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Disturbance decoupling and output stabilization by measurement feedback: a combined approach

by

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Mei 1986
Abstract.

In this paper we shall solve a very general control problem in the context of the geometric approach to linear multivariable control theory. The problem to be considered will be general in the sense that it contains as a special case not only the well-known disturbance decoupling problem by measurement feedback (cf. Schumacher [4]) and the problem of stabilizing the output with respect to the disturbances by means of a state feedback (cf. Hautus [3]), but for instance also the extension of the latter problem to the situation in which only the use of measurement feedback is allowed. Furthermore, in this paper connections with and extensions of recent work of Trentelman [5] will be given.

In the paper a fruitful use will be made of a merge of concepts arising from the geometric approach to control theory on the one hand and the frequency domain approach to control theory on the other hand.
1. Introduction.

In this paper we shall consider an extension of the disturbance decoupling problem with measurement feedback (DDPM).

The latter problem deals with the situation in which by means of a measurement feedback compensator we want to achieve decoupling between exogeneous disturbances entering the system and external, or to-be-controlled, outputs leaving the system. This problem has been studied thoroughly and is now well-understood (cf. Willems & Commault [6], Schumacher [4], Akashi & Imai [1]).

In the present paper we are concerned with situations in which the above-mentioned problem may not be solvable. That is, there may not exist a measurement feedback compensator such that the exogeneous disturbances are decoupled from the external outputs. In such a case a natural question to ask is the following: does there exists a part of the exogeneous disturbances that can be decoupled from the external outputs by means of an suitable measurement feedback compensator, while the remaining part of the exogeneous disturbances only influences the to-be-controlled outputs in a stable sense.

In the latter context, it is also possible to consider the dual problem in which the to-be-controlled outputs are assumed to be decomposed into two parts. The question that arises in that case is whether it is possible to find a measurement feedback compensator such that the first part of the to-be-controlled outputs remains unaffected by the exogeneous disturbances entering the system, while the second part of the controlled outputs only depends on the incoming disturbances in a stable fashion.

In this paper we shall consider a combination of the two above mentioned control problems. That is, we shall assume that both the exogeneous disturbance as well as the external outputs are decomposed into two parts.
The main problem of this paper can then be formulated as follows: is it possible to find a measurement feedback compensator such that the two control problems mentioned above are solved, i.e. such that the first part of the exogeneous disturbances is decoupled from the external outputs, the first part of the external outputs is decoupled from the exogeneous disturbances, the second part of the exogeneous disturbances affects the external outputs in a stable fashion and the second part of the external outputs only depends in a stable manner on the exogeneous disturbances.

It may be clear that the disturbance decoupling problem with measurement feedback can be considered as a special case of our problem. Moreover, in the sequel it will become clear that our problem also generalizes the problem of stabilizing the output with respect to disturbances, (OSDP), as considered in Hautus [3], and the disturbance decoupling problem with output stabilization, (DDPOS), as considered in Trentelman [5]. We note that in both of these problems only state feedback was allowed.

The outline of this paper will be as follows. In section 2 the above motivated problem is stated in a mathematically more rigorous formulation. Furthermore, section 2 will contain the notation that we shall use in this paper. In section 3 and 4 some important special cases of our problem are considered. Section 5 will contain some preliminary results that will be needed for the solution of our problem. In section 6 necessary and sufficient conditions for the solvability of our problem will be given. In section 7 we shall consider the extensions towards measurement feedback of the problems (OSDP) and (DDPOS) as mentioned above. In section 8 we reconsider the main problem of this paper with the additional requirement of internal stability.

In this paper we consider the linear time invariant system given by

\[
\begin{align*}
\dot{x} &= Ax + Bu + G_1 d_1 + G_2 d_2 \\
y &= Cx \\
z_1 &= H_1 x, \quad z_2 = H_2 x
\end{align*}
\] (1a) (1b) (1c)

with \( x \in \mathbb{R}^n \), the state, \( u \in \mathbb{R}^m \), the control input, \( d_1 \in \mathbb{R}^{q_1} \) and \( d_2 \in \mathbb{R}^{q_2} \), the disturbance inputs, \( y \in \mathbb{R}^p \), the measurement, \( z_1 \in \mathbb{R}^{r_1} \) and \( z_2 \in \mathbb{R}^{r_2} \), the to-be-controlled outputs.

\( A, B, C, G_1, G_2, H_1 \) and \( H_2 \) are real matrices of appropriate dimensions.

We shall denote the system described by (1a), (1b) and (1c) as

\[ \sum_{i=1}^2 (A,B,C,G_1,G_2,H_1,H_2). \]

Throughout this paper we shall assume that

\[ \text{im} \, G_1 \subseteq \text{im} \, G_2 \quad \text{and} \quad \ker H_1 \supseteq \ker H_2. \] (2)

Further on, we shall argue that in the context of the problem treated in this paper assumption (2) is no restriction, and therefore can be made without harming the problem formulation.

The reason for making assumption (2) is to keep formulations in the remainder of this paper compact.

See also Trentelman [5], where a similar assumption is made.
Suppose that the linear system as given in (1a), (1b) and (1c) is controlled by means of a feedback compensator of the form

$$\dot{w} = Kw + Ly, \quad u = Mw + Ny,$$

with \( w \) in \( \mathbb{R}^k \), the state of the compensator. We shall denote a compensator given by (3) as \( \mathcal{L}_{fb}(K,L,M,N) \). Application of this compensator results in the following closed loop system

$$
\begin{aligned}
&\begin{cases}
\dot{x} = (A + BNC BM) x + \begin{pmatrix} G_1 \\ 0 \end{pmatrix} d_1 + \begin{pmatrix} G_2 \\ 0 \end{pmatrix} d_2,
\dot{w} = (LC + K) w
\end{cases},
\end{aligned}
$$

which we shall write compactly as

$$
\begin{aligned}
&\mathcal{L}_{cl}
\begin{cases}
z_1 = [H_1 \ 0] \begin{pmatrix} x \\ w \end{pmatrix},
z_2 = [H_2 \ 0] \begin{pmatrix} x \\ w \end{pmatrix},
\end{cases}
\end{aligned}
$$

where

$$
\begin{aligned}
&x^e = A^e x + G_1^e d_1 + G_2^e d_2, \\
&z_1^e = H_1^e x^e, \quad z_2^e = H_2^e x^e
\end{aligned}
$$

and

$$
H_i^e = [H_i \ 0], \quad i = 1,2.
$$
We shall denote the system given in (5a) and (5b) as

\[ \sum_{\mathcal{L}}(A^e, C_1^e, C_2^e, H_1^e, H_2^e). \]

We are now in position to formulate the main problem that will be considered in this paper.

Problem 1.

Given a linear system \( \sum(A, B, C, G_1, G_2, H_1, H_2) \), with \( \text{im } G_1 \subset \text{im } G_2 \) and \( \ker H_1 \supset \ker H_2 \), find a measurement feedback compensator \( L_{\text{fb}}(K, L, M, N) \) such that the resulting closed loop system \( \sum_{\mathcal{L}}(A^e, C_1^e, C_2^e, H_1^e, H_2^e) \) satisfies:

(i) \[ H_1^e (sI - A^e)^{-1} G_1^e = 0 \],
(ii) \[ H_2^e (sI - A^e)^{-1} G_1^e = 0 \],
(iii) \[ H_1^e (sI - A^e)^{-1} G_2^e = 0 \],
(iv) \[ H_2^e (sI - A^e)^{-1} G_2^e \] has no poles outside \( \Phi_g \).

Here \( \Phi_g \) denotes a given subset of the complex plane \( \Phi \) that is symmetric with respect to the real axis and that contains at least one point on the real axis.

We shall now argue that the assumption made in (2) in the context of problem 1 is no restriction. Therefore we assume that (2) may not hold and we consider matrices \( G_3 \) and \( H_3 \) such that

(i) \[ \text{im } G_1 + \text{im } G_2 = \text{im } G_3 \],
(ii) \[ \ker H_1 \cap \ker H_2 = \ker H_3 \].
Clearly we now have that \( \text{im } G_1 \subset \text{im } G_3 \) and \( \ker H_1 \supset \ker H_3 \).

We may now state the following equivalence.

Problem 1 is solvable for the system given by \( \Sigma(A,B,C,G_1,G_2,H_1,H_2) \)

if and only if

problem 1 is solvable for the system given by \( \Sigma(A,B,C,G_1,G_3,H_1,H_2) \).

The proof of this equivalence is straightforward using the formulation of problem 1 as stated above.

It may now be clear that if we are considering a system such that assumption (2) does not hold it is always possible, by a suitable redefinition of the disturbances and the to-be-controlled outputs, to obtain a system such that (2) does hold. Therefore, in the context of problem 1, assumption (2) can be made without loss of generality.

In the remainder of this section we shall introduce some notations we use in this paper. Furthermore we shall recall some important concepts from geometric control theory together with their frequency domain descriptions.

1. Throughout this paper we shall use lower case letters for vectors, capitals for matrices and linear mappings, and scripts for linear subspaces and vector spaces. Linear mappings will be identified with their matrix representation.

The kernel of a mapping \( M \) shall be denoted by \( \ker M \), its image by \( \text{im } M \) and its spectrum by \( \sigma(M) \).

The transposed of a matrix \( M \) will be denoted by \( M^T \).
If \( V \) is a linear subspace of the linear space \( X \) satisfying \( M V \subseteq V \), where \( M \) is a mapping from \( X \) into itself, then \( M|_V \) will denote the restriction of the mapping \( M \) to \( V \), and \( M:X/V \) will denote the mapping induced by \( M \) in the factor space \( X/V \).

If \( V \) is a subspace of \( X \) then \( V^\perp \) will denote the orthogonal complement of \( V \) in \( X \).

2. Let \( X \) be a real \( n \)-dimensional space and let \( K \) be a subspace of \( X \). We will denote the set of all \( n \)-vectors whose components are proper (respectively strictly proper) rational functions with real coefficients as \( X(s) \) (respectively \( X_+(s) \)). We will denote by \( K(s) \) (respectively \( K_+(s) \)) the space of all elements \( \xi(s) \) of \( X(s) \) (respectively \( X_+(s) \)) with the property that \( \xi(s) \in K \) for all \( s \).

As mentioned before, in this paper \( \mathcal{g} \) will denote a given subset of the complex plane \( \mathcal{g} \) that is symmetric with respect to the real axis and that contains at least one point on the real axis. We shall call a rational function stable if the function has no poles outside \( \mathcal{g} \).

3. We shall now recall some basic facts from geometric control theory together with some frequency domain concepts.

Consider the dynamical system given by \( \sum \dot{x} = A x + B u \), \( y = C x \), with state space \( X \), control space \( U \) and measurement space \( Y \).

(i) A linear subspace \( V \) in \( X \) is called a controlled invariant or \((A,B)\)-invariant subspace if \( AV \subseteq V + \text{im} B \). It is well-known that the latter subspace inclusion is equivalent to the existence of a mapping \( F:X \to U \), defining a state feedback \( u = F x \) for \( \sum \), such that \((A + BF) V \subseteq V \).
A subspace $V$ in $X$ is called a stabilizability subspace if there exists a mapping $F : X \to U$ such that $(A + BF) V \subseteq V$ and $\sigma(A + BF | V) \subseteq \mathcal{F}_g$.

If $K$ is a linear subspace of $X$ then $V^*(K)$ will denote the largest controlled invariant subspace in $K$. The largest stabilizability subspace in $K$ will be denoted by $V^g(K)$.

Clearly $V^g(K) \subseteq V^*(K) \subseteq K$ and if $K_1$ and $K_2$ are linear subspaces in $X$ such that $K_1 \subseteq K_2$ then $V^*(K_1) \subseteq V^*(K_2)$ and $V^g(K_1) \subseteq V^g(K_2)$.

(ii) A linear subspace $S$ in $X$ is called a conditioned invariant or $(C,A)$-invariant subspace if $A(S \cap \ker C) \subseteq S$. This is equivalent to the existence of a linear mapping $T : Y \to X$, defining an output injection for $\mathcal{F}$, such that $(A + TC) S \subseteq S$. A linear subspace $S$ is called a detectability subspace if there exists a linear mapping $T : Y \to X$ such that $(A + TC) S \subseteq S$ and $\sigma(A + TC : X / S) \subseteq \mathcal{F}_g$.

If $L$ is a linear subspace in $X$ then $S^g(L)$ will denote the smallest conditioned invariant subspace that contains $L$, and $S^g(L)$ will denote the smallest detectability subspace containing $L$.

It is clear that $L \subseteq S^g(L) \subseteq S^*(L)$ and that if $L_1 \subseteq L_2$ then $S^g(L_1) \subseteq S^g(L_2)$ and $S^g(L_1) \subseteq S^*(L_2)$.

(iii) We shall denote the reachable subspace in $X$, i.e.

$\text{im } B + A \text{ im } B + \ldots + A^{n-1} \text{ im } B$, by $< A | \text{ im } B >$, and the unobservable subspace $\ker C \cap \ker CA \cap \ldots \cap \ker CA^{n-1}$, by $< \ker C | A >$.

(iv) Let $p(s) := \det (A - sI)$ be decomposed as $p(s) = p_g(s) p_b(s)$ where $p_g(s)$ has only zeros in $\mathcal{F}_g$ and $p_b(s)$ has no zeros in $\mathcal{F}_g$. 
We shall denote \( X_g(A) := \ker p_g(A) \) and \( X_b(A) := \ker p_b(A) \), and we note that \( X_g(A) \), respectively \( X_b(A) \), is the span of the generalized eigenvectors corresponding to the eigenvalues in \( \mathbb{C}_g \), respectively in the complement of \( \mathbb{C}_g \) in \( \mathbb{C} \).

(v) In this paper we shall use the frequency domain concept of \((\xi, \omega)\) pairs as introduced in Hautus [3]. A given vector \( x \) in \( X \) is said to have a \((\xi, \omega)\) representation if there exist strictly proper rational functions \( \xi(s) \) in \( X_+(s) \) and \( \omega(s) \) in \( U_+(s) \) such that \( x = (sI-A) \xi(s) - B \omega(s) \).

3. Disturbance decoupling with output stabilization.

In this section, we shall consider a special case of our main problem. The results of this section will be needed to derive necessary and sufficient conditions for the solvability of problem 1.

Consider the following linear system:

\[
\begin{align*}
\dot{z}_1 &= A x + B u + G d, \\
z_1 &= H_1 x, \\
z_2 &= H_2 x,
\end{align*}
\]

with state space \( X \) and control space \( U \).

As explained in section 2 we shall assume that \( \ker H_1 \supset \ker H_2 \), i.e. we consider \( z_2 \) to be an 'enlargement' of \( z_1 \).

The disturbance decoupling problem with output stabilization (DDPOS), (cf. Trentelman [5]), consists of finding a mapping \( F: X \rightarrow U \), defining a state feedback \( u = F x \), such that:

\[
H_1(sI-(A + BF))^{-1}G = 0, \text{ and}
\]

\[
H_2(sI-(A + BF))^{-1}G \text{ is a stable rational function.}
\]
It turns out that the following subspace is crucial to the solvability of (DDPOS) (cf. Trentelman [5]).

**Definition 3.1.** \( V_g(\ker H_1, \ker H_2) \) denotes the set of points in \( X \) for which there exists a \((\xi, \omega)\) representation such that \( H_1 \xi(s) = 0 \) (\( \xi(s) \in \ker H_1 \) for all \( s \)), and \( H_2 \xi(s) \) is stable.

The following results now hold.

**Theorem 3.2.** \( V_g(\ker H_1, \ker H_2) = V^*_g(\ker H_1) + V^*_g(\ker H_2) \).

**Theorem 3.3.** There exists a mapping \( F: X \rightarrow U \) such that:

(i) \((A + BF) V_g(\ker H_1, \ker H_2) \subseteq V_g(\ker H_1, \ker H_2)\),

(ii) \((A + BF) V^*_g(\ker H_2) \subseteq V^*_g(\ker H_2)\),

(iii) \( \sigma((A + BF)): V_g(\ker H_1, \ker H_2) / V^*_g(\ker H_2) \subseteq \mathbb{C} \).

For the proofs of these theorems we refer to Trentelman [5].

We may now prove the next theorem.

**Theorem 3.4.** The following statements are equivalent.

(i) \((DDPOS)\) is solvable.

(ii) \( \text{im } G \subseteq V_g(\ker H_1, \ker H_2) \).

(iii) There exist strictly proper rational matrices \( X(s) \) and \( U(s) \) such that

\[
(sI - A) X(s) - B U(s) = G,
\]

\( H_1 X(s) = 0 \) and \( H_2 X(s) \) is stable.

(iv) There exists a strictly proper rational matrix \( W(s) \) such that

\[
H_1(sI - A)^{-1} B W(s) - H_1(sI - A)^{-1} G = 0 ,
\]

\[
H_2(sI - A)^{-1} B W(s) - H_2(sI - A)^{-1} G \text{ is stable.}
\]

**Proof.**

(i) \( \iff \) (ii) See Trentelman [5].

(ii) \( \iff \) (iii) By definition 3.1.

(i) \( \implies \) (iv) Define \( W(s) := F(sI - (A + BF))^{-1} G \).

(iv) \( \implies \) (iii) Define \( U(s) := -W(s) \) and

\[
X(s) := -(sI - A)^{-1} (B W(s) - G) .
\]
Remark 3.5. From theorem 3.3 (i) it follows that $V_g(ker H_1, ker H_2)$ is an $(A, B)$-invariant subspace.

Remark 3.6. Note that $V^*(ker H_2) \subset V_g(ker H_1, ker H_2) \subset ker H_1$ and that $V^*(ker H_2) \subset ker H_2 \subset ker H_1$.

Take the mapping $F: X \mapsto U$ as defined in theorem 3.3, and decompose the state $X$ as $X = X_1 \oplus X_2 \oplus X_3$, such that $X_1 = V^*(ker H_2)$ and $X_1 \oplus X_2 \neq V_g(ker H_1, ker H_2)$. Furthermore, choose a basis in the state space adapted to this decomposition. With respect to this basis the matrices $(A + BF), H_1$ and $H_2$ have the following form.

$$(A + BF) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix},$$

$H_1 = \begin{bmatrix} 0 & 0 & H_{13} \end{bmatrix}$, $H_2 = \begin{bmatrix} 0 & H_{22} & H_{23} \end{bmatrix}$.

Note that due to theorem 3.3 (iii), $\sigma(A_{22}) \subseteq \Phi_g$.

Therefore, with respect to this basis we have the following.

$$H_1(sI - (A + BF))^{-1} = \begin{bmatrix} 0 & 0 & P_3(s) \end{bmatrix},$$

$$H_2(sI - (A + BF))^{-1} = \begin{bmatrix} 0 & Q_2(s) & Q_3(s) \end{bmatrix},$$

where $P_3(s)$ and $Q_3(s)$ are strictly proper rational matrices and $Q_2(s)$ is a stable strictly proper rational matrix.


The objective of this section is to solve the dual of (DDPOS) as studied in the previous section. The results of this section may be obtained simply by dualizing the results of section 3.
It is possible to give an interpretation in terms of observers of the problem considered in this section. However, since this subject is beyond the scope of this paper it is omitted.

The system considered in this section is given by

\[ \sum \dot{x} = A x + G_1 d_1 + G_2 d_2 , \quad y = C x , \quad z = H x , \]

with state space \( X \) and measurement space \( Y \).

As explained in section 2 we shall assume that \( \text{im } G_1 \subseteq \text{im } G_2 \).

The problem considered in this section consists of finding a mapping \( T: Y \rightarrow X \), defining an output injection for \( \sum \), such that

\[ \text{H(sI-(A + TC))}^{-1} G_1 = 0 , \]

\[ \text{H(sI-(A + TC))}^{-1} G_2 \text{ is stable.} \]

The next definition is the result of dualizing definition 3.1.

**Definition 4.1.** \( S_g (\text{im } G_1 , \text{im } G_2 ) = (V (\ker G_1^T , \ker G_2^T))^\perp \), where \( V (\ker G_1^T , \ker G_2^T) \) is computed relative \( A^T \) and \( C^T \).

Note that \( S_g (\text{im } G_1 , \text{im } G_2 ) \) depends only on \( A,C,G_1 \) and \( G_2 \).

The following theorems may now be obtained from theorem 3.2, 3.3 and 3.4 by pure dualization.

**Theorem 4.2.** \( S_g (\text{im } G_1 , \text{im } G_2 ) = S^*_g (\text{im } G_1 ) \cap S^*_g (\text{im } G_2 ) . \)

**Theorem 4.3.** There exists a mapping \( T: Y \rightarrow X \) such that

(i) \( (A + TC) S_g (\text{im } G_1 , \text{im } G_2 ) \subseteq S_g (\text{im } G_1 , \text{im } G_2 ) , \)

(ii) \( (A + TC) S^*_g (\text{im } G_2 ) \subseteq S^*_g (\text{im } G_2 ) , \)

(iii) \( \sigma((A + TC)): S^*_g (\text{im } G_2 ) / S_g (\text{im } G_1 , \text{im } G_2 ) \subseteq \phi_g . \)
Theorem 4.4. The following statements are equivalent.

(i) The problem of this section is solvable.
(ii) \( \ker H \supset S_g(\text{im } G_1, \text{im } G_2) \).
(iii) There exist strictly proper rational matrices \( X(s) \) and \( Y(s) \) such that
\[
X(s) (sI-A) - Y(s) C = H,
X(s) G_1 = 0 \text{ and } \text{im } G_2 \text{ is stable}.
\]
(iv) There exists a strictly proper rational matrix \( W(s) \) such that
\[
W(s) C(sI-A)^{-1} G_1 - H(sI-A)^{-1} G_1 = 0,
W(s) C(sI-A)^{-1} G_2 - H(sI-A)^{-1} G_2 \text{ is stable}.
\]

Remark 4.5. From theorem 4.3 (i) it follows that \( S_g(\text{im } G_1, \text{im } G_2) \) is a \((C,A)\)-invariant subspace.

Remark 4.6. Note that \( \text{im } G_1 \subset S_g(\text{im } G_1, \text{im } G_2) \subset S^*(\text{im } G_2) \) and \( \text{im } G_1 \subset \text{im } G_2 \subset S^*(\text{im } G_2) \).

Take any mapping \( T: Y \to X \) as described in theorem 4.3, and decompose the state space \( X \) as \( X = X_1 \oplus X_2 \oplus X_3 \), such that \( X_1 = S_g(\text{im } G_1, \text{im } G_2) \) and \( X_1 \oplus X_2 = S^*(\text{im } G_2) \). In the same way as was done in remark 3.6 we may prove that on a basis adapted to the above state space decomposition we have

\[
\begin{align*}
(i) \quad (sI-(A+TC))^{-1} G_1 &= \begin{bmatrix} R_1(s) \\ 0 \\ 0 \end{bmatrix}, \\
(ii) \quad (sI-(A+TC))^{-1} G_2 &= \begin{bmatrix} U_1(s) \\ U_2(s) \\ 0 \end{bmatrix},
\end{align*}
\]

where \( R_1(s) \) and \( U_1(s) \) are strictly proper rational matrices and \( U_2(s) \) is a stable strictly proper rational matrix.
5. Preliminaries.

The results of this section will be instrumental in the proof of problem 1 as stated in section 2, but are also interesting for their own right.

Consider the dynamical system

\[ \dot{x} = A x + B u + G d, \quad y = C x, \quad z = H x. \]

And assume that \( \dot{x} \) is controlled by means of the feedback controller given by

\[ \begin{align*}
\dot{w} &= (A + BF + TC - BWC) w + (BW - T) y, \\
\dot{c}_{fb} &= u = (F - WC) w + Wy,
\end{align*} \]

resulting in the closed loop system given by

\[ \begin{align*}
\dot{x}_e &= A^e x_e + G_e d, \\
z &= H^e x_e,
\end{align*} \]

where

\[ x_e = \begin{bmatrix} x \\ w \end{bmatrix}, \quad A^e = \begin{bmatrix} A + BWC & BF - BWC \\
BWC - TC & A + BF + TC - BWC \end{bmatrix}, \quad G^e = \begin{bmatrix} G \\ 0 \end{bmatrix} \]

and

\[ H^e = [H \quad 0]. \]

We may now prove the following result.
Proposition 5.1.

(i) \( \sigma(A^e) = \sigma(A + BF) \cup \sigma(A + TC) \) where '\( \cup \) ' denotes the disjoint union (cf. Wonham [7]).

(ii) \( H(sI-A)^{-1}G^e = H(sI-(A + BF))^{-1}G + H(sI-(A + TC))^{-1}G - H(sI-(A + BF))^{-1}(sI-(A + BWC)) (sI-(A + TC))^{-1}G \).

Proof.

Note that for any regular matrix \( S \) in \( \mathbb{R}^{2nx2n} \) we have

\[ \sigma(A^e) = \sigma(S A^e S^{-1}) \text{ and } H(sI-A)^{-1}G^e = H S^{-1}(sI-S A^e S^{-1})^{-1}S G^e. \]

In particular with

\[
S = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \text{ and } S^{-1} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}
\]

we obtain

\[
\sigma(A^e) = \sigma(\begin{pmatrix} A + BF & BF - BWC \\ 0 & A + TC \end{pmatrix}) = \sigma(A + BF) \cup \sigma(A + TC),
\]

which proves part (i).

Furthermore we have that \( H S^{-1}(sI-S A^e S^{-1})^{-1}S G^e = \)

\[
= [H \ O] \begin{pmatrix} A + BF & BF - BWC \\ 0 & A + TC \end{pmatrix}^{-1} \begin{pmatrix} G \\ -G \end{pmatrix} = H(sI-(A + BF))^{-1}G - H(sI-(A + BWC))^{-1}(BF - BWC) (sI-(A + TC))^{-1}G.
\]

Part (ii) may now be proven using the fact that

\[(BF - BWC) = (sI-(A + BWC)) - (sI-(A + BF)).\]
In the following proposition a connection is made between the input-output behavior of a linear dynamical system and the external behavior of the autonomous part of the system with respect to the initial conditions. For that purpose we consider the linear system

\[ \dot{x} = Ax + Bu, \quad y = Cx, \]

with state space \( X \). Then we may prove.

**Proposition 5.2.**

(i) \( C(sI-A)^{-1}B = 0 \) if and only if for every vector \( p \) in \( \langle A \mid \text{im } B \rangle \): \( C(sI-A)^{-1}p = 0 \).

(ii) \( C(sI-A)^{-1}B \) is stable if and only if for every vector \( p \) in \( \langle A \mid \text{im } B \rangle \): \( C(sI-A)^{-1}p \) is stable.

**Proof.**

Since \( \langle A \mid \text{im } B \rangle = \text{im } B + A \text{im } B + \ldots + A^{n-1} \text{im } B \) the (if)-part of (i) as well as of (ii) is obvious.

(Only if).

Note that \( C(sI-A)^{-1}B = \sum_{i=1}^{\infty} \frac{CA^{i-1}B}{s^i} \), from which it is immediate that

\[ sC(sI-A)^{-1}B - CB = C(sI-A)^{-1}AB \]

and that \( C(sI-A)^{-1}B = 0 \) if and only if for all \( i \geq 1 \): \( CA^{i-1}B = 0 \).

Therefore, we may conclude that if \( C(sI-A)^{-1}B = 0 \) then \( C(sI-A)^{-1}AB = 0 \). It is also easy to see that if \( C(sI-A)^{-1}B \) is stable then \( C(sI-A)^{-1}AB \) is stable. Repeated use of the previous arguments yields that if \( C(sI-A)^{-1}B = 0 \) then for all \( k \geq 0 \): \( C(sI-A)^{-1}A^kB = 0 \), and if \( C(sI-A)^{-1}B \) is stable then for all \( k \geq 0 \): \( C(sI-A)^{-1}A^kB \) is stable.
By the definition of $\langle A | \ker B \rangle$ and the fact that the sum of stable rational matrices is stable, the proof of the (only if)-part is completed.

Again as in proposition 5.2 we consider the linear dynamical system

$$\dot{x} = A x + B u, \quad y = C x,$$

with state space $X$, control space $U$ and measurement space $Y$. We assume that $S_1$ and $S_2$ are $(C,A)$-invariant subspaces in $X$, and that $V_1$ and $V_2$ are $(A,B)$-invariant subspaces in $X$ satisfying $S_1 \subseteq S_2 \subseteq V_2$ and $S_1 \subseteq V_1 \subseteq V_2$.

We may now state the following.

**Proposition 5.3.** There exists a mapping $W: Y \rightarrow U$, defining a static measurement feedback $u = W y$ for $\dot{x}$, such that $(A + BWC) S_1 \subseteq V_1$ and $(A + BWC) S_2 \subseteq V_2$.

**Proof.**

See Schumacher [4], lemma 3.6.

6. **The main problem.**

We are now in position to state the main result of this paper.

The result provides necessary and sufficient conditions for the solvability of problem 1 as stated in section 2.

**Theorem 6.1.**

Problem 1 is solvable if and only if

$$S^\ast(\ker H_1, \ker H_2) \subseteq V^\ast(\ker H_2).$$
Proof. (Only if).
Assume that there are matrices $K, L, M$ and $N$ defining a measurement feedback compensator which solves problem 1. Therefore we have that

\[
H_1^e(sI-A^e)^{-1}C_1^e = 0, \quad H_2^e(sI-A^e)^{-1}C_1^e = 0,
\]

\[
H_1^e(sI-A^e)^{-1}C_2^e = 0 \text{ and } H_2^e(sI-A^e)^{-1}C_2^e \text{ is stable.} \tag{6}
\]

Define the subspace $V^e$ in $X \oplus \mathcal{W}$ as $V^e := \langle A^e \mid \text{im} \ G_2^e \rangle$.

From proposition 5.2 and (6) it follows that for every vector $x^e$ in $V^e$ we have $H_1^e(sI-A^e)^{-1}x^e = 0$ and $H_2^e(sI-A^e)^{-1}x^e$ is stable. \tag{7}

Define subspaces $S$ and $V$ in $X$ as follows.

$S$ is the set of vectors in $X$ such that \[
\begin{bmatrix}
    x \\
    0
\end{bmatrix}
\] is a vector in $V^e$.

$V$ is the set of vectors $x$ in $X$ for which there exists a vector $w$ in $\mathcal{W}$ such that \[
\begin{bmatrix}
    x \\
    w
\end{bmatrix}
\] is a vector in $V^e$.

Clearly $S \subseteq V$. It may be proven that $S$ is a $(C,A)$-invariant subspace and that $V$ is an $(A,B)$-invariant subspace (cf. Schumacher [4]).

Since $G_2^e = \begin{bmatrix} G_2 \\ 0 \end{bmatrix}$, it is clear that $\text{im} \ G_2 \subseteq S \subseteq V$. And therefore we have

$\text{im} \ G_2 \subseteq S^*(\text{im} \ G_2) \subseteq V$, for $S^*(\text{im} \ G_2)$ is the smallest $(C,A)$-subspace containing $\text{im} \ G_2$. 

Take a vector \( x \) in \( \mathcal{V} \). Then according to the definition of \( \mathcal{V} \) there exists a vector \( w \) in \( \mathcal{W} \) such that \[
abla = \begin{pmatrix} x \\ w \end{pmatrix}
\] is in \( \mathcal{V}^e \).

Define the strictly proper rational vectors \( \xi(s) \) and \( \lambda(s) \) by

\[
\begin{pmatrix} \xi(s) \\ \lambda(s) \end{pmatrix} = (sI-A)^{-1} \begin{pmatrix} x \\ w \end{pmatrix} = (sI-\begin{pmatrix} A+BNCBM \\ LC \\ K \end{pmatrix})^{-1} \begin{pmatrix} x \\ w \end{pmatrix}.
\]

Then we have \( x = (sI-A) \xi(s) - B(NC\xi(s) + M\lambda(s)) \), and since \( H_i^e = [H_i \ 0] \), \( i = 1,2 \), by (7) it follows that \( H_1 \xi(s) = 0 \) and \( H_2 \xi(s) \) is stable.

By definition 3.1 we may conclude that \( x \) is in \( \mathcal{V} \) (ker \( H_1 \), ker \( H_2 \)). Consequently we have \( S^*(\text{im } G_2) \subset S \subset \mathcal{V} \) (ker \( H_1 \), ker \( H_2 \)).

By dual reasoning we may derive \( \mathcal{S}_g(\text{im } G_1, \text{im } G_2) \subset \mathcal{V}^*(\text{ker } H_2) \).

If during this proof we denote \( \mathcal{V}^g = \mathcal{V} \) (ker \( H_1 \), ker \( H_2 \)), \( \mathcal{V}^* = \mathcal{V}^*(\text{ker } H_2) \), \( \mathcal{S}_g = \mathcal{S}_g(\text{im } G_1, \text{im } G_2) \) and \( \mathcal{S}^* = \mathcal{S}^*(\text{im } G_2) \).

Therefore, we now have that \( \mathcal{S}_g \subset \mathcal{V}^* \) and \( \mathcal{S}^* \subset \mathcal{V}^g \), from which we shall derive a measurement feedback compensator that solves problem 1.

Since \( \mathcal{S}_g \) and \( \mathcal{S}^* \) are \((C,A)\)-invariant subspaces and \( \mathcal{V}^* \) and \( \mathcal{V}^g \) are \((A,B)\)-invariant subspaces satisfying \( \mathcal{S}_g \subset \mathcal{V}^* \subset \mathcal{V}^g \) and \( \mathcal{S}^* \subset \mathcal{S}^* \subset \mathcal{V}^g \) by proposition 5.3 there exists a matrix \( W \) such that \( (A + BWC) \mathcal{S}_g \subset \mathcal{V}^* \) and \( (A + BWC) \mathcal{S}^* \subset \mathcal{V}^g \).

Take the matrix \( F \) as indicated in theorem 3.3 and the matrix \( T \) as indicated in theorem 4.3, and define

\[
\begin{align*}
K & := A + BF + TC - BWC, \\
L & := BW - T, \\
M & := F - WC, \\
N & := W.
\end{align*}
\]
We now claim that the matrices \( \{K,L,M,N\} \) constitute a measurement feedback compensator \( \sum_{\text{fb}}(K,L,M,N) \) that solves problem 1. Therefore we remark the following.

1. Note that with \( \sum_{\text{fb}}(K,L,M,N) \) as defined above the closed loop system becomes

\[
\begin{bmatrix}
    \dot{x} \\
    \dot{w}
\end{bmatrix}
= \begin{bmatrix}
    A + BWC & BF - BWC \\
    BWC - TC & A + BF + TC - BWC
\end{bmatrix}
\begin{bmatrix}
    x \\
    w
\end{bmatrix}
+ \begin{bmatrix}
    G_1 \\
    0
\end{bmatrix} d_1 + \begin{bmatrix}
    G_2 \\
    0
\end{bmatrix} d_2
\]

\[
z_1 = [H_1 \ 0] \begin{bmatrix}
    x \\
    w
\end{bmatrix}, \quad z_2 = [H_2 \ 0] \begin{bmatrix}
    x \\
    w
\end{bmatrix}.
\]

By proposition 5.2 (ii) we now know that in the closed loop system the transfer function between disturbances \( d_j \) and to-be-controlled output \( z_i \) equals

\[
H_i(sI - (A + BF))^{-1} G_j + H_i(sI - (A + TC))^{-1} G_j
- H_i(sI - (A + BF))^{-1}(sI - (A + BWC))(sI - (A + TC))^{-1} G_j \text{ where } i,j = 1,2.
\]

2. Note that we have the following subspace inclusions

(i) \( V^* \subset V^g \subset \ker H_1 \)

(ii) \( V^* \subset \ker H_2 \subset \ker H_1 \)

(iii) \( \im G_1 \subset S^g \subset S^* \)

(iv) \( \im G_1 \subset \im G_2 \subset S^* \)

Consider the following state space decomposition of \( X \)

\[
X = X_1 \oplus X_2 \oplus X_3 \oplus X_4 \oplus X_5 \oplus X_6 \text{ where}
\]

\[
X_1 = S^g,
X_1 \oplus X_2 \oplus X_3 = V^*,
X_1 \oplus X_2 \oplus X_4 = S^*,
X_1 \oplus X_2 \oplus X_3 \oplus X_4 \oplus X_5 = V^g.
\]
By remark 3.6 it follows that

\[
H_1(sI-(A + BF))^{-1} = \begin{bmatrix}
0 & 0 & 0 & 0 & P_6(s)
\end{bmatrix},
\]

\[
H_2(sI-(A + BF))^{-1} = \begin{bmatrix}
0 & 0 & Q_4(s) & Q_5(s) & Q_6(s)
\end{bmatrix},
\]

where \(P_6(s)\) and \(Q_6(s)\) are strictly proper rational matrices and \(Q_4(s)\) and \(Q_5(s)\) are stable strictly proper rational matrices.

Analogous, by remark 4.6 it follows that on a basis adapted to the chosen state space decomposition

\[
(sI-(A + TC))^{-1}G_1 = \begin{bmatrix}
R_1(s) \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix},
\]

\[
(sI-(A + TC))^{-1}G_2 = \begin{bmatrix}
U_1(s) \\
U_2(s) \\
0 \\
U_4(s) \\
0
\end{bmatrix},
\]

where \(R_1(s)\) and \(U_1(s)\) are strictly proper rational matrices, and \(U_2(s)\) and \(U_4(s)\) are stable strictly proper rational matrices.
Furthermore, with respect to the chosen decomposition the matrices $G_1$, $G_2$, $H_1$ and $H_2$ may be written as

$$
G_1 = \begin{bmatrix}
G_{11} & 0 \\
0 & G_{22} \\
0 & 0 \\
0 & 0
\end{bmatrix}, \quad G_2 = \begin{bmatrix}
G_{12} \\
0 \\
0 \\
0
\end{bmatrix}, \quad H_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & H_{16}
\end{bmatrix}, \quad H_2 = \begin{bmatrix}
0 & 0 & H_{24} & H_{25} & H_{26}
\end{bmatrix}
$$

The matrix $A + BWC$ with respect to this decomposition becomes

$$
\begin{bmatrix}
A_{11} & A_{12} & * & A_{14} & * & * \\
A_{21} & A_{22} & * & A_{24} & * & * \\
A_{31} & A_{32} & * & A_{34} & * & * \\
0 & A_{42} & * & A_{44} & * & * \\
0 & A_{52} & * & A_{54} & * & * \\
0 & 0 & * & 0 & * & *
\end{bmatrix}
$$

where '*' denotes an unknown matrix block.

3. It is now easy to see that for $(i,j) = (1,1)$, $(1,2)$ and $(2,1)$

$$
H_i(sI-(A + BF))^{-1}G_j = 0, \quad H_i(sI-(A + TC))^{-1}G_j = 0 \quad \text{and} \quad H_i(sI-(A + BWC)(sI-(A + TC))^{-1}G_j = 0.
$$

For $(i,j) = (2,2)$ we have

$$
H_2(sI-(A + BF))^{-1}G_2 = Q_4(s)G_{42}, \quad H_2(sI-(A + TC))^{-1}G_2 = H_{24}U_4(s) \quad \text{and} \quad H_2(sI-(A + BWC))(sI-(A + TC))^{-1}G_2 = Q_4(s)(-A_{42}U_2(s) + (sI-A_{44})U_4(s)) + Q_5(s)(-A_{52}U_2(s) - A_{54}U_4(s)).
$$
where $Q_4(s)$, $Q_5(s)$, $U_2(s)$ and $U_4(s)$ are stable strictly proper rational matrices.

By proposition 5.1 we now may conclude that in the closed loop system the transfer function between the disturbance $d_i$ and the to-be-controlled output $z_i$ equals zero, where $(i,j) = (1,1), (1,2)$ and $(2,1)$. Furthermore, by proposition 5.1, it is easy to see that the transfer function between disturbance $d_2$ and the to-be-controlled output $z_2$ is a stable strictly proper rational function.

This completes the proof of theorem 6.1.

Remark 6.2. Note, as it should be, that the two subspace inclusions appearing in theorem 6.1 are each others dual with respect to taking orthogonal complements and corresponding transposition.

Similar to the theorems 3.4 and 4.4 we can extend the results of theorem 6.1 as follows.

**Theorem 6.3.** The following statements are equivalent.

(i) Problem 1 is solvable.

(ii) $S^g(\text{im } G_2) \subseteq U^g(\ker H_1 \cup \ker H_2)$ and $S^g(\text{im } G_1, \text{im } G_2) \subseteq U^g(\ker H_2)$.

(iii) There exists a strictly proper matrix $X(s)$ and a proper matrix $U(s)$ such that for $(i,j) = (1,1), (1,2)$ and $(2,1)$ $H_1X(s)G_j = 0$, while $H_2X(s)G_2$ is stable, and $X(s)$ and $U(s)$ satisfy

\[
(sI-A) = (sI-A) X(s)(sI-A) + B U(s) C.
\]

(iv) There exists a proper matrix $T(s)$ such that for $(i,j) = (1,1), (1,2)$ and $(2,1)$

\[
H_1(sI-A)^{-1}B T(s) C(sI-A)^{-1}G_j + H_1(sI-A)^{-1}G_j = 0, \text{ and}
\]

\[
H_2(sI-A)^{-1}B T(s) C(sI-A)^{-1}G_2 + H_2(sI-A)^{-1}G_2 \text{ is stable.}
\]
7. Special cases of theorem of problem 1.

In this section we shall consider some special cases of problem 1.

(I) The problem of stabilizing the output with respect to disturbances using measurement feedback, abbreviated (OSDPM).

This problem, (OSDPM), can be formulated as follows.

Given the linear system

\[ \dot{x} = Ax + Bu + Gd, \quad y = Cx, \quad z = Hx, \]  \hspace{1cm} (8)

determine a feedback controller given by

\[ \dot{w} = Kw + Ly, \quad u = Mw + Ny, \]

such that in the resulting closed loop system, compactly given by

\[ \dot{x}^e = A^e x^e + G^e d, \quad z = H^e x^e, \]

\(H^e(sI-A^e)^{-1}G^e\) is stable.

This problem can be seen as a generalization of the problem of stabilizing the output with respect to disturbances using state feedback. The latter problem, abbreviated as (OSDP), was introduced and solved in Hautus [3] and is formulated as follows.

Given a system described by (8), find a feedback \(u = Fx\) such that in the resulting closed loop system \(H(sI-(A+BF))^{-1}G\) is stable.

The problem considered in this paragraph, (OSDPM), can be obtained from problem 1 by setting \(C_1 = 0, C_2 = G, H_1 = 0\) and \(H_2 = H\).

By theorem 6.1 we may conclude.
Corollary 7.1.

(OSDPM) is solvable if and only if
\[ S^*(\text{im } G) \subseteq V_g (X, \text{ker } H) \quad \text{and} \quad S_g (\{0\}, \text{im } G) \subseteq V^*(\text{ker } H). \]

Now it is well-known (cf. Schumacher [4]) that \( V^*_g (X) = < A \mid \text{im } B > + X (A). \)
Hence, by theorem 3.2, we have \( V_g (X, \text{ker } H) = V^*_g (\text{ker } H) + < A \mid \text{im } B > + X (A). \)
In this formulation it is easy to prove that \( A V_g (X, \text{ker } H) \subseteq V_g (X, \text{ker } H). \)
Using the fact that \( \text{im } G \subseteq S^*(\text{im } G) \subseteq < A \mid \text{im } G > \) we have

\[ S^*(\text{im } G) \subseteq V_g (X, \text{ker } H) \quad \text{if and only if} \quad \text{im } G \subseteq V_g (X, \text{ker } H). \]

By dual reasoning it follows that

\[ S_g (\{0\}, \text{im } G) \subseteq V^*_g (\text{ker } H) \quad \text{if and only if} \quad S_g (\{0\}, \text{im } G) \subseteq \text{ker } H. \]

Therefore we may conclude

Corollary 7.2.

(OSDPM) is solvable if and only if
\[ \text{im } G \subseteq V_g (X, \text{ker } H) \quad \text{and} \quad S_g (\{0\}, \text{im } G) \subseteq \text{ker } H. \]

We note that \( \text{im } G \subseteq V_g (X, \text{ker } H) \) is necessary and sufficient for the solvability of (OSDP) (cf. Hautus [3]).
Remark 7.3.
The remarkable fact shows up that, due to the $A$-invariance of $V_g(X, \ker H)$ and $S_g(\{0\}, \im G)$, the solvability of (OSDPM) is equivalent to the solvability of (OSDP) and (OSDP)*, where (OSDP)* is the dual of (OSDP) and is formulated as follows. Given the linear system described by (9), find a matrix $T$, defining an output injection for (9), such that in the resulting closed loop system $H(sI-(A + TC))^{-1}G$ is stable.

Remark 7.4.
Note that due to the $A$-invariance of both $V_g(X, \ker H)$ and $S_g(\{0\}, \im G)$ in the construction as indicated in the proof of theorem 6.1 the mapping $W: Y \rightarrow U$ can taken to be zero. Hence, we do not need a static measurement feedback $u = W y$, and consequently the transfer function of the measurement feedback controller solving (OSDPM) is a strictly proper rational matrix.

Remark 7.5.
From theorem 6.3 a number of equivalent statements can be derived concerning the solvability of (OSDPM). Furthermore, by the fact that the solvability of (OSDPM) is equivalent to the solvability of both (OSDP) and (OSDP)* the connection to theorems 4.3 and 4.4 will yield an additional number of statements equivalent to the solvability of (OSDPM).

(II) The second problem we shall consider in this section can be seen as a generalization of the above-mentioned problem, (OSDPM), but also as an extension of (DDPOS) as described in section 3.

For that reason we consider the following linear system

$$\dot{x} = A x + B u + G d, \quad y = C x, \quad z_1 = H_1 x, \quad z_2 = H_2 x,$$

where we assume that $\ker H_1 \supset \ker H_2$. 
The problem of stabilizing the output with respect to the disturbances by means of a measurement feedback (DDPOSM) now consists of finding a feedback compensator given by

\[ \sum_{fb} \dot{w} = K w + L y, \quad u = M w + N y \]

such that the resulting closed loop system, which may be written compactly by

\[ \sum_{cl} x = A^e x + G^e d, \quad z_1 = H_1^e x^e, \quad z_2 = H_2^e x^e, \]

satisfies

\[
\begin{align*}
H_1^e (sI - A^e)^{-1} G^e &= 0 \\
H_2^e (sI - A^e)^{-1} G^e &\text{is stable.}
\end{align*}
\]

(DDPOSM) may be obtained from problem 1 by setting \( G_1 = 0 \) and \( G_2 = G \).

From theorem 6.1 we may conclude

**Corollary 7.6.**

(DDPOSM) is solvable if and only if

\[
S^* (\text{im } G) \subset V (\ker H_1, \ker H_2) \quad \text{and} \quad S^* (\{0\}, \text{im } G) \subset V^* (\ker H_2).
\]

As we showed in the previous paragraph we also have the following equivalence.

(DDPOSM) is solvable if and only if

\[
S^* (\text{im } G) \subset V (\ker H, \ker H) \quad \text{and} \quad S^* (\{0\}, \text{im } G) \subset \ker H_2.
\]
But, since in general $V_g(\ker H_1, \ker H_2)$ is not $A$-invariant no further simplification is possible. Moreover, in constructing a measurement feedback compensator as described in the proof of theorem 6.1, in general, we do need the mapping $W: Y \rightarrow U$, such that

$$\begin{align*}
(A + BWC) S^*(\text{im } G) & \subset V_g(\ker H_1, \ker H_2) \text{ and } \\
(A + BWC) S_g(\{0\}, \text{im } G) & \subset V^*(\ker H_2).
\end{align*}$$

This implies that the measurement feedback controller solving (DDPOSM), in general, has a proper transfer function.

**Remark 7.7.**

Again, it is possible, by application of theorem 6.3, to obtain a number of statements equivalent to the solvability of (DDPOSM).

(III) We will conclude this section by considering the well-known disturbance decoupling problem by measurement feedback (DDPM) (cf. Willems & Commault [6], Schumacher [4], Akashi & Imai [1]).

The latter problem may be obtained from problem 1 by setting $G_1 = G_2 = G$ and $H_1 = H_2 = H$ in the formulation of problem 1. Moreover, if we set in the proof of theorem 6.1 $G_1 = G_2 = G$ and $H_1 = H_2 = H$, we obtain a new, straightforward and elegant proof of the following well-known equivalence:

**Corollary 7.8.**

(DDPM) is solvable if and only if $S^*(\text{im } G) \subset V^*(\ker H)$.

Although (DDPM) is solved in a nice way nothing can be said with respect to internal stabilization or pole placement. Therefore, in the following section we shall discuss the extension of problem 1 to the case that in addition to problem 1 internal stability is required.
8. Problem 1 with internal stabilization.

In this section we shall reconsider problem 1 as described in section 2. In addition to the mentioned requirements involving disturbance decoupling and output stabilization with respect to disturbances, we now also require internal stabilization.

For that reason we make the following decomposition of the complex plane $\mathcal{C}$, $\mathcal{C}_f \subset \mathcal{C}_s \subset \mathcal{C}$. Where $\mathcal{C}_f$ and $\mathcal{C}_s$ are symmetric with respect to the real axis and contain at least one point on the real axis. With respect to such a decomposition of the complex plane we call a rational function $f$-stable, (resp. $s$-stable) if the rational function has no poles outside $\mathcal{C}_f$ (resp. $\mathcal{C}_s$).

The problem considered in this section now consists of the following.

**Problem 2.**

Given a linear system $\sum_2(A, B, C, G_1, G_2, H_1, H_2)$, as described in section 2, with $\text{im } G_1 \subset \text{im } G_2$ and $\ker H_1 \subset \ker H_2$, find matrices $K, L, M, N$ defining a measurement feedback compensator $\sum_{\text{fb}}(K, L, M, N)$, as defined in section 2, such that the resulting closed loop system $\sum_{\text{cl}}(A^e, G_1^e, G_2^e, H_1^e, H_2^e)$ satisfies:

(i) $H_1^e(sI-A^e)^{-1}G_j^e = 0$ for $(i, j) = (1, 1), (1, 2), \text{ and } (2, 1)$,

(ii) $H_2^e(sI-A^e)^{-1}G_2^e$ is $f$-stable,

(iii) $(sI-A^e)^{-1}$ is $s$-stable.

The solution of problem 2 closely resembles the solution of problem 1, as given in section 6, and will therefore not be given in detail. Some elementary results, however, will be mentioned. Before doing this, some additional notation shall be introduced.
Consider the linear system \( \dot{x} = A x + B u \), \( y = C x \), with state space \( X \), control space \( U \) and measurement space \( Y \).

Throughout this section we shall call the pair \((A, B)\) s-stabilizable if for every \( \lambda \) in \( \mathcal{C} / \mathcal{C}_s \) (the complement of \( \mathcal{C}_s \) in \( \mathcal{C} \)) \( \text{rank} \ [A - \lambda I, B] = n \) (cf. Hautus [2]).

It is well-known that this rank condition is equivalent to the existence of a mapping \( F: X \to U \) such that \( \sigma(A + BF) \subseteq \mathcal{C}_s \).

We shall call the pair \((C, A)\) s-detectable if the pair \((A^T, C^T)\) is s-stabilizable.

Given a linear subspace \( K \) in \( X \), we call \( V_f(K) \) (resp. \( V_s(K) \)) the largest subspace \( V \) in \( K \) for which there exists a mapping \( F: X \to U \) such that

\[ (A + BF) V \subseteq V \text{ and } \sigma(A + BF|V) \text{ is in } \mathcal{C}_f \text{ (resp. } \mathcal{C}_s). \]

Given a linear subspace \( L \) in \( X \), \( S_f(L) \) (resp. \( S_s(L) \)) denotes the smallest subspace \( S \) containing \( L \) for which there exists a mapping \( T: Y \to X \) such that \( (A + TC) S \subseteq S \) and \( \sigma(A + TC:X/S) \) is in \( \mathcal{C}_f \) (resp. \( \mathcal{C}_s ). \)

Another way to characterize \( V_s(K) \) is the following (cf. Hautus [3]).

\( V_s(K) \) is the subspace of points in \( X \) for which there exists a s-stable \((\xi, \omega)\) representation with \( \xi(s) \) in \( K \). Here we note that a \((\xi, \omega)\) representation is called s-stable if both \( \xi(s) \) and \( \omega(s) \) are strictly proper s-stable rational functions.

We may now consider the problem of section 3 with the additional requirement of internal stabilization \((\text{DDPOS})'\).

\((\text{DDPOS})'\) : Given the linear system

\[ \dot{x} = A x + B u + G d, \quad z_1 = H_1 x, \quad z_2 = H_2 x, \]

with state space \( X \) and control space \( U \) and with the assumption \( \ker H_1 \supset \ker H_2 \), find a state feedback \( u = F x \) such that
\[ H_1(sI-(A + BF))^{-1}G = 0, \]
\[ H_2(sI-(A + BF))^{-1}G \text{ is } f\text{-stable and} \]
\[ (sI-(A + BF))^{-1} \text{ is } s\text{-stable.} \]

We shall proceed analogously as in section 3.

**Definition 8.1.**
\[ \mathcal{V}_{fs}(\ker H_1, \ker H_2) \] is the set of points in \( X \) for which there exists a 
\( s\)-stable \((\xi, \omega)\) representation with \( \xi(s) \) in \( \ker H_1 \) and \( H_2\xi(s) \) is \( f\)-stable.

The following theorems generalize the theorems 3.2, 3.3 and 3.4.

**Theorem 8.2.**
\[ \mathcal{V}_{fs}(\ker H_1, \ker H_2) = \mathcal{V}_{fs}^*(\ker H_1) + \mathcal{V}_{fs}^*(\ker H_2). \]

**Theorem 8.3.**
Assume that \((A, B)\) is \( s\)-stabilizable. Then there exists a state feedback 
\( u = Fx \) such that:

(i) \((A + BF) \mathcal{V}_{fs}(\ker H_1, \ker H_2) \subseteq \mathcal{V}_{fs}(\ker H_1, \ker H_2),\)

(ii) \((A + BF) \mathcal{V}_{fs}^*(\ker H_2) \subseteq \mathcal{V}_{fs}^*(\ker H_2),\)

(iii) \(\sigma(A + BF) \subseteq \mathcal{F}_s,\)

(iv) \(\sigma(A + BF; \mathcal{V}_{fs}(\ker H_1, \ker H_2) / \mathcal{V}_{fs}^*(\ker H_2)) \subseteq \mathcal{F}_f.\)
Theorem 8.4.

The following statements are equivalent.

(i) \((\text{DDPOS})'\) is solvable.

(ii) \(\text{im } G \subset \mathcal{V}^s_\text{fs}(\ker H_1, \ker H_2)\), and \((A,B)\) is s-stabilizable.

(iii) There exists s-stable strictly proper rational matrices\(X(s)\) and \(U(s)\) such that

\[
(sI - A) X(s) - B U(s) = G, \quad H_1 X(s) = 0 \quad \text{and} \quad H_2 X(s) \text{ is f-stable}
\]

and \((A,B)\) is s-stabilizable.

(iv) (And if \((H_2,A)\) is s-detectable).

There exists a strictly proper s-stable rational matrix \(T(s)\) such that

\[
H_1 (sI - A)^{-1} B T(s) - H_1 (sI - A)^{-1} G = 0,
\]

\[
H_2 (sI - A)^{-1} B T(s) - H_2 (sI - A)^{-1} G \text{ is f-stable},
\]

and \((A,B)\) is s-stabilizable.

Remark 8.5.

The necessity of the detectability assumption in (iv) may be shown by means of an example (cf. Hautus [3]).

By pure dualization of the previous the problem mentioned in section 4 with the additional requirement of internal stabilization may be solved.

The result of this dualization will be omitted. We now come up with the main result of this section.
Theorem 8.6.

Problem 2 described in this section is solvable if and only if

$$S_s^* (\text{im } G) \subseteq V_{fs} (\ker H_1, \ker H_2),$$
$$S_{fs} (\text{im } G_1, \text{im } G_2) \subseteq V_s^* (\ker H_2),$$

(A,B) is s-stabilizable and

(C,A) is s-detectable.

Where $$S_{fs} (\text{im } G_1, \text{im } G_2) = (V_{fs} (\ker G_1^T \ker G_2^T))^T$$ with

$$V_{fs} (\ker G_1, \ker G_2)$$ computed relative AT and CT.

Remark 8.7.

Analogous, as is done in section 6 a number of results concerning the solvability of problem 2 stated in frequency domain terms can be derived.

Furthermore it is possible to derive results involving (OSDPM) and (DDPOSM), as mentioned in section 7, with the additional requirement of internal stabilization. Let (OSPDM)',(resp. (DDPOSM)') be the problem obtained from problem 2 by setting $$G_1 = 0, H_1 = 0, G_2 = G$$ and $$H_2 = H,$$ (resp. $$G_1 = 0, G_2 = G$$).

Note that (OSDPM)' (resp. (DDPOSM)') obtained in this sense is the extension of (OSDPM) (resp. (DDPOSM)) to the case of additional internal stabilization.

From theorem 8.6 we can conclude.

Corollary 8.7.

(OSDPM)' is solvable if and only if :

$$S_s^* (\text{im } G) \subseteq V_{fs} (X, \ker H),$$
$$S_{fs} ([0], \text{im } G) \subseteq V_s^* (\ker H),$$

(A,B) is s-stabilizable and

(C,A) is s-detectable.

Note that corollary 8.7 is the generalization of corollary 7.1. In the same way corollary 7.2 was derived from corollary 7.1 we can obtain the following.
Corollary 8.8.

(OSDPM) is solvable if and only if:

\[ \text{im } G \subseteq \mathcal{V}_{fs}(X, \ker H), \]
\[ S_{fs}\{0\}, \text{im } G \subseteq \ker H, \]
\[ (A,B) \text{ is s-stabilizable and} \]
\[ (C,A) \text{ is s-detectable}. \]

The following result is the generalization of corollary 7.6.

Corollary 8.9.

(DDPOS) is solvable if and only if:

\[ S^*_s(\text{im } G) \subseteq \mathcal{V}_{fs}(\ker H_1, \ker H_2), \]
\[ S_{fs}\{0\}, \text{im } G \subseteq \mathcal{V}^*(\ker H_2), \]
\[ (A,B) \text{ is s-stabilizable and} \]
\[ (C,A) \text{ is s-detectable}. \]
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