \infty.$$ 

From this it follows that

$$g \in D^\infty(Q;X_1) \text{ and } \forall n \in \mathbb{Z} : (\mathcal{P} g)_{H^1_{1,1}} \in D^\infty(Q;X_1).$$

By Corollary 3.4.3 this is equivalent with
For the converse, suppose (ii) holds. We first consider the case that $\alpha_n = 0$ for all but a finite number of $n$'s. For each $n \in \mathbb{Z}$ we have

$$T_n f = \alpha_n g.$$ 

Let $n \in \mathbb{Z}$ with $\alpha_n \neq 0$. Then there exists $f_n \in S_{\text{even}}(\mathbb{R})$ such that

$$g(r) = r^{\lceil n \rceil} f_n(r), \quad r > 0.$$ 

Hence for all $k \in \mathbb{N}^0$

$$\| r^k T_n f \|_{\infty} = \| \alpha_n \| _{\mathbb{N}^0} < \infty.$$ 

Furthermore, for all $l \in \mathbb{N}^0$

$$\| r^l \mathcal{H}_{n,1} T_n f \|_{\infty} = \| \alpha_n \| _{\mathbb{N}^0} \| r^{l+\lceil n \rceil} \mathcal{H}_{n,1} f_n \|_{\infty} < \infty$$

because of the $\mathcal{H}_{n,2r+1}$-invariance of $S_{\text{even}}(\mathbb{R})$. From Theorem 3.4.21 we conclude that $f \in S(\mathbb{R}^2)$.

Next we consider the case that $\alpha_n \neq 0$ for all but a finite number of $n$'s. Applying Lemma 3.5.1 we obtain for each $k \in \mathbb{N}^0$

$$\| (Q_1^2 + Q_2^2) f \|_{L_2(\mathbb{R}^2)} = \| r^{2k} g(r) \|_{L_2} \| h(\phi) \|_{L_2} \| h \|_{L_2([-\pi, \pi])} < \infty.$$ 

Let $l \in \mathbb{N}^0$. Then, by Lemma 3.5.2, there exist real numbers $a_{i,j,k}$, $0 \leq k \leq l$, $0 \leq i \leq 2l - 2k$ such that

$$(\mathcal{L}(P_1^2 + P_2^2)) f (r, \phi) = \left[ -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right]^l g(r) h(\phi) =$$

$$= \sum_{k=0}^{l} \sum_{i=0}^{2l-2k} a_{i,j,k} r^{i-2l} g^{(i)}(r) h^{(2k)}(\phi), \quad r > 0, \quad \phi \in [-\pi, \pi].$$

So by Lemma 3.5.1 we immediately see

$$\| (P_1^2 + P_2^2) f \|_{L_2(\mathbb{R}^2)} < \infty.$$ 

Hence $f \in D^\infty(Q_1^2 + Q_2^2) \cap D^\infty(P_1^2 + P_2^2) = S(\mathbb{R}^2)$. 

We denote the space of all radial symmetric functions in $S(\mathbb{R}^2)$ by $S_{\text{radsym}}(\mathbb{R}^2)$. As a consequence of the above theorem we state

**Corollary 3.5.4.**

Let $f \in L_2(\mathbb{R}^2)$ be radial symmetric. Define $g : \mathbb{R}^+ \rightarrow \mathcal{C}$ by
\[ g(r) = (IP f) (r, 0) \quad , r \geq 0. \]

Then

\[ f \in S_{\text{radsym}}(IR^2) \text{ if and only if } g \in S_{\text{even}}(IR^+). \]

**Definition 3.5.5.**

Let \( f \) be a complex valued function with domain \( D_f \subseteq IR \). Let \( c \in D_f \). The function \( f \) is called "right-differentiable in \( c \)" if \( \lim_{h \downarrow 0} \frac{f(c+h)-f(c)}{h} \) exists. This number is called the right-derivative of \( f \) in \( c \) and we denote this number by \( f^+_*(c) \).

**Definition 3.5.6.**

By \( C^\infty(IR^+) \) we denote the space of all functions \( f : [0, \infty) \rightarrow C \) for which

(i) \( f \) is infinitely differentiable on \((0, \infty)\), and

(ii) \( f \) is infinitely right-differentiable in \( 0 \).

**Lemma 3.5.7.**

Let \( f \in C^\infty(IR^+) \), and suppose \( \forall k,l \in \mathbb{N}_0 : \sup_{r>0} |r^k f^{(l)}(r)| < \infty \). Then

\[ f \in S_{\text{even}}(IR^+) \text{ if and only if } \forall_{l \in \mathbb{N}_0} : f^{(2l+1)}(0) = 0. \]

**Proof.**

Suppose \( f \in S_{\text{even}}(IR^+) \), i.e. \( IE f \in S_{\text{even}}(IR) \). Let \( l \in \mathbb{N}_0 \), then \( (IE f)^{(2l+1)} \in S_{\text{odd}}(IR) \). So \( (IE f)^{(2l+1)}(0) = 0 \) and therefore,

\[ f^{(2l+1)}(0) = \lim_{r \downarrow 0} f^{(2l+1)}(r) = \lim_{r \downarrow 0} (IE f)^{(2l+1)}(r) = 0. \]

For the converse, suppose \( f^{(2l+1)}(0) = 0 \) for each \( l \in \mathbb{N}_0 \). Clearly \( IE f \) is infinitely differentiable on \( IR \setminus \{0\} \), with

\[
(IE f)^{(l)}(r) = \begin{cases} 
  f^{(l)}(r) & \text{if } r > 0 \\
  (-1)^l f^{(l)}(-r) & \text{if } r < 0.
\end{cases} \quad (l \in \mathbb{N}_0).
\]

And we obtain for each \( l \in \mathbb{N}_0 \),

\[
\lim_{r \downarrow 0} (IE f)^{(l)}(r) = \lim_{r \uparrow 0} (IE f)^{(l)}(r) = \begin{cases} 
  0 & \text{if } l \text{ is odd} \\
  f^{(l)}(0) & \text{if } l \text{ is even}.
\end{cases}
\]

So \( IE f \) is infinitely differentiable in \( 0 \) with
\( (\mathcal{E}f)^{(0)}(0) = \begin{cases} 
0 & \text{if } l \text{ is odd} \\
 f^{(0)}(0) & \text{if } l \text{ is even.} 
\end{cases} \)

Furthermore, for each \( k, l \in \mathbb{N}_0 \)

\[
\sup_{x \in \mathbb{R}_0} \left| x^k (\mathcal{E}f)^{(l)}(x) \right| = \sup_{r > 0} r^k f^{(l)}(r) < \infty.
\]

Hence \( \mathcal{E}f \in S(\mathbb{R}) \).

Lemma 3.5.8.

Let \( f \in C^\infty(\mathbb{R}^+) \), and suppose \( \forall k \in \mathbb{N}_0 : \sup_{r > 0} r^k f^{(l)}(r) < \infty \).

Let \( n \in \mathbb{N} \). Then equivalent are

(i) \( f \in \mathcal{R}_e^{2n} \)

(ii) \( f^{(j)}(0) = 0 \), \( j = 0, \ldots, 2n - 1 \) and \( f^{(2n+1)}(0) = 0 \), \( l = n, n+1, \ldots. \)

Proof.

The proof is elementary and is based on the Taylor remainder formula for infinitely differentiable functions \( g \), which reads

\[
g(r) = \sum_{j=0}^{k} \frac{g^{(j)}(0)}{j!} r^j + \frac{1}{k!} \int_0^r (r-x)^k g^{(k+1)}(x) \, dx , \ r \geq 0 ,
\]

for each \( k \in \mathbb{N}_0 \).

Corollary 3.5.9.

Let \( f \in C^\infty(\mathbb{R}^+) \) and suppose \( \forall k \in \mathbb{N}_0 : \sup_{r > 0} r^k f^{(l)}(r) < \infty \).

Let \( n \in \mathbb{N}_0 \). Then equivalent are

(i) \( f \in \mathcal{R}_e^{2n+1} \)

(ii) \( f^{(j)}(0) = 0 \), \( j = 0, \ldots, 2n \) and \( f^{(2n+2)}(0) = 0 \), \( l = n+1, n+2, \ldots. \)

Proof.

Define \( g \in C^\infty(\mathbb{R}^+) \) by

\[
g(r) = r f(r) , \ r \geq 0 .
\]

Then \( f \in \mathcal{R}_e^{2n+1} \) if and only if \( g \in \mathcal{R}_e^{2n+2} \). Observing that

\[
g^{(j)}(0) = 0 \text{ and } g^{(j)}(0) = j f^{(j-1)}(0) , \ j = 1,2,\ldots ,
\]

the equivalence of assertions (i) and (ii) follows from Lemma 3.5.8. [1]
Summarizing we state

**Theorem 3.5.10.**

Let \( f \in C^{\infty}(\mathbb{R}^+) \) and suppose \( \forall_{k \in \mathbb{N}_0} : \sup_{r \geq 0} | r^k f^{(l)}(r) | < \infty \). Then equivalent are

1. \( \exists_{n, m \in \mathbb{N}_0} : f \in r^{2n} S_{\text{even}}(\mathbb{R}^+) \) and \( f \in r^{2m+1} S_{\text{even}}(\mathbb{R}^+) \)
2. \( \forall_{n \in \mathbb{N}_0} : f \in r^n S_{\text{even}}(\mathbb{R}^+) \)
3. \( \forall_{l, n \in \mathbb{N}_0} : f^{(l)}(0) \in r^n S_{\text{even}}(\mathbb{R}^+) \)
4. \( \forall_{l \in \mathbb{N}_0} : f^{(l)}(0) = 0. \)

**Proof.**

The proof is a consequence of Lemma 3.5.1, Lemma 3.5.8 and Corollary 3.5.9.

We arrive at the main result covering a characterization of \( g \) and \( h \) in order that \( \mathcal{P}^*(g(r)h(\phi)) \in S(\mathbb{R}^2) \).

**Theorem 3.5.11.**

Let \( f \in L_2(\mathbb{R}^2) \) for which there exist functions \( g \in X_1 \) and \( h \in L_2([\pi, \pi]) \) such that

\[
(\mathcal{P} f)(r, \phi) = g(r) h(\phi).
\]

(i) If \( h \) has a finite Fourier series,

\[
h(\phi) = \sum_{n=-N}^N \alpha_n e_n(\phi),
\]

then it is clear that \( h \) extends to a function in \( C_{2\pi-\text{per}}^\infty(\mathbb{R}) \).

First suppose there exist \( n_0, m_0 \in \mathbb{Z} \) such that

\[ \alpha_{2n} \neq 0 \quad \text{and} \quad \alpha_{2m} \neq 0. \]

Then equivalent are

1. \( f \in S(\mathbb{R}^2) \)
2. \( g \in r^{12n} S_{\text{even}}(\mathbb{R}^+) \) and \( g \in r^{12m+11} S_{\text{even}}(\mathbb{R}^+) \)
3. \( g \in r^l S_{\text{even}}(\mathbb{R}^+) \) for each \( l \in \mathbb{N}_0 \)
4. \( g \) extends to a function in \( S_{\text{even}}(\mathbb{R}) \). Furthermore,

\[ \forall_{l, m \in \mathbb{N}_0} : g^{(m)} \in r^l S_{\text{even}}(\mathbb{R}^+). \]
(v) \( g \in C^\infty(\mathbb{R}^+) \) with the properties

\[
\forall_{k \in \mathbb{N}_0} \sup_{r > 0} r^k |g^{(l)}(r)| < \infty \quad \text{and} \\
\forall_{l \in \mathbb{N}_0} : g^{(l)}(0) = 0.
\]

Next suppose for all \( n_0, m_0 \in \mathbb{Z} \),

\[
\alpha_{2n_0} = 0 \quad \text{or} \quad \alpha_{2m_0+1} = 0.
\]

Suppose \( h \neq 0 \) and let \( l_0 := \max \{ l \in \mathbb{N}_0 \mid \alpha_l \neq 0 \lor \alpha_{l+1} \neq 0 \} \).

If \( l_0 = 2k_0 \), then equivalent are

(i) \( f \in S(\mathbb{R}^2) \)

(ii) \( g \in r^{2k_0} S_{\text{even}}(\mathbb{R}^+) \)

(iii) \( g \in r^{2j} S_{\text{even}}(\mathbb{R}^+) \), \( j = 0, \ldots, k_0 \)

(iv) \( g \in C^\infty(\mathbb{R}^+) \) with the properties

\[
\forall_{k,l \in \mathbb{N}_0} \sup_{r > 0} r^k |g^{(l)}(r)| < \infty \quad \text{and} \\
g^{(j)}(0) = 0 , j = 0, \ldots, 2k_0 - 1 \\
g^{(2l+1)}(0) = 0 , l = k_0, k_0 + 1, \ldots.
\]

If \( l_0 = 2k_0 + 1 \), then equivalent are

(i) \( f \in S(\mathbb{R}^2) \)

(ii) \( g \in r^{2k_0+1} S_{\text{even}}(\mathbb{R}^+) \)

(iii) \( g \in r^{2j+1} S_{\text{even}}(\mathbb{R}^+) \), \( j = 0, \ldots, k_0 \)

(iv) \( g \in C^\infty(\mathbb{R}^+) \) with the properties

\[
\forall_{k,l \in \mathbb{N}_0} \sup_{r > 0} r^k |g^{(l)}(r)| < \infty \quad \text{and} \\
g^{(j)}(0) = 0 , j = 0, \ldots, 2k_0 \\
g^{(2l+2)}(0) = 0 , j = k_0 + 1, k_0 + 2, \ldots.
\]

(II) If \( h \) has an infinite Fourier series, then equivalent are

(i) \( f \in S(\mathbb{R}^2) \)

(ii) \( h \) extends to a function in \( C^\infty_{2\pi-\text{per}}(\mathbb{R}) \) and

\[
\forall_{l \in \mathbb{N}_0} : g \in r^l S_{\text{even}}(\mathbb{R}^+).
\]
(iii) $h$ extends to a function in $C^\infty_{2\pi\text{-per}}(\mathbb{R})$ and $g$ extends to a function in $S_{\text{even}}(\mathbb{R})$ with
\[
\forall l, m \in \mathbb{N}_0 : g^{(m)} \in r^l S_{\text{even}}(\mathbb{R}^+)
\]

(iv) $h$ extends to a function in $C^\infty_{2\pi\text{-per}}(\mathbb{R})$ and $g \in C^\infty(\mathbb{R}^+)$ with the properties
\[
\forall k, l \in \mathbb{N}_0 : \sup_{r > 0} | r^k g^{(l)}(r) | < \infty \quad \text{and} \quad \forall l \in \mathbb{N}_0 : g^{(l)}(0) = 0.
\]
Chapter II Gel’fand - Shilov spaces.

Before we start the discussion of the Gel’fand - Shilov spaces in two variables, we devote some attention to the Gel’fand - Shilov spaces in one variable. We present the definitions and some characterizations and properties. For the original introduction see [GS 2, pp. 166-237].

1. The Gel’fand - Shilov spaces $S_{\alpha}(\mathbb{R}), S^{\beta}(\mathbb{R})$ and $S_{\alpha}^{\beta}(\mathbb{R})$.

1.1. Introduction

For each $\alpha \geq 0, \beta \geq 0$ the spaces $S_{\alpha}(\mathbb{R}), S^{\beta}(\mathbb{R})$ and $S_{\alpha}^{\beta}(\mathbb{R})$ are the subspaces of the Schwartz space $S(\mathbb{R})$ in this order defined in the following way.

Definition 1.1.1. 

(i) The space $S_{\alpha}(\mathbb{R})$ consists of all functions $\phi \in S(\mathbb{R})$ which satisfy

$$\exists A > 0 \forall k \in \mathbb{N}_0 \exists B_k > 0 \forall x \in \mathbb{R} : \sup_{x \in \mathbb{R}} |x^k \phi^{(l)}(x)| \leq B_k A^k k^{\alpha}$$

where the constants $A$ and $B_k$ depend on $\phi$.

(ii) The space $S^{\beta}(\mathbb{R})$ consists of all functions $\phi \in S(\mathbb{R})$ which satisfy

$$\exists B > 0 \forall k \in \mathbb{N}_0 \exists A_k > 0 \forall x \in \mathbb{R} : \sup_{x \in \mathbb{R}} |x^k \phi^{(l)}(x)| \leq A_k B^l l^{\beta}$$

where the constants $B$ and $A_k$ depend on $\phi$.

(iii) The spaces $S_{\alpha}^{\beta}(\mathbb{R})$ consists of all functions $\phi \in S(\mathbb{R})$ which satisfy

$$\exists A, B, C > 0 \forall k \in \mathbb{N}_0 \forall x \in \mathbb{R} : \sup_{x \in \mathbb{R}} |x^k \phi^{(l)}(x)| \leq C A^k B^l k^{\alpha} l^{\beta}$$

where the constants $A, B$ and $C$ depend on $\phi$.

In the above definitions $k^k$ and $l^l$ may be replaced by $k!$ and $l!$ respectively, which follows from the inequalities

$$e^{-m} m^m \leq m! \leq m^m, \quad m \in \mathbb{N}_0 . \quad (1.1)$$

Partly as a consequence of Sobolev’s lemma the supremum norm in the above definitions can be replaced by the $L_2(\mathbb{R}^2)$-norm (cf. Chapter I, Section 2.1 for a proof). So the Gel’fand - Shilov spaces can also be defined as follows.
Theorem 1.1.2.
Let \( \alpha \geq 0 \), \( \beta \geq 0 \) and let \( \phi \in S(\mathbb{R}) \). Then

(i) \( \phi \in S_\alpha(\mathbb{R}) \iff \exists A > 0 \, \forall x \in N_\alpha \, \exists \epsilon > 0 \, \forall y \in N_\epsilon \, : \|Q^k P^l \phi\|_2 \leq A B^l (k!)^\alpha \)

(ii) \( \phi \in S^\beta(\mathbb{R}) \iff \exists B > 0 \, \forall x \in N_\beta \, \forall y \in N_\epsilon \, : \|Q^k P^l \phi\|_2 \leq B A^l (k!)^\beta \)

(iii) \( \phi \in S_\alpha^\beta(\mathbb{R}) \iff \exists \gamma > 0 \, \forall x \in N_\beta \, \forall y \in N_\epsilon \, : \|Q^k P^l \phi\|_2 \leq \gamma C A^k B^l (k!)^\alpha (l!)^\beta \).

Of course we are only interested in those \( S_\alpha^\beta \)-spaces which are nontrivial, i.e. which contain a function \( \phi \neq 0 \). From [GS 1] we obtain

Theorem 1.1.3.

(i) For each \( \alpha \geq 0 \) the space \( S_\alpha(\mathbb{R}) \) is nontrivial:
   Every infinitely differentiable function of compact support belongs to these spaces.

(ii) For each \( \beta \geq 0 \) the space \( S^\beta(\mathbb{R}) \) is nontrivial:
   The Fourier transform of an infinitely differentiable function of compact support belongs to
   these spaces.

(iii) The space \( S_\alpha^\beta(\mathbb{R}) \) is nontrivial iff \( \beta > 1 \),
   The space \( S_\alpha^\beta(\mathbb{R}) \) is nontrivial iff \( \alpha > 1 \),
   The space \( S_\alpha^\beta(\mathbb{R}) \) with \( \alpha > 0 \), \( \beta > 0 \) is nontrivial iff \( \alpha + \beta \geq 1 \).

Kashpirovskii proved in [Ka] an important intersection result

\[ S_\alpha(\mathbb{R}) \cap S^\beta(\mathbb{R}) = S_\alpha^\beta(\mathbb{R}) \]  \hspace{1cm} (1.2)

for each \( \alpha \), \( \beta > 0 \) with \( \alpha + \beta \geq 1 \).

Each of the spaces \( S_\alpha(\mathbb{R}) \), \( S^\beta(\mathbb{R}) \) and \( S_\alpha^\beta(\mathbb{R}) \) remains invariant under the following operations, the multiplication operation

\[ Q : \phi(x) \rightarrow x \phi(x), \]

the differentiation operation

\[ P : \phi(x) \rightarrow i \phi'(x), \]

the translation operation
$T_h : \phi(x) \to \phi(x-h) \ , \ h \in \mathbb{R}$,

the dilatation operation

$Z_\lambda : \phi(x) \to \sqrt{|\lambda|} \ \phi(\lambda x) \ , \ \lambda \in \mathbb{R}\setminus\{0\}$,

the parity operation

$\Pi : \phi(x) \to \phi(-x)$.

The operators $T_h$, $Z_\lambda$ and $\Pi$ are bijections.

The pointwise defined Fourier transformation satisfies

$\mathcal{F}(S_a(\mathbb{R})) = S^a(\mathbb{R})$, \quad $\mathcal{F}(S^b(\mathbb{R})) = S_b(\mathbb{R})$, \quad $\mathcal{F}(S^\alpha_\alpha(\mathbb{R})) = S^\alpha_\alpha(\mathbb{R})$.

For each $f \in C^\infty(\mathbb{R})$ the multiplier $M_f : S(\mathbb{R}^2) \to S(\mathbb{R}^2)$ is defined by

$$(M_f \phi)(x) = f(x) \phi(x) \ , \ \phi \in S(\mathbb{R}^2) , \ x \in \mathbb{R} ,$$

see Chapter I, Section 2.1.

Let $f \in C^\infty(\mathbb{R})$. Then $M_f$ maps $S_a(\mathbb{R})$ into itself if

$$\exists \varepsilon > 0 \ \forall \alpha \in \mathbb{N}_0 \ \exists C_{\alpha, \varepsilon} > 0 : \sup_{x \in \mathbb{R}} |f^{(\alpha)}(x) \exp \left(-\varepsilon |x|^{1/\alpha}\right)| \leq C_{\alpha, \varepsilon} \ (\alpha > 0) ,$$

$M_f$ maps $S^\beta(\mathbb{R})$ into itself if

$$\exists \varepsilon > 0 \ \forall \beta \in \mathbb{N}_0 : \sup_{x \in \mathbb{R}} |f^{(\beta)}(x) \cdot (1 + |x|^8)^{-1}| \leq C_B \ B_0 \ l^{1/\beta} ,$$

$M_f$ maps the space $S^\alpha_\alpha(\mathbb{R})$ into itself if

$$\exists \varepsilon > 0 \ \exists C_{\alpha, \beta} > 0 \ \forall \iota \in \mathbb{N}_0 : \sup_{x \in \mathbb{R}} |f^{(\iota)}(x) \exp \left(-\varepsilon |x|^{1/\alpha}\right)| \leq C_\varepsilon \ e^\varepsilon q^q_{q^\beta} \ (\alpha > 0) .$$

In particular, the function

$$f(x) = e^{ibx} \quad (b \in \mathbb{R})$$

is a multiplier in each of the spaces

$S_a(\mathbb{R}) \quad (\alpha > 0)$

$S^\beta(\mathbb{R}) \quad (\beta > 0)$

$S^\alpha_\beta(\mathbb{R}) \quad (\alpha, \beta > 0, \alpha + \beta \geq 1)$.
1.2. Functional analytic characterizations of the spaces $S_\alpha(\mathbb{R})$, $S^\beta(\mathbb{R})$ and $S^{\beta}_\alpha(\mathbb{R})$.

In Chapter I, Section 2.2, we introduced the notion of $C^\omega$-vector for self-adjoint operators in a Hilbert space. In addition here we introduce the notion of analytic vector for self-adjoint operators in a Hilbert space.

Let $A$ denote a self-adjoint operator in a Hilbert space $X$. A vector $v \in D^\omega(A)$ is said to be an analytic vector for $A$ if

$$\exists C, D > 0 \quad \forall n \in \mathbb{N} : \|A^n v\|_X \leq D \ C^n n!$$

(1.4)

where $C$ and $D$ depend on $v$.

The set of analytic vectors for $A$ is called the analyticity domain of $A$ and is denoted by $D^\omega(A)$.

We mention the following characterizations

Let $v \in X$. Then

$$v \in D^\omega(A) \quad \text{iff} \quad \exists \epsilon > 0 \quad \exists \omega \in X : v = e^{-i \lambda \omega}.$$ \quad (1.5)

Let $v \in D^\omega(A)$. Then for all $\alpha > 0$,

$$v \in D^\omega(|A|^{1/\alpha}) \quad \text{iff} \quad \exists C, D > 0 \quad \forall n \in \mathbb{N} : \|A^n v\|_X \leq D \ C^n (n!)^{1/\alpha}.$$ \quad (1.6)

For the proofs we refer to [EG, Sections I.1 and II.2 respectively].

As we have seen, $S(\mathbb{R}) = D^\omega(P) \cap D^\omega(Q)$. Van Eijndhoven proved corresponding statements for the spaces $S_\alpha(\mathbb{R})$, $S^\beta(\mathbb{R})$ and $S^{\beta}_\alpha(\mathbb{R})$:

Characterization 1.2.1.

$$\begin{align*}
S_\alpha(\mathbb{R}) &= D^\omega(P) \cap D^\omega(|Q|^{1/\alpha}) \quad (\alpha > 0) \\
S^\beta(\mathbb{R}) &= D^\omega(|P|^{1/\beta}) \cap D^\omega(Q) \quad (\beta > 0) \\
S^{\beta}_\alpha(\mathbb{R}) &= D^\omega(|P|^{1/\beta}) \cap D^\omega(|Q|^{1/\alpha}) \quad (\alpha, \beta > 0, \alpha + \beta \geq 1)
\end{align*}$$

The proofs of these functional analytic characterizations can be found in [E3].

Kashpirovskii's result, $S_\alpha(\mathbb{R}) \cap S^{\beta}(\mathbb{R}) = S^{\beta}_\alpha(\mathbb{R})$, is a consequence of the above characterizations. Furthermore, Van Eijndhoven deduced from his characterizations

Theorem 1.2.2.

(i) A function $\phi \in S(\mathbb{R})$ belongs to $S_\alpha(\mathbb{R})$, $\alpha > 0$, if and only if

(a) $\exists C, D > 0 : \sup_{x \in \mathbb{R}} |\phi(x)| \exp(C \ |x|^{1/\alpha}) \leq D$ and
(b) \( \forall k \in \mathbb{N} : \) the function \( x \mapsto x^k (IF \psi) (x) \) is bounded on \( \mathbb{R} \).

(ii) A function \( \psi \in S (\mathbb{R}) \) belongs to \( S^\beta (\mathbb{R}) \), \( \beta > 0 \), if and only if

(a) \( \forall k \in \mathbb{N} : \) the function \( x \mapsto x^k \psi(x) \) is bounded on \( \mathbb{R} \) and

(b) \( \exists C, D > 0 : \sup_{x \in \mathbb{R}} \| (IF \psi)(x) \exp(C \|x\|^{1/\beta}) \| \leq D \).

(iii) A function \( \chi \in S (\mathbb{R}) \) belongs to \( S^a_\alpha (\mathbb{R}) \), \( \alpha > 0 \), \( \beta > 0 \), \( \alpha + \beta \geq 1 \), if and only if

(a) \( \exists c_1, D_1 > 0 : \sup_{x \in \mathbb{R}} \| \chi(x) \exp(c_1 \|x\|^{1/\alpha}) \| \leq D_1 \) and

(b) \( \exists c_2, D_2 > 0 : \sup_{x \in \mathbb{R}} \| (IF \chi)(x) \exp(C_2 \|x\|^{1/\beta}) \| \leq D_2 \).

In the monograph [GS 2] Gel'fand and Shilov do not define topologies in the \( S^a_\alpha \)-spaces. However, they introduce a definition of sequential convergence in the \( S^a_\alpha \)-spaces. In the sequel, we endow that Gel'fand - Shilov space with a topology and we present several descriptions for sequential convergence with respect to the topology. Later on we shall see that our descriptions are equivalent with the definition suggested by Gel'fand and Shilov.

From Characterization 1.2.1 we know that each Gel'fand - Shilov space can be written as the intersection of two well-described spaces \( X_1 \) and \( X_2 \). We endow each of the spaces \( X_1 \) and \( X_2 \) with a locally convex topology and after that we endow the Gel'fand - Shilov space \( X_1 \cap X_2 \) with the intersection topology \( T \cap \). We use topological concepts which are described in the Appendices B and C.

Let \( A \) be a self-adjoint operator in a Hilbert space \( X \). For each \( \alpha, c > 0 \) we define the subspace of \( D^\infty (A) \),

\[
D^\alpha (1A \| 1/\alpha, c) := \{ v \in D^\infty (A) \mid \exists D > 0 \ \forall x \in \mathbb{R} : \| A^n v \|_X \leq D e^{c(n!)} \}.
\]

(1.7)

It is clear that

\[
D^\alpha (1A \| 1/\alpha, c) \subset D^\alpha (1A \| 1/\alpha, c') \quad \text{for} \quad 0 < c < c' \quad \text{and that}
\]

\[
D^\alpha (1A \| 1/\alpha, \alpha) = \bigcup_{c > 0} D^\alpha (1A \| 1/\alpha, c).
\]

(1.8)

We endow the space \( D^\alpha (1A \| 1/\alpha, c) \) \( (c > 0) \) with a topology induced by the norm

\[
q_{\alpha, c} (\phi) := \sup_{k \in \mathbb{N}_0} \frac{\| A^k \phi \|_X}{e^{k(k!)^\alpha}}, \quad \phi \in D^\alpha (1A \| 1/\alpha, c).
\]

(1.9)

Then the space \( (D^\alpha (1A \| 1/\alpha, c), q_{\alpha, c}) \) is a Banach space and we arrive at the inductive system

\[
(D^\alpha (1A \| 1/\alpha), \{(D^\alpha (1A \| 1/\alpha, c), q_{\alpha, c}) \mid c > 0\}).
\]

(1.10)

We endow the space \( D^\alpha (1A \| 1/\alpha) \) with the inductive limit topology \( \sigma_{\text{ind}} \) brought about by the above inductive system.
Lemma 1.2.3.
Let \( \alpha, c > 0 \) and suppose \( \phi \in D^{\alpha}(1A \mid 1/\alpha; c) \). Then there exists \( M > 0 \) such that
\[
q_{A, \alpha c} (\phi) \leq \| e^{tA1/\alpha} \phi \|_X \leq M \cdot q_{A, \alpha c} (\phi)
\]
where
\[
d = (2e)^\alpha c \quad \text{and} \quad s = \frac{\alpha}{2} e^c \cdot e^{-1/\alpha}.
\]
For a proof we refer to [EL, Chapter I].
With the aid of this lemma we are able to connect the space \((D^{\alpha}(1A \mid 1/\alpha), \sigma_{\text{ind}})\) with the theory of \(S_{X,A}\)-spaces, as described in [EG]. By definition we have
\[
S_{X, I A \mid 1/\alpha} := \bigcup_{s>0} e^{-s I A \mid 1/\alpha} (X)
\]
where the space \(e^{-s I A \mid 1/\alpha} (X)\) consists of all functions \(\phi \in X\) such that \(e^{-s I A \mid 1/\alpha} \phi \in X\). With the inner product
\[
(\phi, \psi)_X := \langle \phi, e^{-s I A \mid 1/\alpha} \psi \rangle_X, \quad \phi, \psi \in e^{-s I A \mid 1/\alpha} (X),
\]
the space \(e^{-s I A \mid 1/\alpha} (X)\) becomes a Hilbert space. The space \(S_{X, I A \mid 1/\alpha}\) equals \(D^{\alpha}(1A \mid 1/\alpha)\) as a vector space. In [EG] the space \(S_{X, I A \mid 1/\alpha}\) is topologized with the inductive limit topology \(\sigma_{\text{ind}}\), induced by the Hilbert spaces \(e^{-s I A \mid 1/\alpha}(X)\). It is the finest locally convex topology for which all the inclusion maps \(e^{-s I A \mid 1/\alpha} \hookrightarrow S_{X, I A \mid 1/\alpha}\) are continuous.

Lemma 1.2.4.
For each \(\alpha > 0\)
\[
(S_{X, I A \mid 1/\alpha}, \sigma_{\text{ind}}) = (D^{\alpha}(1A \mid 1/\alpha), \sigma_{\text{ind}}).
\]
Proof
The topological equality follows from Lemma 1.2.3. and [Co, Chapter IV, Proposition 5.8].

In [EG] the topological structure of the \(S_{X,A}\) spaces is investigated. In particular we state

Theorem 1.2.5.
Let \((\phi_n)_{n \in \mathbb{N}}\) be a sequence in \(D^{\alpha}(1A \mid 1/\alpha)\) \((\alpha > 0)\). The following statements are mutually equivalent.
(i) \(\phi_n \to 0\) \((n \to \infty)\) in \((D^{\alpha}(1A \mid 1/\alpha), \sigma_{\text{ind}})\).
(ii) \(\exists t > 0 : \| e^{tI A \mid 1/\alpha} \phi_n \|_X \to 0\) \((n \to \infty)\).
Proof

The equivalence of statements (i) and (ii) follows from Lemma 1.2.4. and [EG, Corollary 1.1.12]. The equivalence of statements (ii) and (iii) follows from Lemma 1.2.3.

Next we consider the space \( D^{\infty}(A) = \bigcap_{n \in \mathbb{N}_0} D(A^n) \). For each \( n \in \mathbb{N}_0 \) we endow the space \( D(A^n) \) with the topology defined by the norm

\[
\| A^k \phi \|_X \quad \| \phi \| \in D(A^n) .
\]

Then \( (D(A^n), \| p_n \|) \) is a Banach space, and we arrive at the projective system

\[
(D^{\infty}(A), \{ (D(A^n), \| p_n \|) \mid n \in \mathbb{N}_0 \}) .
\]

We endow the space \( D^{\infty}(A) \) with the projective limit topology \( \tau_{proj} \) brought about by this projective system. Note that \( \tau_{proj} \) equals the topology in \( D^{\infty}(A) \) defined by the norms \( p_n \) restricted to \( D^{\infty}(A) \) \( (n \in \mathbb{N}_0) \). So \( (D^{\infty}(A), \tau_{proj}) \) is a complete countably normed space.

The inclusion maps

\[
D^{\infty}(1A^{1/\alpha}; c) \subset D^{\infty}(A) , \quad c > 0 ,
\]

are continuous. So by [Co, Chapter IV, Proposition 5.7], the inclusion map

\[
D^{\infty}(1A^{1/\alpha}) \subset D^{\infty}(A)
\]

is continuous.

As promised we now define intersection topologies in the Gel’fand - Shilov spaces.

An intersection topology for \( S_{\alpha}(\mathbb{R}) \)

Let \( \alpha > 0 \). Then we have by Characterization 1.2.1,

\[
S_{\alpha}(\mathbb{R}) = D^{\infty}(1Q^{1/\alpha}) \cap D^{\infty}(P) .
\]

We endow the space \( S_{\alpha}(\mathbb{R}) \) with the intersection topology \( T_{\alpha, \cap} \) which is the weakest topology in \( S_{\alpha}(\mathbb{R}) \) for which the inclusion maps

\[
S^{\infty}(\mathbb{R}) \subset D^{\infty}(1Q^{1/\alpha}) \quad \text{and} \quad S_{\alpha}(\mathbb{R}) \subset D^{\infty}(P)
\]

are continuous.

Let \( A > 0 \). We define the space \( S_{\alpha,A}(\mathbb{R}) \) by

\[
S_{\alpha,A}(\mathbb{R}) := \{ \phi \in S_{\alpha}(\mathbb{R}) \mid \forall k \in \mathbb{N}_0 \exists B_{k, > 0} \forall \lambda \in \mathbb{N}_0 : \| Q^k P^l \phi \|_{L^1(\mathbb{R})} \leq B_l A^k(1)^{\alpha} \} .
\]

In the space \( S_{\alpha,A}(\mathbb{R}) \) we define a system of norms, \( \| \cdot \|_{\alpha,A,l} , l \in \mathbb{N}_0 \), by
This family of norms defines a locally convex topology in $S_{\alpha A}(\mathbb{R})$ which we denote by $\text{Top}_{\alpha A}$.

In this way $(S_{\alpha A}(\mathbb{R}), \text{Top}_{\alpha A})$ becomes a complete countably normed space.

Note that the inclusion maps

$$S_{\alpha A}(\mathbb{R}) \subset D^\alpha(Q^{1/\alpha}; A)$$

$$S_{\alpha A}(\mathbb{R}) \subset D^\alpha(P)$$

and

$$S_{\alpha A}(\mathbb{R}) \subset S_{\alpha A_j}(\mathbb{R}) \quad (A_1 < A_2)$$

are continuous. Furthermore, since each function in $S_a(\mathbb{R})$ belongs to some $S_{\alpha A}(\mathbb{R})$ we obtain

$$S_a(\mathbb{R}) = \bigcup_{A \geq 1} S_{\alpha A}(\mathbb{R}).$$

Lemma 1.2.6.

Let $\alpha > 0$ and $A \geq 1$. Let $\phi \in S_{\alpha A}(\mathbb{R})$. Then we have

(i) $\sup_{k \in N_0} \frac{\|Q^k \phi\|_{L^\alpha_{\mathbb{R}}}}{A^k (k!)^{\alpha}} = \|\phi\|_{\alpha 0}$ and

$$\|P^l \phi\|_{L^\alpha_{\mathbb{R}}} \leq \|\phi\|_{\alpha A, l} \quad \text{for all } l \in N_0.$$

(ii) $\|\phi\|_{\alpha A, l} \leq \sqrt{2l-1} \left[ \sup_{k \in N_0} \frac{\|Q^k \phi\|_{L^\alpha_{\mathbb{R}}}}{A^k (k!)^{\alpha}} + \sum_{j=1}^{2l} \|P^l \phi\|_{L^\alpha_{\mathbb{R}}} \right].$

for all $l \in N_0$. Here $A = A \cdot 2^{\alpha+1}$.

Proof

Assertion (i) is a direct consequence of the definitions. We prove assertion (ii). For each $k \in N_0$ and $0 \leq j \leq 2k$ we have

$$\binom{2k}{j} \leq \frac{2^{2k}}{(A \cdot 2^{\alpha+1})^{2k}} \left( \sum_{j=0}^{2k} \frac{(2k)!}{j!} \right)^{\alpha} = \frac{1}{A^{2k} (2k)!^{\alpha}} \leq \frac{1}{A^{2k-j} (2k-j)!^{\alpha}}.$$

Applying Leibnitz's differentiation rule, we estimate for all $k, l \in N_0$.
Whence we obtain the result. \[\square\]

The following theorem contains several descriptions of sequential convergence in \(S_\alpha(\mathbb{R})\).

**Theorem 1.2.7.**

Let \((\phi_n)_{n \in \mathbb{N}}\) be a sequence in \(S_\alpha(\mathbb{R})\) \((\alpha > 0)\). Then equivalent are

(i) \(\phi_n \to 0 (n \to \infty)\) in \((S_\alpha(\mathbb{R}), T_{\alpha, \cap})\).

(ii) \(\phi_n \to 0 (n \to \infty)\) both in \((D_\alpha^a(\mathbb{R}), \sigma_{\text{ind}})\) and in \((D_\infty^a(P), \tau_{\text{proj}})\)

(iii) \(\exists_{l > 0} \forall_{n \in \mathbb{N}} : \phi_n \in e^{-lQ^{1\alpha}}(L_2^a(\mathbb{R})) \cap D_\infty^a(P)\)

and \(\phi_n \to 0 (n \to \infty)\) in the corresponding intersection topology, i.e.

\[
\|e^{-lQ^{1\alpha}} \phi_n\|_{L_2^a(\mathbb{R})} \to 0 (n \to \infty) \quad \text{and} \\
\|p^l \phi_n\|_{L_2^a(\mathbb{R})} \to 0 (n \to \infty) \\
\text{for all} \ l \in \mathbb{N}_0.
\]

(iv) \(\exists_{A > 0} \forall_{n \in \mathbb{N}} : \phi_n \in D_\alpha^a(\mathbb{R}, A) \cap D_\infty^a(P)\)

and \(\phi_n \to 0 (n \to \infty)\) in the corresponding intersection topology, i.e.

\[
\sup_{k \in \mathbb{N}_0} \|Q^k \phi_n\|_{L_2^a(\mathbb{R})} \to 0 (n \to \infty) \quad \text{and} \\
\|p^l \phi_n\|_{L_2^a(\mathbb{R})} \to 0 (n \to \infty) \\
\text{for all} \ l \in \mathbb{N}_0.
\]

(v) \(\exists_{A > 0} \forall_{n \in \mathbb{N}} : \phi_n \in S_{\alpha, A}(\mathbb{R})\)

and \(\phi_n \to 0 (n \to \infty)\) in the topology \(T_{\alpha, A}\) of \(S_{\alpha, A}(\mathbb{R})\), i.e.
Proof
By Lemma B.1 (see Appendix B), assertions (i) and (ii) are equivalent, the equivalence of assertions (ii), (iii) and (iv) is a consequence of Theorem 1.2.5. The equivalence of assertions (iv) and (v) follows from Lemma 1.2.6.

Remark 1.2.8.
Applying standard techniques it can be seen that the $L_2(\mathbb{R})$-norms in the above theorem may be replaced by supremum norms.

Now we are able to show that convergence of a sequence in $(S_\alpha(\mathbb{R}), T_{\alpha,r})$ is equivalent with the definition of convergence given by Gel'fand and Shilov.

Let $\alpha, A > 0$. Define $S_{\alpha,A;\infty}(\mathbb{R})$ by

$$S_{\alpha,A;\infty}(\mathbb{R}) = \{ \phi \in S_\alpha(\mathbb{R}) \mid \forall \alpha \in \mathbb{N}_0, \exists B > 0 \forall k \in \mathbb{N}_0 : \sup_{x \in \mathbb{R}} x^k \psi^{(l)}(x) \leq B \cdot A^k (k!)^\alpha \}.$$  

By $T_{\alpha,A;\infty}$ we denote the locally convex topology in $S_{\alpha,A;\infty}(\mathbb{R})$ defined by the family of norms

$$\| \phi \|_{\alpha,A;\infty} := \sup_{k \in \mathbb{N}_0, x \in \mathbb{R}} x^k \psi^{(l)}(x) \leq A^k (k!)^\alpha, \quad l \in \mathbb{N}_0, \phi \in S_{\alpha,A;\infty}(\mathbb{R}).$$

Gel'fand and Shilov presented the following definition of sequential convergence.

Definition 1.2.9. (Gel'fand, Shilov)

A sequence $(\phi_n)_{n \in \mathbb{N}} \subset S_\alpha(\mathbb{R})$ converges to zero if all the functions $\phi_n, n \in \mathbb{N}$, belong to some space $S_{\alpha,A;\infty}(\mathbb{R})$, where they converge to zero in the topology of $S_{\alpha,A;\infty}(\mathbb{R})$.

Or equivalently, if

for any $l$, the functions $\phi_n^{(l)}$ converge uniformly to zero in any segment $1 \leq x_0 < \infty$ and for some $A$ and $B_l$, independent on $n$, the inequalities

$$x^k \psi^{(l)}(x) \leq B_l A^k (k!)^\alpha$$

are satisfied.

Utilizing Remark 1.2.8 we obtain that our description of convergent sequences in $S_\alpha(\mathbb{R})$ agrees with the definition suggested by Gel'fand and Shilov.

An intersection topology for $S_\beta(\mathbb{R})$.

Let $\beta > 0$. Then we have by Characterization 1.2.1,
We endow the space $S^\beta(\mathbb{R})$ with the intersection topology $T^\wedge$, which is the weakest topology in $S^\beta(\mathbb{R})$ for which the inclusion maps
\[ S^\beta(\mathbb{R}) \hookrightarrow D^{\infty}(Q) \quad \text{and} \quad S^\beta(\mathbb{R}) \hookrightarrow D^{\alpha}(1_{P}\,1^{1/\beta}) \] (1.24)
are continuous.

Let $B > 0$. We define the space $S^{B,B}(\mathbb{R})$ by
\[ S^{B,B}(\mathbb{R}) := \{ \phi \in S^{B}(\mathbb{R}) \mid \exists_{k \in \mathbb{N}_0} \exists_{e \in \mathbb{N}_0} \forall_{e \in \mathbb{N}_0} : \| Q^k \, P^j \phi \|_{L^2(\mathbb{R})} \leq A_k \, B^{j} \} \] (1.25)
In the space $S^{B,B}(\mathbb{R})$ we define a system of norms, $\| \cdot \|_{B,k}^{B,B}$, $k \in \mathbb{N}_0$ by
\[ \| \phi \|_{B,k}^{B,B} := \sup_{t \in \mathbb{N}_0} \frac{\| Q^k \, P^j \phi \|_{L^2(\mathbb{R})}}{B^{j} \, (1)^B}, \quad k \in \mathbb{N}_0, \phi \in S^{B,B}(\mathbb{R}). \] (1.26)
This family of norms defines a locally convex topology in $S^{B,B}(\mathbb{R})$ which we denote by $\text{Top}^{B,B}$. In this way $(S^{B,B}(\mathbb{R}), \text{Top}^{B,B})$ becomes a complete countably normed space.

Note that the inclusion maps
\[ S^{B,B}(\mathbb{R}) \hookrightarrow D^{\infty}(Q), \]
\[ S^{B,B}(\mathbb{R}) \hookrightarrow D^{\alpha}(1_{P}\,1^{1/\beta};B) \quad \text{and} \]
\[ S^{B,B_1}(\mathbb{R}) \hookrightarrow S^{B,B_2}(\mathbb{R}) \quad (B_1 < B_2) \] (1.27)
are continuous. Furthermore, since each function in $S^{B}(\mathbb{R})$ belongs to some $S^{B,B}(\mathbb{R})$ we obtain
\[ S^{B}(\mathbb{R}) = \bigcup_{B \geq 1} S^{B,B}(\mathbb{R}). \] (1.28)

Lemma 1.2.10.

Let $\beta > 0$ and $B \geq 1$. Let $\phi \in S^{B,B}(\mathbb{R})$. Then we have

(i) $\sup_{t \in \mathbb{N}_0} \frac{\| P^j \phi \|_{L^2(\mathbb{R})}}{B^{j} \, (1)^B} = \| \phi \|_{B,0}^{B,B}$ and
\[ \| Q^k \phi \|_{L^2(\mathbb{R})} \leq \| \phi \|_{B,k}^{B,B} \]
for all $k \in \mathbb{N}_0$. (ii) $\| \phi \|_{B,B,k}^{B,B} \leq 2^{2k-1}(2k)! \left( \sup_{t \in \mathbb{N}_0} \frac{\| P^j \phi \|_{L^2(\mathbb{R})}}{B^{j} \, (1)^B} + \sum_{j=0}^{2k} \| Q^j \phi \|_{L^2(\mathbb{R})} \right)$. 

for all $k \in \mathbb{N}_0$. Here $\tilde{B} = B \cdot 2^{\delta+1/2}$.

Proof
The proof runs similarly to the proof of Lemma 1.2.8. and is omitted therefore.

The following theorem contains several descriptions of sequential convergence in $S^B(\mathbb{R})$.

Theorem 1.2.11.
Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence in $S^B(\mathbb{R})$ ($\beta > 0$). Then equivalent are
(i) $\phi_n \to 0$ ($n \to \infty$) in $(S^B(\mathbb{R}), T^B_\beta)$.
(ii) $\phi_n \to 0$ ($n \to \infty$) both in $(D^\omega(1P\|1^\beta), \sigma_{\text{ind}})$ and in $(D^\omega(Q), \tau_{\text{proj}})$.
(iii) $\exists \delta > 0 \forall n \in \mathbb{N} : \phi_n \in e^{-\delta}1P\|1^\beta \cap D^\omega(Q)$
    and $\phi_n \to 0$ ($n \to \infty$) in the corresponding intersection topology, i.e.
    $\|e^{i1P\|1^\beta} \phi_n\|_{L^1(\mathbb{R})} \to 0$ ($n \to \infty$) and
    $\|Q^k \phi_n\|_{L^1(\mathbb{R})} \to 0$ ($n \to \infty$) for all $k \in \mathbb{N}_0$.
(iv) $\exists \delta > 0 \forall n \in \mathbb{N} : \phi_n \in D^\omega(1P\|1^\beta; B) \cap D^\omega(Q)$
    and $\phi_n \to 0$ ($n \to \infty$) in the corresponding intersection topology, i.e.
    $\sup_{l \in \mathbb{N}_0} \frac{\|P^l \phi_n\|_{L^1(\mathbb{R})}}{B^l (1)^\beta} \to 0$ ($n \to \infty$) and
    $\|Q^k \phi_n\|_{L^1(\mathbb{R})} \to 0$ ($n \to \infty$) for all $k \in \mathbb{N}_0$.
(v) $\exists \delta > 0 \forall n \in \mathbb{N} : \phi_n \in S^\beta,B(\mathbb{R})$
    and $\phi_n \to 0$ ($n \to \infty$) in the topology $\text{Top}^\beta,B$ of $S^\beta,B(\mathbb{R})$, i.e.
    $\sup_{l \in \mathbb{N}_0} \frac{\|Q^k P^l \phi_n\|_{L^1(\mathbb{R})}}{B^l (1)^\beta} \to 0$ ($n \to \infty$) for all $k \in \mathbb{N}_0$.

Proof
By Lemma B.1 (see Appendix B), assertions (i) and (ii) are equivalent. The equivalence of assertions (ii), (iii) and (iv) is a consequence of Theorem 1.2.5. The equivalence of assertions (iv) and (v) follows from Lemma 1.2.10.

Remark 1.2.12.
Applying standard techniques it can be seen that the $L^2(\mathbb{R})$-norms in the above theorem may be replaced by supremum norms.
We show that sequential convergence in \((S^\beta(IR), T^\beta)\) is equivalent with the definition of convergence given by Gel'fand and Shilov.

Let \(\beta, B > 0\). Define \(S^{\beta, B, \infty}(IR)\) by

\[
S^{\beta, B, \infty}(IR) = \{ \phi \in S^\beta(IR) \mid \forall k \in \mathbb{N}, \exists A_k > 0 \mid \forall x \in IR \mid x^k \phi^{(l)}(x) \mid \leq A_k B^l (1!)^\beta \}. \tag{1.29}
\]

By \(Top^{\beta, B, \infty}\) we denote the locally convex topology in \(S^{\beta, B, \infty}(IR)\) defined by the family of norms

\[
\| \phi \|_{\beta, B, k} := \sup\limits_{l \in \mathbb{N}, x \in IR} \frac{l x^k \phi^{(l)}(x)}{B^l (1!)^\beta}, \quad k \in \mathbb{N}, \phi \in S^{\beta, B, \infty}(IR). \tag{1.30}
\]

Gel'fand and Shilov presented the following definition of sequential convergence.

**Definition 1.2.13.** (Gel'fand, Shilov)

A sequence \((\phi_n)_{n \in \mathbb{N}} \subset S^\beta(IR)\) converges to zero if all functions \(\phi_n, n \in \mathbb{N}\), belong to some space \(S^{\beta, B, \infty}(IR)\), where they converge to zero in the topology of \(S^{\beta, B, \infty}(IR)\).

Or equivalently, if

for any \(l\), the functions \(\phi_n^{(l)}\) converge uniformly to zero in any segment \(1 x \mid \leq x_0 < \infty\) and for some \(B\) and \(A_k\), independent on \(n\), the inequalities

\[
| x^k \phi^{(l)}(x) | \leq A_k B^l (1!)^\beta
\]

are satisfied. \[
\]

Utilizing Remark 1.2.12, we obtain that sequential convergence in \((S^\beta(IR), T^\beta)\) agrees with the definition suggested by Gel'fand and Shilov.

**Remark 1.2.14.**

The Fourier transformation \( IF \) is a homeomorphism of \(D^{\infty}(Q)\) onto \(D^{\infty}(P)\). Also \( IF \) is a homeomorphism of \(D^{\infty}(Q, 11/alpha)\) onto \(D^{\infty}(P, 11/alpha)\). Hence \( IF \) is a homeomorphism of \( S_\alpha(IR) \) onto \( S_\alpha(IR) \). That means that the spaces \( S_\alpha(IR) \) and \( S^{\alpha}(IR) \) differ only in the nature of their elements, but topologically they can be considered essentially identical. \[
\]

**An intersection topology for \(S^\alpha(IR)\).**

Let \(\alpha, \beta > 0\), with \(\alpha + \beta \geq 1\). Then we have by Characterization 1.2.1,

\[
S^\alpha(IR) = D^{\alpha}(1Q 11/\alpha) \cap D^{\infty}(1P 11/\beta).
\]

We endow the space \(S^\alpha(IR)\) with the intersection topology \(T^\alpha\cap\) which is the weakest topology in \(S^\alpha(IR)\) for which the inclusion maps

\[
S^\alpha(IR) \hookrightarrow D^{\alpha}(1Q 11/\alpha) \text{ and } S^\alpha(IR) \hookrightarrow D^{\infty}(1P 11/\beta)
\]

(1.31)

are continuous.

Let \(\alpha, \beta > 0\). We define the space \(S^{\beta, \alpha}(IR)\) by
In the space \( S_{\alpha, A}(\mathbb{R}) \) we define the norm \( \| \cdot \|_{\alpha, A}^{B, \beta} \) by
\[
\| \phi \|_{\alpha, A}^{B, \beta} := \sup_{k, l \in \mathbb{N}_0} \frac{\| \phi^{(k)} \|_{L^2(\mathbb{R})}}{A^k B^l (k!)^\beta} \quad \phi \in S_{\alpha, A}(\mathbb{R}).
\]

Then \( \left( S_{\alpha, A}(\mathbb{R}), \| \cdot \|_{\alpha, A}^{B, \beta} \right) \) is a Banach space.

Note that the inclusion maps
\[
S_{\alpha, A}(\mathbb{R}) \supset \supset D^{\alpha}(1Q^{1/\alpha}; A),
S_{\alpha, A}(\mathbb{R}) \supset \supset D^{\alpha}(1P^{1/\beta}; B),
S_{\alpha, A}(\mathbb{R}) \supset \supset S_{\alpha, A}(\mathbb{R})
\]
and
\[
S_{\alpha, A^1}(\mathbb{R}) \supset \supset S_{\alpha, A^1}(\mathbb{R}) \quad (A_1 < A_2, B_1 < B_2)
\]
are continuous. Furthermore, since each function in \( S_{\alpha, A}(\mathbb{R}) \) belongs to some \( S_{\alpha, A}^{B, \beta}(\mathbb{R}) \) we obtain
\[
S_{\alpha, A}(\mathbb{R}) = \bigcup_{A \geq 1, B \geq 1} S_{\alpha, A}^{B, \beta}(\mathbb{R}).
\]

Lemma 1.2.15.
Let \( \alpha, \beta > 0 \) with \( \alpha + \beta \geq 1 \) and let \( A, B \geq 1 \). Let \( \phi \in S_{\alpha, A}(\mathbb{R}) \).

Then we have
\[
(i) \quad \sup_{k \in \mathbb{N}_0} \frac{\| Q^k \phi \|_{L^2(\mathbb{R})}}{A^k (k!)^\alpha} \leq \| \phi \|_{\alpha, A}^{B, \beta} \quad \text{and}
\sup_{l \in \mathbb{N}_0} \frac{\| P^l \phi \|_{L^2(\mathbb{R})}}{B^l (l!)^\beta} \leq \| \phi \|_{\alpha, A}^{B, \beta},
\]
\[
(ii) \quad \| \phi \|_{\alpha, A}^{B, \beta} \leq \frac{1}{2 \sqrt{2}} \left( \sup_{k \in \mathbb{N}_0} \frac{\| Q^k \phi \|_{L^2(\mathbb{R})}}{A^k (k!)^\alpha} + \sup_{l \in \mathbb{N}_0} \frac{\| P^l \phi \|_{L^2(\mathbb{R})}}{B^l (l!)^\beta} \right),
\]
here \( \tilde{A} = A \cdot 2^{\alpha+1} \) and \( \tilde{B} = B \cdot 2^{\beta+1} \).

Proof.
Obviously assertion (i) is true. We prove assertion (ii). For each \( k, l \in \mathbb{N}_0 \) we have
\[
\min_{j=0}^{2k,l} \left( \begin{array}{c} 2k \vspace{0.5em} \hline j \end{array} \right) \left( \begin{array}{c} l \\
 j \end{array} \right) j! (2k-j)!^\alpha (2l-j)!^\beta A^{2k-j} B^{2l-j} \leq
\]
\[
\leq \left( \begin{array}{c} 2k \\
 j \end{array} \right) \left( \begin{array}{c} l \\
 j \end{array} \right) j! (2k-j)!^\alpha (2l-j)!^\beta A^{2k-j} B^{2l-j} \leq
\]
\[
\leq \left( \begin{array}{c} 2k \\
 j \end{array} \right) \left( \begin{array}{c} l \\
 j \end{array} \right) j! (2k-j)!^\alpha (2l-j)!^\beta A^{2k-j} B^{2l-j} \leq
\]
\[
\leq A^{2k} B^{2l} (2k)!^a (2l)!^b \sum_{j=0}^{\min(2k,l)} \binom{2k}{j} \binom{l}{j} j!^a \binom{(2l-j)!}{(2k)!}^b \leq
\]

\[
\leq (A \cdot 2^a)^{2k} (B \cdot 2^b)^{2l} (k!)^{2a} (l!)^{2b} \sum_{j=0}^{\min(2k,l)} \binom{2k}{j} \binom{l}{j} j!^a \binom{(2l-j)!}{(2k)!} \leq
\]

\[
\leq (A \cdot 2^a)^{2k} (B \cdot 2^b) (k!)^{2a} (l!)^{2b} \sum_{j=0}^{\min(2k,l)} \binom{2k}{j} \binom{l}{j} j!^a \binom{(2l-j)!}{(2k)!} \leq
\]

\[
\leq (A \cdot 2^{a+1})^{2k} (B \cdot 2^{b+1})^{2l} (k!)^{2a} (l!)^{2b}
\]

Applying Leibnitz's differentiation rule, we estimate for all \( k, l \in \mathbb{N}_0 \)

\[
\|Q^k P^l \phi\|^{2k}_{L^2(\mathbb{R})} \leq \sum_{j=0}^{\min(2k,l)} \binom{2k}{j} \binom{l}{j} j!^a \|Q^{2k-j} \phi\|_{L^2(\mathbb{R})} \|P^{2l-j} \phi\|_{L^2(\mathbb{R})} \leq
\]

\[
\leq \sup_{k \in \mathbb{N}_0} \frac{\|Q^k \phi\|_{L^2(\mathbb{R})}}{A^k (k!)^a} \cdot \sup_{l \in \mathbb{N}_0} \frac{\|P^l \phi\|_{L^2(\mathbb{R})}}{B^l (l!)^b} \cdot \min_{j=0}^{\min(2k,l)} \binom{2k}{j} \binom{l}{j} j! (2k-j)!^a (2l-j)!^b A^{2k-j} B^{2l-j} \leq
\]

\[
\leq (A \cdot 2^{a+1})^{2k} (B \cdot 2^{b+1})^{2l} (k!)^{2a} (l!)^{2b} \cdot \frac{1}{2} \left[ \sup_{k \in \mathbb{N}_0} \frac{\|Q^k \phi\|_{L^2(\mathbb{R})}}{A^k (k!)^a} + \sup_{l \in \mathbb{N}_0} \frac{\|P^l \phi\|_{L^2(\mathbb{R})}}{B^l (l!)^b} \right]^2
\]

Whence we obtain the result. \( \square \)

The following theorem contains several descriptions of sequential convergence in \( S^p_b (\mathbb{R}) \).

**Theorem 1.2.16.**

Let \( (\phi_n)_{n \in \mathbb{N}} \) be a sequence in \( S^p_b (\mathbb{R}) \) \( (\alpha, \beta > 0, \alpha + \beta \geq 1) \). Then equivalent are

(i) \( \phi_n \to 0 \ (n \to \infty) \) in \( (S^p_b (\mathbb{R}), T^p_{\alpha, \beta}) \).

(ii) \( \phi_n \to 0 \ (n \to \infty) \) both in \( (D^\alpha (L^1(\mathbb{R})), \sigma_{ind}) \) and in \( (D^\beta (L^1(\mathbb{R})), \sigma_{ind}) \).

(iii) \( \phi_n \to 0 \ (n \to \infty) \) both in \( (S_a (\mathbb{R}), T^p_{\alpha, \beta}) \) and in \( (S^p_b (\mathbb{R}), T^p_{\alpha, \beta}) \).

(iv) \( \exists \gamma > 0 \forall n \in \mathbb{N} : \phi_n \in e^{-\gamma Q^1 L^2(\mathbb{R})} \cap e^{-\gamma P^1 L^2(\mathbb{R})} \) and \( \phi_n \to 0 \ (n \to \infty) \) in the corresponding intersection topology, i.e.

\[
\| e^{\gamma Q^1 L^2} \phi_n \|_{L^2(\mathbb{R})} \to 0 \ (n \to \infty)
\]

and

\[
\| e^{\gamma P^1 L^2} \phi_n \|_{L^2(\mathbb{R})} \to 0 \ (n \to \infty).
\]

(v) \( \exists A, B > 0 \forall n \in \mathbb{N} : \phi_n \in D^\alpha (L^1(\mathbb{R}); A) \cap D^\beta (L^1(\mathbb{R}); B) \) and \( \phi_n \to 0 \ (n \to \infty) \) in the corresponding intersection topology, i.e.
\[
\sup_{k \in N \setminus I} \frac{\|Q^k \phi_n\|_{L^2(\mathbb{R})}}{A^k (k!)^\alpha} \to 0 \ (n \to \infty) \quad \text{and} \quad \\
\sup_{l \in N \setminus I} \frac{\|P^l \phi_n\|_{L^2(\mathbb{R})}}{B^l (l!)^\beta} \to 0 \ (n \to \infty).
\]

(vi) \( \exists_{A,B > 0} \forall_{n \in N} : \phi_n \in S^A_B(\mathbb{R}) \)
and \( \phi_n \to 0 \ (n \to \infty) \) in the topology of \( S^A_B(\mathbb{R}) \), i.e.
\[
\sup_{k,l \in N \setminus I} \frac{\|Q^k P^l \phi_n\|_{L^2(\mathbb{R})}}{A^k B^l (k!)^\alpha (l!)^\beta} \to 0 \ (n \to \infty).
\]

Proof.

By Lemma B.1 (see Appendix B), assertions (i) and (ii) are equivalent. Since the inclusion maps \( D^\infty(Q) \subset D^\infty(Q) \) and \( D^\infty(P) \subset D^\infty(P) \) are continuous, the equivalence of assertions (ii) and (iii) is settled. The equivalence of assertions (ii), (iv) and (v) is a consequence of Theorem 1.2.5. Finally, assertions (v) and (vi) are equivalent by Lemma 1.2.15.

Remark 1.2.17.

Applying standard techniques it can be seen that the \( L^2(\mathbb{R}) \)-norms in the above theorem may be replaced by supremum norms. For instance, assertion (iv) contains a very useful characterization of sequential convergence. It says that a sequence \( (\phi_n)_{n \in N} \subset S^A_B(\mathbb{R}) \) converges to zero if there exists \( t > 0 \) such that
\[
\sup_{x \in \mathbb{R}} |e^{i A x^{1 + \alpha}} \phi_n(x)| \to 0 \ (n \to \infty) \quad \text{and} \quad \\
\sup_{x \in \mathbb{R}} |e^{i A x^{1 + \beta}} (iF \phi_n)(x)| \to 0 \ (n \to \infty).
\]

Finally we show that sequential convergence in \( (S^A_B, T_{\alpha,\beta}) \) is equivalent with the definition of convergence given by Gel'fand and Shilov. Let \( \alpha, \beta > 0 \) with \( \alpha + \beta \geq 1 \) and let \( A,B > 0 \). Define \( S^A_B, T_{\alpha,\beta}(\mathbb{R}) \) by
\[
S^A_B, T_{\alpha,\beta}(\mathbb{R}) = \{ \phi \in S^A_B(\mathbb{R}) \mid \exists_{C > 0} \forall_{k,l \in N} : \sup_{x \in \mathbb{R}} |x^k \phi^{(l)}(x)| \leq C A^k B^l (k!)^\alpha (l!)^\beta \}.
\]

Gel'fand and Shilov presented the following definition of sequential convergence in \( S^A_B(\mathbb{R}) \).

Definition (Gel'fand, Shilov) 1.2.18.
A sequence \((\phi_n)_{n \in \mathbb{N}} \subseteq S_0^b(\mathbb{R})\) converges to zero if all the functions \(\phi_n\), \(n \in \mathbb{N}\), belong to some space \(S_0^{b,\infty}(\mathbb{R})\), where they converge to zero in the topology of \(S_0^{b,\infty}(\mathbb{R})\).

Or equivalently, if

for any \(I\), the functions \(\phi^{(l)}_n\) converge uniformly to zero in any segment \(lx| \leq x_0 < \infty\) and for some \(A, B, C\), independent on, the inequalities

\[
|x^k \phi^{(l)}_n(x)| \leq C A^k B^l (k!)^\alpha (l!)^\beta
\]

are satisfied.

Utilizing Remark 1.2.17 we obtain that sequential convergence in \((S_0^b(\mathbb{R}), \mathbf{T}_0^b)\) agrees with the definition suggested by Gel'fand and Shilov.

**Remark 1.2.19.**

The Fourier transformation \(\mathbf{F}\) is a homeomorphism of \(S_0^b(\mathbb{R})\) onto \(S_0^b(\mathbb{R})\).

We know that \(S(\mathbb{R}) = D^\infty(P^2 + Q^2)\). Furthermore, we have a characterization of \(S(\mathbb{R})\) in terms of the Hermite functions \(\psi_n\) (cf. Chapter I, Characterization 2.3.2). Zhang Gong-Zhing proved similar assertions for the spaces \(S_0^a(\mathbb{R})\), \(a \geq \frac{1}{2}\):

\[
S_0^a(\mathbb{R}) = D^\infty \{ (p^2 + Q^2)^{\frac{1}{2a}} \} , \quad a \geq \frac{1}{2}. \quad (1.37)
\]

The proof can be found in [Zh].

With the aid of relation (1.5) we can reformulate the above characterization of \(S_0^a(\mathbb{R})\) in terms of the Hermite expansion coefficients of its elements.

**Characterization 1.2.20.**

Let \(a \geq \frac{1}{2}\).

If \(f \in S_0^a(\mathbb{R})\), then there exists \(t > 0\) such that

\[
\sum_{n=0}^{\infty} \exp(t n^{\frac{2a}{3a}}) |(f, \psi_n)_{L_2(\mathbb{R})}|^2 < \infty.
\]

Conversely, if a sequence \((a_n) \in \ell^{\infty}\) satisfies \(\sum_{n=0}^{\infty} \exp(t n^{\frac{2a}{3a}}) |a_n|^2 < \infty\) for certain \(t > 0\) then \(\sum a_n \psi_n\) converges (in the topology of \(S_0^a(\mathbb{R})\)) to a function in \(S_0^a(\mathbb{R})\).

Put differently,
A function \( f \in L_2(\mathbb{R}) \) belongs to \( S_0^\alpha(\mathbb{R}) \) if and only if
\[
\exists l > 0 : (f, \psi_n)_{L_2(\mathbb{R})} = O(\exp(-l n^{2\alpha})) \quad (n \to \infty).
\]

1.3. The even functions in the Gel’fand-Shilov spaces

Let \( R \) be one of the spaces \( S_\alpha(\mathbb{R}), S_\beta(\mathbb{R}) \) and \( S_0^\alpha(\mathbb{R}) \). By \( R_{\text{even}} \) we denote the subspace of \( R \) which consists of all even functions belonging to \( R \).

The space \( R_{\text{even}} \) remains invariant under the operations
\[
\begin{align*}
Q^2 & : \phi(x) \mapsto x^2 \phi(x) \\
-x^{-1}D & : \phi(x) \mapsto \frac{1}{x} \phi'(x) \\
M_f & : \phi(x) \mapsto f(x) \phi(x) \quad , f \text{ is an even multiplier in } R, \\
T_h + T_{-h} & : \phi(x) \mapsto \phi(x-h) + \phi(x+h) \quad , h \in \mathbb{R} \\
Z_\lambda & : \phi(x) \mapsto \sqrt{\lambda} \phi(\lambda x) \quad , \lambda \in \mathbb{R} \setminus \{0\}, \\
\Pi & : \phi(x) \mapsto \phi(-x).
\end{align*}
\]

The pointwise defined Fourier transformation satisfies
\[
\begin{align*}
\mathcal{F}(S_{\alpha,\text{even}}(\mathbb{R})) &= S_{\text{even}}^{\alpha}(\mathbb{R}) \quad , \quad \mathcal{F}(S_{\beta,\text{even}}^3(\mathbb{R})) = S_{\beta,\text{even}}(\mathbb{R}) \quad \text{and} \\
\mathcal{F}(S_{\alpha,\text{even}}^0(\mathbb{R})) &= S_{\beta,\text{even}}^0(\mathbb{R}).
\end{align*}
\]

In [EB] similar results are proved for the Hankel transformations \( \mathcal{H}_{v,2v+1} \): For each \( v \geq -\frac{1}{2} \) we have
\[
\begin{align*}
\mathcal{H}_{v,2v+1}(S_{\alpha,\text{even}}(\mathbb{R})) &= S_{\text{even}}^{\alpha}(\mathbb{R}) \quad , \quad \mathcal{H}_{v,2v+1}(S_{\text{even}}^3(\mathbb{R})) = S_{\beta,\text{even}}(\mathbb{R}) \quad \text{and} \\
\mathcal{H}_{v,2v+1}(S_{\alpha,\text{even}}^0(\mathbb{R})) &= S_{\beta,\text{even}}^0(\mathbb{R}).
\end{align*}
\]

As a consequence the operator \( -x^{-1}D \) is invertible on \( S_{\alpha,\text{even}}(\mathbb{R}), S_{\text{even}}^{\alpha}(\mathbb{R}) \) and \( S_{\alpha,\text{even}}^0(\mathbb{R}), \) where
\[
(x^{-1}D)^{-1} = -\int_0^\infty t \phi(t) \, dt.
\]

Moreover, in [EB] characterizations are given for the spaces \( S_{\alpha,\text{even}}(\mathbb{R}), S_{\text{even}}^{\alpha}(\mathbb{R}) \) and \( S_{\alpha,\text{even}}^0(\mathbb{R}) \) in terms of decay estimates on even functions \( \phi \in L_1(\mathbb{R}) \) and their Hankel transforms \( \mathcal{H}_{v,2v+1} \phi \).

Characterization 1.3.1.\\nLet \( \alpha,\beta > 0 \), \( v \geq -\frac{1}{2} \) and let \( \phi \in L_1(\mathbb{R}) \) be even. Then
Using the same techniques as in the proof of the above characterization we can prove the following.

Characterization 1.3.2.

Let $\alpha, \beta > 0$, $\nu \geq -\frac{1}{2}$ and let $\phi \in L_1(\mathbb{R})$ be even. Then

(i) $\phi \in S_{\alpha, \text{even}}(\mathbb{R})$ iff $\exists \gamma > 0 : \sup_{x \geq 0} \exp(\nu x^{1/\alpha}) |\phi(x)| < \infty$ and $\forall f \in \mathcal{L}_{\nu} : \sup_{x \geq 0} \left| x^j (\mathcal{H}_{\nu, 2v+1} \phi) (x) \right| < \infty$.

(ii) $\phi \in S_{\alpha, \text{even}}^0(\mathbb{R})$ iff $\exists \gamma > 0 : \sup_{x \geq 0} \left| x^j \phi(x) \right| < \infty$ and $\exists \gamma > 0 : \sup_{x \geq 0} \exp(\nu x^{1/\beta}) |(\mathcal{H}_{\nu, 2v+1} \phi) (x)| < \infty$.

(iii) $\phi \in S^{\beta}_{\alpha, \text{even}}(\mathbb{R})$ iff $\exists \gamma > 0 : \sup_{x \geq 0} \exp(\nu x^{1/\alpha}) |\phi(x)| < \infty$ and $\exists \gamma > 0 : \sup_{x \geq 0} \exp(\nu x^{1/\beta}) |(\mathcal{H}_{\nu, 2v+1} \phi) (x)| < \infty$.

With the aid of the operator $\mathcal{E}$ we can reformulate the above characterization.

Characterization 1.3.3.

Let $\alpha, \beta > 0$ and $\nu \geq -\frac{1}{2}$. Let $\phi \in X_1$, then

(i) $\mathcal{E} \phi \in S_{\alpha, \text{even}}(\mathbb{R})$ iff $\phi \in D^\alpha(1Q^{1/\alpha} : X_1)$ and $\mathcal{H}_{\nu, 2v+1} \phi \in D^\nu(Q)$.  

(ii) $\mathcal{E} \phi \in S_{\text{even}}^0(\mathbb{R})$ iff $\phi \in D^\nu(Q) \quad \text{and} \quad \mathcal{H}_{\nu, 2v+1} \phi \in D^\nu(1Q^{1/\beta} : X_1)$.

(iii) $\mathcal{E} \phi \in S^{\beta}_{\alpha, \text{even}}(\mathbb{R})$ iff $\phi \in D^\alpha(1Q^{1/\alpha} : X_1)$ and $\mathcal{H}_{\nu, 2v+1} \phi \in D^\nu(1Q^{1/\beta} : X_1)$.

For each $\alpha \geq 0$ we introduce the space $S_{\alpha, \text{even}}(\mathbb{R}^+)$, which consists of all functions $f : \mathbb{R}^+ \to \mathcal{C}$ such that $\mathcal{E}f \in S_{\alpha, \text{even}}(\mathbb{R})$. The spaces $S^0_{\alpha, \text{even}}(\mathbb{R}^+)$ and $S^{\beta}_{\alpha, \text{even}}(\mathbb{R}^+)$ are defined in a similar way.

Note that $\mathcal{H}_{\nu, 2v+1} Q^2 U_{\nu, 2v+1} = D_{\nu, 2v+1}$ (see (1.19), Chapter I), so an other formulation of Characterization 1.3.3. reads

Characterization 1.3.4.

Let $\alpha, \beta > 0$ and $\nu \geq -\frac{1}{2}$. Then
As a consequence of Characterization 1.3.3 we prove the following properties (cf. Corollary 1.3.4.3).

**Corollary 1.3.5.**

Let $\alpha, \beta > 0$, let $\nu > -1$ and $\mu \geq 0$. Let $g \in X_\mu$.

(I) The following assertions are equivalent.

(i) $g \in D^\alpha(\mid Q \mid^{1/\alpha}; X_1)$ and $H_{v, \mu} g \in D^\alpha(\mid Q \mid^\beta; X_1)$

(ii) $r^{-\nu-\mu-1} g \in D^\alpha(\mid Q \mid^{1/\alpha}; X_1)$ and $r^{-\nu-\mu-1} H_{v, \mu} g \in D^\alpha(\mid Q \mid^\beta; X_1)$

(iii) $E(r^{-\nu+1} \mu^{-1}) g \in S_{\alpha, \text{even}}(\mathbb{R})$

(iv) $g \in r^{-\nu+1} \mu^{-1} S_{\alpha, \text{even}}(\mathbb{R}^+)$

(II) The following assertions are equivalent

(i) $g \in D^\alpha(\mid Q \mid^{1/\alpha}; X_1)$ and $H_{v, \mu} g \in D^\alpha(\mid Q \mid^{1/\beta}; X_1)$

(ii) $r^{-\nu+1} \mu^{-1} g \in D^\alpha(\mid Q \mid^{1/\alpha}; X_1)$ and $r^{-\nu+1} \mu^{-1} H_{v, \mu} g \in D^\alpha(\mid Q \mid^{1/\beta}; X_1)$

(iii) $E(r^{-\nu+1} \mu^{-1}) g \in S_{\alpha, \text{even}}(\mathbb{R})$

(iv) $g \in r^{-\nu+1} \mu^{-1} S_{\alpha, \text{even}}(\mathbb{R}^+)$

(III) If, in addition, $\alpha + \beta \geq 1$ then equivalent are

(i) $g \in D^\alpha(\mid Q \mid^{1/\alpha}; X_1)$ and $H_{v, \mu} g \in D^\alpha(\mid Q \mid^{1/\beta}; X_1)$

(ii) $r^{-\nu+1} \mu^{-1} g \in D^\alpha(\mid Q \mid^{1/\alpha}; X_1)$ and $r^{-\nu+1} \mu^{-1} H_{v, \mu} g \in D^\alpha(\mid Q \mid^{1/\beta}; X_1)$

(iii) $E(r^{-\nu+1} \mu^{-1}) g \in S_{\alpha, \text{even}}(\mathbb{R})$

(iv) $g \in r^{-\nu+1} \mu^{-1} S_{\alpha, \text{even}}(\mathbb{R}^+)$

**Proof.**

(I) Let us first assume that $-\nu+1/2 \mu - 1/2 < 0$. We prove that (i) implies (ii). Let $g \in D^\alpha(\mid Q \mid^{1/\alpha}; X_1)$ and $H_{v, \mu} g \in D^\alpha(\mid Q \mid^{1/\beta}; X_1)$. Since $g \in D^\alpha(\mid Q \mid^{1/\alpha}; X_1)$ there exists $t > 0$ such that

$$\int_0^\infty e^{-t r |\alpha|} g(r)^2 r \, dr < \infty.$$ 

Define $h : \mathbb{R}^+ \to \mathbb{C}$ by
Because $h_{v,\mu} g \in D^\infty(Q;X_1)$, $h$ extends to an even function in $C^\infty(\mathbb{R})$. Furthermore, $g(r) = r^{-\frac{1}{2} \mu + \frac{1}{2}} h(r)$, for each $r > 0$.

So it readily follows that

$$
\int_0^\infty e^{tr} h(r) |^2 r \, dr \leq \int_0^\infty e^{tr} h(r) |^2 r \, dr + \int_1^\infty e^{tr} g(r) |^2 r \, dr < \infty.
$$

Hence $h \in D^\alpha(Q;X_1)$ and therefore, $r^{-\nu + \frac{1}{2} \mu - \frac{1}{2}} g \in D^\alpha(Q;X_1)$. Note that $g \in D^\alpha(Q;X_1) \subset D^\infty(Q;X_1)$ and also $h_{v,\mu} g \in D^\infty(Q;X_1)$, so from Corollary 1.3.4.3, it follows that

$$
r^{-\nu + \frac{1}{2} \mu - \frac{1}{2}} h_{v,\mu} g \in D^\infty(Q;X_1).
$$

So assertion (ii) holds.

The converse implication, (ii) $\Rightarrow$ (i), is obvious.

Next assume that $-\nu + \frac{1}{2} \mu - \frac{1}{2} \geq 0$. Then the proof of '(i) $\Leftrightarrow$ (ii)' runs similarly. However, in this case the implication (ii) $\Rightarrow$ (i) is nontrivial.

The equivalence of assertions (ii) and (iii) is a consequence of Characterization 1.3.3 (i) and the identity

$$
r^{-\nu + \frac{1}{2} \mu + \frac{1}{2}} h_{v,2v+1} r^{-\nu + \frac{1}{2} \mu - \frac{1}{2}} = h_{v,\mu}.
$$

Obviously, (iii) and (iv) are equivalent.

(II) Noting that $h_{v,\mu}(h_{v,\mu} g) = g$ we can replace $g$ by $h_{v,\mu} g$ and $\alpha$ by $\beta$ in the proof of the equivalence (i) $\Leftrightarrow$ (ii) in (I) and we obtain the equivalence (i) $\Leftrightarrow$ (ii) in the present case (II).

The equivalence (ii) $\Leftrightarrow$ (iii) is a consequence of Characterization 1.3.3 (ii) and identity (*).

Obviously (iii) $\Leftrightarrow$ (iv).

(III) Follows from (I) and (II).

In addition to Zhang Gong-Zhing’s characterization of $S^\alpha_n(\mathbb{R})$ in terms of the Hermite functions $\psi_n$, we can characterize functions in $S^\alpha_n,\text{even}(\mathbb{R})$ in terms of Laguerre functions $L^{(v,2v+1)}_n$.

**Characterization 1.3.6.**

Let $\alpha \geq \frac{1}{2}$ and $v \geq -\frac{1}{2}$. Then
An even function $f \in L^2(\mathbb{R})$ belongs to $S^2_{\alpha,\text{even}}(\mathbb{R})$ if and only if

$$\exists t > 0 : (f, L_n^{(\nu, 2\nu + 1)})_{\text{bvi}} = O\left(\exp\left(-t\frac{n}{2\alpha}\right)\right) \quad (n \to \infty).$$

For a proof we refer to [E2, p. 18].
2. The case of two independent variables

2.1. Introduction
We use the multi-index notation which is already introduced in Section 3.1, Chapter I. We add some new notation besides.

Let \( n \in \mathbb{N} \), \( n \geq 2 \), \( k, m \in \mathbb{N}_0^n \), \( x \in \mathbb{R}^n \), \( \alpha \in \mathbb{R}_+^n \) and let \( A \in (C^{\mathbb{N}_0^n})^n \). Then

\[
\begin{align*}
  k + m &= (k_1 + m_1, k_2 + m_2, \ldots, k_n + m_n) \\
  k - m &= (k_1 - m_1, k_2 - m_2, \ldots, k_n - m_n) \\
  (x^\alpha)^m &= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\
  (m!)^\alpha &= (m_1!)^{\alpha_1} (m_2!)^{\alpha_2} \cdots (m_n!)^{\alpha_n} \\
  A(m) &= A_1(m_1) \cdot A_2(m_2) \cdots A_n(m_n).
\end{align*}
\]

In this section we consider the case \( n = 2 \). Let us define the Gel'fand-Shilov spaces in two independent variables.

Definition 2.1.1.
For each \( \alpha, \beta \in \mathbb{R}_+^2 \) the spaces \( S_{\alpha}(\mathbb{R}^2) \), \( S_{\beta}(\mathbb{R}^2) \) and \( S^0_{\alpha}(\mathbb{R}^2) \) are subspaces of the Schwartz space \( S(\mathbb{R}^2) \). They are defined in the following way.

(i) The space \( S_{\alpha}(\mathbb{R}^2) \) consists of all functions \( \phi \in S(\mathbb{R}^2) \) which satisfy

\[
\exists A \in \mathbb{R}^*_+ : \forall l \in \mathbb{N}_0^2 \exists B_l > 0 \forall k \in \mathbb{N}_0^n : \sup_{x \in \mathbb{R}^2} |x^k D^l \phi(x)| \leq B_l A^k (k^l)^\alpha
\]

where the constants \( A \) and \( B_l \) depend on \( \phi \).

(ii) The space \( S_{\beta}(\mathbb{R}^2) \) consists of all functions \( \phi \in S(\mathbb{R}^2) \) which satisfy

\[
\exists B \in \mathbb{R}^*_+ : \forall k \in \mathbb{N}_0^n \exists B_k > 0 \forall l \in \mathbb{N}_0^2 : \sup_{x \in \mathbb{R}^2} |x^k D^l \phi(x)| \leq A_k B^l (l^k)^\beta
\]

where the constants \( B \) and \( A_k \) depend on \( \phi \).

(iii) The space \( S^0_{\alpha}(\mathbb{R}^2) \) consists of all functions \( \phi \in S(\mathbb{R}^2) \) which satisfy

\[
\exists A, B \in \mathbb{R}^*_+ : \exists C > 0 \forall k, l \in \mathbb{N}_0^n : \sup_{x \in \mathbb{R}^2} |x^k D^l \phi(x)| \leq C A^k B^l (k^l)^\alpha (l^k)^\beta
\]

where the constants \( A, B \) and \( C \) depend on \( \phi \).

In the above definitions \( k^k \) and \( l^l \) may be replaced by \( k! \) and \( l! \) respectively (cf. the case of one variable). Furthermore, \( A^k \) and \( B^l \) may be replaced by \( |A|^{1,k_1} \) and \( |B|^{1,l_1} \) respectively. And \( A_k \) and \( B_l \) may be replaced by \( A_k^{(1)} \cdot A_k^{(2)} \) and \( B_l^{(1)} \cdot B_l^{(2)} \) respectively. The latter can be seen as follows. The constant \( A_k = A_{k_1,k_2} \) can be chosen increasingly in both variables \( k_1 \) and \( k_2 \) with \( A_k > 1 \) for all \( k \in \mathbb{N}_0^n \). So we have \( A_k \leq A_{(k_1,k_1)} \cdot A_{(k_1,k_2)} \). The converse is obvious.
Also the following observation is useful. If $\alpha_1 = \alpha_2$ then in the above definitions ($k^\alpha$ may be replaced by $|k|^{k|\alpha|}$ which follows from the inequalities
\[
\left(\frac{x+y}{2}\right)^{x+y} \leq x^y y^x (x+y)^{x+y}, \text{ for all } x, y > 0. \tag{2.2}
\]
Of course $|k|^{k|\alpha|}$ may be replaced by $|k|^{k|\alpha|}$ (cf. the case of one variable). Similarly, if $\beta_1 = \beta_2$ then $|l|^\beta$ may be replaced by $|l|^\beta$ or by $|l|^\beta$.

As in the one dimensional case, in stead of the supremum norm we may use the $L_2(\mathbb{R}^2)$ norm, $\| \cdot \|_{L_2(\mathbb{R}^2)}$. So an alternative definition of the $S^8_{\alpha}$-spaces in two variables is the following.

**Definition 2.1.2.**

Let $\alpha, \beta \in \mathbb{R}_+^2$ and let $\phi \in S(\mathbb{R}^2)$. Then

(i) $\phi \in S_\alpha(\mathbb{R}^2)$ if and only if

\[
\exists A > 0 \forall x \in B_1 ^{\alpha} \exists B > 0 \forall y \in B_2 ^{\alpha} : \| Q_k P_1 \phi \|_{L_2(\mathbb{R}^2)} \leq B A^{k|\alpha|}(k!)^\alpha
\]

(ii) $\phi \in S_\beta(\mathbb{R}^2)$ if and only if

\[
\exists B > 0 \forall x \in B_1 ^{\beta} \exists A > 0 \forall y \in B_2 ^{\beta} : \| Q_k P_1 \phi \|_{L_2(\mathbb{R}^2)} \leq A B^{k|\beta|}(l!)^\beta
\]

(iii) $\phi \in S^8_{\alpha}(\mathbb{R}^2)$ if and only if

\[
\exists A, B, C > 0 \forall x \in B_1 ^{\alpha} \exists y \in B_2 ^{\alpha} : \| Q_k P_1 \phi \|_{L_2(\mathbb{R}^2)} \leq C A^{k|\alpha|} B^{k|\beta|}(l!)^\beta
\]

where we use the notation
\[
Q_k = (Q_1, Q_2)^{(k_1, k_2)} = Q_1^{k_1} Q_2^{k_2} \text{ and } P_1 = (P_1, P_2) = P_1^{l_1} P_2^{l_2}.
\]

The operators $Q_i$ and $P_i$, $i = 1, 2$, are defined in Section 3.1, Chapter 1.

We define the extended real number system $\mathbb{R}_{+, \infty}$ by
\[
\mathbb{R}_{+, \infty} := \mathbb{R}_+ \cup \{ \infty \}. \tag{2.3}
\]

Let $\alpha_2 \in \mathbb{R}$ and $\beta \in \mathbb{R}_+^2$, then we define the space $S_{(\infty, \alpha_2)}(\mathbb{R}^2)$ as follows.

\[
\phi \in S_{(\infty, \alpha_2)}(\mathbb{R}^2) \text{ iff } \phi \in S(\mathbb{R}^2) \quad \text{ and } \quad \exists A, B > 0 \forall x \in \mathbb{R}_+ \exists C > 0 \forall y \in B_1 ^{\alpha_2} \forall z \in B_2 ^{\alpha_2} : \sup_{x \in \mathbb{R}_+} x k D^j \phi(x) \leq C A^{k|\alpha_2|} B^{k|\beta|}(l!)^\beta. \tag{2.4}
\]

Similarly we can define the $S^8(\mathbb{R}^2)$-spaces for $\alpha, \beta \in \mathbb{R}_+^2$. We leave the details to the reader.

From these definitions we obtain for each $\alpha, \beta \in \mathbb{R}_+^2$, $\mathbb{R}_+^2$. We leave the details to the reader.
The question of nontriviality of the $S^0_\alpha(I\!\!R^2)$-spaces can be solved by utilizing the following

**Remark 2.1.3.**

(i) For each $\alpha \in I\!\!R^2_+$ the space $S_\alpha(I\!\!R^2)$ is nontrivial: Every infinitely differentiable function of compact support belongs to these spaces.

(ii) For each $\beta \in I\!\!R^2_+$ the space $S^0(I\!\!R^2)$ is nontrivial: The Fourier transform of an infinitely differentiable function of compact support belongs to these spaces (Paley-Wiener).

(iii) For each $\alpha, \beta \in I\!\!R^2_+$ the space $S^0_\alpha(I\!\!R^2)$ is nontrivial if and only if the spaces $S^0_\alpha(I\!\!R)$ and $S^0_\beta(I\!\!R)$ are nontrivial: Suppose $S^0_\alpha(I\!\!R)$ is nontrivial, $i = 1, 2$. Let $\phi_i \in S^0_\alpha(I\!\!R)$, $\phi_i \neq 0$, $i = 1, 2$. Then the function $(x_1, x_2) \mapsto \phi_1(x_1)\phi_2(x_2)$ is not identically zero and belongs to the space $S^0_\alpha(I\!\!R^2)$. Conversely, suppose $S^0_\alpha(I\!\!R)$ is trivial for some $i \in \{1, 2\}$. Without loss of generality, assume $S^0_\alpha(I\!\!R)$ is trivial. Let $\phi \in S^0_\alpha(I\!\!R^2)$. Then for each fixed $x_2 \in I\!\!R$ the function $x_1 \mapsto \phi(x_1, x_2)$ belongs to $S^0_\alpha(I\!\!R)$. Hence $\phi \equiv 0$.

Kashpirovskii's intersection result in one variable has an analog in the two dimensional case,

$$S_\alpha(I\!\!R^2) \cap S^0(I\!\!R^2) = S^0_\alpha(I\!\!R^2)$$

for each $\alpha, \beta \in I\!\!R^2_+$, with $\alpha + \beta \geq 1$.

This result can be proved with the same techniques as Kashpirovskii used in his paper [Ka]. It will also follow from functional analytic results which we derive in Section 2.2.

We define the twist operator $T : S(I\!\!R^2) \to S(I\!\!R^2)$ by

$$(T\phi)(x_1, x_2) := \phi(x_2, x_1), \phi \in S(I\!\!R^2), (x_1, x_2) \in I\!\!R^2.$$  

(2.7)

Clearly $T$ extends to a unitary operator on $L_2(I\!\!R^2)$. Note that $T = Z_\Lambda$ with $\Lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

**Theorem 2.1.4.**

Let $\alpha, \beta \in I\!\!R^2_+$. Then

(i) $T(S^{(\phi_1, \phi_2)}(I\!\!R^2)) = S^{(\phi_2, \phi_1)}(I\!\!R^2)$
Proof.
Let $\phi \in S(\mathbb{R}^2)$.

(i) For each $k,l \in \mathbb{N}_0$, we have

$$Hence the result.

(ii) Let $k,l \in \mathbb{N}_0$. By Leibnitz's differentiation rule we have

$$P_1^k P_2^l \phi = \sum_{j=0}^{\min(k,l)} \frac{k!}{(k-j)!} \frac{l!}{(l-j)!} P_1^{k-j} P_2^{l-j} \phi.$$

Using this formula, using the identities $P_1 Q_1 = P_1$ and $Q_1 P_1 = P_1$ and using the fact that $P_2$ is an isometry, we obtain

$$\||Q|^k P_1^l \phi\|_{L^2(\mathbb{R}^2)} \leq \sum_{j=0}^{\min(k,l)} \frac{k!}{(k-j)!} \frac{l!}{(l-j)!} \||Q|^j P_1^{k-j} P_2^{l-j} \phi\|_{L^2(\mathbb{R}^2)}.$$

Now standard estimations yield the result.

Note that

$$\|I2\| = T \circ I1 \circ T \quad \text{and} \quad \|I\| = I1 \circ I2.$$  \quad (2.8)

So as a consequence of the above theorem we obtain

Corollary 2.1.5.

Let $\alpha, \beta \in \mathbb{R}_{+\infty}^2$. Then

(i) $I2(S_{(\alpha, \alpha)}(\mathbb{R}^2)) = S_{(\alpha, \alpha)}(\mathbb{R}^2)$

(ii) $I2(S_{\alpha}^0(\mathbb{R}^2)) = S_{\beta}^0(\mathbb{R}^2)$.  \quad \Box

Each of the spaces $S_{\alpha}(\mathbb{R}^2)$, $S_{\beta}^0(\mathbb{R}^2)$ and $S_{\alpha}^0(\mathbb{R}^2)$ remains invariant under the following operations

the multiplication operations

$$Q_1 : \phi(x) \rightarrow x_1 \phi(x)$$

$$Q_2 : \phi(x) \rightarrow x_2 \phi(x)$$

the differentiation operations
\[ P_1 : \phi(x) \to i \frac{\partial}{\partial x_1} \phi(x) \]
\[ P_2 : \phi(x) \to i \frac{\partial}{\partial x_2} \phi(x) \]
the phase-shifts
\[ \Phi_b : \phi(x) \to e^{i(b-x)} \phi(x) \quad b \in \mathbb{R}^2. \]
the translations
\[ T_h : \phi(x) \to \phi(x-h) \quad h \in \mathbb{R}^2. \]
the parity operation
\[ \Pi : \phi(x) \to \phi(-x). \]
The operators \( Q_1, Q_2, \Phi_b, T_h \) and \( \Pi \) are bijections.
Moreover, if \( R \) is one of the spaces \( S_\alpha(\mathbb{R}^2) \), \( S_\beta(\mathbb{R}^2) \) or \( S_\gamma(\mathbb{R}^2) \), then
\[ \text{grad maps } R \text{ into } R \times R, \]
\[ \text{div maps } R \times R \text{ into } R, \]
\[ \Delta \text{ maps } R \text{ into } R, \]
where
\[ \text{grad } \phi = \left( \frac{\partial}{\partial x_1} \phi, \frac{\partial}{\partial x_2} \phi \right), \quad \text{div } \phi = \frac{\partial}{\partial x_1} \phi_1 + \frac{\partial}{\partial x_2} \phi_2 \] and
\[ \Delta \phi = \frac{\partial^2}{\partial x_1^2} \phi + \frac{\partial^2}{\partial x_2^2} \phi. \]
Let \( f \in C^\infty(\mathbb{R}^2) \). Then the multiplier \( M_f \) maps \( S_\alpha(\mathbb{R}^2) \) into itself if
\[ \exists \epsilon \in \mathbb{R^2} : \forall \epsilon \in \mathbb{N^2} : \exists C_{l,\epsilon} > 0 : \sup_{x \in \mathbb{R}^2} | \exp\{-\epsilon_1 \|x_1\|^{1/\alpha_1} - \epsilon_2 \|x_2\|^{1/\alpha_2}\} D^i f(x) | \leq C_{l,\epsilon} , \alpha_1, \alpha_2 > 0. \]
\( M_f \) maps \( S_\beta(\mathbb{R}^2) \) into itself if
\[ \exists \epsilon \in \mathbb{R^2} : \exists C, \gamma > 0 : \forall \epsilon \in \mathbb{N^2} : \sup_{x \in \mathbb{R}^2} | (1 + \|x_1\|^{h_1})^{-1} (1 + \|x_2\|^{h_2})^{-1} D^i f(x) | \leq C B_0^{1/\gamma} (l^1)^{\beta}. \]
\( M_f \) maps the space \( S_\gamma(\mathbb{R}^2) \) into itself if
\[ \exists \epsilon \in \mathbb{R^2} : \exists C, \gamma > 0 : \forall \epsilon \in \mathbb{N^2} : \sup_{x \in \mathbb{R}^2} | \exp\{-\epsilon_1 \|x_1\|^{1/\alpha_1} - \epsilon_2 \|x_2\|^{1/\alpha_2}\} D^i f(x) | \leq C \epsilon l^{1/\gamma}, \alpha_1, \alpha_2 > 0. \]
Cf. [GS 2, p. 239].
For the dilatation operators \( Z_A, A \in GL(\mathbb{R}^2) \), we prove the following

Theorem 2.1.6.
Let \( \alpha, \beta \in \mathbb{R}^2_+ \) and suppose that \( \alpha_1 = \alpha_2 \) and \( \beta_1 = \beta_2 \). Let \( A = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix} \in GL(\mathbb{R}^2) \), then
(i) $Z_A$ maps $S_a(\mathbb{R}^2)$ bijectively onto itself

(ii) $Z_A$ maps $S^\beta(\mathbb{R}^2)$ bijectively onto itself

(iii) $Z_A$ maps $S^\alpha_a(\mathbb{R}^2)$ bijectively onto itself

Proof.

(i) Let $\phi \in S_a(\mathbb{R}^2)$. Then

$$\exists A > 0 \ \forall z \in \mathbb{N}^2 \ \exists \beta > 0 \ \forall z \in \mathbb{N}^2 : \sup_{x \in \mathbb{R}^2} |x^k D^l \phi(x)| \leq B A^{k \lambda^2} |k|^{k \lambda^2}.$$

Let $R_2 := \max\{ |\lambda_j| : j = 1, \ldots, 4 \}$ and $R_1 = R_2 / \| \det A \|$. Then we obtain by means of Theorem 1.3.1.1, the following estimation for all $k, l \in \mathbb{N}^2$.

$$\sup_{x \in \mathbb{R}^2} |x^k D^l \phi(xA)| \leq$$

$$\leq R_1^{k \lambda^2} (2R_2)^l \sum_{m=0}^{l_i} \sum_{j=0}^{l_2} \sum_{i=0}^{k \lambda^2} \left[ \frac{|k|}{m} \right] |\xi(m, k) - m| D^{(i+j, l1-i-j)} \phi(\xi) | \leq$$

$$\leq R_1^{k \lambda^2} (2R_2)^l \sum_{m=0}^{l_i} \sum_{j=0}^{l_2} \sum_{i=0}^{k \lambda^2} B^{(i+j, l1-i-j)} A^{k \lambda^2} |k|^{k \lambda^2} =$$

$$= (2 AR_1)^{k \lambda^2} (2R_2)^l \sum_{m=0}^{l_i} \sum_{j=0}^{l_2} B^{(i+j, l1-i-j)} =$$

$$= \tilde{B} A^{k \lambda^2} |k|^{k \lambda^2},$$

where $\tilde{A} := 2 AR_1$, $\tilde{B} := (2R_2)^l \sum_{i=0}^{l_i} \sum_{j=0}^{l_2} B^{(i+j, l1-i-j)}$.

Hence $Z_A \phi \in S_a(\mathbb{R}^2)$. Since $Z_A Z_A^{-1} = I$, statement (i) is settled.

(ii) The operator $Z_A$ is unitarily equivalent with $Z(\Lambda^{-1} A)$, indeed (cf. Theorem 1.3.1.3)

$$IF Z(\Lambda^{-1} A) IF^* = Z_A.$$

Since $IF(S^\beta(\mathbb{R}^2)) = S^\beta(\mathbb{R}^2)$ we derive with the aid of statement (i) that

$$Z_A(S^\beta(\mathbb{R}^2)) = Z_A IF(S^\beta(\mathbb{R}^2)) = IF Z(\Lambda^{-1} A)(S^\beta(\mathbb{R}^2)) = IF(S^\beta(\mathbb{R}^2)) = S^\beta(\mathbb{R}^2).$$

(iii) This statement is a consequence of Kashpirovskii's intersection result (2.6) and the above statements (i) and (ii).

Let us now consider the general case $\alpha, \beta \in \mathbb{R}^2_{+\infty}$.

Theorem 2.1.7.

Let $\alpha, \beta \in \mathbb{R}^2_{+\infty}$. Let $\Lambda = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix} \in GL(\mathbb{R}^2)$. 
Then we obtain the following

(i) If \(\lambda_2 = \lambda_3 = 0\) and \(\lambda_1, \lambda_4 \neq 0\) then \(Z_A(S^\alpha_0 (\mathbb{R}^2)) = S^\beta_0 (\mathbb{R}^2)\).

(ii) If \(\lambda_1 = \lambda_4 = 0\) and \(\lambda_2, \lambda_3 \neq 0\) then \(Z_A(S^\alpha_0 (\mathbb{R}^2)) = S^{(\beta_2, \beta_1)}_0 (\mathbb{R}^2)\).

(iii) If \(\lambda_4 = 0\) and \(\lambda_1, \lambda_2, \lambda_3 \neq 0\) then \(Z_A(S^\alpha_0 (\mathbb{R}^2)) \subset S^{(\max(\beta_2), \beta_1)}_0 (\mathbb{R}^2)\).

(iv) If \(\lambda_3 = 0\) and \(\lambda_1, \lambda_2, \lambda_4 \neq 0\) then \(Z_A(S^{(\max(\beta_2), \beta_1)}_0 (\mathbb{R}^2)) = S^\alpha_0 (\mathbb{R}^2)\).

(v) If \(\lambda_2 = 0\) and \(\lambda_1, \lambda_3, \lambda_4 \neq 0\) then \(Z_A(S^\alpha_0 (\mathbb{R}^2)) \subset S^{(\max(\beta_2), \max(\beta_1))}_0 (\mathbb{R}^2)\).

(vi) If \(\lambda_1 = 0\) and \(\lambda_2, \lambda_3, \lambda_4 \neq 0\) then \(Z_A(S^\alpha_0 (\mathbb{R}^2)) \subset S^{(\max(\beta_2), \max(\beta_1))}_0 (\mathbb{R}^2)\).

(vii) If \(\lambda_1, \lambda_2, \lambda_3, \lambda_4 \neq 0\) then \(Z_A(S^\alpha_0 (\mathbb{R}^2)) \subset S^{(\max(\beta_2), \max(\beta_1))}_0 (\mathbb{R}^2)\).

In (iii) up to and including (vii) the operators \(Z_A\) are bijections if and only if \(\alpha_1 = \alpha_2\) and \(\beta_1 = \beta_2\).

**Proof.**

For each \(\phi \in S(\mathbb{R}^2)\) we obtain by means of the transformation \(\xi = x \Lambda\) the following equality for all \(k, l \in \mathbb{N}_0\),

\[
\sup_{x \in \mathbb{R}^2} |x^k D^l \phi(x \Lambda)| = \sup_{\xi \in S(\mathbb{R}^2)} \left| \sum_{(p, i, j, l_1, l_2) \in \mathbb{N}_0^2} \left( \sum_{k_1 \in \mathbb{N}_0} \lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \lambda_4^{l_4} \lambda_4^{l_4-j} \cdot \mu_1 \mu_2 \mu_3 \mu_4 \right) \phi(\xi) \right|
\]

where \(\mu_j = \lambda_j / \det \Lambda\), \(j = 1, \ldots, 4\).

The assertions (i) and (ii) follow from (*) at once.

Next we prove the assertions (iii) up to and including (vi) in case \(\beta_1 = \beta_2 = \infty\).

(iii) Suppose \(\lambda_4 = 0\) and \(\lambda_1, \lambda_2, \lambda_3 \neq 0\).

Let \(\phi \in S_\alpha(\mathbb{R}^2)\). Suppose \(\alpha_1, \alpha_2 < \infty\). Then

\[
\exists \alpha > 0 \forall \nu \in \mathbb{N}^2 \exists \nu > 0 \forall \xi \in \mathbb{N}^2 \colon \sup_{x \in \mathbb{R}^2} |x^k D^l \phi(x \Lambda)| \leq B_1 A^{k+1} (k^\alpha).
\]

Let \(k, l \in \mathbb{N}_0^2\). Then we obtain by (*),

\[
\sup_{x \in \mathbb{R}^2} |x^k D^l \phi(x \Lambda)| = \sup_{\xi \in S(\mathbb{R}^2)} \left| \sum_{(p, i, j, l_1, l_2) \in \mathbb{N}_0^2} \left( \sum_{k_1 \in \mathbb{N}_0} \lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \lambda_4^{l_4-j} \cdot \mu_1 \mu_2 \mu_3 \mu_4 \right) \phi(\xi) \right| \leq...
\]
where $R_2 = \max \{ |\lambda_i| : i = 1,2,3,4 \}$ and $R_1 = R_2 / |\det \Lambda|$. For each $\rho \in \{0,...,k_2\}$ we estimate $\rho^{\alpha_1} (1 k l - p)^{(l k l - p)\alpha_2}$ on the one side by 

$$\rho^{\alpha_1} (1 k l - p)^{(l k l - p)\alpha_2} \leq k_2^{k_1 \alpha_1} |k k l - p|^{(l k l - p)\alpha_2} \leq 2^{k_1 k_2 \alpha_1}$$

and on the other hand by 

$$\rho^{\alpha_1} (1 k l - p)^{(l k l - p)\alpha_2} \leq |k k l - p|^{(l k l - p)\alpha_2} \leq |k k l - p|^{\max(\alpha)}.$$

So we obtain

$$\sup_{x \in \mathbb{R}^2} |x k D^l \phi(x \Lambda)| \leq (2^{(l k l - p)\alpha_2})^{1 k l - p} \sum_{i=0}^{l} \left( \sum_{i=0}^{l} \frac{k_1 \alpha_1}{k_2} R_2^{l l} \right) B_{(l k l - p)}$$

and also

$$\sup_{x \in \mathbb{R}^2} |x k D^l \phi(x \Lambda)| \leq (2 A R_1)^{1 k l - p} \sum_{i=0}^{l} \left( \sum_{i=0}^{l} \frac{k_1 \alpha_1}{k_2} R_2^{l l} \right) B_{(l k l - p)}.$$

Hence

$$Z_\Lambda \phi \in S_{(\alpha_1,\alpha_1)}(\mathbb{R}^2) \cap S_{(\max(\alpha),\max(\alpha))}(\mathbb{R}^2) = S_{(\alpha_1,\max(\alpha))}(\mathbb{R}^2).$$

The latter equality follows from Theorem 2.2.6.

Also if one or both of the $\alpha_i$ equal $\infty$, a similar reasoning proves the validity of assertion (iii).

(iv) Suppose $\lambda_3 = 0$ and $\lambda_1,\lambda_2,\lambda_4 \neq 0$.

Then $\Lambda = \tilde{\Lambda} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $\tilde{\Lambda} = \begin{bmatrix} \lambda_2 & \lambda_1 \\ \lambda_4 & 0 \end{bmatrix}$. So by (ii) and (iii)

$$Z_\Lambda(S_\alpha(\mathbb{R}^2)) = Z_{\tilde{\Lambda}}(T(S_\alpha(\mathbb{R}^2))) = Z_{\tilde{\Lambda}}(S_{(\alpha_1,\alpha_1)}(\mathbb{R}^2)) \subset S_{(\alpha_1,\max(\alpha))}(\mathbb{R}^2).$$

(v) Suppose $\lambda_2 = 0$ and $\lambda_1,\lambda_3,\lambda_4 \neq 0$.

Then $\Lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $\Lambda = \begin{bmatrix} \lambda_3 & \lambda_4 \\ \lambda_1 & 0 \end{bmatrix}$. So by (i) and (iii)

$$Z_\Lambda(S_\alpha(\mathbb{R}^2)) = T \circ Z_{\tilde{\Lambda}}(S_\alpha(\mathbb{R}^2)) \subset T(S_{(\alpha_1,\alpha_1)}(\mathbb{R}^2)) = S_{(\alpha_1,\max(\alpha))}(\mathbb{R}^2).$$

(vi) Suppose $\lambda_1 = 0$ and $\lambda_2,\lambda_3,\lambda_4 \neq 0$.

Then $\Lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $\Lambda = \begin{bmatrix} \lambda_3 & \lambda_4 \\ 0 & \lambda_2 \end{bmatrix}$. So by (i) and (iv)
Next we prove assertions (iii) up to and including (vi) in case $\alpha_1 = \alpha_2 = \infty$ and $\beta_1, \beta_2$ arbitrarily.

(iii) Suppose $\lambda_4 = 0$ and $\lambda_1, \lambda_2, \lambda_3 \neq 0$.

Then $\Lambda^{-1} = \frac{1}{\det \Lambda} \begin{pmatrix} 0 & -\lambda_2 \\ -\lambda_3 & \lambda_1 \end{pmatrix}$. Since $Z_\Lambda = IF Z_{(\Lambda^{-1})^{T}} IF$ and $IF(S_\beta(I^2)) = S^\beta(I^2)$ we derive with the aid of the $S_\alpha$-case of (vi),

$$Z_\Lambda(S^\beta(I^2)) = Z_\Lambda IF(S_\beta(I^2)) = IF(Z_{(\Lambda^{-1})^{T}} S_\beta(I^2)) \subseteq \langle IF(S_{(\max(0),\beta_3)}(I^2)) = S^{(\max(0),\beta_3)}(I^2) \rangle.$$

(iv) Suppose $\lambda_3 = 0$ and $\lambda_1, \lambda_2, \lambda_4 \neq 0$.

Then $\Lambda^{-1} = \frac{1}{\det \Lambda} \begin{pmatrix} \lambda_4 & 0 \\ -\lambda_3 & \lambda_1 \end{pmatrix}$. So with the aid of the $S_\alpha$-case of (v) we derive

$$Z_\Lambda(S^\beta(I^2)) = IF(Z_{(\Lambda^{-1})^{T}} S_\beta(I^2)) \subseteq \langle IF(S_{(\max(0),\beta_3)}(I^2)) = S^{(\max(0),\beta_3)}(I^2) \rangle.$$

(v) Suppose $\lambda_2 = 0$ and $\lambda_1, \lambda_3, \lambda_4 \neq 0$.

Then $\Lambda^{-1} = \frac{1}{\det \Lambda} \begin{pmatrix} \lambda_4 & -\lambda_2 \\ -\lambda_3 & 0 \end{pmatrix}$. So with the aid of the $S_\alpha$-case of (iv) we obtain

$$Z_\Lambda(S^\beta(I^2)) = IF(Z_{(\Lambda^{-1})^{T}} S_\beta(I^2)) \subseteq \langle IF(S_{(\max(0),\beta_3)}(I^2)) = S^{(\max(0),\beta_3)}(I^2) \rangle.$$

(vi) Suppose $\lambda_1 = 0$ and $\lambda_2, \lambda_3, \lambda_4 \neq 0$.

Then $\Lambda^{-1} = \frac{1}{\det \Lambda} \begin{pmatrix} \lambda_4 & -\lambda_2 \\ -\lambda_3 & 0 \end{pmatrix}$. So with the aid of the $S_\alpha$-case of (iii) we obtain

$$Z_\Lambda(S^\beta(I^2)) = IF(Z_{(\Lambda^{-1})^{T}} S_\beta(I^2)) \subseteq \langle IF(S_{(\max(0),\beta_3)}(I^2)) = S^{(\max(0),\beta_3)}(I^2) \rangle.$$

The $S_\alpha$-case of the assertions (iii) up to and including (vi) follows from Kashpirovskii’s intersection result and the above results.

(vii) Suppose $\lambda_i \neq 0$, $i = 1, \ldots, 4$.

Then $\Lambda = \Lambda_1 \Lambda_2$ with $\Lambda_1 = \begin{pmatrix} \lambda_2/\lambda_4 & 1 \\ 1 & 0 \end{pmatrix}$ and $\Lambda_2 = \begin{pmatrix} \lambda_3 & \lambda_4 \\ \lambda_1 - \lambda_2 \lambda_3/\lambda_4 & 0 \end{pmatrix}$.

So by (iii) we obtain

$$Z_\Lambda(S_0^\beta(I^2)) = Z_{\Lambda_1} \circ Z_{\Lambda_2} \subseteq \langle Z_{\Lambda_1}(S_{(\max(0),\beta_3)}(I^2)) \subseteq \langle S_{(\max(0),\max(0))}(I^2) \rangle.$$

If $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ then the operators $Z_\Lambda$ are bijections by Theorem 2.1.6. For the converse, suppose $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$. We consider the case $\alpha_1 = \alpha_2 = \infty$ and $\lambda_i \neq 0$, $i = 1, \ldots, 4$. Let $\phi \in S_{(\max(0),\max(0))}^\beta(I^2)$. Then $Z_\Lambda(Z_{\Lambda^{-1}} \phi) = \phi$ with $Z_{\Lambda^{-1}} \phi \in S_{(\max(0),\max(0))}^\beta(I^2) \subseteq S_\alpha^\beta(I^2)$. The other cases run similarly.
2.2. Functional analytic characterizations of the spaces $S_{\alpha}(\mathbb{R}^2)$, $S_{\beta}(\mathbb{R}^2)$ and $S_{\alpha}(\mathbb{R}^2)$

Let $A, B$ denote self-adjoint operators in a Hilbert space $X$. In Section 1.2 we introduced the analyticity domain $D^{\alpha}(A)$. In addition here we introduce the joint analyticity domain $D^{\alpha}(A,B)$ which consists of all $v \in D^{\alpha}(A,B)$ with the properties

$$\exists_{C,D,E>0} \forall_{n \in \mathbb{N}} \forall_{k,l \in \mathbb{N}^2} : \|A^{k_1} B^{l_1} \cdots A^{k_n} B^{l_n} v\|_X \leq C D^{k_1} E^{l_1} \|k\| \|l\|.$$  

(2.10)

The subspace $D^{\alpha}(A,B)$ will be trivial, in general. We mention the following characterization.

Characterization 2.2.1.

Let $A$ and $B$ be strongly commuting self-adjoint operators. Let $v \in D^{\alpha}(A,B)$. Then for each $\alpha, \beta > 0$,

$$v \in D^{\alpha}(|A|^\alpha, |B|^\beta)$$

if and only if $\exists_{C, D, E > 0} \forall_{k, l \in \mathbb{N}_0}$:

$$\|A^{k_1} B^{l_1} v\|_X \leq C D^{k_1} E^{l_1} \|k\| \|l\|.$$  

This characterization is the analog of (1.6).

For all $\alpha, \beta > 0$ the space $D^{\alpha}(|A|^\alpha, |B|^\beta)$ is nontrivial, due to the spectral theorem. In the one-variable case Van Eijndhoven’s intersection results

$$S_{\alpha}(\mathbb{R}) = D^{\alpha}(P) \cap D^{\alpha}(1Q^{1/\alpha}) \quad (\alpha > 0)$$

$$S_{\beta}(\mathbb{R}) = D^{\alpha}(1P^{1/\beta}) \cap D^{\alpha}(Q) \quad (\beta > 0)$$

$$S_{\alpha}(\mathbb{R}) = D^{\alpha}(1P^{1/\beta}) \cap D^{\alpha}(1Q^{1/\alpha}) \quad (\alpha, \beta > 0, \alpha + \beta \geq 1)$$

are valid. Here we prove the corresponding statements for the two-variable case. In the proof we do not utilize Kashpirovskii’s intersection result, but we deduce it from the characterizations, which we present in the next theorem.

Theorem 2.2.2.

Let $\alpha, \beta \in \mathbb{R}_+^2$ with $\alpha_i, \beta_i > 0 \quad (i = 1, 2)$. Then

(i) $S_{\alpha}(\mathbb{R}^2) = D^{\alpha}(1Q_1^{1/\alpha_1}, 1Q_2^{1/\alpha_2}) \cap D^{\alpha}(P_1, P_2)$

(ii) $S_{\beta}(\mathbb{R}^2) = D^{\alpha}(Q_1, Q_2) \cap D^{\alpha}(1P_1^{1/\beta_1}, 1P_2^{1/\beta_2})$

(iii) $S_{\alpha}(\mathbb{R}^2) = D^{\alpha}(1Q_1^{1/\alpha_1}, 1Q_2^{1/\alpha_2}) \cap D^{\alpha}(1P_1^{1/\beta_1}, 1P_2^{1/\beta_2}),$

provided that $\alpha + \beta \geq 1$.

Proof.

Let $\psi \in S(\mathbb{R}^2)$. By Leibnitz’s differentiation rule we have for all $k, l \in \mathbb{N}_0^2$, 

$$\|A^{k_1} B^{l_1} \psi\|_X \leq C D^{k_1} E^{l_1} \|k\| \|l\|.$$  

(2.10)
Note that multi-index notation is used.

Let $k, l \in \mathbb{N}_0$. We estimate

\[
p^I Q^k p^l \psi = \sum_{j=0}^{\min(2k,l)} \binom{l}{j} \binom{2k}{j} j!(i)^j \sup_{\mathbb{R}^2} |Q^{2k-j} p^{2l-j} \psi|.
\]

We use this inequality in the proofs of statements (i) and (iii).

(i) Since the operators $Q_1$ and $Q_2$ strongly commute, it follows from Characterization 2.2.1 that for each $\phi \in D^\infty(Q_1, Q_2)$ we have

\[
\phi \in D^\omega(1 Q_1 \cdot 1 Q_2) \quad \text{if and only if} \quad \exists C, D > 0 \forall k \in \mathbb{N}_0^2 : \|Q^k \phi\|_{L^2(\mathbb{R}^2)} \leq C D^{k!} (k!)^a.
\]

Let $\phi \in S_0(\mathbb{R}^2)$. Then $\phi \in S(\mathbb{R}^2) \subset D^\infty(P_1, P_2)$. Furthermore, from Definition 2.1.2 we obtain

\[
\exists A, B > 0 \forall k \in \mathbb{N}_0^2 : \|Q^k \phi\|_{L^2(\mathbb{R}^2)} \leq B A^{k!} (k!)^a.
\]

So $\phi \in D^\omega(1 Q_1 \cdot 1 Q_2) \subset D^\infty(1 Q_1, 1 Q_2) \cap D^\infty(Q_1, Q_2)$. Since $\phi \in D^\omega(1 Q_1 \cdot 1 Q_2)$ there exist $C, D > 0$ such that

\[
\forall k \in \mathbb{N}_0^2 : \|Q^k \phi\|_{L^2(\mathbb{R}^2)} \leq C D^{k!} (k!)^a.
\]

Let $k, l \in \mathbb{N}_0^2$. Using (*) we estimate

\[
\|Q^k p^l \phi\|_{L^2(\mathbb{R}^2)} \leq \sum_{j=0}^{\min(2k,l)} \binom{l}{j} \binom{2k}{j} j! \|P^{2l-j} \phi\|_{L^2(\mathbb{R}^2)} \|Q^{2k-j} \phi\|_{L^2(\mathbb{R}^2)} \leq
\]

\[
\leq \sum_{j=0}^{\min(2k,l)} \binom{l}{j} 2^{2l!} j! \|P^{2l-j} \phi\|_{L^2(\mathbb{R}^2)} \cdot C D^{2k-j!} (2k-j!)^a \leq
\]

\[
\leq C (2D)^{2l!} ((2k!)^a \sum_{j=0}^{l} \binom{l}{j} j! \|P^{2l-j} \phi\|_{L^2(\mathbb{R}^2)}.
\]

Since $(2n)! \leq 2^{2n} (n!)^2$ for all $n \in \mathbb{N}_0$, the above estimation yields

\[
\|Q^k p^l \phi\|_{L^2(\mathbb{R}^2)} \leq B A^{k!} (k!)^a
\]

where
A = 2^{|a|+1} D$ and $B_l = \left( C \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} j! \| P^{2l-j} \phi \|_{L_2(L^2)} \right)^{1/2}.

Note that $B_l < \infty$ because $\phi \in D^\infty(P_1, P_2)$. Hence $\phi \in S_\alpha(R^2)$, by Definition 2.1.2.

(ii) We obtain characterization (ii) from (i) by applying Fourier transformation.

(iii) By Definition 2.1.1 it is clear that $S_\alpha^0(R^2) \subset S_\alpha(R^2) \cap S^0(R^2)$.

So characterizations (i) and (ii) yield

$$S_\alpha^0(R^2) \subset D^\infty(|Q_1 1^{1/\alpha}|, |Q_2 1^{1/\alpha}|) \cap D^\infty(|P_1 1^{1/\beta_1}|, |P_2 1^{1/\beta_2}|).$$

Conversely, let $\phi \in D^\infty(|Q_1 1^{1/\alpha}|, |Q_2 1^{1/\alpha}|) \cap D^\infty(|P_1 1^{1/\beta_1}|, |P_2 1^{1/\beta_2}|)$. Then there exist $D, E > 0$ such that for all $k, l \in N_0$

$$\|Q^k \phi\|_{L_2(R^2)} \leq D E^{1/k} (k!)^\alpha$$

and

$$\|P^l \phi\|_{L_2(R^2)} \leq D E^{1/l} (l!)^\beta.$$

Let $k, l \in N_0$. Using (*) we estimate

$$\|Q^k P^l \phi\|_{L_2(R^2)}^2 \leq \sum_{j=0}^{\min(2k, l)} \begin{bmatrix} l \\ j \end{bmatrix} \begin{bmatrix} 2k \\ j \end{bmatrix} j! \| P^{2l-j} \phi \|_{L_2(R^2)} \| Q^{2k-j} \phi \|_{L_2(R^2)} \leq \sum_{j=0}^{\min(2k, l)} \begin{bmatrix} l \\ j \end{bmatrix} \begin{bmatrix} 2k \\ j \end{bmatrix} j! \frac{(2k-j)!!}{((2l-j)!!)^{\alpha}} \cdot \frac{(2l-j)!!}{((2l)!!)^{\beta}} \leq D^2 E^{2((k+1)l)/(k+1)} \frac{((2k)!!)^{\alpha}}{((2l)!!)^{\beta}} \sum_{j=0}^{\min(2k, l)} \begin{bmatrix} l \\ j \end{bmatrix} \begin{bmatrix} 2k \\ j \end{bmatrix} \begin{bmatrix} 2k \\ j \end{bmatrix} j! \frac{(2k-j)!!}{((2l)!!)^{\beta}} \leq D^2 E^{2((k+1)l)/(k+1)} 2^{((2\alpha+1+\beta+1)/2)l} (l!)^{2\alpha} \sum_{j=0}^{\min(2k, l)} \begin{bmatrix} l \\ j \end{bmatrix} \begin{bmatrix} 2k \\ j \end{bmatrix} j! \frac{(2k-j)!!}{((2l)!!)^{\beta}} \leq D^2 E^{2((k+1)l)/(k+1)} 2^{((2\alpha+1+\beta+1)/2)l} (l!)^{2\alpha} \leq$$

Thus we find that

$$\|Q^k P^l \phi\|_{L_2(R^2)} \leq D (E \cdot 2^{1/2+1/k})^{1/(k+1)} (E \cdot 2^{1/2+1/\alpha})^{1/(1/\alpha)} (l!)^{\alpha} (l!)^\beta.$$

Hence $\phi \in S_\alpha^0(R^2)$, by Definition 2.1.2. 

As a consequence of the preceding theorem we obtain Kashpirovskii's intersection result

$$S_\alpha(R^2) \cap S^0(R^2) = S_\alpha^0(R^2).$$

Lemma 2.2.3.
Let \( \alpha, \beta \in \mathbb{R}^2_+ \) with \( \alpha_i, \beta_i > 0 \) \((i = 1, 2)\). Then

(i) \( D^{\alpha}(l Q_1^1 | 1^{1/\alpha_1}) \cap D^{\alpha}(l Q_2^1 | 1^{1/\alpha_2}) = D^{\alpha}((Q_1^2 + Q_2^2)^{2 \alpha_1}) \)

(ii) \( D^{\alpha}(l Q_1^1 | 1^{1/\alpha_1}) \cap D^{\alpha}(l Q_2^2 | 1^{1/\alpha_2}) = D^{\alpha}((Q_1^2 + Q_2^2)^{2 \alpha_2}) \)

(iii) \( D^{\alpha}(l P_1^1 | 1^{1/\beta_1}) \cap D^{\alpha}(l P_2^2 | 1^{1/\beta_2}) = D^{\alpha}((P_1^2 + P_2^2)^{2 \beta_1}) \)

(iv) \( D^{\alpha}(l P_1^2 | 1^{1/\beta_1}) \cap D^{\alpha}(l P_2^1 | 1^{1/\beta_2}) = D^{\alpha}((P_1^2 + P_2^2)^{2 \beta_2}) \)

Proof.

(i) It is clear that \( D^{\alpha}(l Q_1^1 | 1^{1/\alpha_1}) \cap D^{\alpha}(l Q_2^1 | 1^{1/\alpha_2}) \subset D^{\alpha}(l Q_1^1 | 1^{1/\alpha_1}) \cap D^{\alpha}(l Q_2^2 | 1^{1/\alpha_2}) \).

For the converse, suppose \( \phi \in D^{\alpha}(l Q_1^1 | 1^{1/\alpha_1}) \cap D^{\alpha}(l Q_2^2 | 1^{1/\alpha_2}) \). Then there exist \( C, D > 0 \) such that for all \( n \in \mathbb{N}_0 \)

\[
\| Q_i^k \phi \|_{L_2(\mathbb{R}^2)} \leq C D^{\alpha}(n^k) \quad ; i = 1, 2.
\]

Let \( k \in \mathbb{N}_0^2 \). Then we estimate

\[
\| Q_i^k \phi \|_{L_2(\mathbb{R}^2)} = (Q_i^{2k_1} \phi, Q_i^{2k_2} \phi)_{L_2(\mathbb{R}^2)} \leq \| Q_i^{2k_1} \phi \|_{L_2(\mathbb{R}^2)} \| Q_i^{2k_2} \phi \|_{L_2(\mathbb{R}^2)} \leq C D^{2k_1}(2k_1)^{1/\alpha_1} C D^{2k_2}(2k_2)^{1/\alpha_2}.
\]

Since \((2n)! \leq 2^{2n}(n!)^2\) we obtain from the above estimation

\[
\| Q_i^k \phi \|_{L_2(\mathbb{R}^2)} \leq C(2D)^{k_1} (k_1!)^\alpha.
\]

So \( \phi \in D^{\alpha}(l Q_1^1 | 1^{1/\alpha_1}) \cap D^{\alpha}(l Q_2^2 | 1^{1/\alpha_2}) \).

(ii) Obviously \( D^{\alpha}((Q_1^2 + Q_2^2)^{2 \alpha_1}) \subset D^{\alpha}(l Q_1^1 | 1^{1/\alpha_1}) \cap D^{\alpha}(l Q_2^2 | 1^{1/\alpha_2}) \).

For the converse, let \( \phi \in D^{\alpha}(l Q_1^1 | 1^{1/\alpha_1}) \cap D^{\alpha}(l Q_2^2 | 1^{1/\alpha_2}) \). Then there exists \( t > 0 \) such that the functions

\[
x \mapsto \exp(t \cdot l_{x_i} | 1^{1/\alpha_i}) \phi(x) \quad , i = 1, 2,
\]

belong to \( L_2(\mathbb{R}^2) \). Therefore,

\[
\iint_{\mathbb{R}^2} \left| \exp(t \cdot 2^{-2\alpha_1} (x_1^2 + x_2^2)^{2\alpha_1}) \phi(x) \right|^2 dx \leq \iint_{l_{x_1} l_{x_2}} \left| \exp(t \cdot 2^{-2\alpha_1} (2x_1^2)^{2\alpha_1}) \phi(x) \right|^2 dx +
\]

\[
\leq \iint_{l_{x_1} l_{x_2}} \left| \exp(t \cdot 2^{-2\alpha_1} (2x_1^2)^{2\alpha_1}) \phi(x) \right|^2 dx +
\]

\[
\int_{|x_1|, |x_2| < \frac{1}{2a_1}} \left| \exp(t \cdot 2^{2a_1} (2x_1^2 + 2x_2^2) \right| \phi(x) \, dx \leq \int_{\mathbb{R}^2} \left| \exp(t \cdot |x_1|^{1/a_1}) \right| \phi(x) \, dx + \int_{\mathbb{R}^2} \left| \exp(t \cdot |x_2|^{1/a_2}) \right| \phi(x) \, dx < \infty.
\]

Hence \( \phi \in D^\alpha((Q_1^2 + Q_2^2)^{2a_1}). \)

The assertions (iii) and (iv) follow from assertions (i) and (ii) respectively by applying Fourier transformation.

As a consequence we present the following intersection characterizations for the Gel'fand-Shilov spaces in two variables.

**Characterization 2.2.4.**

Let \( \alpha, \beta \in \mathbb{R}^2 \), \( \alpha_i, \beta_i > 0 \), \( i = 1,2 \). Then

**I**

(i) \( S_\alpha(\mathbb{R}^2) = D^\alpha(1|Q_1|^{1/a_1}, 1|Q_2|^{1/a_2}) \cap D^\alpha(P_1, P_2) \)

(ii) \( S_\alpha(\mathbb{R}^2) = D^\alpha(1|Q_1|^{1/a_1}) \cap D^\alpha(1|Q_2|^{1/a_2}) \cap D^\alpha(P_1) \cap D^\alpha(P_2) \)

(iii) \( S(\alpha, \alpha)(\mathbb{R}^2) = D^\alpha((Q_1^2 + Q_2^2)^{2a_1}) \cap D^\alpha(P_1^2 + P_2^2) \)

**II**

(i) \( S_\beta(\mathbb{R}^2) = D^\beta(Q_1, Q_2) \cap D^\beta(P_1^{1/\beta_1}, P_2^{1/\beta_2}) \)

(ii) \( S_\beta(\mathbb{R}^2) = D^\alpha(Q_1) \cap D^\alpha(Q_2) \cap D^\alpha(P_1^{1/\beta_1}) \cap D^\alpha(P_2^{1/\beta_2}) \)

(iii) \( S(\beta, \beta)(\mathbb{R}^2) = D^\alpha(Q_1^2 + Q_2^2) \cap D^\alpha(P_1^2 + P_2^2)^{2\beta_1}. \)

**III**

Let, in addition \( \alpha + \beta \geq 1 \). Then

(i) \( S_{(\alpha_1, \alpha_2)}(\mathbb{R}^2) = D^\alpha(1|Q_1|^{1/a_1}, 1|Q_2|^{1/a_2}) \cap D^\alpha(1|P_1|^{1/\beta_1}, 1|P_2|^{1/\beta_2}) \)

(ii) \( S_{(\alpha_1, \alpha_2)}(\mathbb{R}^2) = D^\alpha(1|Q_1|^{1/a_1}) \cap D^\alpha(1|Q_2|^{1/a_2}) \cap D^\alpha(1|P_1|^{1/\beta_1}) \cap D^\alpha(1|P_2|^{1/\beta_2}) \)

(iii) \( S_{(\beta_1, \beta_2)}(\mathbb{R}^2) = D^\alpha((Q_1^2 + Q_2^2)^{2\alpha_1}) \cap D^\alpha((P_1^2 + P_2^2)^{2\beta_1}). \)

Using the same techniques as in [EE], the following theorem can be proved.

**Theorem 2.2.5.**

Let \( \alpha \in \mathbb{R}^2_+ \) with \( \alpha_1, \alpha_2 \geq \frac{1}{2} \). Then

\[
S_\alpha(\mathbb{R}^2) = D^\alpha((Q_1^2 + P_1^2)^{2\alpha_1}) \cap D^\alpha((Q_2^2 + P_2^2)^{2\alpha_2}).
\]

If in addition \( \alpha_1 = \alpha_2 \), then

\[
S_\alpha(\mathbb{R}^2) = D^\alpha((Q_1^2 + P_1^2)^{2\alpha_1}) \cap D^\alpha((Q_2^2 + P_2^2)^{2\alpha_2}).
\]
Next we present two applications of Characterization 2.2.4.

**Theorem 2.2.6.**

Let $\alpha, \beta, \delta, \gamma \in IR^2_+$. Then

(i) $S_\alpha(IR^2) \cap S_\delta(IR^2) = S_{\alpha \min \delta}(IR^2)$

(ii) $S^\beta(IR^2) \cap S^\gamma(IR^2) = S^\beta \min \gamma(IR^2)$

(iii) $S^\alpha_\alpha(IR^2) \cap S^\delta_\delta(IR^2) = S^{\alpha \min \delta}(IR^2)$.

Here

$\alpha \ min \beta := (\min(a_1, b_1), \min(a_2, b_2))$

for each $a, b \in IR^2$.

We used the results of this theorem in the proof of Theorem 2.1.7 already. Note that Theorem 2.2.6 is an immediate consequence of the functional analytic Characterizations 2.2.4. However, they do not follow easily from the definitions of the Gel'fand Shilov spaces.

**Theorem 2.2.7.**

Let $\alpha, \beta \in IR^2_+$. $\alpha_i, \beta_i \neq 0$ ($i = 1, 2$).

I. Let $\phi \in C^\infty(IR^2)$. Then equivalent are

(i) $\phi \in S_\alpha(IR^2)$

(ii) $\exists c_1, c_2 > 0 : \sup_{x \in IR^2} | \exp(c_1 |x_1|^{1/\alpha_1} + c_2 |x_2|^{1/\alpha_2}) \phi(x)| < \infty$ and

$\forall k \in N_+ : \sup_{x \in IR^2} | x^k (\langle F \phi \rangle (x)| < \infty$.

II. Let $\psi \in C^\infty(IR^2)$. Then equivalent are

(i) $\psi \in S^\beta(IR^2)$

(ii) $\forall k \in N_+ : \sup_{x \in IR^2} | x^k \psi(x)| < \infty$ and

$\exists c_1, c_2 > 0 : \sup_{x \in IR^2} | \exp(c_1 |x_1|^{1/\beta_1} + c_2 |x_2|^{1/\beta_2}) (\langle F \psi \rangle (x)| < \infty$. 
III. Let \( \chi \in C^\infty (\mathbb{R}^2) \) and suppose \( \alpha + \beta \geq 1 \). Then equivalent are

(i) \( \phi \in S_0^\alpha (\mathbb{R}^2) \)

(ii) \( \exists c_1, c_2 > 0 : \sup_{x \in \mathbb{R}^2} | \exp(c_1 |x_1|^{1/\alpha} + c_2 |x_2|^{1/\alpha}) \chi(x) | < \infty \) and

\[
\exists d_1, d_2 > 0 : \sup_{x \in \mathbb{R}^2} | \exp(d_1 |x_1|^{1/\beta} + d_2 |x_2|^{1/\beta}) (\mathcal{F} \chi)(x) | < \infty.
\]

**Proof.**

(i) Suppose \( \phi \in S_0(\mathbb{R}^2) \). Then also the functions \( P_1 \phi, P_2 \phi \) and \( P_1 P_2 \phi \) belong to \( S_0(\mathbb{R}^2) \).

From Characterization 2.2.4.1 (ii) it follows that

\[
S_0(\mathbb{R}^2) \subset D^\alpha (|Q_1|^{1/\alpha}) \cap D^\alpha (|Q_2|^{1/\alpha}).
\]

So there exist \( c_1, c_2 > 0 \) such that for each \( f \in \{ \phi, \phi_x, \phi_x, \phi_x x, \phi_x x \} \),

\[
\left\| \int_{\mathbb{R}^2} | \exp(2 c_1 |x_1|^{1/\alpha} + c_2 |x_2|^{1/\alpha}) f(x) |^2 \, dx \right\| < \infty (i = 1, 2).
\]

Hence we obtain by the Cauchy-Schwarz inequality,

\[
\left\| \int_{\mathbb{R}^2} | \exp(c_1 |x_1|^{1/\alpha} + c_2 |x_2|^{1/\alpha}) f(x) |^2 \, dx \right\| < \infty
\]

for each \( f \in \{ \phi, \phi_x, \phi_x, \phi_x x, \phi_x x \} \).

Let \( x \in \mathbb{R}^2 \). Consider the following estimation

\[
\exp(2 c_1 |x_1|^{1/\alpha}) | \phi(x) |^2 = -2 \exp(2 c_1 |x_1|^{1/\alpha}) \int_{\mathbb{R}^2} \text{Re}(\phi(t, x_2) \phi_x(t, x_2)) \, dt \leq
\]

\[
\leq 2 \int_{\mathbb{R}^2} \exp(2 c_1 |t|^{1/\alpha}) | \phi(t, x_2) \phi_x(t, x_2) | \, dt \leq
\]

\[
\leq 2 \left( \int_{\mathbb{R}^2} \exp(2 c_1 |t|^{1/\alpha}) | \phi(t, x_2) |^2 \, dt \cdot \int_{\mathbb{R}^2} \exp(2 c_1 |t|^{1/\alpha}) | \phi_x(t, x_2) |^2 \, dt \right)^{1/2} \leq
\]

\[
\leq \sqrt{2} \left( \int_{\mathbb{R}^2} \exp(2 c_1 |t|^{1/\alpha}) | \phi(t, x_2) |^2 \, dt + \int_{\mathbb{R}^2} \exp(2 c_1 |t|^{1/\alpha}) | \phi_x(t, x_2) |^2 \, dt \right).
\]

Applying this technique once more we obtain for each \( x \in \mathbb{R}^2 \),

\[
\exp(2 c_1 |x_1|^{1/\alpha} + 2 c_2 |x_2|^{1/\alpha}) | \phi(x) |^2 \leq
\]

\[
\leq 2 \int_{\mathbb{R}^2} \exp(2 c_1 |t|^{1/\alpha} + 2 c_2 |s|^{1/\alpha}) | \phi(t) |^2 \, ds \int_{\mathbb{R}^2} \exp(2 c_1 |t|^{1/\alpha} + 2 c_2 |s|^{1/\alpha}) | \phi(s) |^2 \, dt.
\]
Note that $M < \infty$ and that $M$ does not depend on $x$. Hence

$$\sup_{x \in \mathbb{R}^2} \left| \exp(c_1 \|x\|^{1/\alpha} + c_2 \|x_2\|^{1/\alpha}) \phi(x) \right| \leq \sqrt{2M < \infty}.$$ 

Furthermore, since $\phi \in S_\alpha(\mathbb{R}^2)$ we have $IF \phi \in S_\alpha(\mathbb{R}^2)$ and also $Q_k IF \phi \in S_\alpha(\mathbb{R}^2)$ for each $k \in \mathbb{N}_0$. So

$$\sup_{x \in \mathbb{R}^2} \left| x^k(IF \phi)(x) \right| < \infty \quad \text{for each } k \in \mathbb{N}_0^2.$$ 

For the converse, suppose (ii) holds. There exist $c_1, c_2 > 0$ such that

$$\sup_{x \in \mathbb{R}^2} \left| \exp(c_1 \|x\|^{1/\alpha} + c_2 \|x_2\|^{1/\alpha}) \phi(x) \right| = \hat{M} < \infty.$$ 

So

$$\begin{align*}
\int_{\mathbb{R}^2} \left| \exp\left(\frac{1}{2} c_1 \|x\|^{1/\alpha} + c_2 \|x_2\|^{1/\alpha}\right) \phi(x) \right|^2 \, dx &= \int_{\mathbb{R}^2} \left| \exp\left(-\frac{1}{2} c_1 \|x\|^{1/\alpha} - c_2 \|x_2\|^{1/\alpha}\right) \phi(x) \right|^2 \, dx \\
&\leq \hat{M}^2 \int_{\mathbb{R}^2} \exp(-c_1 \|x\|^{1/\alpha}) \exp(-c_2 \|x_2\|^{1/\alpha}) \, dx < \infty.
\end{align*}$$

Thus we obtain that $\phi \in D^{\alpha}(1 \|x\|^{1/\alpha})$ and similarly $\phi \in D^{\alpha}(1 \|x_2\|^{1/\alpha})$.

Let $k \in \mathbb{N}_0$. There exists $B_k > 0$ such that

$$\sup_{x \in \mathbb{R}^2} \left| x^k(1 + x_1^2)(1 + x_2^2)(IF \phi)(x) \right| < B_k.$$ 

Therefore,

$$\begin{align*}
\int_{\mathbb{R}^2} \left| x^k(IF \phi)(x) \right|^2 \, dx &\leq \left\{ \int_{\mathbb{R}^2} \left| x^k(1 + x_1^2)(1 + x_2^2)(IF \phi)(x) \right|^4 \left(\frac{1}{(1 + x_1^2)(1 + x_2^2)}\right)^2 \, dx \right\}^{\frac{1}{4}}.
\end{align*}$$
Thus we obtain $\mathcal{IF} \phi \in D^\omega(Q_1, Q_2)$, or equivalently $\phi \in D^\omega(P_1, P_2)$. Hence $\phi \in D^\omega(IQ_1 \cap Q_2) \cap D^\omega(Q_1 \cap \I) \cap D^\omega(P_1, P_2) = S_\alpha(\mathbb{R}^2)$.

(II) Noting that $\mathcal{IF}(S^\beta(\mathbb{R}^2)) = S_\beta(\mathbb{R}^2)$ and that $(\mathcal{IF} \psi)(x) = \psi(-x)$ for all $x \in \mathbb{R}^2$, the equivalence of (i) and (ii) follows from I at once.

(III) This equivalence is a consequence of assertions I and II and Kashpirovskii’s intersection result.
2.3. Topological structure in the two-variable Gel'fand-Shilov spaces

Consider the functional analytic characterizations,

\[ S_\alpha(\mathbb{R}^2) = D^\alpha(1Q_1|^{1/\alpha}_1) \cap D^\alpha(1Q_2|^{1/\alpha}_2) \cap D^\alpha(P_1) \cap D^\alpha(P_2) \]
\[ S_\beta(\mathbb{R}^2) = D^\alpha(Q_1) \cap D^\alpha(Q_2) \cap D^\alpha(P_1|^{1/\beta}_1) \cap D^\alpha(P_2|^{1/\beta}_2) \]
\[ S_\delta(\mathbb{R}^2) = D^\alpha(1Q_1|^{1/\alpha}_1) \cap D^\alpha(1Q_2|^{1/\alpha}_2) \cap D^\alpha(1P_1|^{1/\delta}_1) \cap D^\alpha(1P_2|^{1/\delta}_2) \]

(cf. Characterization 2.2.4).

In Section 1.2, the spaces \( D^\alpha(1Q_1|^{1/\alpha}_1), D^\alpha(1P_1|^{1/\beta}_1), D^\alpha(Q_1), D^\alpha(P_1) (i = 1, 2) \) are topologyed. So, as in the one-variable case, it seems natural to endow a two-variable Gel'fand-Shilov space with an intersection topology. In the one-variable case we discussed the topological constructions at length. Therefore, in the two-variable case our treatment is rather brief.

An intersection topology for \( S_\alpha(\mathbb{R}^2) \)

Let \( \alpha \in \mathbb{R}_+^2 \) (\( \alpha_i > 0 \), \( i = 1, 2 \)). Then we have by Characterization 2.2.4,

\[ S_\alpha(\mathbb{R}^2) = D^\alpha(1Q_1|^{1/\alpha}_1) \cap D^\alpha(1Q_2|^{1/\alpha}_2) \cap D^\alpha(P_1) \cap D^\alpha(P_2). \]

We endow the space \( S_\alpha(\mathbb{R}^2) \) with the intersection topology \( T_{\alpha,\cap} \) which is the weakest topology in \( S_\alpha(\mathbb{R}^2) \) for which the inclusion maps \( (2.12) \) are continuous.

Let \( A > 0 \). We define the space \( S_{\alpha,A}(\mathbb{R}^2) \) by

\[ S_{\alpha,A}(\mathbb{R}^2) := \{ \phi \in S_\alpha(\mathbb{R}^2) \mid \forall l \in \mathbb{N}_0^2 \exists \beta_i > 0 \forall k \in \mathbb{N}_0^2 : \|Q^k P^l \phi\|_{L^2(\mathbb{R}^2)} \leq B_i A^{1/k}(k!)^\alpha \}. \]

In the space \( S_{\alpha,A}(\mathbb{R}^2) \) we introduce a system of norms, \( \| \cdot \|_{\alpha,A,l} \), \( l \in \mathbb{N}_0^2 \), by

\[ \| \phi \|_{\alpha,A,l} := \sup_{k \in \mathbb{N}_0} \frac{\|Q^k P^l \phi\|_{L^2(\mathbb{R}^2)}}{A^{1/k}(k!)^\alpha}, l \in \mathbb{N}_0^2, \phi \in S_{\alpha,A}(\mathbb{R}^2). \]

This family of norms induces a locally convex topology in \( S_{\alpha,A}(\mathbb{R}^2) \) which we denote by \( \text{Top}_{\alpha,A} \).

In this way, \( (S_{\alpha,A}(\mathbb{R}^2), \text{Top}_{\alpha,A}) \) becomes a complete countably normed space.

Note that the inclusion maps \( (2.15) \) are continuous. Furthermore, since each function in \( S_\alpha(\mathbb{R}^2) \) belongs to some \( S_{\alpha,A}(\mathbb{R}^2) \) we obtain
Lemma 2.3.1.
Let \( \alpha \in \mathbb{R}^2 \) (\( \alpha_i > 0 \), \( i = 1, 2 \)) and let \( A \geq 1 \). Let \( \phi \in S_{\alpha A}(\mathbb{R}^2) \). Then we have

\[
(i) \quad \sup_{k \in B^\alpha_0} \frac{\|Q^k \phi\|_{L^\alpha_2(\mathbb{R}^2)}}{A^{k_1}(k_1)^\alpha} = \|\phi\|_{\alpha A, (0,0)} \quad \text{and} \quad \|P^l \|_{L^\alpha_2(\mathbb{R}^2)} \leq \|\phi\|_{\alpha A, l} \quad \text{for all } l \in B^\alpha_0.
\]

\[
(ii) \quad \|\phi\|_{\alpha A, l} \leq \sqrt{2^{l/1^l-1}} \left( \sup_{k \in B^\alpha_0} \frac{\|Q^k \phi\|_{L^\alpha_2(\mathbb{R}^2)}}{A^{k_1}(k_1)^\alpha} + \sum_{j=l}^{2l} \|P^j \|_{L^\alpha_2(\mathbb{R}^2)} \right),
\]

for all \( l \in B^\alpha_0 \). Here \( \hat{A} = A \cdot 2^{\max(\alpha)+1} \) with \( \max(\alpha) = \max \{ \alpha_i \mid i = 1, 2 \} \).

Proof.
The proof runs similarly to the proof of Lemma 1.2.6. In the estimations we apply the techniques used in the proof of Theorem 2.2.2. \( \square \)

Lemma 2.3.2.
Let \( \alpha \in \mathbb{R}^2 \) (\( \alpha_i > 0 \), \( i = 1, 2 \)) and let \( A \geq 1 \). Let \( \phi \in S_{\alpha A}(\mathbb{R}^2) \). Then we have

\[
(i) \quad \sup_{k \in B^\alpha_0} \frac{\|Q^k_1 \phi\|_{L^\alpha_2(\mathbb{R}^2)}}{A^{k_1}(k_1)^\alpha} \leq \sup_{k \in B^\alpha_0} \frac{\|Q^k_2 \phi\|_{L^\alpha_2(\mathbb{R}^2)}}{A^{k_1}(k_1)^\alpha}, \quad (i = 1, 2),
\]

\[
(ii) \quad \sup_{k \in B^\alpha_0} \frac{\|Q^k_1 \phi\|_{L^\alpha_2(\mathbb{R}^2)}}{(2A)^{k_1}(k_1)^\alpha} \leq \frac{1}{\sqrt{2}} \left( \sup_{k_2 \in B^\alpha_0} \frac{\|Q^k_1 \phi\|_{L^\alpha_2(\mathbb{R}^2)}}{A^{k_1}(k_1)^\alpha} + \sup_{k_2 \in B^\alpha_0} \frac{\|Q^k_2 \phi\|_{L^\alpha_2(\mathbb{R}^2)}}{A^{k_2}(k_2)^\alpha} \right),
\]

\[
\|P^l \|_{L^\alpha_2(\mathbb{R}^2)} \leq \frac{1}{\sqrt{2}} (\|P^{2l}_1 \|_{L^\alpha_2(\mathbb{R}^2)} + \|P^{2l}_2 \|_{L^\alpha_2(\mathbb{R}^2)}) \quad \text{for all } l \in B^\alpha_0.
\]

Proof.
We only proof the first inequality of assertion (ii). Let \( k \in B^\alpha_0 \). Then

\[
\|Q^k \|_{L^\alpha_2(\mathbb{R}^2)} = (Q^{2k_1}_1 \phi, Q^{2k_2}_2 \phi)_{L^\alpha_2(\mathbb{R}^2)} \leq \|Q^{2k_1}_1 \|_{L^\alpha_2(\mathbb{R}^2)} \cdot \|Q^{2k_2}_2 \|_{L^\alpha_2(\mathbb{R}^2)}.
\]

Since for all \( m \in B^\alpha_0 \), \( (m!)^2 = \frac{(2m)!}{2m} \) we estimate

\[
\left[ \frac{2m!}{2^m m!} \right] \geq \frac{(2m)!}{2^m m!}.
\]
Whence we obtain the result. 

The following theorem contains several descriptions of sequential convergence in $S_\alpha(\mathbb{R}^2)$.

**Theorem 2.3.3.**

Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence in $S_\alpha(\mathbb{R}^2)$ ($\alpha_1, \alpha_2 > 0$). Then equivalent are

(i) $\phi_n \to 0$ ($n \to \infty$) in $(S_\alpha(\mathbb{R}^2), T_{\alpha, n})$

(ii) $\phi_n \to 0$ ($n \to \infty$) in each of the spaces $(D^\infty(1 Q_1 \overline{1}^{1/\alpha_1}), \sigma_{\text{ind}})$ and $(D^\infty(P_i), \tau_{\text{proj}})$ ($i = 1, 2$).

(iii) $\exists_{\lambda > 0} \forall_{n \in \mathbb{N}}: \phi_n \in \bigcap_{i=1}^2 (e^{-1} Q_1 \overline{1}^{1/\alpha_i} (L_2(\mathbb{R}^2)) \cap D^\infty(P_i))$

and $\phi_n \to 0$ in the corresponding intersection topology, i.e.

\[ \|e^{-1} Q_1 \overline{1}^{1/\alpha_i} \|_{L_2(\mathbb{R}^2)} \to 0 \quad (n \to \infty) \] and

\[ \|P_i^I \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \quad (n \to \infty) \] for all $i \in \mathbb{N}_0$, ($i = 1, 2$).

(iv) $\exists_{\lambda > 0} \forall_{n \in \mathbb{N}}: \phi_n \in \bigcap_{i=1}^2 (D^\infty(1 Q_1 \overline{1}^{1/\alpha_i}; A) \cap D^\infty(P_i))$

and $\phi_n \to 0$ ($n \to \infty$) in the corresponding intersection topology, i.e.

\[ \sup_{k \in \mathbb{N}_0} \|Q_k^I \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \quad (n \to \infty) \] and

\[ \|P_i^I \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \quad (n \to \infty) \] for all $i \in \mathbb{N}_0$, ($i = 1, 2$).

(v) $\exists_{\lambda > 0} \forall_{n \in \mathbb{N}}: \phi_n \in \bigcap_{i=1}^2 (D^\infty(1 Q_1 \overline{1}^{1/\alpha_i}; A) \cap D^\infty(P_i))$

and

\[ \sup_{k \in \mathbb{N}_0} \|Q_k^I \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \quad (n \to \infty) \] and

\[ \|P_i^I \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \quad (n \to \infty) \] for all $i \in \mathbb{N}_0$. 


(vi) \( \exists \alpha_0 > 0, \forall n \in \mathbb{N} : \phi_n \in S_{\alpha, A}(\mathbb{R}^2) \)
and \( \phi_n \to 0 \quad (n \to \infty) \) in the topology \( \text{Top}_{\alpha, A} \) of \( S_{\alpha, A}(\mathbb{R}^2) \), i.e.
\[
\sup_{k \in \mathbb{N}^2} \frac{\|Q_k P^l \phi_n\|_{L_2(\mathbb{R}^2)}}{A^{lk} (k!)^a} \to 0 \quad (n \to \infty) \quad \text{for all } l \in \mathbb{N}_0.
\]

If, in addition \( \alpha_1 = \alpha_2 \) then each of the above assertions is equivalent with

(vii) \( \exists \alpha_0 > 0, \forall n \in \mathbb{N} : \phi_n \in e^{-\ell(Q_1^2 + Q_2^2)}(L_2(\mathbb{R}^2)) \cap D^\infty(P_1^2 + P_2^2) \)
and \( \phi_n \to 0 \quad (n \to \infty) \) in the corresponding intersection topology, i.e.
\[
\|e^{\ell(Q_1^2 + Q_2^2)} \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \quad (n \to \infty) \quad \text{and}
\|P_1^2 + P_2^2 \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \quad (n \to \infty) \quad \text{for all } l \in \mathbb{N}_0.
\]

\[
\|
\]

**Remark 2.3.4.**

Applying standard techniques it can be seen that the \( L_2(\mathbb{R}^2) \)-norms in the above theorem may be replaced by supremum norms.

### An intersection topology for \( S^\beta(\mathbb{R}^2) \)

Let \( \beta \in \mathbb{R}_+^2 \) \((\beta_i > 0, i = 1, 2)\). Then we have by Characterization 2.2.4,
\[
S^\beta(\mathbb{R}^2) = D^\infty(Q_1) \cap D^\infty(Q_2) \cap D^\infty(1 P_1^1 1^{\beta_1}) \cap D^\infty(1 P_2^1 1^{\beta_2}).
\]

We endow the space \( S^\beta(\mathbb{R}^2) \) with the intersection topology \( T^\beta \), which is the weakest topology in \( S^\beta(\mathbb{R}^2) \) for which the inclusion maps
\[
S^\beta(\mathbb{R}^2) \hookrightarrow D^\infty(Q_i) \quad \text{and} \quad S^\beta(\mathbb{R}^2) \hookrightarrow D^\infty(1 P_i^1 1^{\beta_i}) \quad , \quad i = 1, 2,
\]
are continuous.

Let \( B > 0 \). We define the space \( S^{B, \beta}(\mathbb{R}^2) \) by
\[
S^{B, \beta}(\mathbb{R}^2) := \{ \phi \in S^\beta(\mathbb{R}^2) \mid \forall k \in \mathbb{N}_0^2 : \exists \alpha_k > 0, \forall \ell \in \mathbb{N}_0^2 : \|Q_k P^l \phi\|_{L_2(\mathbb{R}^2)} \leq A_k B^{1/1} (1^{\beta})^\beta \}.
\]

In the space \( S^{B, \beta}(\mathbb{R}^2) \) we introduce a system of norms, \( \| \cdot \|_{B, \beta, k} \), \( k \in \mathbb{N}_0^2 \), by
\[
\| \phi \|_{B, \beta, k} := \sup_{l \in \mathbb{N}_0^2} \frac{\|Q_k P^l \phi\|_{L_2(\mathbb{R}^2)}}{B^{1/1} (1^{\beta})^\beta}, \quad k \in \mathbb{N}_0^2 \quad , \quad \phi \in S^{B, \beta}(\mathbb{R}^2).
\]

This family of norms induces a locally convex topology in \( S^{B, \beta}(\mathbb{R}^2) \) which we denote by \( \text{Top}^{B, \beta} \).

In this way, \( (S^{B, \beta}(\mathbb{R}^2), \text{Top}^{B, \beta}) \) becomes a complete countably normed space.
Note that the inclusion maps

\[
S^{B,\beta}(\mathbb{R}^2) \hookrightarrow D^\infty(Q_i) \quad (i = 1, 2),
\]
\[
S^{B,\beta}(\mathbb{R}^2) \hookrightarrow D^\infty(1P_1^{1/\beta}; B) \quad (i = 1, 2) \quad \text{and}
\]
\[
S^{B,\beta_1}(\mathbb{R}^2) \hookrightarrow S^{B,\beta_2}(\mathbb{R}^2) \quad (B_1 < B_2)
\]

are continuous. Furthermore, since each function in \(S^{B}(\mathbb{R}^2)\) belongs to some \(S^{B,\beta}(\mathbb{R}^2)\) we obtain

\[
S^{B}(\mathbb{R}^2) = \bigcup_{B \geq 1} S^{B,\beta}(\mathbb{R}^2).
\] (2.21)

**Lemma 2.3.5.**

Let \(\beta \in \mathbb{R}_+^2 (\beta_i > 0, \ i = 1, 2)\) and let \(B \geq 1\). Let \(\phi \in S^{B,\beta}(\mathbb{R}^2)\). Then we have

(i) \[
\sup_{i \in \mathbb{N}^2} \frac{\|P^i \phi\|_{L^2(\mathbb{R}^2)}}{B^{1/1(1)^\beta}} = \|\phi\|_{B, (0, 0)}
\]

and

\[
\|Q^k \phi\|_{L^2(\mathbb{R}^2)} \leq \|\phi\|_{B, k} \quad \text{for all } k \in \mathbb{N}_0^2.
\]

(ii) \[
\|\phi\|_{B, k} \leq \sqrt{2^{2k} \cdot 1 - (2k)!} \left( \sup_{i \in \mathbb{N}_0} \frac{\|P^i \phi\|_{L^2(\mathbb{R}^2)}}{B^{1/1(1)^\beta}} + \sum_{j=0}^{2k} \|Q^j \phi\|_{L^2(\mathbb{R}^2)} \right),
\]

for all \(k \in \mathbb{N}_0^2\). Here \(B = B \cdot 2^{\max(\beta)+\frac{3}{2}}\) with \(\max(\beta) = \max\{\beta_i \mid i = 1, 2\}\).

**Proof.**

The proof runs similarly to the proof of Lemma 2.3.1.

**Lemma 2.3.6.**

Let \(\beta \in \mathbb{R}_+^2 (\beta_i > 0, \ i = 1, 2)\) and let \(B \geq 1\). Let \(\phi \in S^{B,\beta}(\mathbb{R}^2)\). Then we have

(i) \[
\sup_{i \in \mathbb{N}_0} \frac{\|P^i \phi\|_{L^2(\mathbb{R}^2)}}{B^{1/1(1)^\beta}} \leq \sup_{i \in \mathbb{N}_0} \frac{\|P^i \phi\|_{L^2(\mathbb{R}^2)}}{B^{1/1(1)^\beta}} \quad (i = 1, 2),
\]

\[
\|Q^k_1 \phi\|_{L^2(\mathbb{R}^2)} = \|Q^{(k_1, 0)} \phi\|_{L^2(\mathbb{R}^2)}, \ k_1 \in \mathbb{N}_0^2,
\]

\[
\|Q^k_2 \phi\|_{L^2(\mathbb{R}^2)} = \|Q^{(0, k_2)} \phi\|_{L^2(\mathbb{R}^2)}, \ k_2 \in \mathbb{N}_0^2.
\]

(ii) \[
\sup_{i \in \mathbb{N}_0} \frac{\|P^i \phi\|_{L^2(\mathbb{R}^2)}}{(2B)^{1/1(1)^\beta}} \leq \frac{1}{\sqrt{2}} \left( \sup_{i \in \mathbb{N}_0} \frac{\|P^i_1 \phi\|_{L^2(\mathbb{R}^2)}}{B^{1/1(1)^\beta}} + \sup_{i \in \mathbb{N}_0} \frac{\|P^i_2 \phi\|_{L^2(\mathbb{R}^2)}}{B^{1/1(1)^\beta}} \right),
\]

\[
\|Q^k \phi\|_{L^2(\mathbb{R}^2)} \leq \frac{1}{\sqrt{2}} (\|Q^{2k_1} \phi\|_{L^2(\mathbb{R}^2)} + \|Q^{2k_2} \phi\|_{L^2(\mathbb{R}^2)}), \ \text{for all } k \in \mathbb{N}_0^2.
\]

**Proof.**
The proof runs similarly to the proof of Lemma 2.3.2.

The following theorem contains several descriptions of sequential convergence in $S^B(\mathbb{R}^2)$.

**Theorem 2.3.7.**

Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence in $S^B(\mathbb{R}^2)$ ($\beta_1, \beta_2 > 0$). Then equivalent are

(i) $\phi_n \to 0 \ (n \to \infty)$ in $(S^B(\mathbb{R}^2), T^B_\alpha)$

(ii) $\phi_n \to 0 \ (n \to \infty)$ in each of the spaces $(D^\alpha(1P_1^{1/\beta_1}), \sigma_{\text{ind}})$ and $(D^\alpha(Q_i), \tau_{\text{proj}}) \ (i = 1, 2)$.

(iii) $\exists \varepsilon > 0 \ \forall n \in \mathbb{N} : \phi_n \in \bigcap_{i=1}^{2} (e^{\varepsilon i P_1^{1/\beta_1}} (L_2(\mathbb{R}^2)) \cap D^\alpha(Q_i))$

and $\phi_n \to 0 \ (n \to \infty)$ in the corresponding intersection topology, i.e.

$$\|e^{\varepsilon i P_1^{1/\beta_1}} \phi_n\|_{L_2(\mathbb{R}^2)} = \|e^{\varepsilon i Q_1^{1/\beta_1}} \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \ (n \to \infty)$$

$$\|Q_i^k \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \ (n \to \infty) \ \text{for all} \ k \in \mathbb{N}_0 \ , \ (i = 1, 2).$$

(iv) $\exists B > 0 \ \forall n \in \mathbb{N} : \phi_n \in \bigcap_{i=1}^{2} (D^\alpha(1P_1^{1/\beta_1}; B) \cap D^\alpha(Q_i))$

and $\phi_n \to 0 \ (n \to \infty)$ in the corresponding intersection topology, i.e.

$$\sup_{i \in \mathbb{N}_0} \|P_i^k \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \ (n \to \infty)$$

and

$$\|Q_i^k \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \ (n \to \infty) \ \text{for all} \ k \in \mathbb{N}_0 \ , \ (i = 1, 2).$$

(v) $\exists B > 0 \ \forall n \in \mathbb{N} : \phi_n \in \bigcap_{i=1}^{2} (D^\alpha(1P_1^{1/\beta_1}; B) \cap D^\alpha(Q_i))$

and

$$\sup_{i \in \mathbb{N}_0} \|P_i^k \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \ (n \to \infty)$$

and

$$\|Q_i^k \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \ (n \to \infty) \ \text{for all} \ k \in \mathbb{N}_0^2.$$
(vii) \( \exists x > 0 \) \( \forall n \in \mathbb{N} : \phi_n = e^{-\left( t^2 + P_2^2 \right)^{3/4}} \) \( L_2(\mathbb{R}^2) \cap \mathcal{D}^{\alpha}(Q^1_{3} + Q^2_{3}) \) \\
and \( \phi_n \to 0 \) \( (n \to \infty) \) in the corresponding intersection topology, i.e.

\[
\| e^{-\left( t^2 + P_2^2 \right)^{3/4}} \phi_n \|_{L^2(\mathbb{R}^2)} = \| e^{-\left( t^2 + P_2^2 \right)^{3/4}} \mathcal{F} \phi_n \|_{L^2(\mathbb{R}^2)} \to 0 \ (n \to \infty) \text{ and}
\]

\[
\| (Q^1_{3} + Q^2_{3})^k \phi_n \|_{L^2(\mathbb{R}^2)} \to 0 \ (n \to \infty) \text{ for all } k \in \mathbb{N}_0.
\]

\[\square\]

Remark 2.3.8.

Applying standard techniques it can be seen that the \( L_2(\mathbb{R}^2) \)-norms in the above theorem may be replaced by supremum norms.

The Fourier transform \( \mathcal{F} \) is a homeomorphism of \( \mathcal{S}_\alpha(\mathbb{R}^2) \) onto \( \mathcal{S}^\alpha(\mathbb{R}^2) \). (2.22)

**An intersection topology for \( \mathcal{S}^\alpha_0(\mathbb{R}^2) \)**

Let \( \alpha, \beta \in \mathbb{R}_+^2 \) \( (\alpha_i, \beta_i > 0, \ i = 1, 2) \) with \( \alpha + \beta \geq 1 \). Then we have by Characterization 2.2.4,

\[
\mathcal{S}^\alpha_0(\mathbb{R}^2) = D^{\alpha_1}(1Q_1 1^{1/\alpha_1}) \cap D^{\alpha_2}(1Q_2 1^{1/\alpha_2}) \cap D^\alpha(1P_1 1^{1/\beta_1}) \cap D^\alpha(1P_2 1^{1/\beta_2}).
\]

We endow the space \( \mathcal{S}^\alpha_0(\mathbb{R}^2) \) with the intersection topology \( T^\alpha_0 \cap \) which is the weakest topology in \( \mathcal{S}^\alpha_0(\mathbb{R}^2) \) for which the inclusion maps

\[
\mathcal{S}^\alpha_0(\mathbb{R}^2) \hookrightarrow D^{\alpha_1}(1Q_1 1^{1/\alpha_1}) \text{ and } \mathcal{S}^\alpha_0(\mathbb{R}^2) \hookrightarrow D^{\alpha_2}(1P_1 1^{1/\beta_1})
\]

\( (i = 1, 2) \) (2.23)

are continuous.

Let \( A, B > 0 \). We define the space \( \mathcal{S}^{\alpha, \beta}_{\alpha, \beta}(\mathbb{R}^2) \) by

\[
\mathcal{S}^{\alpha, \beta}_{\alpha, \beta}(\mathbb{R}^2) := \{ \phi \in \mathcal{S}^\alpha_0(\mathbb{R}^2) \mid \exists C > 0 \ \forall k, l \in \mathbb{N}_0 : \| Q^k P^l \phi \|_{L_2(\mathbb{R}^2)} \leq C A^{1k_1} B^{1l_1} (k!)^\alpha (l!)^\beta \}.
\] (2.24)

In the space \( \mathcal{S}^{\alpha, \beta}_{\alpha, \beta}(\mathbb{R}^2) \) we define the norm \( \| \phi \|_{\mathcal{S}^{\alpha, \beta}_{\alpha, \beta}(\mathbb{R}^2)} \) by

\[
\| \phi \|_{\mathcal{S}^{\alpha, \beta}_{\alpha, \beta}(\mathbb{R}^2)} := \sup_{k, l \in \mathbb{N}_0} \frac{\| Q^k P^l \phi \|_{L_2(\mathbb{R}^2)}}{A^{1k_1} B^{1l_1} (k!)^\alpha (l!)^\beta}, \ \phi \in \mathcal{S}^{\alpha, \beta}_{\alpha, \beta}(\mathbb{R}^2).
\] (2.25)

Then \( (\mathcal{S}^{\alpha, \beta}_{\alpha, \beta}(\mathbb{R}^2), \| \cdot \|_{\mathcal{S}^{\alpha, \beta}_{\alpha, \beta}(\mathbb{R}^2)} ) \) is a Banach space.

Note that the inclusion maps

\[
\mathcal{S}^{\alpha, \beta}_{\alpha, \beta}(\mathbb{R}^2) \hookrightarrow D^{\alpha_i}(1Q_i 1^{1/\alpha_i}; A) \quad (i = 1, 2),
\]

\[
\mathcal{S}^{\alpha, \beta}_{\alpha, \beta}(\mathbb{R}^2) \hookrightarrow D^{\alpha_i}(1P_i 1^{1/\beta_i}; B) \quad (i = 1, 2),
\]

\[
\mathcal{S}^{\alpha, \beta}_{\alpha, \beta}(\mathbb{R}^2) \hookrightarrow \mathcal{S}_{\alpha, \beta}(\mathbb{R}^2)
\]

\[
\mathcal{S}^{\alpha, \beta}_{\alpha, \beta}(\mathbb{R}^2) \hookrightarrow \mathcal{S}^{\alpha, \beta}_{\alpha, \beta}(\mathbb{R}^2) \quad (A_1 < A_2, B_1 < B_2)
\]

(2.26)
are continuous. Furthermore, since each function in $S_0^0 (\mathbb{R}^2)$ belongs to some $S_{\alpha \beta}^0 (\mathbb{R}^2)$ we obtain

$$S_0^0 (\mathbb{R}^2) = \bigcup_{\alpha \geq 1, \beta \geq 1} S_{\alpha \beta}^0 (\mathbb{R}^2).$$  \hspace{1cm} (2.27)

Lemma 2.3.9.

Let $\alpha, \beta \in \mathbb{R}^+ (\alpha_i, \beta_i > 0, i = 1, 2)$ with $\alpha + \beta \geq 1$ and let $A, B \geq 1$.

Let $\phi \in S_{\alpha \beta}^0 (\mathbb{R}^2)$. Then we have

(i) \[
\sup_{k \in \mathbb{N}^2} \frac{\|Q_k \phi\|_{L_2(\mathbb{R}^2)}}{A^{1/k} (k!)^\alpha} \leq \|\phi\|_{S_{\alpha \beta}^0} \quad \text{and} \quad \sup_{i \in \mathbb{N}^2} \frac{\|P_i \phi\|_{L_2(\mathbb{R}^2)}}{B^{1/(1)!} (1!)^\beta} \leq \|\phi\|_{S_{\alpha \beta}^0}.
\]

(ii) \[
\|\phi\|_{S_{\alpha \beta}^0} \leq \frac{1}{\sqrt{2}} \left( \sup_{k \in \mathbb{N}^2} \frac{\|Q_k \phi\|_{L_2(\mathbb{R}^2)}}{A^{1/k} (k!)^\alpha} + \sup_{i \in \mathbb{N}^2} \frac{\|P_i \phi\|_{L_2(\mathbb{R}^2)}}{B^{1/(1)!} (1!)^\beta} \right),
\]

here $\hat{A} = A \cdot 2^{\max(\alpha)+1}$ and $\hat{B} = B \cdot 2^{\max(\beta)+1}$.

Proof.

The proof runs similarly to the proof of Lemma 1.2.15.

The following theorem contains several descriptions of sequential convergence in $S_0^0 (\mathbb{R}^2)$.

Theorem 2.3.10.

Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence in $S_0^0 (\mathbb{R}^2)$ (\(\alpha_i, \beta_i > 0, \alpha_i + \beta_i \geq 1 \quad (i = 1, 2)\)).

Then equivalent are

(i) \[
\phi_n \to 0 \quad (n \to \infty) \quad \text{in} \quad (S_0^0 (\mathbb{R}^2), \mathcal{T}_{0 \cap})
\]

(ii) \[
\phi_n \to 0 \quad (n \to \infty) \quad \text{in each of the spaces} \quad (D^\omega (1Q_i^{1/\alpha_i}), \sigma_{\text{ind}}) \quad \text{and} \quad (D^\omega (1P_i^{1/\beta_i}), \sigma_{\text{ind}}), \quad (i = 1, 2).
\]

(iii) \[
\phi_n \to 0 \quad (n \to \infty) \quad \text{both in} \quad (S_0^0 (\mathbb{R}^2), \mathcal{T}_{0 \cap}) \quad \text{and in} \quad (S^\beta (\mathbb{R}^2), \mathcal{T}_\beta^0).
\]

(iv) \[
\exists \varepsilon > 0 \forall n \in \mathbb{N} : \phi_n \in \bigcap_{i=1}^2 \left( e^{-\varepsilon \Omega_{i}} (L_2 (\mathbb{R}^2)) \cap e^{-\varepsilon \Omega_{i}} (L_2 (\mathbb{R}^2)) \right)
\]

and $\phi_n \to 0 \quad (n \to \infty)$ in the corresponding intersection topology, i.e.

$$\|e^{-\varepsilon \Omega_{i}} \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \quad (n \to \infty), \quad i = 1, 2 \quad \text{and} \quad \|e^{-\varepsilon \Omega_{i}} \phi_n\|_{L_2(\mathbb{R}^2)} \to 0 \quad (n \to \infty), \quad i = 1, 2.$$
(v) \( \exists_{A, B > 0} \forall_{n \in \mathbb{N}} : \phi_n \in \bigcap_{i=1}^{2} (D^{\alpha_i}(\mathbb{R}^2; A) \cap D^{\alpha_i}(\mathbb{R}^2; B)) \)
and \( \phi_n \to 0 \ (n \to \infty) \)
in the corresponding intersection topology, i.e.
\[
\sup_{k \in \mathbb{N}_0} \frac{\|Q^k \phi_n\|_{L^2(\mathbb{R}^2)}}{A^k (k!)^\alpha} \to 0 \ (n \to \infty) \quad , \ i = 1, 2 \quad \text{and}
\]
\[
\sup_{l \in \mathbb{N}_0} \frac{\|P^l \phi_n\|_{L^2(\mathbb{R}^2)}}{B^l (l!)^\beta} \to 0 \ (n \to \infty) \quad , \ i = 1, 2.
\]

(vi) \( \exists_{A, B > 0} \forall_{n \in \mathbb{N}} : \phi_n \in \bigcap_{i=1}^{2} (D^{\alpha_i}(\mathbb{R}^2; A) \cap D^{\alpha_i}(\mathbb{R}^2; B)) \)
and
\[
\sup_{k \in \mathbb{N}_0} \frac{\|Q^k \phi_n\|_{L^2(\mathbb{R}^2)}}{(2A)^k (k!)^\alpha} \to 0 \ (n \to \infty) \quad \text{and}
\]
\[
\sup_{l \in \mathbb{N}_0} \frac{\|P^l \phi_n\|_{L^2(\mathbb{R}^2)}}{(2B)^l (l!)^\beta} \to 0 \ (n \to \infty).
\]

(vii) \( \exists_{A, B > 0} \forall_{n \in \mathbb{N}} : \phi_n \in \mathcal{S}_{0, A}^{0, B}(\mathbb{R}^2) \)
and \( \phi_n \to 0 \ (n \to \infty) \)
in the topology of \( \mathcal{S}_{0, A}^{0, B}(\mathbb{R}^2) \), i.e.
\[
\sup_{k, l \in \mathbb{N}_0} \frac{\|Q^k P^l \phi_n\|_{L^2(\mathbb{R}^2)}}{A^k B^l (k!)^\alpha (l!)^\beta} \to 0 \ (n \to \infty).
\]

If, in addition \( \alpha_1 = \alpha_2 \) and \( \beta_1 = \beta_2 \) then each of the above assertions is equivalent with

(viii) \( \exists_{t > 0} \forall_{n \in \mathbb{N}} : \phi_n \in e^{-t(Q_t^1 + Q_t^2)} \mathcal{S}_{0, A}^{0, B}(\mathbb{R}^2) \)
and \( \phi_n \to 0 \ (n \to \infty) \)
in the corresponding intersection topology, i.e.
\[
\|e^{t(Q_t^1 + Q_t^2)} \phi_n\|_{L^2(\mathbb{R}^2)} \to 0 \ (n \to \infty) \quad \text{and}
\]
\[
\|e^{t(Q_t^1 + Q_t^2)} \phi_n\|_{L^2(\mathbb{R}^2)} \to 0 \ (n \to \infty).
\]

Remark 2.3.11.
Applying standard techniques it can be seen that the \( L^2(\mathbb{R}^2) \)-norms in the above theorem may be replaced by supremum norms. For instance, assertion (viii) contains a very useful characterization of sequential convergence. It says that a sequence \( \phi_n \in \mathcal{S}_{0, A}^{0, B}(\mathbb{R}^2) \) converges to zero if there exists \( t > 0 \) such that
The Fourier transformation \( \mathcal{F} \) is a homeomorphism of \( \mathcal{S}^0_0(\mathbb{R}^2) \) onto \( \mathcal{S}^0_0(\mathbb{R}^2) \).

Corollary 2.3.12. (Gel'fand-Shilov)

Convergence of a sequence \( (\phi_n)_{n \in \mathbb{N}} \) in a Gel'fand-Shilov space in the corresponding intersection topology agrees with the definition of convergence in [GS 2]. In the last mentioned book the following is stated.

(i) A sequence \( (\phi_n)_{n \in \mathbb{N}} \subset \mathcal{S}_0(\mathbb{R}^2) \) converges to zero if and only if for any \( l \in \mathbb{N}_0 \), the sequence \( (D^l \phi_n)_{n \in \mathbb{N}} \) converges uniformly to zero in any rectangle \( |x_i| \leq x_{0i} < \infty \), \( i = 1,2 \), and for some \( A \) and \( B_i \), independent on \( n \), the inequalities

\[
|x_k D^l \phi_n(x)| \leq B_i A^{1/k} (k!)^a
\]

are satisfied.

(ii) A sequence \( (\phi_n)_{n \in \mathbb{N}} \subset \mathcal{S}^0(\mathbb{R}^2) \) converges to zero if and only if for any \( l \in \mathbb{N}_0 \), the sequence \( (D^l \phi_n)_{n \in \mathbb{N}} \) converges uniformly to zero in any rectangle \( |x_i| \leq x_{0i} < \infty \), \( i = 1,2 \), and for some \( B \) and \( A_k \), independent on \( n \), the inequalities

\[
|x_k D^l \phi_n(x)| \leq A_k B^{1/l} (l!)^b
\]

are satisfied.

(iii) A sequence \( (\phi_n)_{n \in \mathbb{N}} \subset \mathcal{S}_0^0(\mathbb{R}^2) \) converges to zero if and only if for any \( l \in \mathbb{N}_0 \), the sequence \( (D^l \phi_n)_{n \in \mathbb{N}} \) converges uniformly to zero in any rectangle \( |x_i| \leq x_{0i} < \infty \), \( i = 1,2 \), and for some \( A, B \), and \( C \), independent on \( n \), the inequalities

\[
|x_k D^l \phi_n(x)| \leq C A^{1/k} B^{1/l} (k!)^a (l!)^b
\]

are satisfied.

Let \( \alpha = (\alpha_1, \alpha_2) \) with \( \alpha_1 \geq \frac{1}{2} \). From Theorem 2.2.5 we obtain

\[
\mathcal{S}^0_0(\mathbb{R}^2) = D^\infty \{ (Q_1^2 + Q_2^2 + P_1^2 + P_2^2)^{\frac{1}{2\alpha_1}} \}
\]

With the aid of relation (1.5) we can reformulate the above characterization of \( \mathcal{S}^0_0(\mathbb{R}^2) \) in terms of the Hermite expansion coefficients of its elements.

Theorem 2.3.13.
Let \( \alpha = (\alpha_1, \alpha_1) \) with \( \alpha_1 \geq \frac{1}{2} \).

If \( f \in S^\alpha_0(\mathbb{R}^2) \), then there exists \( t > 0 \) such that
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \exp(\frac{t(n+m)}{2\alpha_1}) \| (f, \psi_{n,m})_{L^2(\mathbb{R}^2)} \|^2 < \infty.
\]

Conversely, if a sequence \((a_{n,m}) \in C^N \) satisfies
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \exp(\frac{t(n+m)}{2\alpha_1}) |a_{n,m}|^2 < \infty
\]
for certain \( t > 0 \) then \( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,m} \psi_{n,m} \) converges (in the topology of \( S^\alpha_0(\mathbb{R}^2) \)) to a function in \( S^\alpha_0(\mathbb{R}^2) \).

Put differently,
A function \( f \in L_2(\mathbb{R}^2) \) belongs to \( S^\alpha_0(\mathbb{R}^2) \) if and only if
\[
\exists t \geq 0: (f, \psi_{n,m})_{L^2(\mathbb{R}^2)} = O(\exp(-t(n+m)^{\frac{1}{2\alpha_1}})) \quad (n,m \to \infty).
\]

2.4. Fourier expansions in the two variable Gel'fand-Shilov spaces

From Section 3.4, Chapter I, we know that each \( f \in S(\mathbb{R}^2) \) has a 'Fourier expansion'
\[
f(r \cos \phi, r \sin \phi) = \sum_{n \in \mathbb{Z}} (T_n f)(r) e_n(\phi), \quad r \geq 0, \phi \in [-\pi, \pi]
\]
where
\[
(T_n f)(r) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(r \cos \phi, r \sin \phi) e^{-n(\phi)} d\phi, \quad n \in \mathbb{Z}
\]
and
\[
e_n(\phi) = e^{in\phi}, \quad n \in \mathbb{Z}.
\]

In Section 3.4 we deduced several types of convergence of the above 'Fourier expansion'.

Let \( \alpha, \beta \in \mathbb{R}_{+,\infty} \). For convenience we introduce the abbreviation \( S^\beta_0(\mathbb{R}^2) := S^0_{(\alpha,\alpha)}(\mathbb{R}^2) \). Since \( S^\beta_0(\mathbb{R}^2) \subset S(\mathbb{R}^2) \), each \( f \) in \( S^\beta_0(\mathbb{R}^2) \) has a 'Fourier expansion'. Here we prove that the corresponding 'Cartesian expansion' converges to \( f \) in the topology of \( S^\beta_0(\mathbb{R}^2) \).

Moreover, we investigate what kind of characterizations the \( T_n f \) satisfy. And conversely, we make conditions on functions \( g_n \), \( n \in \mathbb{Z} \), which ensure that \( \sum_{n \in \mathbb{Z}} g_n(r) e_n(\phi) \) determines a function in \( S^\beta_0(\mathbb{R}^2) \).
The set up is the same as in Section 3.4, Chapter 1.

The space \( S_\alpha(\mathbb{R}^2) \)

We take \( \alpha > 0 \) fixed, and \( S_\alpha(\mathbb{R}^2) = S_{(\alpha, \alpha)}(\mathbb{R}^2) \).

**Lemma 2.4.1.**

Let \( f \in S_\alpha(\mathbb{R}^2) \). Then for each \( k \in \mathbb{N}_0 \) there exist \( t, \tau > 0 \) such that

\[
\begin{align*}
(i) & \quad \sum_{n \in \mathbb{Z}} |n|^{k} \int_{0}^{\infty} |e^{tr_{\mathbb{R}^2}}(T_{n} f)(r)|^2 \, dr < \infty \quad \text{and} \\
(ii) & \quad \sum_{n \in \mathbb{Z}} |n|^{k} \int_{0}^{\infty} |e^{tr_{\mathbb{R}^2}}(T_{n} f)(r)| \, dr < \infty.
\end{align*}
\]

**Proof:**

Let \( k \in \mathbb{N}_0 \). Define \( h : \mathbb{R}^2 \rightarrow \mathbb{C} \) by

\[
h(x_1, x_2) = \left[ x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right]^k f(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2.
\]

Then \( h \in S_\alpha(\mathbb{R}^2) \subset D^\alpha((Q_1^2 + Q_2^2)^{\frac{1}{2\alpha}}) \). So there exists \( t > 0 \) such that the function

\[
g(x_1, x_2) := e^{t(x_1^2 + x_2^2)^{\frac{1}{2\alpha}}} h(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2,
\]

is square integrable. For each \( n \in \mathbb{Z} \) we have

\[
(T_n g)(r) = e^{tr_{\mathbb{R}^2}}(T_n h)(r) = (in)^k e^{tr_{\mathbb{R}^2}}(T_n f)(r), \quad r \geq 0.
\]

Hence

\[
\sum_{n \in \mathbb{Z}} |n|^{k} \int_{0}^{\infty} |e^{tr_{\mathbb{R}^2}}(T_{n} f)(r)|^2 \, dr \leq \sum_{n \in \mathbb{Z}} n^{2k} \|e^{tr_{\mathbb{R}^2}} T_n f\|_{L^2}^2 = \sum_{n \in \mathbb{Z}} \|T_n g\|_{L^2}^2 = \|g\|_{L^2(\mathbb{R}^2)}^2.
\]

This yields the validity of assertion (i). Applying (i) we prove assertion (ii).

Let \( k \in \mathbb{N}_0 \). Choose \( t > 0 \) such that

\[
\sum_{n \in \mathbb{Z}} (|n| + 1)^{2k+2} \int_{0}^{\infty} |e^{tr_{\mathbb{R}^2}}(T_{n} f)(r)|^2 \, dr < \infty.
\]

Let \( \varepsilon \in (0, t) \) and define \( \tau := t - \varepsilon \). Then we estimate
\[ \left\{ \sum_{n \in \mathbb{Z}} |n|^{k} \int_{0}^{\infty} e^{-\nu_{n}r}(T_{n}f)(r) \, r \, dr \right\}^{2} \leq \left\{ \sum_{n \in \mathbb{Z}} (|n|+1)^{k} \int_{0}^{\infty} e^{-\nu_{n}r}(T_{n}f)(r) \, r \, dr \right\}^{2} \leq \left\{ \sum_{n \in \mathbb{Z}} \frac{1}{(n^2+1)^{\nu_{n}}} (|n|+1)^{k+1} \int_{0}^{\infty} e^{-\nu_{n}r}(T_{n}f)(r) \, r \, dr \right\}^{2} \leq \sum_{n \in \mathbb{Z}} \frac{1}{n^2+1} \sum_{n \in \mathbb{Z}} (|n|+1)^{2k+2} \left[ \int_{0}^{\infty} e^{-\nu_{n}r} \left| e^{\nu_{n}r} (T_{n}f)(r) \right| \, \sqrt{r} \, dr \right]^{2} \leq \pi \coth(n) \int_{0}^{\infty} e^{-2\nu_{n}r} \, dr \sum_{n \in \mathbb{Z}} (|n|+1)^{2k+2} \int_{0}^{\infty} \left| e^{\nu_{n}r} (T_{n}f)(r) \right|^{2} \, r \, dr < \infty. \]

This shows the validity of assertion (ii). \( \square \)

**Corollary 2.4.2.**

Let \( f \in S_{\alpha}(\mathbb{R}^{2}) \). Then for all \( n \in \mathbb{Z} \)

\[
T_{n}f \in i^{n} S_{\alpha, \text{even}}(\mathbb{R}^{+}).
\]

**Proof.**

Let \( n \in \mathbb{Z} \). From Lemma 2.4.1 (i) we obtain \( T_{n}f \in D^{\alpha}(iQ_{1}^{1/\alpha} ; X_{1}) \). Since \( f \in S_{\alpha}(\mathbb{R}^{2}) \), we have \( f \in S_{\alpha}(\mathbb{R}^{2}) \subset S(\mathbb{R}^{2}) \). So by Lemma 1.3.4.14 and Lemma 1.3.4.16 (i),

\[
B_{\vert_{1/1}, l_{1}} T_{n}f = i^{n} T_{n}f \in D^{\alpha}(Q_{1} ; X_{1}).
\]

Hence, Corollary 1.3.5 yields

\[
T_{n}f \in i^{n} S_{\alpha, \text{even}}(\mathbb{R}^{+}).
\]

\( \square \)

**Corollary 2.4.3.**

Let \( f \in S_{\alpha}(\mathbb{R}^{2}) \). Then there exist functions \( f_{n} \in S_{\alpha, \text{even}}(\mathbb{R}) \), \( n \in \mathbb{Z} \), such that

\[
f(x_{1}, x_{2}) = \sum_{n \in \mathbb{Z}} f_{n}(x_{1}^{2} + x_{2}^{2}) (x_{1} + \text{sgn}(n) i x_{2})^{1/\alpha}, \quad (x_{1}, x_{2}) \in \mathbb{R}^{2}.
\]

Hence there are also functions \( g_{n} \in S_{2\alpha}(\mathbb{R}) \), \( n \in \mathbb{Z} \), such that

\[
f(x_{1}, x_{2}) = \sum_{n \in \mathbb{Z}} g_{n}(x_{1}^{2} + x_{2}^{2}) (x_{1} + \text{sgn}(n) i x_{2})^{1/\alpha}, \quad (x_{1}, x_{2}) \in \mathbb{R}^{2}.
\]

Both series are pointwise convergent and convergent in \( L_{2}(\mathbb{R}^{2}) \)-sense. For a stronger result cf.
Corollary 2.4.6.

Proof. The proof runs similarly to the proof of Corollary I.3.4.18. Note that the second part of this corollary is a consequence of Theorem H.3 (see Appendix H).

The next theorem provides necessary and sufficient conditions for functions $g_n \in X_1$, $n \in \mathbb{Z}$, which guarantee that

$$IP^k(\sum_{n \in \mathbb{Z}} g_n \otimes e_n) \in S_\alpha(\mathbb{R}^2).$$

Before we state this theorem we mention

Lemma 2.4.4. Let $g \in X_1$ and let $\alpha > 0$. Suppose there exists $r > 0$ such that

$$\sup_{r \geq 0} |e^{ir}\|g(r)\| < \infty.$$ 

Then for each $\varepsilon \in (0, r)$ there exists $K_\varepsilon > 0$ such that

$$\|e^{i\varepsilon r}g\|_{X_1} \leq K_\varepsilon \|e^{ir}g\|_\infty.$$ 

Proof. Let $\varepsilon = (0, r)$ and define $K_\varepsilon := \left( \int_0^\infty e^{-2ir} r \, dr \right)^{1/2}$. Then

$$\|e^{i\varepsilon r}g\|_{X_1}^2 = \int_0^\infty |e^{ir}g(r)|^2 e^{-2ir} r \, dr \leq \sup_{r \geq 0} |e^{ir}g(r)|^2 \int_0^\infty e^{-2ir} r \, dr = K_\varepsilon^2 \|e^{ir}g\|_\infty^2.$$ 

Whence we obtain the result.

Theorem 2.4.5. Let $f \in L_2(\mathbb{R}^2)$. The following assertions are equivalent.

(i) $f \in S_\alpha(\mathbb{R}^2)$

(ii) $\exists r > 0 : \sum_{n \in \mathbb{Z}} \|e^{ir}T_n f\|_{X_1}^2 < \infty$ and $\forall l \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \|r^l H_{n+1,1} T_n f\|_{X_1}^2 < \infty.$
(iii) \[ \exists t > 0 : \sum_{n \in \mathbb{Z}} \| e^{t x^n} T_n f \|_{X}^2 < \infty \quad \text{and} \quad \forall \| e^{t x^n} T_n f \|_{X}^2 < \infty \quad \text{and} \quad \forall \| e^{t x^n} T_n f \|_{X}^2 < \infty \]

(iv) \[ \exists t > 0 : \sum_{n \in \mathbb{Z}} \| e^{t x^n} T_n f \|_{X}^2 < \infty \quad \text{and} \quad \forall \| e^{t x^n} T_n f \|_{X}^2 < \infty \quad \text{and} \quad \forall \| e^{t x^n} T_n f \|_{X}^2 < \infty \]

(v) \[ \exists t > 0 : \sum_{n \in \mathbb{Z}} \| e^{t x^n} T_n f \|_{X}^2 < \infty \quad \text{and} \quad \forall \| e^{t x^n} T_n f \|_{X}^2 < \infty \quad \text{and} \quad \forall \| e^{t x^n} T_n f \|_{X}^2 < \infty \]

Proof.

We prove the implication diagram

\begin{align*}
(ii) & \iff (i) \implies (iii) \implies (ii), \\
(iii) & \iff (iv) \implies (v) \implies (ii), \\
(i) & \implies (iv).
\end{align*}

The implication \( (i) \implies (ii) \) is a consequence of Lemma 2.4.1 (i) and Theorem 1.3.4.21. We prove the converse implication \( (ii) \implies (i) \). Suppose (ii) holds. Take \( t > 0 \) such that

\[ \sum_{n \in \mathbb{Z}} \| e^{t x^n} T_n f \|_{X}^2 < \infty. \]

We show that

\[ f \in D^{w}((Q_1^n + Q_2^n)^{2\alpha}) \quad \text{and} \quad IF f \in D^{w}((Q_1^n + Q_2^n)). \]

We have

\[ IP(e^{t (Q_1^n + Q_2^n)} f) = e^{t x^n} IP f \]

and

\[ \| e^{t x^n} IP f \|_{X}^2 = \sum_{n \in \mathbb{Z}} \| e^{t x^n} T_n f \|_{X}^2 < \infty. \]

Therefore, \( f \in D^{w}((Q_1^n + Q_2^n)^{2\alpha}) \). Furthermore, for each \( l \in \mathbb{N}_0 \)

\[ IP((Q_1^n + Q_2^n)^l IF f) = r^{2l} IP(IF f) \]

and

\[ \| r^{2l} IP(IF f) \|_{X}^2 = \sum_{n \in \mathbb{Z}} \| r^{2l} H_{1/1,1} T_n f \|_{X}^2 < \infty. \]

So \( IF f \in D^{w}((Q_1^n + Q_2^n)). \) By Characterization 2.2.4 we obtain \( f \in S_{\alpha}(\mathbb{R}^2) \). Clearly (iii) \( \implies (ii) \). We prove \( (i) \implies (iii) \). Suppose \( f \in S_{\alpha}(\mathbb{R}^2) \). Then, by Lemma 2.4.1, there exists \( t > 0 \) such that
\[
\sum_{n \in \mathbb{Z}} (n^2 + 1) \int_{0}^{\infty} \left| e^{irw} (T_n f) (r) \right|^2 r \, dr < \infty.
\]

Hence

\[
\sum_{n \in \mathbb{Z}} \| e^{irw} T_n f \|_{L^1} = \sum_{n \in \mathbb{Z}} \left( \int_{0}^{\infty} \left| e^{irw} (T_n f) (r) \right|^2 r \, dr \right)^{1/2} \leq \left( \sum_{n \in \mathbb{Z}} \frac{1}{n^2 + 1} \sum_{n \in \mathbb{Z}} (n^2 + 1) \int_{0}^{\infty} \left| e^{irw} (T_n f) (r) \right|^2 r \, dr \right)^{1/2} < \infty.
\]

Since \( S_\infty (R^2) \subset S (R^2) \) we can apply Theorem 1.3.4.21 leading to the result

\[
\forall \epsilon \in \mathbb{R}^+ : \sum_{n \in \mathbb{Z}} \| r^j \mathcal{H}_{1,n,1} T_n f \|_{L^1} < \infty.
\]

The implication \'(iv) \Rightarrow (iii)' is a consequence of Lemma 2.4.4 and of Lemma 1.3.4.20.

Next we show \'(i) \Rightarrow (iv)' . Let \( f \in S_\infty (R^2) \). Then Theorem 2.2.7 indicates that there exists \( t > 0 \) such that

\[
\| e^{i(Q^1 + Q^2)} \|_{L^\infty} < \infty.
\]

So for all \( n \in \mathbb{Z} \),

\[
\| e^{iQ} T_n f \|_{L^\infty} = \sup_{r \geq 0} \left( e^{irw} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(r \cos \phi, r \sin \phi) e^{-i\phi} \, d\phi \right) \leq \sqrt{2\pi} \| e^{i(Q^1 + Q^2)} \|_{L^\infty} f \|_{L^\infty}.
\]

We define \( g \in S_\infty (R^2) \) by

\[
g(x_1, x_2) = 1 - (x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1})^2 \cdot f(x_1, x_2) , (x_1, x_2) \in R^2.
\]

Then for all \( n \in \mathbb{Z} \),

\[
(T_n g) (r) = (1 + n^2) (T_n f) (r) \quad r \geq 0,
\]

and by (*) we obtain the following. There exists \( \tau > 0 \) such that for all \( n \in \mathbb{Z} \),

\[
\| e^{i\tau \nu} T_n g \|_{L^\infty} \leq \sqrt{2\pi} \| e^{i(Q^1 + Q^2)} \|_{L^\infty} g \|_{L^\infty} < \infty.
\]

Using these inequalities we deduce that
Since $S_{\alpha}(\mathbb{R}^2) \subset S(\mathbb{R}^2)$ we can apply Theorem 1.3.4.21 leading to the result
$$\forall \ell \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \|r^\ell H_{|n,1,1}| T_n f\|_\infty < \infty.$$ 

It is clear that '(iv) $\Rightarrow$ (v)' Finally we prove (v) $\Rightarrow$ (ii). Suppose (v) holds. Take $\ell > 0$ such that
$$\sum_{n \in \mathbb{Z}} \|e^{\ell i \omega} T_n f\|_\infty^2 < \infty.$$ 

Then, by Lemma 2.4.4, there exists $K > 0$ such that
$$\sum_{n \in \mathbb{Z}} \|e^{\ell i \omega} T_n f\|_2^2 \leq K \sum_{n \in \mathbb{Z}} \|e^{\ell i \omega} T_n f\|_\infty^2 < \infty.$$ 

As in the proof of Theorem 1.3.4.21 we also find
$$\forall \ell \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \|r^\ell H_{|n,1,1}| T_n f\|_2^2 < \infty$$

which completes the proof.

As a consequence we mention

**Corollary 2.4.6.**

Let $f \in S_{\alpha}(\mathbb{R}^2)$ and let $(g_n)_{n \in \mathbb{Z}} \subset S_{2\alpha}(\mathbb{R})$ be such that
$$f(x_1,x_2) = \sum_{n \in \mathbb{Z}} g_n(x_1^2 + x_2^2) (x_1 + \text{sgn}(n) i x_2)^{\alpha+1}, (x_1,x_2) \in \mathbb{R}^2.$$ 

The series in the above expression admits the following kinds of convergence

(i) Absolute convergence with respect to the $L_2(\mathbb{R}^2)$-norm.

(ii) Absolute convergence with respect to the supremum norm on $\mathbb{R}^2$.

(iii) Absolute and uniform convergence on $\mathbb{R}^2$.

And most importantly, the series converges in the sense of the topology in $S_{\alpha}(\mathbb{R}^2)$.

**Proof.**

The assertions (i) and (ii) follow from Theorem 2.4.5 (iii) and (iv). Then assertion (iii) follows from Weierstrass' test for uniform convergence. We know that the series
\[
\sum_{n \in \mathbb{Z}} h_n(x_1, x_2) := \sum_{n \in \mathbb{Z}} g_n(x_1^2 + x_2^2) (x_1 + \text{sgn}(n) i x_2)^{1,n}\]

is pointwise convergent, with sum \( f(x_1, x_2) \). We prove that the series converges to \( f \) in the topology of \( S_a(\mathbb{R}^2) \), i.e.

(a) \( \sum_{n=-N}^{M} h_n \to f \quad (N,M \to \infty) \) in the space \( D^a((Q_1^2 + Q_2^2)^{1/2} \cdot \sigma_{\text{ind}}) \)

(b) \( \sum_{n=-N}^{M} h_n \to f \quad (N,M \to \infty) \) in the countably normed space \( D^a(P_1^2 + P_2^2) \).

(a) Let \( t > 0 \) be such that \( \sum_{n \in \mathbb{Z}} e^{-t|n|} T_n f \|_{X_1} < \infty \).

For each \( N,M \in \mathbb{N} \), let \( V(N,M) = \{ n \in \mathbb{Z} \mid n > M \text{ or } n < -N \} \). Then

\[
\| e^{t(Q_1^2 + Q_2^2)^{1/2}} \sum_{n=-N}^{M} h_n \|_{L^1(\mathbb{R}^2)} \leq \sum_{n \in V(N,M)} \| e^{-t|n|} T_n f \|_{X_1} \to 0 \quad (N,M \to \infty).
\]

(b) For each \( l \in \mathbb{N}_0 \),

\[
\| (x_1^2 + x_2^2)^l \sum_{n=-N}^{M} \| F h_n - LF f \|_{L^1(\mathbb{R}^2)} \leq \sum_{n \in V(N,M)} \| r^{2l} h_{1n,1} T_n f \|_{X_1} \to 0 \quad (N,M \to \infty).
\]

So \( \sum_{n \in \mathbb{Z}} h_n \) converges to \( f \) in the topology of \( S_a(\mathbb{R}^2) \).

Remark 2.4.7.

Let \( f \in S_a(\mathbb{R}^2) \) and let \( (g_n)_{n \in \mathbb{Z}} \subseteq S_{2a}(\mathbb{R}) \) be such that

\[
f(x_1, x_2) = \sum_{n \in \mathbb{Z}} g_n(x_1^2 + x_2^2) (x_1 + \text{sgn}(n) i x_2)^{1,n}, \quad (x_1, x_2) \in \mathbb{R}^2.
\]

\[(*)\]

Then for all \( k,l \in \mathbb{N}_0^2 \),

\[
\left[ x_1^k \cdot x_2^l \left( \frac{\partial}{\partial x_1} \right)^{l_1} \left( \frac{\partial}{\partial x_2} \right)^{l_2} f(x_1, x_2) \right] = \sum_{n \in \mathbb{Z}} \left[ x_1^k \cdot x_2^l \left( \frac{\partial}{\partial x_1} \right)^{l_1} \left( \frac{\partial}{\partial x_2} \right)^{l_2} g_n(x_1^2 + x_2^2) (x_1 + \text{sgn}(n) i x_2)^{1,n} \right], \quad (x_1, x_2) \in \mathbb{R}^2
\]

and the series admits the same types of convergence as the series in (*) (cf. Corollary 2.4.6.).
Corollary 2.4.8.
Let $m \in \mathbb{Z}$ and let $f \in L_2(\mathbb{R}^2)$ with the special polar form

$$(Pf)(r,\phi) = g(r) e_m(\phi), \quad r \geq 0, \quad \phi \in [-\pi, \pi].$$

Then

$$f \in S_\alpha(\mathbb{R}^2) \text{ if and only if } g \in r^{1m} S_{\alpha, \text{even}}(\mathbb{R}^+).$$

Proof.
For each $n \in \mathbb{Z}$, we have

$$T_n f = \delta_{n,m} g, \quad n \in \mathbb{Z}.$$ 

So Theorem 2.4.5 yields

$$f \in S_\alpha(\mathbb{R}^2) \text{ if and only if } \exists \gamma > 0: \|re^{-r^2} g\|_{L_1} < \infty \quad \text{and} \quad \forall \gamma > 0: \|g\|_{L_1} < \infty.$$ 

Or equivalently,

$$f \in S_\alpha(\mathbb{R}^2) \text{ if and only if } g \in D^{\alpha} (1Q_{1/\alpha}; X_1) \text{ and } H_{1m} g \in D^{\alpha} (Q; X_1).$$

Applying Corollary 1.3.5 we obtain the desired result.

If we take $m = 0$ in the above corollary we obtain a result for $S_{\alpha, \text{rad sym}}(\mathbb{R}^2)$, the radial symmetric functions in $S_\alpha(\mathbb{R}^2)$.

Corollary 2.4.9.
Let $f \in L_2(\mathbb{R}^2)$ be radial symmetric. Define $g: \mathbb{R}^+ \to \mathcal{C}$ by

$$g(r) = (Pf)(r, 0), \quad r \geq 0.$$ 

Then

$$f \in S_{\alpha, \text{rad sym}}(\mathbb{R}^2) \text{ if and only if } g \in S_{\alpha, \text{even}}(\mathbb{R}^+).$$

The space $S^\beta(\mathbb{R}^2)$
We take $\beta > 0$ fixed, and $S^\beta(\mathbb{R}^2) = S^{\beta, 0}(\mathbb{R}^2)$.

Lemma 2.4.10.
Let $f \in S^\beta(\mathbb{R}^2)$. Then for each $l \in \mathbb{N}_0$ there exist $r, \tau > 0$ such that
Proof.
Since \( f \in S^\beta (\mathbb{R}^2) \), the Fourier transform \( \mathcal{F} f \) belongs to \( S^\beta (\mathbb{R}^2) \). Now both results follow from Lemma 2.4.1 by applying the Hecke-Bochner identities \( T_n \mathcal{F} = (-i)^{|n|} H_{|n|,1} T_n f \), \( n \in \mathbb{Z} \) (cf. Lemma 1.3.4.14). \( \square \)

Corollary 2.4.11.
Let \( f \in S^\beta (\mathbb{R}^2) \). Then for all \( n \in \mathbb{Z} \)

\[
T_n f \in r^{-|n|} S^\beta_{\text{even}} (\mathbb{R}^+) .
\]

Proof.
Let \( n \in \mathbb{Z} \). From Lemma 1.3.4.16 (i) we obtain \( T_n f \in \mathcal{D}^{\infty} (Q;X_1) \). By Lemma 2.4.10 (i) we have \( H_{|n|,1} T_n f \in \mathcal{D}^{\infty} (Q^{1/\beta};X_1) \). Hence, Corollary 1.3.5 yields

\[
T_n f \in r^{-|n|} S^\beta_{\text{even}} (\mathbb{R}^+) .
\]

\( \square \)

The next theorem provides necessary and sufficient conditions for functions \( g_n \in X_1 \), \( n \in \mathbb{Z} \), which guarantee that

\[
IP_\beta^* ( \sum_{n \in \mathbb{Z}} g_n \otimes e_n ) \in S^\beta (\mathbb{R}^2) .
\]

Theorem 2.4.12.
Let \( f \in L_2 (\mathbb{R}^2) \). The following assertions are equivalent.

(i) \( f \in S^\beta (\mathbb{R}^2) \)

(ii) \( \forall k \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^k T_n f \|_{X_1}^2 < \infty \) \quad \text{and} \quad \exists \gamma > 0 : \sum_{n \in \mathbb{Z}} \| \mathcal{F} r^\gamma T_n f \|_{X_1}^2 < \infty \)

(iii) \( \forall k \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^k T_n f \|_{X_1} < \infty \) \quad \text{and} \quad \exists \gamma > 0 : \sum_{n \in \mathbb{Z}} \| \mathcal{F} r^\gamma T_n f \|_{X_1} < \infty \)

(iv) \( \forall k \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^k T_n f \|_{\infty} < \infty \) \quad \text{and} \quad \exists \gamma > 0 : \sum_{n \in \mathbb{Z}} \| \mathcal{F} r^\gamma T_n f \|_{\infty} < \infty \).
(v) \( \forall k \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^k T_n f \|_\infty^2 < \infty \) and \( \exists \epsilon > 0 : \sum_{n \in \mathbb{Z}} \| e^{ir^n} \mathcal{H} T_n f \|_\infty^2 < \infty \).

**Proof.**

The proof is a consequence of Lemma 2.4.5 by applying the Fourier property \( \mathcal{F}(S^p(R^2)) = S^p(R^2) \), the Hecke-Bochner identities \( T_n \mathcal{F} = (-i)^n \mathcal{H} T_n \) (see Lemma 1.3.4.14) and the Hankel relation \( \mathcal{H} T_n \mathcal{H} = I \).

Let us prove the equivalence of (i) and (ii). The following assertions are mutually equivalent

- \( f \in S^p(R^2) \)
- \( \mathcal{F} f \in S^p(R^2) \)
- \( \exists \lambda \geq 0 : \sum_{n \in \mathbb{Z}} \| e^{ir^n} T_n f \|_{\mathcal{H}_1}^2 < \infty \) and \( \forall \lambda \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^n \mathcal{H} T_n f \|_{\mathcal{H}_1}^2 < \infty \)
- \( \exists \lambda \geq 0 : \sum_{n \in \mathbb{Z}} \| e^{ir^n} \mathcal{H} T_n f \|_{\mathcal{H}_1}^2 < \infty \) and \( \forall \lambda \in \mathbb{N}_0 : \sum_{n \in \mathbb{Z}} \| r^n T_n f \|_{\mathcal{H}_1}^2 < \infty \).

Similarly we can prove that (i) is equivalent with each of the assertion (iii), (iv) and (v). \( \square \)

As a consequence we mention

**Corollary 2.4.13.**

Let \( f \in S^p(R^2) \) and let \((f_n)_{n \in \mathbb{Z}} \subset S^p_{\text{even}}(R^2)\) be such that

\[
f(x_1, x_2) = \sum_{n \in \mathbb{Z}} f_n \left( \sqrt{x_1^2 + x_2^2} \right) (x_1 + \text{sgn}(n) i x_2)^n, \quad (x_1, x_2) \in R^2.
\]

The series in the above expression admits the following kinds of convergence

(i) Absolute convergence with respect to the \( L_2(R^2) \)-norm.
(ii) Absolute convergence with respect to the supremum norm.
(iii) Absolute and uniform convergence on \( R^2 \).

And most importantly, the series converges in the sense of the topology in \( S^p(R^2) \).

**Proof.**

The proof runs similarly to the proof of Corollary 2.4.6. \( \square \)

**Corollary 2.4.14.**

Let \( m \in \mathbb{Z} \) and let \( f \in L_2(R^2) \) with the special polar form

\[
(\mathcal{F} f)(r, \phi) = g(r) e_m(\phi), \quad r \geq 0, \quad \phi \in [-\pi, \pi].
\]

Then
The proof runs similarly to the proof of Corollary 2.4.8.

If we take $m = 0$ in the above corollary we obtain a result for $S^{\beta}_{\text{radsym}}(\mathbb{R}^2)$, the radial symmetric functions in $S^{\beta}(\mathbb{R}^2)$.

**Corollary 2.4.15.**

Let $f \in L_2(\mathbb{R}^2)$ be radial symmetric. Define $g : \mathbb{R}^+ \to \mathcal{C}$ by

$$g(r) = (Pf)(r,0), \quad r \geq 0.$$  

Then

$$f \in S^{\beta}_{\text{radsym}}(\mathbb{R}^2) \text{ if and only if } g \in S^{\beta}_{\text{even}}(\mathbb{R}^+).$$

The space $S^{\beta}_{\alpha}(\mathbb{R}^2)$

We take $\alpha, \beta > 0$ fixed, and $S^{\beta}_{\alpha}(\mathbb{R}^2) = S^{(\beta,\beta)}_{(\alpha,\alpha)}(\mathbb{R}^2)$. Since $S^{\beta}_{\alpha}(\mathbb{R}^2) = S_{\alpha}(\mathbb{R}^2) \cap S^{\beta}(\mathbb{R}^2)$, the results follow immediately from the past. We summarize some important results.

**Theorem 2.4.16.**

Let $f \in S^{\beta}_{\alpha}(\mathbb{R}^2)$, then for all $n \in \mathbb{Z}$

$$T_n f \in \mathcal{L}^{n+1}_{\alpha,\text{even}}(\mathbb{R}^+).$$

**Theorem 2.4.17.**

Let $f \in L_2(\mathbb{R}^2)$. The following assertions are equivalent.

1. $f \in S^{\beta}_{\alpha}(\mathbb{R}^2)$
2. $\exists r > 0 : \sum_{n \in \mathbb{Z}} \|e^{pr \cdot x} T_n f \|_{L^2_{\chi_1}} < \infty$ and $\exists r > 0 : \sum_{n \in \mathbb{Z}} \|e^{pr \cdot y} H_{\alpha,1,1} T_n f \|_{L^2_{\chi_1}} < \infty$
3. $\exists r > 0 : \sum_{n \in \mathbb{Z}} \|e^{pr \cdot x} T_n f \|_{X_1} < \infty$ and $\exists r > 0 : \sum_{n \in \mathbb{Z}} \|e^{pr \cdot y} H_{\alpha,1,1} T_n f \|_{X_1} < \infty$
4. $\exists r > 0 : \sum_{n \in \mathbb{Z}} \|e^{pr \cdot x} T_n f \|_{L^\infty} < \infty$ and $\exists r > 0 : \sum_{n \in \mathbb{Z}} \|e^{pr \cdot y} H_{\alpha,1,1} T_n f \|_{L^\infty} < \infty$
Corollary 2.4.18.
Let \( f \in S_0^0(\mathbb{R}^2) \) and let \((f_n)_{n \in \mathbb{Z}} \subset S_0^0(\mathbb{R}^+)^{even}(\mathbb{R}^2)\) be such that

\[
f(x_1, x_2) = \sum_{n \in \mathbb{Z}} f_n(\sqrt{x_1^2 + x_2^2}) (x_1 + \text{sgn}(n)i x_2)^{in^1}, \quad (x_1, x_2) \in \mathbb{R}^2.
\]

The series in the above expression admits the following kinds of convergence

(i) Absolute convergence with respect to the \( L_2(\mathbb{R}^2) \)-norm.
(ii) Absolute convergence with respect to the supremum norm.
(iii) Absolute and uniform convergence on \( \mathbb{R}^2 \).

And most importantly, the series converges in the sense of the topology in \( S_0^0(\mathbb{R}^2) \).

Corollary 2.4.19.
Let \( m \in \mathbb{Z} \) and let \( f \in L_2(\mathbb{R}^2) \) with the special polar form

\[
(P f)(r, \phi) = g(r) e_m(\phi), \quad r \geq 0, \quad \phi \in [-\pi, \pi].
\]

Then

\[
f \in S_0^0(\mathbb{R}^2) \text{ if and only if } g \in r^{in^1} S_0^0(\mathbb{R}^+)^{even}.
\]

Corollary 2.4.20.
Let \( f \in L_2(\mathbb{R}^2) \) be radial symmetric. Define \( g : \mathbb{R}^+ \rightarrow \mathbb{C} \) by

\[
g(r) = (P f)(r, 0), \quad r \geq 0.
\]

Then

\[
f \in S_0^0(\mathbb{R}^2) \text{ if and only if } g \in S_0^0(\mathbb{R}^+)\text{, even}.
\]
2.5. A polar coordinates factorization problem

Let us consider functions \( f \in L_2(\mathbb{R}^2) \) which have the factored polar form

\[
 (P f)(r, \phi) = g(r) h(\phi)
\]  

(2.29)

where \( g \in X_1 \) and \( h \in L_2([-\pi, \pi]) \). In Section 3.5, Chapter 1, we presented necessary and sufficient conditions on the functions \( g \) and \( h \) which ensure that \( f \in S(\mathbb{R}^2) \). Here we want to derive necessary and sufficient conditions on the functions \( g \) and \( h \) in order that \( f \in S_{\alpha, \beta}(\mathbb{R}^2) \).

In each of the cases \( (\alpha > 0, \beta = \infty), (\alpha = \infty, \beta \leq 1), 0 < \alpha \leq \beta \) we solve the problem. However, if \( (\alpha = \infty, \beta > 1) \) or \( 1 < \beta < \alpha \) we only have a rough idea how the conditions look like.

We think that our theorems find their applications in solving differential equations, such as \( \Delta u = v \), in the Gel’fand Shilov spaces by means of the method of separation of variables (cf. [GS 3]).

Before we start dealing with the problem we derive some auxiliary characterizations.

In Chapter I we pointed at the fact that the Hermite functions \( \Psi_{n,m}, n,m \geq 0 \), are eigenfunctions of the differential operator

\[
 -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + x_1^2 + x_2^2
\]  

(2.30)

with respective eigenvalues \( 2(n + m + 1) \). The functions \( \Psi_{n,m}, n,m \geq 0 \), constitute an orthonormal basis in the Hilbert space \( L_2(\mathbb{R}^2) \). In Appendix G we present the solution of the corresponding eigenvalue problem in polar coordinates,

\[
 \left[ -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + r^2 \right] u = \lambda u.
\]  

(2.31)

The eigenvalues are

\[
 \lambda = \lambda_{n,m} = 4n + 2|m| + 2 \quad , \quad n \in \mathbb{N}_0 , m \in \mathbb{Z}
\]  

(2.32)

with respective eigenfunctions

\[
 u(r, \phi) = U_{n,m}(r, \phi) = \left[ \frac{2 \Gamma(n+1)}{\Gamma(m+1)} \right]^{\frac{1}{2}} r^{|m|} e^{-\frac{1}{2}r^2} L_n^{(|m|)}(r^2) \cdot \frac{1}{\sqrt{2\pi}} e^{im\phi} =
\]

\[
 = L_n^{(|m|,1)}(r) e_m(\phi) \quad , \quad n \in \mathbb{N}_0 , m \in \mathbb{Z}.
\]  

(2.33)

The functions \( U_{n,m} \) establish an orthonormal basis in the Hilbert space \( K = L_2((0, \infty) \times [-\pi, \pi], rdrd\phi) \). Let

\[
 T_3 = IP T_3 IP^*
\]  

(2.34)

where \( T_3 = P_1^2 + P_2^2 + Q_1^2 + Q_2^2 \), as in Chapter 1. Being unitarily equivalent with the self-adjoint operator \( T_3 \), the operator \( \tilde{T}_3 \) is self-adjoint, with domain \( D(\tilde{T}_3) = IP D(T_3) \). For each
\( n \in \mathbb{N}_0, m \in \mathbb{Z} \) we have
\[
\tilde{T}_3 U_{n,m} = \lambda_{n,m} U_{n,m}.
\] (2.35)

We define the self-adjoint operator \( U : D(U) \to K \) by
\[
U f = \sum_{n \in \mathbb{N}_0} \lambda_{n,m} (f, U_{n,m})_K U_{n,m}, \quad f \in D(U)
\] (2.36)

with
\[
D(U) = \{ g \in K : \sum_{n \in \mathbb{N}_0} |\lambda_{n,m}|^2 |(g, U_{n,m})_K|^2 < \infty \}.
\] (2.37)

Evidently for all \( n \in \mathbb{N}_0, m \in \mathbb{Z} \) the function \( U_{n,m} \) is an eigenfunction of the operator \( U \) with eigenvalue \( \lambda_{n,m} \). In addition we prove

\textbf{Theorem 2.5.1.}
\[
\tilde{T}_3 = U.
\]

\textit{Proof.}
Let \( f \in D(U) \). For each \( N \in \mathbb{N} \) we define
\[
f_N = \sum_{n=0}^{N} \sum_{m=-N}^{N} (f, U_{n,m})_K U_{n,m}.
\]
Then \( f_N \to f \) in the sense of \( K \) \((N \to \infty)\). Furthermore,
\[
\tilde{T}_3 f_N = \sum_{n=0}^{N} \sum_{m=-N}^{N} \lambda_{n,m} (f, U_{n,m})_K U_{n,m}.
\]
Since \( f \in D(U) \), it follows that
\[
\tilde{T}_3 f_N \to U f \quad \text{in the sense of } K \quad (N \to \infty).
\]

Being self-adjoint, the operator \( \tilde{T}_3 \) is also closed. So we obtain
\[
f \in D(\tilde{T}_3) \quad \text{and} \quad \tilde{T}_3 f = U f.
\]

Now we proved that \( U \subset \tilde{T}_3 \). Since both \( U \) and \( \tilde{T}_3 \) are self-adjoint we obtain \( U = \tilde{T}_3 \).

\[ \square \]

For the remainder of this section we use the abbreviation
\[
S^a_{\alpha}(\mathbb{R}^2) = S^{[0,\alpha]}_{\alpha}(\mathbb{R}^2).
\] (2.38)

From Section 3.3, Chapter I, and Section 2.2 we recall
and the corresponding characterizations of $S(\mathbb{R}^2)$ and $S^\alpha_\alpha(\mathbb{R}^2)$ in terms of the Hermite expansion coefficients of its elements,

A function $f \in L_2(\mathbb{R}^2)$ belongs to $S(\mathbb{R}^2)$ if and only if

$$\forall \lambda \in \mathbb{R}_+ : (f, \Psi_{n,m})_{L_2(\mathbb{R}^2)} = O((n+m)^{-k}) \quad (n,m \to \infty), \quad (2.40)$$

and

A function $f \in L_2(\mathbb{R}^2)$ belongs to $S^\alpha_\alpha(\mathbb{R}^2)$, $\alpha \geq \frac{1}{2}$, if and only if

$$\exists \lambda > 0 : (f, \Psi_{n,m})_{L_2(\mathbb{R}^2)} = O(\exp(-t(n+m)^{-\frac{1}{2\alpha}})) \quad (n,m \to \infty). \quad (2.41)$$

By $IP S^\alpha_\alpha(\mathbb{R}^2)$ we denote all functions $f \in K$ with the property

$$IP^* f \in S^\alpha_\alpha(\mathbb{R}^2). \quad (2.42)$$

From (2.39), the definition of $\tilde{T}_3$ and Theorem 2.5.1 we obtain

$$IP S(\mathbb{R}^2) = D^\infty(\tilde{T}_3) = D^\infty(U) \quad (2.43)$$

and

$$IP S^\alpha_\alpha(\mathbb{R}^2) = D^\infty(\tilde{T}_3^{2\alpha}) = D^\infty(U^{2\alpha}). \quad (2.44)$$

Now we can give characterizations of $IP S(\mathbb{R}^2)$ and $IP S^\alpha_\alpha(\mathbb{R}^2)$ in terms of the $U_{n,m}$-expansion coefficients of its elements.

**Characterization 2.5.2.**

(i) A function $f \in K$ belongs to $IP S(\mathbb{R}^2)$ if and only if

$$\forall \lambda \in \mathbb{R}_+ : (f, U_{n,m})_K = O((n+m)^{-k}) \quad (n, m \to \infty).$$

(ii) A function $f \in K$ belongs to $IP S^\alpha_\alpha(\mathbb{R}^2)$, $\alpha \geq \frac{1}{2}$, if and only if

$$\exists \lambda > 0 : (f, U_{n,m})_K = O(\exp(-t(n+m)^{-\frac{1}{2\alpha}})) \quad (n, m \to \infty).$$

As we have seen, the $U_{n,m}$ constitute an orthonormal basis in $K$. Since $IP^* : K \to L_2(\mathbb{R}^2)$ is a unitary operator, the functions $IP^* U_{n,m}$, $n \in \mathbb{N}_0$, $m \in \mathbb{Z}$, constitute an orthonormal basis in $L_2(\mathbb{R}^2)$. Note that for $(x_1, x_2) \in \mathbb{R}^2$
\[
\left( I^p_{n,m} \right) (x_1, x_2) = \frac{1}{\sqrt{\pi}} \left( \frac{n!}{(1 + n)!} \right) \frac{1}{2} e^{-\frac{1}{2} (x_1^2 + x_2^2)} L_n^{(1 + m)}(x_1^2 + x_2^2) \\
\cdot (x_1 + i \text{sgn}(m) x_2)^{1 + m}, \quad n \in \mathbb{N}_0, \ m \in \mathbb{Z}.
\] (2.45)

Remember that we took already cognizance of the basis \( I^p_{n,m} \) in Section 3.3, Chapter I.

As a consequence, the above characterization admits the following reformulation.

**Corollary 2.5.3.**

(i) A function \( f \in L_2(\mathbb{R}^2) \) belongs to \( S(\mathbb{R}^2) \) if and only if
\[
\forall t \in \mathbb{R}_0 : (f, I^p_{n,m} L_{1}(\mathbb{R}^2)) = O((n + 1)^{-t}) \quad (n, 1 \to \infty).
\]

(ii) A function \( f \in L_2(\mathbb{R}^2) \) belongs to \( I^p S_\alpha^q(\mathbb{R}^2) \), \( \alpha \geq \frac{1}{2} \), if and only if
\[
\exists t > 0 : (f, I^p_{n,m} L_{1}(\mathbb{R}^2)) = O(\exp(-t(n + 1)^{2\alpha})) \quad (n, 1 \to \infty).
\]

From Section 2.4, Chapter I and Section 1.3 we recall characterizations of \( S_{even}(\mathbb{R}) \) and \( S_{\alpha,even}(\mathbb{R}) \) in terms of the Laguerre expansion coefficients of its elements.

Let \( \nu \geq -\frac{1}{2} \). Then

A function \( f \in X_1 \) belongs to \( S_{even}(\mathbb{R}^+) \) if and only if
\[
\forall t \in \mathbb{N}_0 : (f, L_n^{(\nu+1)})_{X_{even}} = O(n^{-t}) \quad (n \to \infty)
\]
and, for \( \alpha \geq \frac{1}{2} \),

A function \( f \in X_1 \) belongs to \( S_{\alpha,even}(\mathbb{R}^+) \) if and only if
\[
\exists t > 0 : (f, L_n^{(\nu+1)})_{X_{even}} = O(\exp(-t n^{2\alpha})) \quad (n \to \infty).
\]

With the aid of the identity
\[
(g, L_n^{(\nu+1)})_{X_{i}} = (r^{-\nu} g, L_n^{(\nu+1)})_{X_{even}}
\]
and the above characterizations we obtain

**Characterization 2.5.4.**

Let \( \nu \geq -\frac{1}{2} \) and \( \alpha \geq \frac{1}{2} \). Then
(i) A function \( g \in X_1 \) belongs to \( r^\nu S_{\text{even}}(\mathbb{R}^+) \) if and only if
\[
\forall k \in N_0 : (g, L_n^{(\nu,1)})_{X_1} = O(n^{-k}) \quad (n \to \infty).
\]

(ii) A function \( g \in X_1 \) belongs to \( r^\nu S_{\alpha, \text{even}}^\omega(\mathbb{R}^+) \) if and only if
\[
\exists \gamma > 0 : (g, L_n^{(\nu,1)})_{X_1} = O((\exp(-t n^2 \gamma)) \quad (n \to \infty).
\]

Using the characterization of the \( S_{\alpha}^\omega(\mathbb{R}) \)-spaces originating from Zhang Gong-Zhing and the classification of these spaces in terms of Laguerre polynomials as derived by Van Eijndhoven [E2] we deduce the following

Characterization 2.5.5.

Let \( \alpha \geq \frac{1}{2} \), let \( V \subset \mathbb{R}^+ \) and let \( g \in X_1 \). The following statements are equivalent.

(i) \( \exists A > 0 \forall \nu \in V : g \in r^\nu S_{\alpha, \text{even}}^{A, \infty}(\mathbb{R}^+) \)

(ii) \( \exists \gamma > 0 \forall \nu \in V : (g, L_n^{(\nu,1)})_{X_1} = O((\exp(-t n^2 \gamma)) \quad (n \to \infty).
\]

As a side-result we mention

Characterization 2.5.6.

Let \( 0 < \alpha \leq \beta \) with \( \alpha + \beta \geq 1 \). Let \( \nu \geq -\frac{1}{2} \) and let \( g \in X_1 \).

The following assertions are equivalent.

(i) \( g \in S_{\alpha, \text{even}}^\beta(\mathbb{R}^+) \)

(ii) \( \exists A > 0 : r \mapsto g(r) \exp(a r^{1/\alpha}) \in X_1 \) and
\[
\exists \gamma > 0 : (g, L_n^{(\nu,2\nu+1)})_{X_1} = O((\exp(-t n^{2\beta})) \quad (n \to \infty).
\]

(iii) \( \exists A > 0 : \sup_{r \geq 0} |g(r)| \exp(a r^{1/\alpha}) < \infty \) and
\[
\exists \gamma > 0 : (g, L_n^{(\nu,2\nu+1)})_{X_1} = O((\exp(-t n^{2\beta})) \quad (n \to \infty).
\]

Proof.

First we prove the equivalence of assertions (i) and (ii). By the characterization of \( S_{\beta, \text{even}}^\omega(\mathbb{R}^+) \) in terms of the Laguerre expansion coefficients of its elements (i.e Characterization 1.3.6) and by
Characterization (1.5) it follows that assertion (ii) is equivalent with
\[ g \in D^\omega (IQ^{1/\alpha}; X_1) \text{ and } \mathcal{E} g \in S_{\alpha,\text{even}}^\phi (IR). \]
And this is equivalent with
\[ \mathcal{E} g \in D^\omega (IQ^{1/\alpha}) \cap D^\omega (IQ^{1/\beta}) \cap D^\omega (IP^{1/\beta}). \]
Noting that \( 0 < \alpha \leq \beta \) and applying Characterization 1.2.1 we derive that assertions (i) and (ii) are equivalent.

Next we prove the implication (ii) \( \Rightarrow \) (iii). Suppose (ii) holds. Since (ii) is equivalent with (i),
\[ \int_0^\infty |g'(\rho)|^2 d\rho < \infty \text{ and } \int_0^\infty |g(\rho) \exp(a \rho^{1/\alpha})|^2 d\rho < \infty \text{ for some } a > 0. \]

For each \( r > 0 \) we estimate
\[
g(r) \exp(a r^{1/\alpha}) = -2 \exp(a r^{1/\alpha}) \int_r^\infty \text{Re}(\overline{g(p)} g'(p)) d\rho \leq \]
\[
\leq 2 \int_r^\infty |g(p) \exp(a p^{1/\alpha})| \cdot |g'(p)| d\rho \leq \]
\[
\leq 2 \left( \int_0^\infty |g(\rho) \exp(a \rho^{1/\alpha})|^2 d\rho \cdot \int_0^\infty |g'(\rho)|^2 d\rho \right)^{1/2}.
\]
So assertion (iii) is true. Finally suppose (iii) holds. Then (ii) follows from the estimation
\[
\int_0^\infty |g(r) \exp(\frac{1}{2} a r^{1/\alpha})|^2 r dr = \int_0^\infty |g(r) \exp(a r^{1/\alpha}) \exp(-\frac{1}{2} a r^{1/\alpha})|^2 r dr \leq \]
\[
\leq \sup_{r > 0} |g(r) \exp(a r^{1/\alpha})|^2 \cdot \int_0^\infty \exp(-a r^{1/\alpha}) r dr \quad (a > 0).
\]

Corollary 2.5.7.
Let \( 0 < \alpha \leq \beta \) with \( \alpha + \beta \geq 1 \). Let \( \nu \geq -\frac{1}{2} \) and let \( g \in X_1 \). The following assertions are equivalent.

(i) \( g \in \nu S_{\alpha,\text{even}}^\phi (IR^+) \)

(ii) \( g = O(\nu^\nu) \) \( (r \downarrow 0) \) \text{ and } \exists_{a > 0} : r \mapsto g(r) \exp(a r^{1/\alpha}) \in X_1 \text{ and } \exists_{r > 0} : (g, L_\nu^{(\nu,1)})_{X_1} = O(\exp(-t n^{2\beta})) \quad (n \to \infty) \)
(iii) \[ g = O(r^\gamma) \quad (r \downarrow 0) \text{ and } \exists \alpha > 0 : \sup_{r > 0} |g(r)\exp(a r^{1/\alpha})| < \infty \text{ and } \exists \gamma > 0 :\]
\[ \forall \in \mathbb{V} : (g, L_n^{(v,1)})_{X_1} = O(\exp(-n^{1/2})) \quad (n \to \infty). \]

Corollary 2.5.8.
Let \( 0 < \alpha \leq \beta \) with \( \alpha + \beta \geq 1 \). Let \( \mathbb{V} \subset \mathbb{R}^+ \) and let \( g \in X_1 \). The following assertions are equivalent.

(i) \[ \exists A, B > 0 : \forall \in \mathbb{V} : g \in r^\gamma S_A^{0, B, \infty} \quad (\mathbb{R}^+) \]

(ii) \[ g = O(r^\gamma) \quad (r \downarrow 0) \text{ and } \exists \alpha > 0 : \forall \in \mathbb{V} : r \mapsto g(r)\exp(a r^{1/\alpha}) \in X_1 \text{ and } \exists \gamma > 0 :\]
\[ \forall \in \mathbb{V} : (g, L_n^{(v,1)})_{X_1} = O(\exp(-n^{1/2})) \quad (n \to \infty) \]

(iii) \[ g = O(r^\gamma) \quad (r \downarrow 0) \text{ and } \exists \alpha > 0 : \forall \in \mathbb{V} : \sup_{r > 0} |g(r)\exp(a r^{1/\alpha})| < \infty \text{ and } \exists \gamma > 0 :\]
\[ \forall \in \mathbb{V} : (g, L_n^{(v,1)})_{X_1} = O(\exp(-n^{1/2})) \quad (n \to \infty). \]

Next we deal with the polar coordinates factorization problem as posed in the beginning of this section.

The factorization problem for \( IP S_0(\mathbb{R}^2) \)
Coming straight to the point we formulate

Theorem 2.5.9.
Let \( f \in L_2(\mathbb{R}^2) \) for which there exist functions \( g \in X_1 \) and \( h \in L_2([-\pi, \pi]) \) such that
\[ (IP f)(r, \phi) = g(r) h(\phi). \]

Let \( \alpha > 0 \). The following assertions are equivalent

(i) \( f \in S_\alpha(\mathbb{R}^2) \)

(ii) \( h \) extends to a function in \( C^{2\alpha}_{2\alpha-\text{per}}(\mathbb{R}) \) and
For convenience we set \( a_m = (h, e_m)_{L_4(\{-\pi, \pi\})} \), \( m \in \mathbb{Z} \).

(i) \( \Rightarrow \) (ii): Suppose \( f \in S_\alpha(\mathbb{R}^2) \subset S(\mathbb{R}^2) \). Then \( h \in C_{2x-per}(\mathbb{R}) \) according to Theorem 1.3.5.3. From Corollary 2.4.2 we obtain for each \( m \in \mathbb{Z} \)

\[
T_m f \in r^{m1} S_{\alpha,even}(\mathbb{R}^+) .
\]

For the converse, suppose (ii) holds. Then \( g \in S_{\alpha,even}(\mathbb{R}^+) \). So there exists \( t > 0 \) such that

\[
\|e^{itx} g\|_{X_1} < \infty .
\]

Hence

\[
\|e^{it(Q_1^2 + Q_2^2)^{\frac{1}{2}}} f\|_{L_4(\mathbb{R}^2)} = \|e^{itx} g\|_{X_1} \cdot \|h\|_{L_4(\{-\pi, \pi\})} < \infty ,
\]

which implies that \( f \in D^{\alpha}((Q_1^2 + Q_2^2)^{\frac{1}{2}}) \). Furthermore, according to Theorem 1.3.5.3 we obtain \( f \in D^{\alpha}(P_1^2 + P_2^2) \). Applying Characterization 2.2.4 we conclude that

\[
f \in D^{\alpha}((Q_1^2 + Q_2^2)^{\frac{1}{2}}) \cap D^{\alpha}(P_1^2 + P_2^2) = S_\alpha(\mathbb{R}^2) .
\]

A consequence of the above theorem is the characterization of \( S_{\alpha,rad sym}(\mathbb{R}^2) \) as stated in Corollary 2.4.9.

**Lemma 2.5.10.**

Let \( f \in C^{\infty}(\mathbb{R}^+) \), let \( \alpha > 0 \) and let \( n \in \mathbb{N} \). Equivalent are

(i) \( f \in r^{2\alpha} S_{\alpha,even}(\mathbb{R}^+) \)

(ii) \( f^{(j)}(0) = 0 \), \( j = 0, \ldots, 2n - 1 \) ,

\[ f^{(2j+1)}(0) = 0, \quad l = n, n + 1, \ldots \] and

\( f \in S_\alpha(\mathbb{R}^+) \).

**Proof.**

The proof is elementary (cf. proof of Lemma 1.3.5.8). \( \Box \)

**Corollary 2.5.11.**

Let \( f \in C^{\infty}(\mathbb{R}^+) \), let \( \alpha > 0 \) and let \( n \in \mathbb{N}_0 \). Equivalent are
(i) \( f \in r^{2n+1} S_{\alpha,\text{even}}(\mathbb{R}^+) \)

(ii) \( f^{(j)}(0) = 0, \quad j = 0, \ldots, 2n \), 
\( f^{(l)}(0) = 0, \quad l = n+1, n+2, \ldots \) and 
\( f \in S_\alpha(\mathbb{R}^+) \).

**Proof.**
The proof runs similarly to the proof of Corollary 1.3.5.9. \[ \]

In Section 2.3 we introduced the spaces \( S_{\alpha,A;\infty}(\mathbb{R}) \), \( \alpha,A > 0 \). In addition we define

\[ S_{\alpha,A;\infty}(\mathbb{R}) := \bigcap_{\beta > 0} S_{\alpha,A+\varepsilon;\infty}(\mathbb{R}). \] (2.49)

Then

\[ S_{\alpha,A;\infty}(\mathbb{R}) \subset S_{\alpha,A';\infty}(\mathbb{R}) \subset S_{\alpha,A+\varepsilon;\infty}(\mathbb{R}) \] for all \( \varepsilon > 0 \). (2.50)

Similar definitions can be formulated for the \( S^p \) - and \( S^p_0 \) -spaces.

**Theorem 2.5.12.**
Let \( f \in C^\infty(\mathbb{R}^+) \) and let \( \alpha > 0 \). The following assertions are equivalent.

(i) \( \exists n,m \in \mathbb{N}_0 : f \in r^{2n} S_{\alpha,\text{even}}(\mathbb{R}^+) \) and \( f \in r^{2m+1} S_{\alpha,\text{even}}(\mathbb{R}^+) \)

(ii) \( \exists A > 0 \forall p \in \mathbb{N}_0 : f \in r^p S_{\alpha,A;\infty,\text{even}}(\mathbb{R}^+) \)

(iii) \( \exists A > 0 \forall p,q \in \mathbb{N}_0 : f^{(q)} \in r^p S_{\alpha,A;\infty,\text{even}}(\mathbb{R}^+) \)

(iv) \( \forall f \in \mathbb{N}_0 : f^{(0)}(0) = 0 \) and \( f \in S_\alpha(\mathbb{R}^+) \).

**Proof.**
Clearly (ii) \( \Rightarrow \) (i). We prove the converse. Suppose (i) holds. By Theorem 1.3.5.10 we obtain that \( f \in r^p S_{\alpha,\text{even}}(\mathbb{R}^+) \) for all \( p \in \mathbb{N}_0 \). Since \( f \in r^{2n} S_{\alpha,\text{even}}(\mathbb{R}^+) \) it follows that \( f \in S_{\alpha,\text{even}}(\mathbb{R}^+) \). So there exists \( A > 0 \) such that \( f \in S_{\alpha,A;\infty,\text{even}}(\mathbb{R}^+) \). For each \( p \in \mathbb{N}_0 \) we define the assertion \( H(p) \) by

\[ H(p) : \text{The function } f \text{ belongs to } r^j S_{\alpha,A;\infty,\text{even}}(\mathbb{R}^+) \quad j = 0, \ldots, p. \]

Then \( H(0) \) is true. Suppose \( H(p) \) is true for some \( p \in \mathbb{N}_0 \). Then there exists \( g \in S_{\alpha,A;\infty}(\mathbb{R}) \) such that \( f(r) = r^p g(r) \), \( r \geq 0 \). The function \( g \) satisfies the estimations

\[ \forall B > 0 \exists k \in \mathbb{N}_0 : \sup_{r \geq 0} |r^k g^{(j)}(r)| \leq B_1 A^k (k!)^p. \]

Write
\[ f(r) = r^{p+1} \left( \frac{1}{r} g(r) \right) , \quad r > 0. \]

Since \( f \in r^{p+1} S_{\text{even}}(\mathbb{R}^+) \) we have

\[ \forall \epsilon \in \mathbb{N}_0 \exists C_i > 0 : \sup_{0 \leq r \leq 1} \frac{1}{r} g(r) \leq C_i. \]

And therefore, for each \( k, l \in \mathbb{N}_0 \)

\[ \sup_{0 \leq r \leq 1} r^{k} \left( \frac{d}{dr} \right)^l \left( \frac{1}{r} g(r) \right) \leq \bar{C}_i A^k(k!)^a, \]

with

\[ \bar{C}_i = \sum_{m=0}^{l} \left( \frac{l}{m} \right) m! B_{i-m}. \]

Hence, for each \( k, l \in \mathbb{N}_0 \) we estimate

\[ \sup_{r \geq 1} r^{k} \left( \frac{d}{dr} \right)^l \left( \frac{1}{r} g(r) \right) \leq \bar{B}_i A^k(k!)^a \]

with \( \bar{B}_i = \max \{ C_i, \bar{C}_i \} \).

Therefore, \( \frac{1}{r} g(r) \in S_{a \alpha A; \infty, \text{even}}(\mathbb{R}^+) \). And since \( f(r) = r^{p+1} \left( \frac{1}{r} g(r) \right) \) we conclude that assertion \( H(p+1) \) is true. Assertion (ii) follows by induction.

Obviously (iii) \( \Rightarrow \) (ii). We prove the converse implication. Suppose (ii) holds. Then \( f^{(q)} \in r^p S_{\text{even}}(\mathbb{R}^+) \) for all \( p, q \in \mathbb{N}_0 \), by Theorem 1.3.5.10. For each \( q \in \mathbb{N}_0 \) we define the assertion \( G(q) \) by

\[ G(q) : \text{The function } f^{(j)} \text{ belongs to } r^p S_{a \alpha A; \infty, \text{even}}(\mathbb{R}^+) \quad j = 0, \ldots, q \text{ and all } p \in \mathbb{N}_0. \]

Then assertion \( G(0) \) is true, by assumption. Suppose \( G(q) \) is true for some \( q \in \mathbb{N}_0 \). Then

\[ \forall \epsilon \in \mathbb{N}_0 \exists \epsilon_{h_{p,q}} \in S_{a \alpha A; \infty}(\mathbb{R}) \forall \epsilon_{r > 0} : \epsilon_{r^{-p}} f^{(q)}(r) = h_{p,q}(r). \]

Fix \( p \in \mathbb{N}_0 \). Define \( h_{p,q+1} : \mathbb{R} \to \mathbb{C} \) by
Then $h_{p,q+1} \in S_{\alpha A^*;\omega,\text{even}}(\mathbb{R})$ (see [GS 2, p. 185 and p. 193]). And, for each $r > 0$, 

$$r^{-p} f^{q+1}(r) = h_{p,q+1}(r).$$

So $G(q+1)$ is true and assertion (iii) follows by induction. The equivalence of assertions (i) and (iv) follows from Lemma 2.5.10 and Corollary 2.5.11.

We arrive at the main result covering necessary and sufficient conditions on functions $g$ and $h$ in order that

$$g(r) h(\phi) \in \mathcal{P} S_{\alpha}(\mathbb{R}^2).$$

**Theorem 2.5.13.**

Let $f \in L_2(\mathbb{R}^2)$ for which there exist functions $g \in X_1$ and $h \in L_2([-\pi, \pi])$ such that

$$(IP f)(r, \phi) = g(r) h(\phi).$$

Let $\alpha > 0$.

(i) If $h$ has a finite Fourier series,

$$h(\phi) = \sum_{n=-N}^{N} a_n e_n(\phi),$$

then it is clear that $h$ extends to a function in $C_{2\pi-\text{per}}^\omega(\mathbb{R})$. First suppose there exist $n_0, m_0 \in \mathbb{Z}$ such that

$$\alpha_{2n_0} \neq 0 \text{ and } \alpha_{2m_0+1} \neq 0.$$

Then equivalent are

(i) $f \in S_{\alpha}(\mathbb{R}^2)$

(ii) $g \in r^{1/2} S_{\alpha,\text{even}}(\mathbb{R}^+) \text{ and } g \in r^{1/2} S_{\alpha,\text{even}}(\mathbb{R}^+)$

(iii) $\exists \alpha_0 \forall p \in \mathbb{N}_0 : g^{(p)} \in r^{p} S_{\alpha A^*;\omega,\text{even}}(\mathbb{R}^+)$

(iv) $\exists \alpha_0 \forall p, q \in \mathbb{N}_0 : g^{(p)} \in r^{p} S_{\alpha A^*;\omega,\text{even}}(\mathbb{R}^+)$

(v) $g \in S_{\alpha}(\mathbb{R}^+)$ with the property $\forall l \in \mathbb{N}_0 : g^{(l)}(0) = 0$.

Next suppose that for all $n_0, m_0 \in \mathbb{Z}$,

$$\alpha_{2n_0} = 0 \text{ or } \alpha_{2m_0+1} = 0.$$ 

Suppose $h \neq 0$ and let $l_0 := \max \{ l \in \mathbb{N}_0 \mid \alpha_l \neq 0 \lor \alpha_{-l} \neq 0 \}$. If $l_0 = 2k_0$, then equivalent are
(i) \( f \in S(\mathbb{R}^2) \)

(ii) \( g \in r^{2k} S_{\alpha, \text{even}}(\mathbb{R}^+) \)

(iii) \( g \in r^{2j} S_{\alpha, \text{even}}(\mathbb{R}^+) \), \( j = 0, \ldots, k_0 \)

(iv) \( g \in S_{\alpha}(\mathbb{R}^+) \) with the properties

\[
g_{+}^{(j)}(0) = 0 \quad , j = 0, \ldots, 2k_0 - 1
\]

\[
g_{+}^{(2l+1)}(0) = 0 \quad , l = k_0, k_0 + 1, \ldots
\]

If \( l_0 = 2k_0 + 1 \), then equivalent are

(i) \( f \in S_{\alpha}(\mathbb{R}^2) \)

(ii) \( g \in r^{2k+1} S_{\alpha, \text{even}}(\mathbb{R}^+) \)

(iii) \( g \in r^{2j+1} S_{\alpha, \text{even}}(\mathbb{R}^+) \), \( j = 0, \ldots, k_0 \)

(iv) \( g \in S_{\alpha}(\mathbb{R}^+) \) with the properties

\[
g_{+}^{(j)}(0) = 0 \quad , j = 0, \ldots, 2k_0
\]

\[
g_{+}^{(2l)}(0) = 0 \quad , j = k_0 + 1, k_0 + 2, \ldots
\]

(II) If \( h \) has an infinite Fourier series,

\[
h(\phi) = \sum_{n \in \mathbb{Z}} \alpha_n e_n(\phi) \quad \text{with} \quad \forall N \in \mathbb{N}_0 \exists m \in \mathbb{Z} : |m| > N : \alpha_m \neq 0,
\]

then equivalent are

(i) \( f \in S_{\alpha}(\mathbb{R}^2) \)

(ii) \( h \text{ extends to a function in } C^{\infty}_{2\pi\text{-per}}(\mathbb{R}) \) and

\[\exists A > 0 \forall p \in \mathbb{N}_0 : g \in r^p S_{\alpha A^{2\pi}, \text{even}}(\mathbb{R}^+)\]

(iii) \( h \text{ extends to a function in } C^{\infty}_{2\pi\text{-per}}(\mathbb{R}) \) and

\[\exists A > 0 \forall p, q \in \mathbb{N}_0 : g^{(q)} \in r^p S_{\alpha A^{2\pi}, \text{even}}(\mathbb{R}^+)\]

(iv) \( h \text{ extends to a function in } C^{\infty}_{2\pi\text{-per}}(\mathbb{R}) \) and

\( g \in S_{\alpha}(\mathbb{R}^+) \) with \( \forall l \in \mathbb{N}_0 : g_{+}^{(l)}(0) = 0. \)
The factorization problem for $L^p S^0(\mathbb{R}^2)$

In the present case the factorization problem is much more difficult than in the preceding case. This emerges in Theorem 2.5.15. For instance, if we want a function $f \in L_2(\mathbb{R}^2)$, with polar factorization $(L^p f)(r,\phi) = g(r) h(\phi)$, to be an element of $S^0(\mathbb{R}^2)$ ($\beta \leq 1$), then $h$ must be an even or an odd trigonometric polynomial, i.e.

$$\exists \alpha \in \mathbb{N}_+ : h = \sum_{m=-N}^{N} \alpha_m e^{2\pi i m r}$$

If $\beta > 1$ we only have an idea how the conditions on the functions $g$ and $h$ should be to ensure that $f \in S^0(\mathbb{R}^2)$ (cf. Conjecture 2.5.20). Let us first prove the following.

**Lemma 2.5.14.**

Let $\beta > 0$ and let $f \in S^0(\mathbb{R}^2)$. Suppose there exist functions $g \in X_1$ and $h \in L_2([-\pi,\pi])$ such that

$$(L^p f)(r,\phi) = g(r) h(\phi).$$

Then $h$ extends to a function in $C_{2\pi-per}^\infty(\mathbb{R})$ and

$$\forall m \in \mathbb{Z} : [(h, e_m)_{L_2([-\pi,\pi])} \neq 0 \Rightarrow g \in L^1 S_{\text{even}}^0(\mathbb{R}^+)].$$

**Proof.**

For convenience we set $\alpha_m = (h, e_m)_{L_2([-\pi,\pi])}$, $m \in \mathbb{Z}$. Since $f \in S^0(\mathbb{R}^2) \subset S(\mathbb{R}^2)$ we obtain from Theorem 1.3.5.3 that $h$ extends to a function in $C_{2\pi-per}^\infty(\mathbb{R})$. Corollary 2.4.11 yields

$$T_m f \in L^1 S_{\text{even}}^0(\mathbb{R}^+), \text{ for all } m \in \mathbb{Z}.$$ 

Noting that $T_m f = \alpha_m g$, $m \in \mathbb{Z}$, the result follows.

**Theorem 2.5.15.**

Let $f \in L_2(\mathbb{R}^2)$ for which there exist functions $g \in X_1$ and $h \in L_2([-\pi,\pi])$ such that

$$(L^p f)(r,\phi) = g(r) h(\phi).$$

Furthermore, assume that $f \neq 0$. Let $0 < \beta \leq 1$. Then the following assertions are equivalent

(i) $f \in S^0(\mathbb{R}^2)$

(ii) $h$ is an even or odd trigonometric polynomial and

$$\forall m \in \mathbb{Z} : [(h, e_m)_{L_2([-\pi,\pi])} \neq 0 \Rightarrow g \in L^1 S_{\text{even}}^0(\mathbb{R}^+)].$$

**Proof.**

(i) $\Rightarrow$ (ii): Let $f \in S^0(\mathbb{R}^2)$. According to Lemma 2.5.14 the function $h$ extends to an element of $C_{2\pi-per}^\infty(\mathbb{R})$ and
\[ \forall m \in \mathbb{Z}, \ (h, e_m)_{L_2(\mathbb{R}^+)} = 0 \Rightarrow g \in r^{1 \to m} S_{\text{even}}^p(\mathbb{R}^+) \].

It remains to show that \( h \) must be an even or an odd trigonometric polynomial. Suppose \( h \) is neither an even nor an odd trigonometric polynomial. Applying Theorem 1.3.5.10 it follows that

\[ g \in r^l S_{\text{even}}^p(\mathbb{R}^+) \text{ for all } l \in \mathbb{N}_0. \]

This implies, by the same theorem, that

\[ g_{<^l}(0) = 0 \text{ for all } l \in \mathbb{N}_0. \] (**)

Moreover, according to (*) we have

\[ g \in S^p(\mathbb{R}^+). \]

Since \( \beta \leq 1 \), there exists at least a strip \( 1 \leq A \) of the complex plane \( z = x + iy \) such that \( g \) may be extended analytically in that strip (see [GS 2 pp. 172/173]). Using (**) it follows that \( g = 0 \) in that strip. This implies that \( f = 0 \) on \( \mathbb{R} \), which is in contradiction with the assumption \( f \) not being the nulfunction.

The converse implication \((\text{ii}) \Rightarrow (\text{i})\) is a consequence of Corollary 2.4.14.

A consequence of the above theorem is the characterization of \( S_{\text{rad}}^p(\mathbb{R}^2) \), \( \beta \leq 1 \), as stated in Corollary 2.4.15. In the case \( \beta > 1 \) we refer to Conjecture 2.5.20 below.

The factorization problem for \( B^p S^p_0(\mathbb{R}^2) \)

First we take \( \alpha = \beta \) and in that case we are able to present a complete solution of the factorization problem. The proofs are based on Characterization 2.5.2 and Characterization 2.5.5 where the Laguerre basis plays a central role. After that, using Kashpirovskii's intersection result profitably, we obtain the complete solution of the factorization problem in each of the cases \( \beta \leq 1 \) and \( \beta \geq \alpha \).

Although we are not able to solve the problem in the case \( 1 < \beta < \alpha \) completely, we conjecture that its solution is of the same form as in the aforementioned cases (cf. Conjecture 2.5.20). We start with a simple estimation.

**Lemma 2.5.16.**

Let \( 0 < v \leq 1 \). Then for all \( x, y > 0 \),

\[ 2^{v-1}(x^v + y^v) \leq (x+y)^v \leq x^v + y^v. \]

**Proof.**

The proof is a consequence of the following inequalities for non-negative concave functions \( f \),
Now we present the solution of the factorization problem for the $S^o_\alpha(\mathbb{R}^2)$ spaces.

Theorem 2.5.17.
Let $f \in L_2(\mathbb{R}^2)$ for which there exist functions $g \in X_1$ and $h \in L_2([-\pi,\pi])$ such that

\[
(I^p f)(r, \psi) = g(r) h(\psi).
\]

Let $\alpha \geq \frac{1}{2}$. Then equivalent are

(i) $f \in S^o_\alpha(\mathbb{R}^2)$

(ii) $h$ extends to a function in $C^{2p}_{2\pi \text{-per}}(\mathbb{R})$ and the Fourier coefficients satisfy the growth estimate

\[
\exists \beta > 0 : (h, e_m)_{L_2([-\pi,\pi])} = O(\exp(-t |m|^{2\alpha})), \quad |m| \to \infty.
\]

Furthermore,

\[
\exists \alpha > 0 \forall m \in \mathbb{Z} : (h, e_m)_{L_2([-\pi,\pi])} \neq 0 \implies g \in r^{1/m} S^o_{\alpha; \text{even}}(\mathbb{R}^+).
\]

Proof.
We use our convenient notation $\alpha_m = (h, e_m)_{L_2([-\pi,\pi])}, \quad m \in \mathbb{Z}$.

Suppose $f \in S^o_\alpha(\mathbb{R}^2)$. By Theorem 2.5.9 and Theorem 2.5.14 we obtain

$h$ extends to a function in $C^{2p}_{2\pi \text{-per}}(\mathbb{R})$ and

\[
\forall m \in \mathbb{Z} : \alpha_m \neq 0 \implies g \in r^{1/m} S^o_{\alpha; \text{even}}(\mathbb{R}^+).
\]

From Characterization 2.5.2 we have

\[
\exists \alpha > 0 \exists \beta > 0 \forall n \in \mathbb{N} \forall m \in \mathbb{Z} : (I^p f, U_{n,m})_K \leq C_1 \exp(-t(n + |m|^{2\alpha})).
\]

Since the $U_{n,m}$ constitute an orthonormal basis in $K$,

\[
I^p f = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{Z}} (I^p f, U_{n,m})_K U_{n,m} = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{Z}} (I^p f, U_{n,m})_K \mathbb{L}^{(1,1)}(1/m,1) \otimes e_m.
\]

The series in the above expression are absolute convergent with respect to the norm in $K$. Therefore, it is allowed to choose any order of summation. In this case we choose
For the function \( h \) we have the Fourier series

\[
h = \sum_{m \in \mathbb{Z}} \alpha_m e_m
\]

and because \( \langle IP f \rangle (r, \phi) = g(r) h(\phi) \) we also have

\[
IP f = \sum_{m \in \mathbb{Z}} \alpha_m g \otimes e_m.
\]

Comparing (2.51) and (2.52) we obtain the equalities

\[
\alpha_m g = \sum_{n \in \mathbb{N}_0} \langle IP f, U_{n,m} \rangle L_n^{(1m,1)}(r), \quad m \in \mathbb{Z},
\]

as functions in \( L_2((0,\infty), r dr) \).

Due to the estimations for the coefficients \( \langle IP f, U_{n,m} \rangle \) the series in (2.53) represents a continuous function on \( \mathbb{R}^+ \). Since \( g \) is also continuous we even have the pointwise equalities

\[
\alpha_m g(r) = \sum_{n \in \mathbb{N}_0} \langle IP f, U_{n,m} \rangle L_n^{(1m,1)}(r), \quad r \geq 0, \quad m \in \mathbb{Z}.
\]

Let \( m \in \mathbb{Z} \) such that \( \alpha_m \neq 0 \) and choose \( r_0 > 0 \) such that \( g(r_0) \neq 0 \). Then

\[
\alpha_m = \frac{1}{g(r_0)} \sum_{n \in \mathbb{N}_0} \langle IP f, U_{n,m} \rangle L_n^{(1m,1)}(r_0) = \frac{\sqrt{2}}{g(r_0)} r_0^{1m} e^{-\frac{r_0^2}{2}} \sum_{n \in \mathbb{N}_0} \langle IP f, U_{n,m} \rangle \left[ \frac{n!}{(1m + n)!} \right]^{\frac{1}{2}} L_n^{(1m,1)}(r_0^2).
\]

For fixed \( x > 0 \) and \( v > -1 \), the generalized Laguerre polynomials \( L_n^{(v)}(x) \) satisfy the asymptotic equality (cf. [MOS, p. 248])

\[
L_n^{(v)}(x) = x^{-\frac{1}{2}v} O(n^{\frac{1}{2}v - \frac{1}{2}}), \quad n \to \infty.
\]

In particular, there exists \( C_2 > 0 \) such that for all \( n \in \mathbb{N}_0 \)

\[
1L_n^{(1m,1)}(r_0^2) \leq C_2 r_0^{-1m - \frac{1}{2}} (\sqrt{n})^{1m - \frac{1}{2}}.
\]

From the asymptotic expression for the \( \Gamma \)-function,

\[
\frac{\Gamma(z + a)}{\Gamma(z + b)} = z^{(a-b)} (1 + O(\frac{1}{z})), \quad z \to \infty,
\]

we derive that there exists \( C_3 > 0 \) such that for all \( n \in \mathbb{N}_0 \)
\[
\left( \frac{n!}{(1m l +n)!} \right)^{\frac{1}{2}} = \left( \frac{\Gamma(n+1)}{\Gamma(1m l +n +1)} \right)^{\frac{1}{2}} \leq C_3 (\sqrt{n})^{-1m l}.
\]

Applying the above asymptotic formulas and using Lemma 2.5.16 we estimate

\[
| \alpha_m | = \frac{\sqrt{\frac{1}{2}}}{g(r_0)} r_0^{m l} e^{-\frac{1}{2} \tau_2} \sum_{n \in \mathbb{N}_0} |(P f, U_{n,m})_K| \left( \frac{n!}{(1m l +n)!} \right)^{\frac{1}{2}} |L^{(m l)}_n(r_0^2)| \leq
\]

\[
\leq \frac{\sqrt{\frac{1}{2}}}{g(r_0)} r_0^{m l} e^{-\frac{1}{2} \tau_2} \cdot \sum_{n \in \mathbb{N}_0} C_1 \exp(-\tau (n + m l) / 2a) \cdot C_3 (\sqrt{n})^{-1m l} \cdot C_2 r_0^{m l \frac{1}{2}} (n!)^{m l \frac{1}{2}} =
\]

\[
= C_4 \sum_{n \in \mathbb{N}_0} \exp(-\tau (n + m l) / 2a) (\sqrt{n})^{-\frac{1}{2}} \leq
\]

\[
\leq C_4 \sum_{n \in \mathbb{N}_0} \exp(-\tau (n + m l) / 2a) =
\]

\[
= C \exp(-\tau m l / 2a)
\]

where \(C_4, C\) and \(\tau\) are suitable constants independent on \(m\). This settles the order estimate for the \(\alpha_m\). It remains to show that there exists \(A > 0\) such that

\[
\forall m \in \mathbb{Z} : \alpha_m \neq 0 : g \in r^{m l} s_{A, \alpha_m, even}(\mathbb{R}_+).
\]

To prove this we use Characterization 2.5.5 which contains a characterization of those spaces in terms of the Laguerre basis. Fix \(m \in \mathbb{Z}\) with \(\alpha_m \neq 0\). Then, by relation (2.53),

\[
g = \frac{1}{\alpha_m} \sum_{n \in \mathbb{N}_0} (P f, U_{n,m})_K L^{(m l,1)}_n.
\]

On the other hand, since \(\{L^{(m l,1)}_n \mid n \in \mathbb{N}_0\}\) is orthonormal basis in \(X_1\) the function \(g\) has also the Laguerre expansion

\[
g = \sum_{n \in \mathbb{N}_0} (g, L^{(m l,1)}_n)_{X_1} L^{(m l,1)}_n.
\]

Hence for each \(n \in \mathbb{N}_0\),

\[
(g, L^{(m l,1)}_n)_{X_1} = \frac{1}{\alpha_m} (P f, U_{n,m})_K.
\]

And so we have the estimations
\( (g, L_n^{(l, m, 1)})_{t} \leq \frac{1}{|\alpha_m|} \exp(-\tau (n + m 1) \frac{1}{2a}) \leq \frac{1}{|\alpha_m|} \exp(-\tau (n \frac{1}{2a} + m 1 \frac{1}{2a})) =
\]
\[
= \frac{1}{|\alpha_m|} \exp(-\tau |m| \frac{1}{2a}) \cdot \exp(-\tau n \frac{1}{2a}) , \quad n \in N_0
\]

where \( \tau = t \cdot 2^{2a} \), independent on \( m \). Now we have proved that
\[
\forall m \in \mathbb{Z} : \alpha_m \neq 0 : (g, L_n^{(l, m, 1)})_{t} = O(\exp(-\tau n \frac{1}{2a})).
\]

According to Characterization 2.5.5 this is equivalent with
\[
\exists A > 0 \forall m \in \mathbb{Z} : \alpha_m \neq 0 : g \in \alpha_m A : \alpha_m \cdot v \in \alpha_m \mathbb{R}^+.
\]

This completes the proof of the implication \( '(i) \Rightarrow (ii)' \).

For the converse, suppose \( (ii) \) holds. Then there exist \( t_1, C_1 > 0 \) such that
\[
\forall m \in \mathbb{Z} : |\alpha_m| \leq C_1 \exp(-t_1 |m| \frac{1}{2a}).
\]

Applying Characterization 2.5.5 we obtain that there exist \( t_2, C_2 > 0 \) such that
\[
\forall m \in \mathbb{Z} : \alpha_m \neq 0 : \forall n \in \mathbb{N}_0 : |(g, L_n^{(l, m, 1)})_{t_1}| \leq C_2 \exp(-t_2 n \frac{1}{2a}).
\]

Let \( t = \min(t_1, t_2) \). We show that
\[
(IP \ f, U_{n,m})_K = O(\exp(-\tau (n + m 1) \frac{1}{2a})) , \quad n, |m| \to \infty.
\]

For all \( n \in \mathbb{N}_0, m \in \mathbb{Z} \)
\[
(IP \ f, U_{n,m})_K = (g, L_n^{(l, m, 1)})_{t_1} (h, c_m)_{L_{l-\alpha, s}} = \alpha_m (g, L_n^{(l, m, 1)})_{t_1}.
\]

Let \( n \in \mathbb{N}_0 \) and \( m \in \mathbb{Z} \).
If \( \alpha_m = 0 \), then \( (IP \ f, U_{n,m})_K = 0 \). If \( \alpha_m \neq 0 \), then we estimate
\[
|\alpha_m| \cdot |(g, L_n^{(l, m, 1)})_{t_1}| \leq
\]
\[
\leq C_1 \exp(-t_1 |m| \frac{1}{2a}) \cdot C_2 \exp(-t_2 n \frac{1}{2a}) \leq
\]
\[
\leq C_1 C_2 \exp(-t(n \frac{1}{2a} + m 1 \frac{1}{2a})) \leq
\]
\[
\leq C_1 C_2 \exp(-t(n + m 1) \frac{1}{2a}).
\]

Note that the latter inequality is a consequence of Lemma 2.5.16. Hence
\[ \left( \mathcal{P} f, U_{a,m}\right)_K = O(\exp(-t(n + |m| 2^a))) \], \ n, |m| \to \infty

and Characterization 2.5.2 (ii) yields \( f \in S_{\alpha}^0(\mathbb{R}^2) \). This makes the proof complete. \( \square \)

If \( \alpha \leq \beta \) we are able to solve the factorization problem taking advantage of Kashpirovskii’s intersection result, Theorem 2.5.9 and Theorem 2.5.17, resulting in

**Theorem 2.5.18.**

Let \( f \in L_2(\mathbb{R}^2) \) for which there exist functions \( g \in X_1 \) and \( h \in L_2([-\pi, \pi]) \) such that

\[ \left( \mathcal{P} f \right)(r, \phi) = g(r) h(\phi). \]

Let \( 0 < \alpha \leq \beta \) such that \( \alpha + \beta \geq 1 \). Then equivalent are

(i) \( f \in S_{\alpha}^0(\mathbb{R}^2) \).

(ii) \( h \) extends to a function in \( C_{2\pi-per}^{\infty}(\mathbb{R}) \) and the Fourier coefficients satisfy the growth estimate

\[ \exists l > 0 : (h, e_m)_L^2([-\pi, \pi]) = O(\exp(-t |m| 2^l)) \], \ |m| \to \infty.

Furthermore,

\[ \exists A, \beta > 0 \ \forall m \in \mathbb{Z} \ [\left( h, e_m\right)_L^2([-\pi, \pi]) \neq 0 \Rightarrow g \in r^{|m|} S_{\alpha; \beta; \infty, even}(\mathbb{R}^+) \].

**Proof.**

Since \( \alpha \leq \beta \) we have \( S_{\alpha}(\mathbb{R}^2) \subset S_{\beta}(\mathbb{R}^2) \). So, applying Kashpirovskii’s intersection result twice we obtain

\[ S_{\alpha}^0(\mathbb{R}^2) = S_{\alpha}(\mathbb{R}^2) \cap S_{\beta}(\mathbb{R}^2) = (S_{\alpha}(\mathbb{R}^2) \cap S_{\beta}(\mathbb{R}^2)) \cap S_{\beta}(\mathbb{R}^2) = S_{\alpha}(\mathbb{R}^2) \cap S_{\beta}(\mathbb{R}^2). \]

And the result follows from the above stated theorems. \( \square \)

A consequence of this theorem is the characterization of \( S_{\alpha; \text{radsym}}^0(\mathbb{R}^2) \), (for \( 0 < \alpha \leq \beta \)), as stated in Corollary 2.4.20.

If \( \beta \leq 1 \), the solution of the factorization problem admits a great simplification, viz. the function \( h \) can only be an even or an odd trigonometric polynomial:

**Theorem 2.5.19.**

Let \( f \in L_2(\mathbb{R}^2) \) for which there exist functions \( g \in X_1 \) and \( h \in L_2([-\pi, \pi]) \) such that
Furthermore, assume that $f \neq 0$. Let $\alpha > 0$, $0 < \beta \leq 1$ such that $\alpha + \beta \geq 1$. Then equivalent are

(i) \hspace{1cm} f \in S^\beta_\theta(\mathbb{R}^2)

(ii) \hspace{1cm} h \text{ is an even or odd trigonometric polynomial and } \\
    \forall m \in \mathbb{Z} \hspace{0.5cm} ((h, e_m)_{L^2([-\pi, \pi])} \neq 0 \implies g \in r^m \mathcal{S}^\beta_{0, \text{even}}(\mathbb{R}^+)).

\textbf{Proof.}

The result is a consequence of Kashpirovskii's intersection result, Theorem 2.5.9 and Theorem 2.5.15.

We conclude this section with the

\textbf{Conjecture 2.5.20.}

In Theorem 2.5.18 the condition $0 < \alpha \leq \beta$ can be replaced by $\alpha > 0$, $\beta > 0$ and we admit $\alpha$ to be $\infty$. 

\hfill \Box
Appendix A

Countably normed spaces

In this appendix we present some terminology about countably normed spaces. A discussion in more detail can be found in [GS 2].

A countably normed space $X$ is a locally convex topological vector space whose topology $\mathcal{T}$ is defined by a countable family of norms $\mathcal{P} = \{p_k \mid k \in \mathbb{N}\}$. Here $\mathcal{T}$ is the topology on $X$ that has as a subbase of neighbourhoods the sets

$$U(x_0; p_k; \varepsilon) := \{x \in X \mid p_k(x - x_0) < \varepsilon\}$$

where $p_k \in \mathcal{P}$, $x_0 \in X$ and $\varepsilon > 0$.

Thus a subset $O$ of $X$ is open if and only if for every $x_0 \in O$ there are $p_1, \ldots, p_n$ in $\mathcal{P}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that

$$U(x_0; p_1, \ldots, p_n; \varepsilon_1, \ldots, \varepsilon_n) := \bigcap_{j=1}^n U(x_0; p_j; \varepsilon_j) \subset O.$$

It can be shown that $X$ endowed with this topology is a locally convex topological vector space.

In a countably normed space $X$ with norms $p_k, k \in \mathbb{N}$, one says $x_n \to x$ ($n \to \infty$) if $p_k(x - x_n) \to 0$ ($n \to \infty$) for each fixed $k \in \mathbb{N}$. Equivalently, convergence can be described by the metric

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} \min(1, p_k(x - y)).$$

If a countably normed space $X$ is endowed with two sets of norms

$$\{p_k \mid k \in \mathbb{N}\} \text{ and } \{q_l \mid l \in \mathbb{N}\}$$

we say that the sets are equivalent if and only if

$$\forall k \in \mathbb{N} \exists c > 0 \exists i_1, \ldots, i_n \in \mathbb{N} \forall f \in X : p_k(f) \leq C (q_{i_1}(f) + \cdots + q_{i_n}(f))$$

and

$$\forall l \in \mathbb{N} \exists d > 0 \exists k_1, \ldots, k_n \in \mathbb{N} \forall f \in X : q_l(f) \leq D (p_{k_1}(f) + \cdots + p_{k_n}(f)).$$

Equivalent sets of norms provide identical notions of convergence, open sets, etc.

A countable family of norms $\{p_k \mid k \in \mathbb{N}\}$ is called directed if

$$\forall k_1, k_2 \in \mathbb{N} \exists c > 0 \forall f \in X : p_{k_1}(f) + p_{k_2}(f) \leq C p_k(f).$$

Direct families are very useful because they provide a simple description of open sets, continuous seminorms and continuous linear mappings:

Let $X$ be a countably normed space endowed with a directed family of norms $\{p_k \mid k \in \mathbb{N}\}$ and let $Y$ be a countably normed space with a family of norms $\{q_l \mid l \in \mathbb{N}\}$. Then
(i) \( O \subset X \) is open if and only if \( \forall x_0 \in O \ \exists k \in \mathbb{N} \ \exists \epsilon > 0 : U(x_0 ; p_k ; \epsilon) \subset O. \)

(ii) A seminorm \( q : X \to \mathbb{R}^+ \) is continuous if and only if
\[
\exists k \in \mathbb{N} \ \exists C > 0 \ \forall x \in X : q(x) \leq C p_k(x).
\]

(iii) A linear mapping \( T : X \to Y \) is continuous if and only if
\[
\forall k \in \mathbb{N} : q_k \circ T \text{ is a continuous seminorm on } X.
\]

We mention that if the \( p_k \) are not directed, a seminorm \( q : X \to \mathbb{R}^+ \) is continuous if and only if
\[
\exists k_1, \ldots, k_n \in \mathbb{N} \ \exists c_1, \ldots, c_n > 0 \ \forall x \in X : q(x) \leq \sum_{i=1}^{n} c_i p_k(x).
\]

We conclude the introduction to countably normed spaces with some definitions.

Let \( X \) be a countably normed space with norms \( p_k, k \in \mathbb{N} \). A sequence \( (x_n) \subset X \) is said to be Cauchy, if it is Cauchy with respect to each of the norms \( p_k, k \in \mathbb{N} \). If every fundamental sequence in \( X \) converges to some element in \( X \), then \( X \) is said to be complete.

A Fréchet space is a locally convex complete topological vector space with a countable basis of zero-neighbourhoods. Hence, if \( X \) is complete then \( X \) is a Fréchet space. A set \( B \subset X \) is bounded, if it is bounded with respect to each of the norms, i.e.
\[
\forall k \in \mathbb{N} : \exists C_k > 0 : \forall x \in B : p_k(x) \leq C_k.
\]

A set \( F \subset X \) is called compact if every family of open sets covering \( F \) contains a finite subfamily, which covers also \( F \).

A set \( G \subset X \) is called relatively compact if the closure of \( G \), \( \text{cl}(G) := \bigcap \{ \tilde{G} \supset G \mid \tilde{G} \text{ is closed in } X \} \), is compact.

The space \( X \) is called semi-Montel if all its bounded sets are relatively compact.
Appendix B

Product topology and intersection topology

Each Gel'fand Shilov space can be written as the intersection of two well described spaces (cf. Characterization 1.2.1). So we think it is natural to endow a Gel'fand Shilov space with an intersection topology. Here we present some fundamental topological features. For proofs and further results we refer to [Si, Chapter IV, Section 22].

Let \((X_1, T_1)\) and \((X_2, T_2)\) be topological spaces. An open base for the product topology \(T_{prod}\) in the product space \(X_1 \times X_2\) is the class

\[ \{ O_1 \times O_2 \mid O_i \in T_i, i = 1,2 \}, \]

i.e. every set in \(T_{prod}\) is a union of sets in this class.

For \(i = 1,2\) we define the projections \(p_i : X_1 \times X_2 \to X_i\) by

\[ p_i(x_1, x_2) = x_i. \]

The topology \(T_{prod}\) can be characterized in terms of these projections, \(T_{prod}\) is the weakest topology in \(X_1 \times X_2\) for which the projections \(p_i, i = 1,2\), are continuous.

We identify the space \(X_1 \cap X_2\), whenever taking intersection makes sense, with the diagonal \(D \subset X_1 \times X_2\) defined by

\[ D := \{(x,x) \mid x \in X_1 \cap X_2\}. \]

Let \(T_{rel}\) be the relative topology in \(D\) of the product space \(X_1 \times X_2\), i.e.

\[ T_{rel} := \{ O \cap D \mid O \in T_{prod}\}. \]

Then \(T_{rel}\) is the weakest topology in \(D\) for which the restrictions of the projections \(p_i\) to \(D\) are continuous. This motivates us to define the intersection topology \(T_\triangle\) in \(X_1 \cap X_2\):

\(T_\triangle\) is the weakest topology in \(X_1 \cap X_2\) for which the inclusion maps \(X_1 \cap X_2 \hookrightarrow X_1\) and \(X_1 \cap X_2 \hookrightarrow X_2\) are continuous.

Let \(V := \{ O_1 \cap X_2, X_1 \cap O_2 \mid O_i \in T_i, i = 1,2\}\). Then \(V\) is an open subbase for the topology \(T_\triangle\) in the sense that \(T_\triangle\) equals the class of all unions of finite intersections of sets in \(V\). So the class \(\{ O_1 \cap O_2 \mid O_i \in T_i, i = 1,2\}\) is an open base for the topology \(T_\triangle\).

The following lemma presents a characterization for sequential convergence in the topological intersection space \((X_1 \cap X_2, T_\triangle)\).

Lemma B.1.

Let \((y_n)_{n \in \mathbb{N}}\) be a sequence in \(X_1 \cap X_2\). Then equivalent are
(i) \( y_n \to 0 \ (n \to \infty) \) in the topology \( T \cap \).

(ii) \( y_n \to 0 \ (n \to \infty) \) in both topologies \( T_1 \) and \( T_2 \).

**Proof.**

Since the inclusion maps \( X_1 \cap X_2 \subseteq X_1 \) and \( X_1 \cap X_2 \subseteq X_2 \) are continuous (i) implies (ii). For the converse suppose \( y_n \to 0 \ (n \to \infty) \) in both topologies \( T_1 \) and \( T_2 \). Let \( O_i \in T_i \ (i = 1, 2) \) and suppose \( 0 \in O_1 \cap O_2 \). Because \( y_n \to 0 \ (n \to \infty) \) in \( T_i \), there exists \( N_i \in \mathbb{N} \) such that \( y_n \in O_i \) for all \( n > N_i \ (i = 1, 2) \). So for all \( n > \max \{N_1, N_2\} \) we obtain \( y_n \in O_1 \cap O_2 \). Therefore, \( y_n \to 0 \) in \( T \cap \). \( \square \)
Appendix C

Inductive and projective limits

In this appendix we describe how to define topologies in unions and intersection of topological vector spaces. For a more detailed treatment we refer to [Co, Chapter IV] and [Sc, Chapter II].

A topological vector space (TVS) is a vector space $V$ endowed with a topology $T$ such that the mappings

$$ (v, w) \mapsto v + w \quad (v, w) \in V \times V \quad \text{and} \quad \lambda v \mapsto \lambda v \quad (\lambda, v) \in \mathbb{C} \times V $$

are continuous (here $V \times V$ and $\mathbb{C} \times V$ are endowed with their respective product topologies).

A TVS $(V, T)$ is called Hausdorff if

$$ \forall_{v, w \in V} \exists_{O_v, O_w \in T} \{ v \in O_v, w \in O_w \ \text{and} \ O_v \cap O_w = \emptyset \}. $$

A subset $W$ of vector space $V$ is called

- convex if $\forall_{v, w \in W} \forall_{t : 0 < t < 1} [t v + (1 - t) w \in W]$.
- balanced if $\forall_{v \in W} \forall_{\lambda \in \mathbb{C} : |\lambda| < 1} [\lambda v \in W]$.

A locally convex space (LCS) is a TVS, such that the topology is locally convex, i.e. for each neighbourhood $U$ of $0$ there exists a convex, balanced and open set $B$ such that $B \subset U$.

It is known that a LCS is a TVS with a topology defined by a family of seminorms $P$; the LCS is Hausdorff if $\bigcap_{p \in P} \{ x \mid p(x) = 0 \} = \{ 0 \}$.

Now we present a definition of inductive limit. A more general definition can be found in Schaefer's.

An inductive system is a pair $(V, \{(V_\alpha, T_\alpha) \mid \alpha \in A\})$, where $V$ is a vector space, $V_\alpha$ is a linear manifold in $V$ that has a topology $T_\alpha$ such that $(V_\alpha, T_\alpha)$ is a LCS; moreover, it is required that

(a) $A$ is a directed set and $V_\alpha \subseteq V_\beta$ if $\alpha \leq \beta$,

(b) the inclusion map $V_\alpha \subset \rightarrow V_\beta$ is continuous ($\alpha \leq \beta$)

(c) $V = \bigcup \{ V_\alpha \mid \alpha \in A \}$.

Note that condition (b) is equivalent to

(b') if $\alpha \leq \beta$ and $U_\beta \in T_\beta$, then $U_\beta \cap V_\alpha \in T_\alpha$.

Suppose $(V, \{(V_\alpha, T_\alpha) \mid \alpha \in A\})$ is an inductive system. Let $B = \{ \text{the collection of all convex balanced sets } W \text{ such that } W \cap V_\alpha \in T_\alpha \text{ for all } \alpha \in A \}$,
\[ T = \text{the collection of all subsets } U \text{ of } V \text{ such that } \forall x \in U \exists w \in B \ | x + W \subseteq U. \]

Then \((V, T)\) is a (not necessarily Hausdorff) LCS, here \(T\) is called the inductive limit topology and \((V, T)\) is said to be the inductive limit of the spaces \(\{(V_\alpha, T_\alpha) \mid \alpha \in A\}\). Note that \(T\) is the finest locally convex topology in \(V\) for which the canonical injections \(V_\alpha \hookrightarrow V\) are continuous.

Next we present a definition of projective limit. A projective system is a pair \((V, \{(V_\alpha, T_\alpha) \mid \alpha \in A\})\), where \(V\) and \(V_\alpha (\alpha \in A)\) are vector spaces such that \(V\) is contained in each \(V_\alpha\) and \(T_\alpha\) is a locally convex topology in \(V_\alpha (\alpha \in A)\). Moreover, it is required that

(a) \(A\) is a directed set and \(V_\beta \subseteq V_\alpha\) if \(\alpha \leq \beta\)

(b) the inclusion map \(V_\beta \hookrightarrow V_\alpha\) is continuous (\(\alpha \leq \beta\))

(c) \(V = \cap \{V_\alpha \mid \alpha \in A\}\).

Let \((V, \{(V_\alpha, T_\alpha) \mid \alpha \in A\})\) be a projective system. By \(\tau\) we denote the weakest topology in \(V\) such that the inclusion maps \(V \hookrightarrow V_\alpha\) are continuous (\(\alpha \in A\)). The topology \(\tau\) is called the projective limit topology and \((V, \tau)\) is said to be the projective limit of the spaces \(\{(V_\alpha, T_\alpha) \mid \alpha \in A\}\). The following observation is useful.

Let \((V, \{(V_\alpha, T_\alpha) \mid \alpha \in A\})\) be a projective system and suppose that for each \(\alpha \in A\) the topology \(T_\alpha\) is defined by a family of seminorms \(P_\alpha\). Then the projective limit topology \(\tau\) equals the topology in \(V\) defined by the seminorms \(\{p \text{ restricted to } V \mid \exists \alpha \in A : p \in P_\alpha\}\). For a proof we refer to [Sc, Chapter II, Section 5].
Appendix D

The Fourier transformation on $L_2(\mathbb{R}^n)$

In this appendix we present some basic notions about the Fourier transformation. All proves are omitted; an interested reader may read them in [We, Chapter 10].

We use the multi-index notation introduced in Section 3.1. Let $n \in \mathbb{N}$, fixed. The Schwartz space $S(\mathbb{R}^n)$, the space of rapidly decreasing functions, is the vector space of $C^\infty$-functions on $\mathbb{R}^n$ for which the following property holds

$$ \forall k \in \mathbb{N}_0^n \exists l \in \mathbb{N}_0^n : \sup_{x \in \mathbb{R}^n} |x^k D^l \phi(x)| < \infty. $$

Since for every $\phi \in S(\mathbb{R}^n)$ and for all $p > 1$ there exists a $C > 0$ such that

$$ |\phi(x)| \leq C(1 + x^2)^{-p}, \quad x \in \mathbb{R}^n, $$

it is clear that for all $p \in [1, \infty)$,

$$ S(\mathbb{R}^n) \subset L_p(\mathbb{R}^n). $$

Let us introduce the Fourier transformation on $S(\mathbb{R}^n)$ and on $L_2(\mathbb{R}^n)$ and $L_1(\mathbb{R}^n)$ thereafter. For all $\phi \in S(\mathbb{R}^n)$ we define the Fourier transformation $F_0$ by the integral

$$ F_0 \phi : x \mapsto (2\pi)^{-n/2} \int_{\mathbb{R}^n} \phi(y) e^{-i(x \cdot y)} \, dy $$

where

$$ (x \cdot y) = \sum_{j=1}^n x_j y_j \quad \text{for } x, y \in \mathbb{R}^n. $$

The transformation $F_0$ is a bijective linear mapping of $S(\mathbb{R}^n)$ onto itself. We have

$$ F_0^{-1} : x \mapsto (2\pi)^{-n/2} \int_{\mathbb{R}^n} \phi(y) e^{i(x \cdot y)} \, dy $$

and $(F_0 \phi)(x) = (F_0^{-1} \phi)(-x)$ for every $\phi \in S(\mathbb{R}^n)$. Moreover

$$ \|F_0 \phi\|_2 = \|F_0^{-1} \phi\|_2, \quad \phi \in S(\mathbb{R}^n), $$

here $\| \cdot \|_2$ denotes the norm in $L_2(\mathbb{R}^n)$. The mappings $F_0$ and $F_0^{-1}$ can be uniquely extended to unitary operators $\mathcal{F}$ and $\mathcal{F}^*$ on $L_2(\mathbb{R}^n)$ with $\mathcal{F}^* \mathcal{F} = \mathcal{F} \mathcal{F}^* = I$. The operator $\mathcal{F}$ is called the Fourier transformation on $L_2(\mathbb{R}^n)$. The mappings $F_0$ and $F_0^{-1}$ can be extended to $L_1(\mathbb{R}^n)$ in the natural way. These unique extensions are defined for each $f \in L_1(\mathbb{R}^n)$ by the absolutely convergent integrals

$$ x \mapsto (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y) e^{-i(x \cdot y)} \, dy $$
\[ x \mapsto (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y) e^{i(x \cdot y)} \, dy \]

which define continuous functions on \( \mathbb{R}^n \).

The Fourier transformation on \( L_1(\mathbb{R}^n) \) is closely allied to the Fourier transformation on \( L_2(\mathbb{R}^n) \). Indeed, for each \( f \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n) \), \( \mathcal{F}f \) and \( \mathcal{F}^{-1}f \) have the continuous representants

\[
(\mathcal{F} f)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y) e^{-i(x \cdot y)} \, dy, \quad x \in \mathbb{R}^n
\]

and

\[
(\mathcal{F}^{-1} f)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y) e^{i(x \cdot y)} \, dy, \quad x \in \mathbb{R}^n.
\]

In general, if \( f \in L_2(\mathbb{R}^n) \), then

\[
(\mathcal{F} f)(x) = \lim_{N \to \infty} (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y) e^{-i(x \cdot y)} \, dy, \quad x \in \mathbb{R}^n,
\]

here "l.i.m. = limit in mean" stands for the limit in \( L_2(\mathbb{R}^n) \). A similar formula holds for \( \mathcal{F}^{-1} \).

Finally on \( S(\mathbb{R}^n) \) we define the Fourier transformation with respect to the \( j \)-th variable \( \mathcal{F}_{0,j} \) by the integral

\[
\mathcal{F}_{0,j} \phi : x \mapsto (2\pi)^{-1/2} \int_{\mathbb{R}} \phi(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n) e^{-iy} \, dy, \quad x \in \mathbb{R}^n, \quad j = 1, 2, \ldots, n.
\]

For each \( j \in \{1, \ldots, n\} \) the transformation \( \mathcal{F}_{0,j} \) is a bijective linear mapping of \( S(\mathbb{R}^n) \) onto itself. We have

\[
\mathcal{F}_{0,j}^{-1} \phi : x \mapsto (2\pi)^{-1/2} \int_{\mathbb{R}} \phi(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n) e^{iy} \, dy
\]

and \( (\mathcal{F}_{0,j} \phi)(x) = (\mathcal{F}_{0,j}^{-1} \phi)(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n) \) for every \( \phi \in S(\mathbb{R}^n) \). Moreover \( \|\mathcal{F}_{0,j} \phi\|_2 = \|\mathcal{F}_{0,j}^{-1} \phi\|_2 = \|\phi\|_2, \phi \in S(\mathbb{R}^n) \).

For every \( i, j \in \{1, \ldots, n\} \) we define the permutation operator \( p_{i,j} \) on \( S(\mathbb{R}^n) \) by

\[
p_{i,j} \phi : x \mapsto \phi(x^{(i,j)}), \quad x \in \mathbb{R}^n,
\]

where \( x^{(i,j)} \in \mathbb{R}^n \) with \( x^{(i,j)}_k = \begin{cases} x_k, & k \neq i, j \\ x_j, & k = i, \quad k = 1, \ldots, n \\ x_i, & k = j. \end{cases} \)

(\( x^{(i,j)} \) is obtained from \( x \) by exchanging the coordinates \( x_i \) and \( x_j \)). Then we can write

\[
\mathcal{F}_{0,j} = p_{1,j} \mathcal{F}_{0,1} p_{1,j}, \quad j = 1, \ldots, n.
\]

Furthermore,

\[
\mathcal{F}_0 = \mathcal{F}_{0,n} \circ \mathcal{F}_{0,n-1} \circ \cdots \circ \mathcal{F}_{0,1}.
\]

The mappings \( \mathcal{F}_{0,j} \) and \( \mathcal{F}_{0,j}^{-1} \) can be uniquely extended to unitary operators \( \mathcal{U}_j \) and \( \mathcal{U}_j^{-1} \) on
$L_2(\mathbb{R}^n)$ with $\mathcal{F}_j = \mathcal{F}_j^* = \mathcal{F}_j^{-1}$ and $\mathcal{F}_j^2 = I$. The operator $\mathcal{F}_j$ is called the Fourier transformation with respect to the $j$-th variable on $L_2(\mathbb{R}^n)$. We mention the useful formula

$$\mathcal{F} = \mathcal{F}_0 \circ \mathcal{F}_{n-1} \circ \cdots \circ \mathcal{F}_1.$$ 

The mappings $F_{0,j}$ and $F_{0,j}^{-1}$ can be extended to $L_1(\mathbb{R}^n)$ in the natural way.
In this report no distinction is made between $F_{0,j}$ and $\mathcal{F}_j$ nor between $F_0$ and $\mathcal{F}$. 


Appendix E

The inverse of a certain matrix

Consider the upper triangular matrix $A$ defined by

$$A_{l,k} = \begin{cases} -2 \left( \frac{\Gamma(k+1)}{\Gamma(k+\frac{1}{2})/ \Gamma(l+\frac{1}{2})} \right)^{1/2}, & 0 \leq l \leq k - 1 \\ -1, & l = k. \end{cases}$$

In this appendix we shall determine the inverse matrix $A^{-1}$. To this end we define the upper triangular matrix $P$ by $P := A + I$. This matrix satisfies

$$P_{l,j} P_{j,k} = -2 P_{l,k}, \quad l + 1 \leq j \leq k - 1.$$ 

For the matrix $P$ we prove the following

**Theorem E.1.**

(i) For each $n \in \mathbb{N}, n \geq 2$, there exists a real number $c_n$ such that

$$(P^n)_{l,k} = c_n P_{l,k} (k-l-(n-1)) \cdot (k-l-(n-2)) \cdot \cdots \cdot (k-l-1), \quad 0 \leq l \leq k - 1$$

(ii) $c_n = \frac{(-2)^{n-1}}{(n-1)!}, \quad n \geq 2.$

**Proof.**

It is obvious that (i) is true for $n = 2$, then $c_2 = -2$.

Now suppose, inductively, that (i) is true for certain $n \geq 2$. Then we derive

$$(P^{n+1})_{l,k} = \sum_{j=l+1}^{k-n} P_{l,j} (P^n)_{j,k} =$$

$$= \sum_{j=l+1}^{k-n} P_{l,j} P_{j,k} \cdot c_n (k-j-(n-1)) \cdot (k-j-(n-2)) \cdot \cdots \cdot (k-j-1) =$$

$$= -2 c_n P_{l,k} \sum_{j=l+1}^{k-n} (k-j-(n-1)) \cdot (k-j-(n-2)) \cdot \cdots \cdot (k-j-1) =$$

$$= -2 c_n P_{l,k} \sum_{j=1}^{k-l-n} j \cdot (j+1) \cdot \cdots \cdot (j+n-2) =$$

$$= 2 c_n P_{l,k} \sum_{j=1}^{k-l-n} (k-l-n) \cdot (k-l-n+1) \cdot \cdots \cdot (k-l-2) \cdot (k-l-1) \cdot \frac{1}{n}, \quad 0 \leq l \leq k - 1.$$ 

Hence we find
\[(P^{n+1})_{l,k} = c_{n+1} \cdot P_{l,k} (k-l-n) (k-l-(n-1)) \cdot \cdots \cdot (k-l-1), \quad 0 \leq l \leq k-1\]

with
\[c_{n+1} = \frac{-2c_n}{n}.
\]

Since \(c_2 = -2\) it follows immediately that \(c_n = \frac{(-2)^n}{(n-1)!}, \quad n \geq 2.\)

The equality (*) follows from the formula in [Jo, p. 8], which reads
\[\sum_{i=1}^{n} i(i+1)(i+2) \cdots (i+m) = \frac{n(n+1)(n+2) \cdots (n+m)(n+m+1)}{m+2}.
\]

The above theorem yields, for every \(n \geq 1\),
\[(P^n)_{l,k} = (-2)^n \left[ \frac{\Gamma(k+1)}{\Gamma(k+\frac{1}{2})} \right]^{\frac{k}{2}} \left[ \frac{\Gamma(l+\frac{1}{2})}{\Gamma(l+1)} \right]^{\frac{l}{2}} \left[ \frac{k-l-1}{n-1} \right], \quad 0 \leq l \leq k-1.
\]

Now we are in a position to invert \(-A\):
\[(-A)^{-1} = (I-P)^{-1} = \sum_{n=0}^{\infty} P^n.
\]

So
\[(-A)_{l,l}^{-1} = 1, \quad l \in \mathbb{N}_0
\]
and for every \(0 \leq l \leq k - 1\) we have
\[(-A)_{l,l}^{-1} = \sum_{n=1}^{\infty} (P^n)_{l,k} =
\]
\[= \sum_{n=1}^{\infty} (-2)^n \left[ \frac{\Gamma(k+1)}{\Gamma(k+\frac{1}{2})} \right]^{\frac{k}{2}} \left[ \frac{\Gamma(l+\frac{1}{2})}{\Gamma(l+1)} \right]^{\frac{l}{2}} \left[ \frac{k-l-1}{n-1} \right]
\]
\[= \left[ \frac{\Gamma(k+1)}{\Gamma(k+\frac{1}{2})} \right]^{\frac{k}{2}} \left[ \frac{\Gamma(l+\frac{1}{2})}{\Gamma(l+1)} \right]^{\frac{l}{2}} \cdot (-2) \cdot \sum_{n=0}^{k-l-1} \left[ \frac{k-l-1}{n} \right] (-2)^n \cdot 1^{(k-l-1)-n}
\]
\[= 2 \left[ \frac{\Gamma(k+1)}{\Gamma(k+\frac{1}{2})} \right]^{\frac{k}{2}} \left[ \frac{\Gamma(l+\frac{1}{2})}{\Gamma(l+1)} \right]^{\frac{l}{2}} (-1)^{k-l}.
\]

So \(A^{-1}\) is the upper triangular matrix defined by
This can also be verified by the following straightforward calculation.

Let \( 0 \leq l \leq k - 1 \), then

\[
(A \cdot A^{-1})_{l,k} = \sum_{j=l}^{k-1} A_{l,j} (A^{-1})_{j,k} = -(A^{-1})_{l,k} = (A^{-1})_{l,k} - \sum_{j=l+1}^{k-1} A_{l,j} (A^{-1})_{j,k} ,
\]

with

\[
\begin{align*}
&\sum_{j=l+1}^{k-1} A_{l,j} (A^{-1})_{j,k} = \sum_{j=l+1}^{k-1} -2 \left[ \frac{\Gamma(j+1) \Gamma(j+\frac{1}{2})}{\Gamma(j+\frac{1}{2}) \Gamma(j+1)} \right]^{\frac{1}{2}} \cdot (-2) \left[ \frac{\Gamma(k+1) \Gamma(j+\frac{1}{2})}{\Gamma(k+\frac{1}{2}) \Gamma(j+1)} \right]^{\frac{1}{2}} \cdot (-1)^{k-j} = \\
&= \left[ \frac{\Gamma(l+\frac{1}{2}) \Gamma(k+1)}{\Gamma(l+1) \Gamma(k+\frac{1}{2})} \right]^{\frac{1}{2}} \cdot 4 \cdot \sum_{j=l+1}^{k-1} (-1)^{k-j} = \\
&= 4 \cdot (\cdots)^{\frac{1}{2}} \cdot \begin{cases} (-1)^{k+l+1} , & \text{if } k - l \text{ is even} \\
0 , & \text{if } k - l \text{ is odd} \
\end{cases}
\end{align*}
\]

If \( k - l \) is odd, then

\[
(A \cdot A^{-1})_{l,k} = 2 \cdot (\cdots)^{\frac{1}{2}} \cdot (-1)^{k-l} + 2 \cdot (\cdots)^{\frac{1}{2}} + 0 = 0.
\]

If \( k - l \) is even, then

\[
(A \cdot A^{-1})_{l,k} = 2 \cdot (\cdots)^{\frac{1}{2}} + 2 \cdot (\cdots)^{\frac{1}{2}} - 4 \cdot (\cdots)^{\frac{1}{2}} = 0.
\]

Furthermore,

\[
(A \cdot A^{-1})_{l,l} = 1 ,
\]

hence \( A \cdot A^{-1} = I \). Analogously \( A^{-1} \cdot A = I \).
Appendix F

The Hermite operator \( P^2 + Q^2 \).

In this appendix we prove that the Hermite operator \( P^2 + Q^2 \) is self-adjoint. First we present some useful identities concerning the Hermite polynomials \( H_n \) and the Hermite functions \( \psi_n \) which are introduced in Section 2.3, Chapter I. From [MOS, p. 252] we obtain the recurrence relations

\[
H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0, \quad n = 0, 1, 2, \ldots
\]

where we take

\[
H_n(x) = 0 \quad \text{if} \quad n < 0.
\]

Then it follows at once that

\[
2x^2 \psi_n(x) = \sqrt{n+1} \psi_{n+1}(x) + \sqrt{n} \psi_{n-1}(x), \quad n = 0, 1, 2, \ldots
\]

Applying this formula once more, we obtain

\[
2x^2 \psi_n(x) = \sqrt{(n+1)(n+2)} \psi_{n+2}(x) + (2n+1) \psi_n(x) + \sqrt{n(n-1)} \psi_{n-2}(x), \quad n = 0, 1, 2, \ldots
\]

Finally, from [SD, Chapter II, formula (2.51)] we obtain

\[
\mathcal{L} \psi_n = (-i)^n \psi_n, \quad n = 0, 1, 2, \ldots
\]

We define the self-adjoint operator \( T \) by

\[
T f = \sum_{n=0}^{\infty} (2n+1) (f, \psi_n)_{L^2(\mathbb{R})} \psi_n, \quad f \in D(T),
\]

with

\[
D(T) = \{ g \in L^2(\mathbb{R}) \mid \sum_{n=0}^{\infty} n^2 \| g, \psi_n \|_{L^2(\mathbb{R})}^2 < \infty \}.
\]

If we take \( f \in < \{ \psi_n \mid n \in \mathbb{N}_0 \} > \), then

\[
T f = (P^2 + Q^2) f
\]

because the \( \psi_n \) are eigenfunctions of the differential operator \( \frac{d^2}{dx^2} + x^2 \) with respective eigenvalues \( 2n + 1 \) \( (n \in \mathbb{N}_0) \). Here we prove the following result.
Theorem F.1.

\[ P^2 + Q^2 = T. \]

Proof.

Since \( P^2 + Q^2 \) is symmetric on its domain we derive for all \( f \in D(P^2) \cap D(Q^2) \),

\[
(P^2 + Q^2)f = \sum_{n=0}^{\infty} ((P^2 + Q^2)f, \psi_n)_{L_2(\mathbb{R})} \psi_n
\]

\[
= \sum_{n=0}^{\infty} (f, (P^2 + Q^2)\psi_n)_{L_2(\mathbb{R})} \psi_n
\]

\[
= \sum_{n=0}^{\infty} (2n+1)(f, \psi_n)_{L_2(\mathbb{R})} \psi_n
\]

so \( f \in D(T) \) and \( (P^2 + Q^2)f = Tf \). Next we prove that \( D(T) \subset D(P^2 + Q^2) \). Let \( f \in D(T) \). Then \( f \in L_2(\mathbb{R}) \) with

\[
\sum_{n=0}^{\infty} n^2 |(f, \psi_n)_{L_2(\mathbb{R})}|^2 < \infty. \tag{8}
\]

For each \( N \in \mathbb{N} \) we define \( f_N := \sum_{n=0}^{N} (f, \psi_n)_{L_2(\mathbb{R})} \psi_n \).

Then \( f_N \in D(Q^2) \) (\( N \in \mathbb{N} \)) and \( f_N \to f \) (\( N \to \infty \)) in \( L_2(\mathbb{R}) \)-sense. Furthermore, we derive with the aid of (3),

\[
Q^2 f_N = \sum_{n=0}^{N} (f, \psi_n)_{L_2(\mathbb{R})} Q^2 \psi_n
\]

\[
= \frac{1}{2} \sum_{n=0}^{N} (f, \psi_n)_{L_2(\mathbb{R})} \left[ \sqrt{n+1} \sqrt{n+2} \psi_{n+2} + (2n+1) \psi_n + \sqrt{n(n-1)} \psi_{n-2} \right].
\]

By (8) it follows that the sequence \( (Q^2 f_N)_{N \in \mathbb{N}} \) converges in the Hilbert space \( L_2(\mathbb{R}) \) with limit

\[
g = \frac{1}{2} \sum_{n=0}^{\infty} (f, \psi_n)_{L_2(\mathbb{R})} \left[ \sqrt{n+1} \sqrt{n+2} \psi_{n+2} + (2n+1) \psi_n + \sqrt{n(n-1)} \psi_{n-2} \right].
\]

Since \( Q^2 \) is self-adjoint, the operator \( Q^2 \) is also closed (cf. [We, Chapter 5, Theorem 5.3]). Therefore, we conclude

\( f \in D(Q^2) \) and \( Q^2 f = g \).

Finally we show that \( f \in D(P^2) \). Since \( f \in D(T) \), also \( IF^* f \in D(T) \) because by (4) we have

\[
\sum_{n=0}^{\infty} n^2 |(IF^* f, \psi_n)_{L_2(\mathbb{R})}|^2 = \sum_{n=0}^{\infty} n^2 |(f, IF \psi_n)_{L_2(\mathbb{R})}|^2 =
\]
The above consideration yields

$$IF^* f \in D(Q^2).$$

So $f \in IF(D(Q^2)) = D(P^2)$. Hence $f \in D(P^2) \cap D(Q^2) = D(P^2 + Q^2)$.

And this completes the proof.
Appendix G

An eigenvalue problem

Consider the eigenvalue problem for polar coordinates

\[
\begin{bmatrix}
- \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + r^2
\end{bmatrix} u = \lambda u.
\]  

(1)

Suppose (1) has a solution of the form

\[
u(r, \phi) = f(r) g(\phi).
\]

(2)

Substituting this in (1) and dividing by \(u\), we obtain

\[
- \frac{f''(r)}{f(r)} - \frac{1}{r} \frac{f'(r)}{f(r)} - \frac{1}{r^2} \frac{g''(\phi)}{g(\phi)} + r^2 = \lambda.
\]

(3)

Thus \(\frac{g''(\phi)}{g(\phi)}\) is independent of \(\phi\), say

\[
g''(\phi) = -\mu g(\phi).
\]

(4)

Note that \(u(r, \pi) = u(r, -\pi)\) and also \(u_\phi(r, \pi) = u_\phi(r, -\pi)\), so we arrive at the eigenvalue problem for \(g\),

\[
\begin{cases}
g''(\phi) = -\mu g(\phi) \\
g(-\pi) = g(\pi), \quad g'(-\pi) = g(\pi)
\end{cases}
\]

(5)

which has the solutions

\[
\mu = m^2, \quad g(\phi) = e^{im\phi} \quad (m \in \mathbb{Z}).
\]

(6)

Fix \(m \in \mathbb{Z}\). If we take \(\frac{g''(\phi)}{g(\phi)} = -m^2\) in (3) we get the eigenvalue problem for \(f\),

\[
-f''(r) - \frac{1}{r} f'(r) + \left[ \frac{m^2}{r^2} + r^2 \right] f(r) = \lambda f(r).
\]

(7)

Let \(f(r) = r^{-\frac{1}{2}} \chi(r)\). Then

\[
-\chi''(r) + \left[ \frac{m^2 - \frac{1}{2}}{r^2} + r^2 \right] \chi(r) = \lambda \chi(r).
\]

(8)

The solutions are (see [Ti, Section 4.16] and also [MOS, p. 243])

\[
\lambda = 4n + 2 \mid m \mid 1 + 2, \quad \chi(r) = r^{\frac{1}{2} + \frac{1}{2}} e^{-\frac{1}{2}r^2} L_n^{(1/2)}(r^2), \quad n = 0, 1, 2, \ldots
\]

(9)

Noting that \(f(r) = r^{-\frac{1}{2}} \chi(r)\) we obtain the solution of (1),
\[
\lambda = 4n + 2 \left| m \right| + 2,
\quad n \in \mathbb{N}_0, \quad m \in \mathbb{Z}. \tag{10}
\]

\[
u(r, \phi) = r^{|m|} e^{-\frac{1}{2} r^2} L_n^{(|m|)}(r^2) e^{im\phi}
\]

Normalizing the eigenfunctions with respect to the norm in \( K = L_2((0, \infty) \times [-\pi, \pi], \ r dr d\phi) \) we obtain:

The eigenvalues of (1) are

\[
\lambda = \lambda_{n,m} = 4n + 2 \left| m \right| + 2, \quad n \in \mathbb{N}_0, \quad m \in \mathbb{Z} \tag{11}
\]

with respective orthonormal eigenfunctions (in \( K \))

\[
u(r, \phi) = U_{n,m}(r, \phi) =
\frac{2 \Gamma(n+1)}{\Gamma(|m|+n+1)} \left( r^{|m|} e^{-\frac{1}{2} r^2} L_n^{(|m|)}(r^2) \right) \frac{1}{\sqrt{2\pi}} e^{im\phi}. \tag{12}
\]
Appendix H

Correspondence between $S_{2\alpha}(\mathbb{R})$ and $S_{\alpha,\text{even}}(\mathbb{R})$.

In [GS 2, p. 172] the following characterization of $S_{\alpha}(\mathbb{R})$ is given.

Characterization H.1.

The space $S_{\alpha}(\mathbb{R})$, $\alpha \neq 0$, consists of those and only those functions $\phi$ for which

$$\exists_\alpha > 0 \forall q \in \mathbb{N}_2 \exists C_q > 0 : \sup_{x \in \mathbb{R}} |\phi^{(q)}(x)| \exp (a |x|^\alpha) \leq C_q.$$  

The constants $a$ and $C_q$ depend on the function $\phi$. \[\square\]

Lemma H.2.

Let $f \in C^\infty(\mathbb{R})$. For each $x > 0$ we define $g(x) = f(\sqrt{x})$. Then $g \in C^\infty(0, \infty)$ and for all $l \in \mathbb{N}_0$

$$g^{(l)}(x) = \sum_{i=0}^{l} b_{i,l} x^{-\frac{i}{2}(2l-i)} f^{(i)}(\sqrt{x}) , \quad x > 0 ,$$

with

$$b_{i,l} = (-1)^{l-i} \left( \frac{1}{2} \right)^{2l-i} \frac{(2l-i-1)!}{(l-i)!(i-1)!} , \quad 0 < i \leq l ,$$

$$b_{0,l} = 0 , \quad l > 0 ,$$

$$b_{0,0} = 1 .$$

Proof. 

Induction arguments yield the validity of this lemma. We observe that the $b_{i,l}$ satisfy the recurrence relations

$$b_{i,l+1} = \frac{1}{2} (b_{i-1,l} - (2l-i) b_{i,l}) , \quad 0 \leq i \leq l + 1$$

with initial values

$$b_{0,0} = 1 \text{ and } b_{i,l} = 0 \text{ if } i < 0 \text{ or } i > l .$$ \[\square\]

Next we present the main theorem of this Appendix,
Theorem H.3.

(i) Let $f \in S_{\alpha,\text{even}}(\mathbb{R})$ ($\alpha > 0$). Then there exists $g \in S_{2\alpha}(\mathbb{R})$ such that

$$f(x) = g(x^2) \quad \text{for all } x \in \mathbb{R}.$$ 

(ii) Let $g \in S_{2\alpha}(\mathbb{R})$ ($\alpha > 0$) and define $f : \mathbb{R} \to \mathbb{C}$ by

$$f(x) = g(x^2) \quad x \in \mathbb{R}.$$ 

Then $f \in S_{\alpha,\text{even}}(\mathbb{R})$.

Proof.

(i) We have $f \in S_{\alpha,\text{even}}(\mathbb{R}) \subset S_{\text{even}}(\mathbb{R})$. Applying Borel's theorem there exists $g \in S(\mathbb{R})$ such that

$$f(x) = g(x^2) \quad \text{for all } x \in \mathbb{R}.$$ 

$$g(x) = 0 \quad \text{for all } x \leq -1.$$ 

We prove that this function $g$ belongs to $S_{2\alpha}(\mathbb{R})$. By Characterization H.1 there are constants $a$ and $C_q$ ($q \in \mathbb{N}_0$) such that $\sup_{x \in \mathbb{R}} |f^{(q)}(x)| \exp(a x |x|^{1/\alpha}) \leq C_q$. Applying Lemma H.2 we derive for all $l \in \mathbb{N}_0$ and $x > 1$

$$|g^{(l)}(x)| \exp(a x^{2\alpha}) \leq \sum_{i=0}^{l} |b_{i,l}| \cdot |f^{(i)}(\sqrt{x})| \exp(a \sqrt{x}^{-\alpha}) \leq \sum_{i=0}^{l} |b_{i,l}| C_i.$$ 

Furthermore, since $g$ is continuous there exist constants $K_l$ ($l \in \mathbb{N}_0$) such that

$$\sup_{x \in [-1,1]} |g^{(l)}(x)| \exp(a x^{2\alpha}) \leq K_l.$$ 

Hence by Characterization H.1 we obtain $g \in S_{2\alpha}(\mathbb{R})$.

(ii) Since $g \in S_{2\alpha}(\mathbb{R})$, there exist constants $a > 0$ and $C_q > 0$ such that

$$\sup_{x \in \mathbb{R}} |g^{(q)}(x)| \exp(a x |x|^{1/\alpha}) < C_q \quad (q \in \mathbb{N}_0).$$ 

For each $q \in \mathbb{N}_0$ there exist polynomials $p_{q,j}$, $j = 0, \ldots, q$, of degree $\leq q$ such that

$$f^{(q)}(x) = \sum_{j=0}^{q} p_{q,j}(x) g^{(j)}(x^2).$$ 

And so there exist constants $K_q > 0$ ($q \in \mathbb{N}_0$) such that for all $x \in \mathbb{R}$

$$|f^{(q)}(x)| \exp(\frac{1}{2} a x |x|^{1/\alpha}) =$$
\[
\begin{align*}
\mathcal{L}(x^2) & \leq 1 \sum_{j=0}^{q} g^{(j)}(x^2) \frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2} a x^2\right) p_{q,j}(x) \exp\left(-\frac{1}{2} a |x|^{1/\alpha}\right) \\
& \leq 1 \sum_{j=0}^{q} C_j \left| p_{q,j}(x) \right| \exp\left(-\frac{1}{2} a |x|^{1/\alpha}\right) \leq K_q.
\end{align*}
\]

Therefore, \( f \in S_{\alpha,\text{even}}(\mathcal{M}) \).

\[
\square
\]
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