An asymmetric shortest queue problem

Citation for published version (APA):

Document status and date:
Published: 01/01/1989

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.
Memorandum COSOR 89-22

An asymmetric shortest queue problem

I.J.B.F. Adan
J. Wessels
W.H.M. Zijm

Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
5600 MB Eindhoven
The Netherlands

Eindhoven, August 1989
The Netherlands
AN ASYMMETRIC SHORTEST QUEUE PROBLEM

I.J.B.F. Adan *
J. Wessels *
W.H.M. Zijm **

University of Technology, Eindhoven

Abstract. In this paper we study a system consisting of two identical servers, each with exponentially distributed service times. Jobs arrive according to a Poisson stream. On arrival a job joins the shortest queue and in case both queues have equal length, he joins queue 1 say with probability $q$ and queue 2 with probability $1 - q$, where $q$ is not necessarily equal to $1/2$ (which corresponds to the symmetric shortest queue problem) but arbitrary between 0 and 1. We show that the stationary queue length distribution can be represented by an infinite sum of product form solutions, which satisfy nice recurrence relations. Due to the recurrence relations, the successive terms of the infinite sum can be easily calculated. Moreover, the convergence of the infinite sum is exponentially fast. Based on these properties, a numerically highly attractive algorithm is obtained.

Keywords: difference equation, product form, queues in parallel, stationary queue length distribution.

Introduction

Consider a system consisting of two identical servers, each with exponentially distributed service times. Jobs arrive according to a Poisson stream. On arrival a job joins the shortest queue and in case both queues have equal length, he joins the first queue with probability $q$ and the second one with probability $1 - q$. This problem is known as the shortest queue problem. Haight [3] originally introduced the problem. Kingman [4] and Flatto and McKean [2] treated the symmetric problem, that is $q = \frac{1}{2}$, by using a generating function analysis. They showed


and

that the generating function for the equilibrium distribution of the lengths of the two queues is a meromorphic function. Then by the decomposition of the generating function into partial fractions, it follows that the equilibrium probabilities can be represented by an infinite sum of product form solutions. However, the decomposition leads to cumbersome formulae for the equilibrium probabilities.

In this paper it will be shown that the equilibrium distribution of the lengths of the two queues can be found in an elementary way directly from the equilibrium equations. The main result is that the equilibrium probabilities can be represented by an infinite sum of product form solutions, which was, as we already mentioned, also recognized by Kingman and Flatto and McKean. Moreover, we provide nice recurrence relations for the terms in the infinite sum. Since the successive terms are easily calculated by means of these recurrence relations and the convergence of the infinite sum is exponentially fast, the analysis leads to a numerically highly attractive algorithm for calculating the equilibrium probabilities.

The paper is organized as follows. In section 1 we formulate the equilibrium equations. Then, in the next section, we treat the symmetric shortest queue problem. Based on these results, we treat the general case in the section 3.

1. Equilibrium equations

For simplicity of notation we suppose that the exponential servers have service times with unit mean and the Poisson arrival process has a rate $2\rho$ with $0 < \rho < 1$. The parallel queue system can be represented by a continuous time Markov process, whose state space consists of the pairs $(m, n), m, n = 0, 1, \ldots$ where $m$ and $n$ are the lengths of the two queues. Let $\{p_{m,n}\}$ denote the equilibrium distribution of the lengths of the two queues. The equilibrium equations become for all $n > m$

$$p_{m,n} (2\rho + 1) = p_{m-1,n} 2\rho + p_{m,n+1} + p_{m+1,n} \quad \text{if } m > 0, n > m+1$$
$$p_{m,m+1} (2\rho + 1) = p_{m-1,m+1} 2\rho + p_{m,m+2} + p_{m+1,m+1} + p_{m,m} 2q\rho \quad \text{if } m > 0, n = m+1$$
$$p_{0,n} (2\rho + 1) = p_{0,n+1} + p_{1,n} \quad \text{if } n > 1$$
$$p_{0,1} (2\rho + 1) = p_{0,2} + p_{1,1} + p_{0,0} 2q\rho$$

and a similar set holds for all $n < m$, and for $n = m$,

$$p_{m,n} (2\rho + 1) = p_{m-1,m} 2\rho + p_{m,m+1} + p_{m,m-1} 2\rho + p_{m,m+1} \quad \text{if } m > 0$$
$$p_{0,0} 2\rho = p_{0,1} + p_{1,0}$$

We will prove that the equilibrium probabilities $p_{m,n}$ can be represented by an infinite sum of product form solutions. That is, there exist parameters $\xi_i$ and $\eta_i$ and coefficients $c_i$, $d_i$ and $e_i$ such that
Because there is a drift along and to the diagonal in this system, it is more convenient to work on coordinate axis in these directions. Therefore, define for all $s \geq 0$ and $r \geq 0$

\[
q_{s+r} = p_{s,s+r},
\]

\[
q_{s,r} = p_{s+r,s}.
\]

Note that by definition $q_{s,0} = q_{s,0}$. In this paper we will derive expressions for the probabilities $q_{s,r}$ and $q_{s,r}$. The set of equations for the probabilities $p_{m,n}$ in the upper triangle $n > m$ is transformed into the following set of equations for the probabilities $q_{s,r}$,

\[
q_{s+r} 2(\rho + 1) = q_{s-1,r+1} 2\rho + q_{s,r+1} + q_{s+1,r-1} 2q \rho
\]

if $s > 0$, $r > 1$

\[
q_{s,1} 2(\rho + 1) = q_{s-1,2} 2\rho + q_{s,2} + q_{s+1,0} + q_{s,0} 2q \rho
\]

if $s > 0$, $r = 1$

\[
q_{0,r} (2\rho + 1) = q_{0,r+1} + q_{1,r-1}
\]

if $r > 1$

\[
q_{0,1}(2\rho + 1) = q_{0,2} + q_{1,0} + q_{0,0} 2q \rho
\]

A similar set is obtained for the probabilities $q_{s,r}$. The equations on the diagonal become

\[
q_{s,0} 2(\rho + 1) = (q_{s-1,1} + q_{s-1,1}) 2\rho + q_{s,1} + q_{s,1}
\]

if $s > 0$

\[
q_{0,0} 2\rho = q_{0,1} + q_{0,1}
\]

We will show that there exist parameters $\alpha_i$ and $\beta_i$ and coefficients $c_i$, $d_i$ and $e_i$ such that

\[
q_{s,r} = \sum_{i=0}^{m} c_i \alpha_i^s \beta_i^r, \quad q_{s,0} = q_{s,0} = \sum_{i=0}^{m} d_i \alpha_i^s \quad \text{and} \quad q_{s,r} = \sum_{i=0}^{m} e_i \alpha_i^s \beta_i^r \quad \text{for} \ s \geq 0 \text{ and } r \geq 1.
\]

Clearly the forms for $p_{m,n}$, $q_{s,r}$ and $q_{s,r}$ are equivalent, with $\alpha_i = \xi_i \eta_i$ and $\beta_i = \eta_i$. Throughout the analysis we will use the trivial, but vital property that the equations, on which the analysis is based, are linear, i.e. if two functions satisfy an equation, then any linear combination also satisfies the equation.

2. The symmetric case

In this section we will analyze the special case $q = \frac{1}{2}$. By symmetry

\[
q_{s,r} = q_{s,r}.
\]
Then the equations on the diagonal become

\[ q_{s,0} (p + 1) = q_{s-1,1} 2\rho + q_{s,1} \quad \text{if } s > 0 \]

\[ \overline{q}_{0,0} \rho = \overline{q}_{0,1} \]

Inserting these equations in (2) and (4), we can eliminate the probabilities \( \overline{q}_{s,0} \), yielding

\[ \overline{q}_{s,1} (p + 1) = \overline{q}_{s-1,2} 2\rho + \overline{q}_{s,2} + (\overline{q}_{s+1,1} + \overline{q}_{s,1} 2\rho) \frac{1}{(p + 1)} \]

\[ + (\overline{q}_{s,1} + \overline{q}_{s-1,1} 2\rho) \frac{\rho}{(p + 1)} \quad \text{if } s > 0, r = 1 \quad (7) \]

\[ \overline{q}_{0,1} (2\rho + 1) = \overline{q}_{0,2} + (\overline{q}_{1,1} + \overline{q}_{0,1} 2\rho) \frac{1}{(p + 1)} + \overline{q}_{0,1} \]

(8)

Hence the analysis can be restricted to the probabilities \( \overline{q}_{s,r} \) with \( s \geq 0 \) and \( r \geq 1 \). These probabilities satisfy the equations (1), (3), (7) and (8). We will investigate whether the probabilities \( \overline{q}_{s,r} \) have some kind of separable structure. Obviously, the equations (1), (3), (7) and (8) do not allow a separable solution of the form \( \overline{q}_{s,r} = \alpha^s \beta^r \), but numerical experiments indicate that there exist \( \alpha \) and \( \beta \) such that

\[ \overline{q}_{s,r} \sim K \alpha^s \beta^r \quad \text{as } s \to \infty \text{ and } r \geq 1, \]

for some \( K \). The question is, what are in general the parameters \( \alpha \) and \( \beta \)? Intuitively, \( \alpha \) stands for the ratio of the probability that there are \( n+2 \) and \( n \) jobs in the system. So a reasonable choice seems \( \alpha = \rho^2 \), which is supported by the numerical experiments. The parameter \( \beta \) follows by observing that the form \( \alpha^s \beta^r \) has to satisfy equation (1). Inserting this form in (1) and dividing both sides by the common term \( \alpha^{-1} \beta^{-1} \) gives a quadratic form for the unknown \( \beta \). This is stated in the following lemma.

**Lemma 1.** The form \( \alpha^s \beta^r \) is a solution of equation (1) if and only if \( \alpha \) and \( \beta \) satisfy

\[ \alpha \beta (p + 1) = \beta^2 2\rho + \alpha \beta^2 + \alpha^2. \quad (9) \]

Putting \( \alpha = \rho^2 \) in (9) we obtain two roots \( \beta = \rho \) and \( \beta = \rho^2/(2 + \rho) \). The root \( \beta = \rho \) is not useful, since it yields the asymptotic solution \( \overline{q}_{s,r} \sim K \rho^{2s} \rho^r \) for some \( K \), which corresponds to the equilibrium distribution of two independent \( M|M|1 \) queues, each with a workload \( \rho \). Therefore the only reasonable choice is \( \beta = \rho^2/(2 + \rho) \), which is also supported by numerical experiments. Let \( \alpha_0 = \rho^2 \) and \( \beta_0 = \rho^2/(2 + \rho) \), then we empirically found that

\[ \overline{q}_{s,r} \sim K \alpha_0^s \beta_0^r \quad \text{as } s \to \infty \text{ and } r \geq 1, \quad (10) \]


The asymptotic solution \( \alpha_0^s \beta_0^r \) perfectly describes the behaviour of the equilibrium probabilities in the interior of the set \( \{(s, r), s \geq 0, r \geq 1\} \) as well as near the boundary \( r = 1 \), but it does not
capture the behaviour near the boundary \( s = 0 \). One easily verifies that \( \alpha_0' \beta_0' \) indeed satisfies equation (1) and also (7) on the boundary \( r = 1 \) and that it violates equation (3) on the boundary \( s = 0 \). Obviously we can further improve the asymptotic solution by adding a term to correct the error on the boundary \( s = 0 \).

Form the linear combination \( \alpha_0' \beta_0' + c_0 \alpha' \beta' \). We will try to choose \( c_0, \alpha \) and \( \beta \) such that this linear combination satisfies equation (3) and (1). Inserting it in equation (3) gives for all \( r > 1 \)

\[
(\beta_0' + c_0 \beta') (2p + 1) = (\beta_0'^{p+1} + c_0 \beta'^{p+1}) + (\alpha_0 \beta_0'^{-1} + c_0 \alpha \beta'^{-1}).
\]

Since this must hold for all \( r > 1 \), we have to put \( \beta = \beta_0 \). Further we want \( \alpha' \beta_0' \) to satisfy equation (1). By virtue of lemma 1 there are two \( \alpha' \)'s such that \( \alpha' \beta_0' \) satisfies equation (1), namely \( \alpha_0 = p^2 \) and \( \alpha_1 = 2p^3/(2 + p)^2 \). So we have to put \( \alpha = \alpha_1 \). Then for any \( c_0 \), the linear combination \( \alpha_0' \beta_0' + c_0 \alpha_1' \beta_0' \) satisfies equation (1), because equation (1) is linear. Finally, dividing the above equation by the common term \( \beta_0'^{-1} \) yields an equation for the unknown \( c_0 \). Hence we can choose the coefficient \( c_0 \) such that the linear combination also satisfies equation (3). In general, the result of this procedure can be stated as

**LEMMA 2.** Let \( x_1 \) and \( x_2 \) be the roots of the quadratic form (9) for fixed \( \beta \). Then the linear combination \( k_1 \ x_1' \beta' + k_2 \ x_2' \beta' \) satisfies the equations (1) and (3) if \( k_1 \) and \( k_2 \) satisfy

\[
k_2 = -\frac{x_2 - \beta}{x_1 - \beta} \ k_1.
\]

Applying this lemma with \( x_1 = \alpha_0, \ x_2 = \alpha_1, \ \beta = \beta_0, \ k_1 = 1 \) and \( k_2 = c_0 \), yields

\[
c_0 = -\frac{\alpha_1 - \beta_0}{\alpha_0 - \beta_0}.
\]

Then \( \alpha_0' \beta_0' + c_0 \alpha_1' \beta_0' \) satisfies the equations (1) and (3). Experiments indicated that this refinement indeed captures the behaviour of \( \bar{q}_{s,r} \) near the boundary \( s = 0 \). We conclude that

\[
\bar{q}_{s,r} \sim K (\alpha_0' \beta_0' + c_0 \alpha_1' \beta_0') \quad \text{as} \quad s + r \to \infty \quad \text{and} \quad r \geq 1,
\]

for some \( K \). Flatto and McKean [2] proved this result, which is stronger than the asymptotic result (10). We added an extra term to compensate the error on the boundary \( s = 0 \). On the other hand we introduced a new error on the boundary \( r = 1 \), since the extra term violates equation (7). Because \( \alpha_1 < \alpha_0 \) the term \( \alpha_1' \beta_0' \) is very small compared to \( \alpha_0' \beta_0' \) even for moderate \( s \). Therefore its disturbing effect near the boundary \( r = 1 \) is practically negligible, except in the neighbourhood of the origin. However we can compensate this second order error on the boundary \( r = 1 \) in the same way as we did on the boundary \( s = 0 \), by again adding a correction term.

Form the linear combination \( \alpha_0' \beta_0' + c_0 \alpha_1' \beta_0' + d_1 \alpha' \beta' \). The term \( \alpha_0' \beta_0' \) already satisfies the equations (7) and (1) and we will try to choose \( d_1, \alpha \) and \( \beta \) such that the linear combination \( c_0 \alpha_1' \beta_0' + d_1 \alpha' \beta' \) also satisfies the equations (7) and (1). Based on similar arguments as
before, we have to put \( \alpha = \alpha_1 \) and \( \beta = \beta_1 \), where \( \beta_0 \) and \( \beta_1 \) are the roots of (9) for \( \alpha = \alpha_1 \). Then for any \( d_1 \), the linear combination \( c_0 \alpha_1 \beta_0 + d_1 \alpha_1 \beta_1 \) satisfies equation (1) and we can choose \( d_1 \) such that the linear combination also satisfies (7). In general, the result can be stated as

**Lemma 3.** Let \( y_1 \) and \( y_2 \) be the roots of the quadratic form (9) for fixed \( \alpha \). Then the linear combination \( k_1 \alpha y_1^2 + k_2 \alpha y_2^2 \) satisfies the equations (1) and (7) if \( k_1 \) and \( k_2 \) satisfy

\[
k_2 = -\frac{(\alpha + \rho)\beta_1 - (\rho + 1)}{(\alpha + \rho)\beta_1 - (\rho + 1)} k_1.
\]

Applying lemma 3 with \( y_1 = \beta_0 \), \( y_2 = \beta_1 \), \( \alpha = \alpha_1 \), \( k_1 = c_0 \) and \( k_2 = d_1 \), yields

\[
d_1 = -\frac{(\alpha_1 + \rho)\beta_1 - (\rho + 1)}{(\alpha_1 + \rho)\beta_0 - (\rho + 1)} c_0.
\]

Then the linear combination \( \alpha_1 \beta_0^2 + c_0 \alpha_1 \beta_0 + d_1 \alpha_1 \beta_1 \) satisfies equation (1) and (7). Now we compensated the error on the boundary \( r = 1 \), but we introduced a new one on the boundary \( s = 0 \), since the compensating term \( \alpha_1 \beta_1 \) violates equation (3). But it is clear how to continue this compensating procedure: for the initial values \( \alpha_0 = \rho^2 \) and \( \beta_0 = \rho^2 / (2 + \rho) \), generate the sequence \( \alpha_i \) and \( \alpha_{i+1} \) are the roots of

\[
\alpha \beta_i 2(\rho + 1) = \beta_i^2 2\rho + \alpha \beta_i^2 + \alpha^2
\]

and \( \beta_i \) and \( \beta_{i+1} \) are the roots of

\[
\alpha_{i+1} \beta 2(\rho + 1) = \beta^2 2\rho + \alpha_{i+1} \beta^2 + \alpha_{i+1}^2.
\]

By virtue of lemma 1 all the solutions \( \alpha_i \) \( \beta_i \) and \( \alpha_{i+1} \) \( \beta_{i+1} \) satisfy equation (1). Because equation (1) is linear, any linear combination also satisfies (1). Now form for all \( s \geq 0 \) and \( r \geq 1 \) the infinite sum

\[
\sum_{i=0}^{\infty} d_i (\alpha_i + c_i \alpha_{i+1}) \beta_i = d_0 \alpha_0 \beta_0 + \sum_{i=0}^{\infty} (d_i c_i \beta_i + d_{i+1} \beta_{i+1}) \alpha_{i+1},
\]

where in the first sum we formed pairs with a common factor \( \beta_i \) and in the second one with a common factor \( \alpha_{i+1} \). Put \( d_0 = 1 \) and successively generate the coefficients \( c_i \) and \( d_{i+1} \) such that \( (\alpha_i + c_i \alpha_{i+1}) \beta_i \) satisfies equation (3) on the boundary \( s = 0 \) and \( (d_i c_i \beta_i + d_{i+1} \beta_{i+1}) \alpha_{i+1} \) satisfies equation (7) on the boundary \( r = 1 \). By virtue of lemma 2 and lemma 3, this yields for all \( i = 0, 1, ... \)

\[
c_i = -\frac{\alpha_{i+1} - \beta_i}{\alpha_i - \beta_i}, \quad \text{and} \quad d_{i+1} = -\frac{(\alpha_{i+1} + \rho)\beta_{i+1} - (\rho + 1)}{(\alpha_{i+1} + \rho)\beta_i - (\rho + 1)} c_i d_i.
\]

The following theorem establishes our main result: for all \( s \geq 0 \) and \( r \geq 1 \) the infinite sum of product form solutions (13) equals the equilibrium probability \( q_{s,r} \) apart from a normalizing
constant $C$.

**THEOREM 1.** If $q = \frac{1}{2}$, then for all $s \geq 0$ and $r \geq 1$

$$\bar{q}_{s,r} = q_{s,r} = C^{-1} \sum_{i=0}^{\infty} d_i (\alpha_i^r + c_i \alpha_{i+1}^r) \beta_i^r,$$

where $C = \rho(2 + p) / 2(1 - p^2)(2 - p)$.

For a detailed proof of the theorem as well as the lemmas we refer to [1]. In [1] it is further proved that the terms $d_i (\alpha_i^r + c_i \alpha_{i+1}^r) \beta_i^r$ in (13) are alternating and exponentially fast and monotonously decreasing in modulus. This makes the solution approach very appropriate from a numerical point of view. Inserting the solution in equation (2) and (4) (or (5) and (6)) and using equation (9) to simplify the expressions, we obtain the representations on the diagonal, i.e. for all $s \geq 0$

$$\bar{q}_{s,0} = C^{-1} \sum_{i=0}^{\infty} d_i \left( \frac{\alpha_i^{r+1}}{\alpha_i + \rho} + c_i \frac{\alpha_{i+1}^{r+1}}{\alpha_{i+1} + \rho} \right). \quad (14)$$

A property that we will use in the following section, is that for all $i$

$$c_i > 0 \quad \text{and} \quad d_i + c_id_i < -1 \quad (\text{see also } [1]). \quad (15)$$

### 3. The general case

In this section we treat the case of an arbitrary $q$. Define for all $s \geq 0$ and $r \geq 0$

$$q_{s,r} = \frac{1}{2} (\bar{q}_{s,r} + q_{s,r}). \quad (16)$$

Then $\{q_{s,r}\}$ satisfies the equilibrium equations for $q = \frac{1}{2}$ and thus for all $s \geq 0$ and $r \geq 1$

$$q_{s,r} = C^{-1} \sum_{i=0}^{\infty} d_i (\alpha_i^r + c_i \alpha_{i+1}^r) \beta_i^r, \quad (17)$$

and for $s \geq 0$ and $r = 0$

$$q_{s,0} = C^{-1} \sum_{i=0}^{\infty} d_i \left( \frac{\alpha_i^{r+1}}{\alpha_i + \rho} + c_i \frac{\alpha_{i+1}^{r+1}}{\alpha_{i+1} + \rho} \right). \quad (18)$$

Immediate from (16) we have that $\bar{q}_{s,0} = q_{s,0} = q_{s,0}$ for all $s \geq 0$. For $\bar{q}_{s,r}$ and $q_{s,r}$ with $s \geq 0$ and $r \geq 1$, we will show that we can choose coefficients $\bar{d}_i$ and $d_i$ (with $\frac{1}{2} (\bar{d}_i + d_i) = d_i$) such that

$$\bar{q}_{s,r} = C^{-1} \sum_{i=0}^{\infty} \bar{d}_i (\alpha_i^r + c_i \alpha_{i+1}^r) \beta_i^r \quad \text{and} \quad q_{s,r} = C^{-1} \sum_{i=0}^{\infty} d_i (\alpha_i^r + c_i \alpha_{i+1}^r) \beta_i^r. \quad (19)$$

First note that each term $(\alpha_i^r + c_i \alpha_{i+1}^r) \beta_i^r$ satisfies the equilibrium equations for $s \geq 0$ and $r > 1$ for the special case $q = \frac{1}{2}$. But the probability $q$ may be arbitrary, since it does not occur in the equilibrium equations for $s \geq 0$ and $r > 1$. Hence, by the linearity of the equilibrium equations,
for any choice of the coefficients $d_i$ and $d_j$, the linear combinations (19) satisfy the equations for $s \geq 0$ and $r > 1$ for an arbitrary $q$. Now we have to choose $d_i$ and $d_j$ such that the equations for $s \geq 0$ and $r = 1$ are also satisfied. Inserting (18) and (19) in equation (2) and using equation (9) to simplify the expressions, yields for $s > 0$

$$
\sum_{i=0}^{\infty} \bar{d}_i (\alpha_i^{s+1} + c_i \alpha_i^{s+1}) = \sum_{i=0}^{\infty} d_i \left( \frac{\alpha_i + 2qp}{\alpha_i + \rho} \alpha_i^{s+1} + c_i \frac{\alpha_i + 2qp}{\alpha_i + \rho} \alpha_i^{s+1} \right)
$$

and taking terms with a common factor $\alpha_i^{s+1}$, then

$$
\bar{d}_0 \alpha_0^{s+1} + \sum_{i=0}^{\infty} (\bar{d}_i c_i + \bar{d}_{i+1}) \alpha_i^{s+1} = d_0 \alpha_0 + 2qp \alpha_0^{s+1} + \sum_{i=0}^{\infty} (d_i c_i + d_{i+1}) \frac{\alpha_i + 2qp}{\alpha_i + \rho} \alpha_i^{s+1}.
$$

Inserting (18) and (19) in equation (4) the same equation is obtained for $s = 0$. Since the above equation must hold for all $s \geq 0$ we have to put for $i = 0, 1, \ldots$

$$
\bar{d}_{i+1} = (d_i c_i + d_{i+1}) \frac{\alpha_i + 2qp}{\alpha_i + \rho} - \bar{d}_i c_i
$$

with initial value $\bar{d}_0 = d_0 (\alpha_0 + 2qp) / (\alpha_0 + \rho)$. Similarly, for $i = 0, 1, \ldots$

$$
d_{i+1} = (d_i c_i + d_{i+1}) \frac{\alpha_i + 2(1-q)p}{\alpha_i + \rho} - d_i c_i
$$

with initial value $d_0 = d_0 (\alpha_0 + 2(1-q)p) / (\alpha_0 + \rho)$. Then the infinite sums (18) and (19) formally satisfy all the equilibrium equations. To complete the proof we have to show that the infinite sums (19) converge absolutely for all $s \geq 0$ and $r \geq 1$. By definition (16) we have that

$$
\frac{1}{2} (\bar{d}_i + d_i) = d_i.
$$

From (15), (20) and (21) it follows by induction that $\bar{d}_i$, $d_i$, and $d_i$ have the same sign. Then by (22),

$$
|\bar{d}_i| \leq |d_i| \text{ and } |d_i| \leq |\bar{d}_i|.
$$

Hence the sums (19) are bounded by the sum (17), which converges absolutely for all $s \geq 0$ and $r \geq 1$. This proves

**Theorem 2.** For all $s \geq 0$ and $r \geq 1$

$$
\bar{q}_{s,r} = C^{-1} \sum_{i=0}^{\infty} \bar{d}_i (\alpha_i^{s} + c_i \alpha_i^{s+1}) \beta_i^s, \quad q_{s,r} = C^{-1} \sum_{i=0}^{\infty} d_i (\alpha_i^{s} + c_i \alpha_i^{s+1}) \beta_i^s
$$

and for all $s \geq 0$ and $r = 0$

$$
\bar{q}_{s,0} = \bar{q}_{s,0} = C^{-1} \sum_{i=0}^{\infty} d_i \left( \frac{\alpha_i^{s+1}}{\alpha_i + \rho} + c_i \frac{\alpha_i^{s+1}}{\alpha_i + \rho} \right)
$$

where the coefficients $\bar{d}_i$ and $d_i$ are generated according to (20) and (21).
4. Concluding remarks

In the previous sections we showed that the equilibrium distribution of the shortest queue problem can be represented by an infinite sum of product form solutions. This solution is very appropriate for numerical analysis. Our present research is concerned with related problems, which may have a similar product form structure; we already obtained results for the shortest queue problem with non-identical servers (i.e. with different service rates).

References


