Optimal Control of One-Warehouse Multi-Retailer Systems with Discrete Demand

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October 5, 2004

Abstract
This paper considers a two-echelon distribution system which consists of a single warehouse serving \( N \) (possibly nonidentical) retailers that face discrete stochastic demand of the customers. The warehouse orders from an exogenous supplier with ample stock. Orders arrive at the warehouse after a fixed leadtime. The warehouse satisfies, if possible, the replenishment requests from the retailers. In case of insufficient stock, the available stock is allocated to the retailers. The shipments from the warehouse reach their final destinations after a further fixed leadtime. Excess demand at the retailers is backlogged, and linear holding and penalty costs are incurred. We assume periodic review and centralized control, and the objective is to minimize the average expected inventory holding and penalty costs. Under the so-called balance assumption (also known as the allocation assumption), we show that base stock policies are optimal. Actually, we extend the optimality of base stock policies for continuous demand models under the balance assumption to the discrete demand case. Further, we derive newsboy inequalities for the optimal base stock levels and develop an efficient algorithm for the computations of an optimal policy.

Subject classifications: Inventory/Production: multi-echelon, periodic review, stochastic discrete demand, optimal policy, newsboy characterizations.

Area of review: Manufacturing, Service and Supply Chain Operations.

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1. INTRODUCTION

This paper treats a two-echelon distribution system under periodic review and centralized control. There are several retailers supplied by shipments from a warehouse, which in return orders from an exogenous supplier with ample stock. There are fixed leadtimes between the supplier and the warehouse, and between the warehouse and the retailers. The retailers face discrete stochastic demand of the customers. Excess demand is backlogged and linear penalty costs are incurred. There are no fixed costs and the objective is to minimize the average inventory holding and penalty costs of the system in the long-run.

Clark and Scarf [1960] were the first to consider the inventory control problem in a distribution system facing continuous demands. It was not possible to apply their decomposition approach to these systems due to the so-called allocation (rationing) problem. In each period, there is an allocation decision of how to distribute the physical stock at the warehouse among all the stock points has to be made.

Although the optimal policy for the inventory control of a distribution system is unknown, there are some approximate approaches. The key assumption (in these approaches) is that the inventories of the retailers are balanced, i.e., the allocation policy may apportion negative quantities to the retailers. The balance assumption allows one to apply Clark and Scarf’s decomposition to distribution systems.

The distribution model under consideration with continuous demands is well studied in the literature; see Clark and Scarf [1960], Eppen and Schrage [1981], Federgruen and Zipkin [1984a,b,c], Diks and De Kok [1998]. See also Van Houtum, Inderfurth, and Zijm [1996], and Axsäter [2003] for extensive reviews. However, the discrete demand case has not been studied up to now. Discrete demand processes are important since it makes it possible to handle positive probability mass at any point in the demand distribution, particularly at zero. This is highly important in case of intermittent demand.

Our contribution in this study is threefold. First, under the balance assumption, we extend the optimality of base stock policies to two-echelon distribution systems facing discrete demands. The proof is not a trivial extension because the proof in the continuous demand case heavily depends on identifying equalities that have to be satisfied by the optimal control parameters (base stock levels and allocation functions). In contrast, in
the discrete demand case optimal control parameters have to satisfy certain inequalities. Second, we show that the optimal base-stock levels satisfy newsboy inequalities. Under continuous demand, newsboy equalities have been derived for multi-echelon serial and distribution systems by Van Houtum and Zijm [1991], and Diks and De Kok [1998], respectively. Our newsboy inequalities extend the newsboy equalities of Diks and De Kok, and to the best of our knowledge are the first newsboy characterization for multi-echelon systems under discrete demand. Third, we develop an efficient algorithm for computation of an optimal policy.

Following the same line of thought as used in this study, our results may be extended to distribution systems with more than two echelons, but this is left as a subject for further research.

This paper is organized as follows. In §2, we introduce the model. The complete analysis is presented in §3; §3.1 and §3.2 set the stage for the proof, which is conducted in §3.3 and §3.4. Newsboy inequalities and the algorithm for the computation of an optimal policy are discussed in §3.5 and §3.6, respectively.

2. MODEL

Let us consider a two-stage distribution system that consists of a single warehouse serving $N$ retailers. The warehouse (indexed as stock point 0) orders from an exogenous supplier with ample stock and the retailers are supplied by shipments from the warehouse. Retailers face stochastic and independent demands of the customers. Demands in different periods are i.i.d., discrete nonnegative random variables. Any unfulfilled demand at a retailer is backlogged. Time is divided into periods of equal length and the following sequence of events takes place during a period: (i) inventory levels are observed and the current period’s ordering/shipment decisions are made considering the arrival of the orders/shipments given before (at the beginning of the period), (ii) orders/shipments arrive following their respective leadtimes (at the beginning of the period), (iii) demand occurs, (iv) holding and penalty costs are assessed on the period ending inventory and backorder levels (at the end of the period). Leadtimes of orders (between the supplier and the warehouse), and shipments (between the warehouse and the retailers) are fixed. Costs consist of linear holding and penalty costs. Finally, we assume that the system is centrally controlled and the objective is to minimize the expected holding and penalty costs of the system in the long-run.

Let us go over some basic definitions for the sake of completeness. Echelon stock of a stock point is the
stock on hand at that point plus in transit to or on hand at any successor stock point minus the backorders of external customers. **Echelon inventory position** of a stock point is the echelon stock of that stock point plus all the orders that are in transit to that stock point. Here is the basic notation for this study:

\[
\begin{align*}
Z &= \text{set of integer numbers. } Z^- = \{..., -2, -1\}, \ Z^+ = \{1, 2, \ldots\}, \text{ and } Z_0^+ = Z^+ \cup \{0\} \\
t &= \text{index for time. Period } t \text{ is defined as the time interval between epochs } t \text{ and } t + 1 \text{ for } t \in Z_0^+. \\
N &= \text{number of retailers, } N \in Z^+. \\
i &= \text{index for stock points, } i = 0 \text{ is the warehouse, and } i = 1, 2, \ldots, N \text{ are the retailers.} \\
J &= \text{set of retailers, i.e., } J = \{1, 2, \ldots, N\}. \\
h_i &= \text{additional inventory holding cost parameter for stock point } i. \text{ At the end of a period: } \\
&\quad \text{(i) cost } h_0 \text{ is charged for each unit on stock at the warehouse or in transit to any retailers,} \\
&\quad \quad h_0 \geq 0, \\
&\quad \text{(ii) costs } h_0 + h_i \text{ are charged for each unit on stock at retailer } i, \ h_i \geq 0 \ \forall i \in J. \\
p_i &= \text{penalty cost parameter for retailer } i. \text{ A cost } p_i \text{ is charged for each unit of backlog at the end of a period at retailer } i. \ p_i > 0 \ \forall i \in J \\
l_i &= \text{leadtime parameter for stock point } i. \ l_i \in Z_0^+ \ \forall i \in J \text{ and } l_0 \in Z^+ \\
\mu_i &= \text{mean of one-period demand faced by retailer } i. \\
\mu_0 &= \text{mean of one-period demand faced by the system, i.e., } \mu_0 = \sum_{i=1}^{N} \mu_i. \\
D_i(t, t + s) &= \text{discrete random variable denoting the demand faced by retailer } i \text{ during the periods } t, t + 1, \ldots, t + s \text{ for } t, s \in Z_0^+. \\
D_0(t, t + s) &= \text{discrete random variable denoting the aggregate demand faced by the system during the periods } t, t + 1, \ldots, t + s, \text{ i.e., } D_0(t, t + s) = \sum_{i=1}^{N} D_i(t, t + s) \text{ for } t, s \in Z_0^+. \\
D_i^{(l)} &= \text{discrete random variable denoting } l\text{-period demand faced by retailer } i, \ l \in Z_0^+. \\
D_0^{(l)} &= \text{discrete random variable denoting } l\text{-period aggregate demand faced by the system, } l \in Z_0^+. \\
F_i^{(l)} &= \text{cumulative distribution function of } l\text{-period demand of retailer } i \text{ defined over } Z_0^+. \\
F_0^{(l)} &= \text{cumulative distribution function of } l\text{-period demand faced by the system defined over } Z_0^+, \text{ i.e., } F_0^{(l)} = F_1^{(l)} * F_2^{(l)} * \ldots * F_N^{(l)}. \\
I_i(t) &= \text{echelon stock of stock point } i \text{ at the beginning of period } t \text{ just after the receipt of the incoming order/shipment.} \\
\hat{I}_i(t) &= \text{echelon stock of stock point } i \text{ at the end of period } t.
\end{align*}
\]
\[ IP_i(t) = \text{echelon inventory position of stock point } i \text{ at the beginning of period } t \text{ just before ordering (if } i = 0) \text{ or shipment (if } i \in J). \]

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## 3. ANALYSIS

We first discuss the ordering and allocation decisions and their impacts on the costs, and introduce the optimization problem under study in §3.1. Then we analyze the allocation decision and introduce the balance assumption in §3.2. This constitutes the basis for the analysis of a single ordering cycle, and the derivation of an average cost optimal policy, in §3.3 and §3.4, respectively. Finally, we discuss the newsboy inequalities in §3.5 and conclude with an efficient algorithm for computations in §3.6.

### 3.1 Dynamics of the System

The total cost of the system at the end of an arbitrary period, \( t \), is equal to

\[
 h_0 \left( \hat{I}_0(t) - \sum_{i \in J} \hat{I}_i(t) \right) + \sum_{i \in J} (h_0 + h_i) \hat{I}_i^+(t) + \sum_{i \in J} p_i \hat{I}_i^-(t)
\]

where \( a^+ = \max \{0, a\} \) and \( a^- = -\min \{0, a\} \) for \( a \in \mathbb{R} \). Substituting \( \hat{I}_i(t) = \hat{I}_i^+(t) - \hat{I}_i^-(t) \) first, rearranging the terms, and then using the identity \( \hat{I}_i^+(t) = \hat{I}_i(t) + \hat{I}_i^-(t) \) leads to the following result:

\[
 h_0(\hat{I}_0(t) - \sum_{i \in J} \hat{I}_i(t)) + \sum_{i \in J} (h_0 + h_i) \hat{I}_i^+(t) + \sum_{i \in J} p_i \hat{I}_i^-(t) = h_0 \hat{I}_0(t) + \sum_{i \in J} h_i \hat{I}_i^+(t) + \sum_{i \in J} (h_0 + p_i) \hat{I}_i^-(t) = h_0 \hat{I}_0(t) + \sum_{i \in J} h_i \hat{I}_i(t) + \sum_{i \in J} (h_0 + h_i + p_i) \hat{I}_i^-(t).
\]

We define \( h_0 \hat{I}_0(t) \) as the cost attached to the echelon of the warehouse (echelon of stock point 0) at the end of period \( t \). This cost is denoted by \( C_0(t) \). We define \( h_i \hat{I}_i(t) + (h_0 + h_i + p_i) \hat{I}_i^-(t) \) as the cost attached to the echelon of retailer \( i \) at the end of period \( t \), and denote it by \( C_i(t) \).

Now, consider the following two connected decisions and the resulting effects on the expected costs, which start with an order given to the supplier in period \( t, t \in \mathbb{Z}_0^+ \). Figure 1 illustrates the dependence among these decisions and the resulting cost consequences.
**Ordering Decision:** Assume that at the beginning of period $t$ the warehouse gives an order that raises the inventory position of the system to some level $y_0$, i.e., $IP_0(t) = y_0$. The order materializes at the beginning of period $t + l_0$ and the echelon stock of the warehouse at that epoch is $y_0 - D_0(t, t + l_0 - 1)$. There are two consequences of the ordering decision:

- It directly determines the expected value of the cost attached to the echelon of the warehouse at the end of period $t + l_0$,

\[
E[C_0(t + l_0)|IP_0(t) = y_0] = E[h_0(y_0 - D_0(t, t + l_0))]
\]

\[
= h_0(y_0 - (l_0 + 1)\mu_0).
\]

- It limits the shipment quantities to the retailers. In other words, it puts an upper bound on the level to which one can increase the aggregate echelon inventory positions of the retailers in period $t + l_0$,

\[
\sum_{i=1}^{N} IP_i(t + l_0) \leq y_0 - D_0(t, t + l_0 - 1).
\]

**Allocation Decision:** At the beginning of period $t + l_0$, the system-wide stock is rationed among all stock points. In other words, the shipment quantities to the retailers are determined; as a result, the decision of how much stock to retain at the warehouse is made. At epoch $t + l_0$, the inventory position of retailer $i$ is increased to some level $z_i$ such that $\sum_{i \in J} z_i \leq y_0 - D_0(t, t + l_0 - 1)$ and $z_i \geq \hat{IP}_i(t + l_0)$ for all $i \in J$. These decisions directly affect the cost of echelon $i$ at the end of period $t + l_0 + l_i$, for all $i \in J$. The expected value

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**Figure 1:** The consequences of the order given to the supplier in period $t$
of the cost attached to echelon $i$ is

$$
\mathbb{E}[C_i(t + l_0 + l_i)| IP_i(t + l_0) = z_i] = \mathbb{E}[h_i(z_i - D_i(t + l_0, t + l_0 + l_i) + (h_0 + h_i + p_i)(z_i - D_i(t + l_0, t + l_0 + l_i))]
$$

$$
= h_i(z_i - (l_i + 1)\mu_i) + (h_0 + h_i + p_i)\mathbb{E}[D_i(t + l_0, t + l_0 + l_i) - z_i]_v.
$$

We define the expected costs as a consequence of the ordering and allocation decisions that begin with the warehouse’s order given at epoch $t$ as the cycle cost of period $t$ and denote it by $C_{cyc}(t)$.

$$
C_{cyc}(t) = C_0(t + l_0) + \sum_{i \in J} C_i(t + l_0 + l_i)
$$

Let $\Pi$ and $g(\pi)$ denote the set of all ordering policies and the average expected cost of ordering policy $\pi$, respectively. The expected long-run average cost of any policy $\pi \in \Pi$ is simply the average of the expected value of the sum of costs over all cycles:

$$
g(\pi) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left[ \sum_{t=0}^{T-1} \sum_{i=0}^{N} C_i(t) \right]
$$

$$
= \lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left[ \sum_{t=0}^{T-1} \sum_{i \in J} C_0(t) + \sum_{t=0}^{T-1} \sum_{i \in J} C_i(t) + \sum_{t=0}^{T-1} C_{cyc}(t) - \sum_{t=T}^{T+l_0-1} C_0(t) - \sum_{t=T}^{T+l_0+l_i-1} \sum_{i \in J} C_i(t) \right]
$$

$$
= \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[C_{cyc}(t)].
$$

The optimization problem that we are after is

$$
\min_{\pi \in \Pi} \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[C_{cyc}(t)]. \tag{1}
$$

The minimization problem given above is intricate since the decisions are highly interdependent. In the next subsection, we introduce the myopic allocation problem and discuss the balance assumption.

### 3.2 Analysis of the Allocation Decision

In this subsection, we discuss the Allocation Decision described in §3.1. Let us consider the sequence of decisions and the resulting costs as a result of increasing the inventory position of the system up to $y_0$ at the beginning of period $t$, $t \in \mathbb{Z}_0^+$. Suppose the echelon stock of the warehouse at the beginning of period $t + l_0$ (i.e., $y_0 - D_0(t, t + l_0 - 1)$) is distributed among all stock points such that the sum of the expected holding and penalty costs of the retailers in the periods the allocated quantities reach their destinations (i.e., period $t + l_0 + l_i$ for retailer $i$) is minimized. This way of rationing is called myopic allocation because the effect of
the allocation decisions on the subsequent periods is not considered. The mathematical formulation of the problem is as follows:

$$\min_{z_i, i \in J} \sum_{i \in J} E[C_i(t + l_0 + l_i)|IP_i(t + l_0) = z_i]$$ \hspace{1cm} (2)

subject to:

$$\sum_{i \in J} z_i \leq y_0 - D_0(t, t + l_0 - 1)$$ \hspace{1cm} (3)

$$\hat{IP}_i(t + l_0) \leq z_i \hspace{0.5cm} \forall \hspace{0.2cm} i \in J$$ \hspace{1cm} (4)

Both constraints serve for the physical balance of the stocks. While (4) assures that no negative quantity is allocated to the retailers, (3) compels that the sum of the allocated quantities cannot exceed the available stock in the system.

Although myopic allocation allows the allocation decisions to be made independent of the future allocation and ordering decisions, it still depends on previous periods’ decisions due to (4). Now, consider a relaxed version of the myopic allocation problem where (4) is omitted. This is equivalent to assuming that the quantities allocated to the retailers may be negative. We refer to this assumption as the balance assumption.

In the absence of (4), $C_{cy}(t)$ depends only on the ordering and allocation decisions that start with an order given by the warehouse in period $t$, not on decisions of other periods. Now, we focus on how to minimize the cycle cost of period $t$.

### 3.3 Analysis of a Single Cycle

First of all, we give the definition of convexity for functions defined over $\mathbb{Z}$. Let $\Delta f(x)$ and $\Delta^2 f(x)$ denote first and second order difference equations for a function $f$ where $\Delta f(x) = f(x+1) - f(x)$ and $\Delta^2 f(x) = \Delta f(x+1) - \Delta f(x) = f(x+2) - 2f(x+1) + f(x)$. A discrete function is convex over $\mathbb{Z}$ if $\Delta^2 f(x) \geq 0$ for all $x \in \mathbb{Z}$.

For retailer $i$, define $G_i(y_i)$ as the expected cost attached to echelon $i$ at the end of period $t + l_i$ when the inventory position at the beginning of period $t$ is increased to $y_i$ for $y_i \in \mathbb{Z}$, and $t \in \mathbb{Z}_0^+$, i.e., $G_i(y_i) = E[G_i(t + l_i)|IP_i(t) = y_i]$. Now, we analyze the function $G_i(\cdot)$ in Lemma 1.

**Lemma 1** For all $i \in J$:

(i) $G_i(y_i) = h_i (y_i - (l_i + 1) \mu_i) + (h_0 + h_i + p_i)E[D_i(t, t + l_i) - y_i]_+$, $y_i \in \mathbb{Z}$,
(ii) $\Delta G_i(y_i) = (h_0 + h_i + p_i) F_i^{(l+1)}(y_i) - (h_0 + p_i), \; y_i \in \mathbb{Z}$.

(iii) $G_i(y_i)$ is convex over $\mathbb{Z}$.

(iv) $G_i(y_i)$ is minimized at all $y_i \in Y_i^* = \{y_i^*, y_i^* + 1, \ldots, y_i^*\}$ where

$$y_i^* = \min \{ y_i | F_i^{(l+1)}(y_i) \geq \frac{h_0 + p_i}{h_0 + h_i + p_i} \}, \; \text{and} \; y_i^* = \min \{ y_i | F_i^{(l+1)}(y_i) > \frac{h_0 + p_i}{h_0 + h_i + p_i} \}.$$

If $\{ y_i | F_i^{(l+1)}(y_i) > \frac{h_0 + p_i}{h_0 + h_i + p_i} \} = \emptyset$ then $y_i^* = \infty$.

(Note that $y_i^* = \infty$ if $h_i = 0$. Moreover, $y_i^* = \infty$ if $h_i = 0$ and $F_i^{(1)}$ has an infinite support.)

Proof: The proof is straightforward and left to the reader.

Now, let us focus on how to characterize an optimal solution for the myopic allocation problem under the balance assumption, i.e., for (5)-(6).

Lemma 2 Let $x \in \mathbb{Z}$.

(i) If $x \geq \sum_{i \in J} y_i^*$, then $z_i^*(x) \in Y_i^*$ for all $i \in J$ such that $\sum_{i \in J} z_i^*(x) \leq x$.

(ii) If $x \leq \sum_{i \in J} y_i^*$, then (6) is binding, i.e., $\sum_{i \in J} z_i^*(x) = x$.

Proof: See the Appendix.
Notice that the optimal solution of (5)-(6) for \( x \geq \sum_{i \in J} y_i^* \) is fully characterized by Lemma 2. The following lemma identifies optimal solutions for \( x < \sum_{i \in J} y_i^* \).

**Lemma 3** Let \( x < \sum_{i \in J} y_i^* \), and \( x \in \mathbb{Z} \).

(i) A given solution \( \{z_i^*(x)\}_{i \in J} \) is optimal if and only if:

\[
\Delta G_i(z_i^*(x)) \geq \Delta G_j(z_j^*(x) - 1) \quad \forall i, j \in J, \ i \neq j.
\]

(ii) Given an optimal solution \( \{z_i^*(x)\}_{i \in J} \) for \( x \), an optimal solution \( \{z_i^*(x + 1)\}_{i \in J} \) for \( x + 1 \) is given by

\[
z_k^*(x + 1) = z_k^*(x) + 1, \text{ where } k \in \{i \in J | \Delta G_i(z_i^*(x)) = \min_{j \in J} \Delta G_j(z_j^*(x))\}, \text{ and } z_j^*(x + 1) = z_j^*(x) \quad \forall j \in J \setminus \{k\}.
\]

(iii) Given an optimal solution \( \{z_i^*(x)\}_{i \in J} \) for \( x \), an optimal solution \( \{z_i^*(x - 1)\}_{i \in J} \) for \( x - 1 \) is given by

\[
z_k^*(x - 1) = z_k^*(x) - 1, \text{ where } k \in \{i \in J | \Delta G_i(z_i^*(x) - 1) = \max_{j \in J} \Delta G_j(z_j^*(x) - 1)\}, \text{ and } z_j^*(x - 1) = z_j^*(x) \quad \forall j \in J \setminus \{k\}.
\]

**Proof**: See the Appendix.

We are now prepared for the main result in regard to optimal allocation functions of the retailers.

**Theorem 1** There exist optimal allocation functions \( \{z_i^*(x)\}_{i \in J} \), such that \( \Delta z_i^*(x) \geq 0 \) for \( x \in \mathbb{Z} \).

**Proof**: Distinguish two cases: (i) \( \sum_{i \in J} y_i^* \) is finite, (ii) \( \sum_{i \in J} y_i^* \) is infinite. In case (i), \( z_i^*(\sum_{i \in J} y_i^*) = y_i^* \forall i \in J \) due to Lemma 2. Starting from this optimal solution, optimal solutions for \( x < \sum_{i \in J} y_i^* \) can be obtained using part (iii) of Lemma 3, which leads to \( z_i^*(x + 1) - z_i^*(x) = \Delta z_i^*(x) \in \{0, 1\} \) for all \( i \in J \). For \( x > \sum_{i \in J} y_i^* \), take \( z_i^*(x) = y_i^* \forall i \in J \); as a result \( \Delta z_i^*(x) = 0 \) for all \( i \in J \). In case (ii), an optimal solution \( \{z_i^*(x)\}_{i \in J} \) of (5)-(6) can be determined for some given \( x \in \mathbb{Z} \) by Lagrange relaxation (see Everett [1963]). Based on \( \{z_i^*(x)\}_{i \in J} \), optimal solutions for \( x + 1 \) and \( x - 1 \) can be constructed utilizing parts (ii) and (iii) of Lemma 3, respectively. Continuing in this manner, \( z_i^*(x) \) is determined for all \( x \in \mathbb{Z} \) and \( i \in J \) such that \( \Delta z_i^*(x) \in \{0, 1\} \). \( \square \)
Remark 1: Theorem 1 shows the existence of nondecreasing optimal allocation functions, but not all optimal allocation functions have to be nondecreasing. Consider the case with three identical retailers, i.e., three retailers with identical leadtimes, cost parameters and demand distributions. Define \( \bar{z}^*(x) := (z_1^*(x), z_2^*(x), z_3^*(x)) \).

There are three optimal alternatives for rationing 4 units: \( \bar{z}^*(4) \in \{(2, 1, 1), (1, 2, 1), (1, 1, 2)\} \). For \( x = 5 \), \( \bar{z}^*(5) \in \{(2, 2, 1), (1, 2, 2), (2, 1, 2)\} \). Given \( z^*(4) = (2, 1, 1) \), if one follows part (ii) of Lemma 3, then \( z^*(5) \) is (2, 2, 1) or (2, 1, 2), which leads to nondecreasing optimal allocation functions at \( x = 4 \). Now, consider the following optimal allocations: \( z^*(4) = (2, 1, 1) \) and \( z^*(5) = (1, 2, 2) \); observe that then \( z_1^*(x) \) is decreasing for \( x = 4 \).

We can show the following properties of the function \( H^*(\cdot) \).

**Lemma 4** Let \( x \in \mathbb{Z} \).

(i) \( \Delta H^*(x) < 0 \) for \( x < \sum_{i \in J} y_i^* \),

(ii) \( \Delta H^*(x) = 0 \) for \( x \geq \sum_{i \in J} y_i^* \),

(iii) \( H^*(x) \) is convex in \( x \).

**Proof**: See the Appendix.

Lemma 4 gives the shape of the optimal objective function of the myopic allocation problem given in (5)–(6) as a function of the amount to allocate. \( H^*(x) \) is convex, strictly decreasing in the region \( (-\infty, \sum_{i=1}^N y_i^*) \), and constant over \( [\sum_{i=1}^N y_i^*, +\infty) \).

Define

\[ z = \text{set of allocation functions, i.e., } \{z_i(x), \ x \in \mathbb{Z}\}_{i \in J}. \]

\[ z^* = \text{set of optimal allocation functions, i.e., } \{z_i^*(x), \ x \in \mathbb{Z}\}_{i \in J} \text{ such that } z_i^*(x) \text{ is optimal for all } i \in J. \]

\[ G_0(y_0) = \text{expected value of the cost attached to the echelon of the warehouse at the end of period } t + l_0 \text{ given } \text{IP}_0(t) = y_0 \text{ for } y_0 \in \mathbb{Z}, \text{ and } t \in \mathbb{Z}_0^+, \text{ i.e.,} \]

\[ G_0(y_0) = \mathbb{E} [C_0(t + l_0) | \text{IP}_0(t) = y_0] = h_0[y_0 - (l_0 + 1)\mu_0]. \]

\[ \tau(y_0) = \left[ y_0 - \sum_{i \in J} y_i^* + 1 \right]^+, y_0 \in \mathbb{Z}. \]
Under the balance assumption, let us denote the expected cycle cost of period \( t \) given \( IP_0(t) = y_0 \) and \( z \) for \( y_0 \in \mathbb{Z}, t \in \mathbb{Z}_0^+ \), by \( G_{cyc}(y_0, z) \). Thus,

\[
G_{cyc}(y_0, z) = \mathbb{E} \left[ C_0(t + l_0) + \sum_{i \in J} C_i(t + l_0 + l_i) \mid IP_0(t) = y_0, z \right]
\]

\[
= G_0(y_0) + \sum_{x = 0}^{\infty} \sum_{i \in J} G_i(z_i(y_0 - x)) \Pr\{D_0^{(l_i)} = x\}. \quad (7)
\]

**Lemma 5.**

(i) \( G_{cyc}(y_0, z^*) = G_0(y_0) + \sum_{x = 0}^{\infty} H^*(y_0 - x) \Pr\{D_0^{(l_i)} = x\}, \ y_0 \in \mathbb{Z}, \)

(ii) \( G_{cyc}(y_0, z^*) \leq G_{cyc}(y_0, z), \ y_0 \in \mathbb{Z}, \)

(iii) \( \Delta G_{cyc}(y_0, z^*) = h_0 + \sum_{x = \tau(y_0)}^{\infty} \Delta H^*(y_0 - x) \Pr\{D_0^{(l_i)} = x\}, \ y_0 \in \mathbb{Z}, \)

(iv) \( G_{cyc}(y_0, z^*) \) is convex in \( y_0 \),

(v) \( G_{cyc}(y_0, z^*) \) is minimized at all \( y_0 \in Y_0^* = \{y_0^*, y_0^* + 1, ..., \overline{y}_0\} \) where \( y_0^* = \min\{y_0 | \Delta G_{cyc}(y_0, z^*) \geq 0\}, \) and \( \overline{y}_0^* = \min\{y_0 | \Delta G_{cyc}(y_0, z^*) > 0\}. \)

**Proof:** See the Appendix.

Part (ii) of Lemma 5 implies that whatever the ordering decision is made at the beginning of the cycle, utilizing \( z^* \) for allocation leads to expected cycle costs as good as any other set of allocation functions. The expressions for the optimal order-up-to levels minimizing \( G_{cyc}(y_0, z^*) \) are given in part (v); note that if \( \Delta G_{cyc}(y_0, z^*) = 0 \) then \( |Y_0^*| > 1 \).

**Corollary 1** Under the balance assumption, the minimum expected cycle cost of an arbitrary period \( t \in \mathbb{Z}_0^+ \) is \( G_{cyc}(\underline{y}_0^*, z^*). \)

### 3.4 Analysis of the Infinite Horizon Problem

In the previous section, we studied the cycle cost of an arbitrary period, which is shown to be convex under the balance assumption. Now, we return to the infinite horizon problem given in (1) and study it under the balance assumption.
Denote a base stock policy by a tuple \((y_0, z)\), where \(y_0\) is the target echelon inventory position of the warehouse, and \(\{z_i(x)\}_{i \in J}\) are the (state-dependent) target inventory positions of the retailers when the system-wide on-hand stock (state) is \(x\). The decisions are made so that, at the beginning of each period \(t\):

- the echelon inventory position of the warehouse is increased up to \(y_0\), i.e., \(IP_0(t) = y_0\),
- the inventory position of retailer \(i\) is raised to \(z_i(I_0(t))\), i.e., \(IP_i(t) = z_i(I_0(t))\) \(\forall i \in J\).

**Theorem 2** Under the balance assumption, the minimization of the average expected cost of the system in an infinite horizon (see (1)) can be accomplished by following a base stock policy \((y_0, z^*)\) with \(y_0 \in Y_0^*\).

**Proof:** Corollary 1 shows that base stock policy \((y_0 \in Y_0^*, z^*)\) minimizes the expected cycle cost of period \(t\). Due to the fact that warehouse order-up-to level \((y_0 \in Y_0^*)\), and optimal allocation functions \((z^*)\) are independent of time, the proposed control policy can be applied to optimize each period’s cycle cost within the horizon; as a result minimizing (1). □

**Remark 2:** There are two causes for an imbalance situation. One one hand, the retailers might face disproportionate demands in the previous period and the amount of stock at the warehouse (at the beginning of the current period) is not enough to preclude the allocation of a negative quantity to at least one retailer. On the other hand, imbalance may emanate from decreasing allocation functions. Recall the example in Remark 1. Take some period \(t \in \mathbb{Z}_0^+\). Assume that: (i) at the beginning of period \(t\), the amount of stock to allocate is \(4\), (ii) the amount of stock the warehouse will receive in period \(t + 1\) is \(1\), (iii) \(z^*(4) = (2, 1, 1)\), \(z^*(5) = (1, 2, 2)\). If no demand occurs at any of the retailers in period \(t\), then an imbalance occurs in period \(t + 1\) due to decreasing \(z_1^*(x)\) at \(x = 4\). This kind of imbalance can be prevented by using a base stock policy \((y_0 \in Y_0^*, \hat{z}^*)\) with nondecreasing allocation functions \(\hat{z}^*\) (i.e., \(\{\hat{z}^*_i(x), x \in \mathbb{Z}\}_{i \in J}\) such that \(\hat{z}^*_i(x)\) is optimal and nondecreasing for all \(i \in J\)).
3.5 Newsboy Inequalities

The optimality of base stock policies has been proved in §3.4. In this subsection, we identify necessary conditions for an optimal warehouse base stock level, which constitute newsboy inequalities.

Define
\[ P_i(y_0, z) = \text{probability of non-stockout at retailer } i \text{ in period } t + l_0 + l_i \text{ given } z, \text{ and } IP_0(t) = y_0 \text{ for } y_0 \in \mathbb{Z} \text{ and } t \in \mathbb{Z}_0^+ , \text{ i.e.,} \]
\[ P_i(y_0, z) = \sum_{x=0}^{\infty} F_i^{l_i+1}(z_i(y_0 - x)) \Pr\{D_0^{l_0} = x\}. \quad (8) \]

\[ \tilde{z}^* = \text{ set of nondecreasing optimal allocation functions (i.e., } \{\tilde{z}_i^*(x), \ x \in \mathbb{Z}\}_{i \in J} \text{ such that } \tilde{z}_i^*(x) \text{ is nondecreasing and optimal for all } i \in J \text{ with the additional property that for all } i \in J \text{ with } |Y_i^*| > 1, \tilde{z}_i^*(x) \in Y_i^* \setminus \{y_i^*\} \text{ for } x > \sum_{i \in J} Y_i^*. \]

The existence of nondecreasing allocation functions with the additional property can easily be verified. In fact, the optimal allocation functions constructed in the proof of Theorem 1 do have the additional property.

Now, we derive upper and lower bounds on \( \Delta G_{cyc}(y_0, \tilde{z}^*) \).

**Lemma 6** For all \( i \in J \), and \( y_0 \in \mathbb{Z} \):
\[
\left[ (h_0 + p_i) - (h_0 + h_i + p_i) F_i^{l_i+1}(y_i^*) \right] F_0^{l_0}(\tau(y_0) - 1) - p_i \\
+ (h_0 + h_i + p_i) P_i(y_0, \tilde{z}^*) - \sum_{x=\tau(y_0)}^{\infty} \Pr\{D_i^{l_i+1} = \tilde{z}_i^*(y_0 - x)\} \Pr\{D_0^{l_0} = x\} \leq \Delta G_{cyc}(y_0, \tilde{z}^*) \leq \\
\left[ (h_0 + p_i) - (h_0 + h_i + p_i) F_i^{l_i+1}(y_i^*) \right] F_0^{l_0}(\tau(y_0) - 1) - p_i + (h_0 + h_i + p_i) P_i(y_0, \tilde{z}^*). 
\]

**Proof:** See the Appendix.

Recall from Lemma 1 that if material availability is always guaranteed by the warehouse then the optimal order-up-to levels at the retailers satisfy newsboy inequalities. Utilizing the result of Lemma 6, similar newsboy inequalities can also be derived for an optimal warehouse order-up-to level.
Theorem 3 For each $y_0 \in Y_0^*$:

$$P_i(y_0, \tilde{z}^*) \geq \frac{p_i}{h_0 + h_i + p_i} + \left[ F_i^{(l_i+1)}(y^*) - \frac{h_0 + p_i}{h_0 + h_i + p_i} \right] F_0^{(l_0)}(\tau(y_0) - 1) \quad \forall i \in J, \tag{9}$$

$$P_i(y_0^* - 1, \tilde{z}^*) < \frac{p_i}{h_0 + h_i + p_i} + \left[ F_i^{(l_i+1)}(y^*) - \frac{h_0 + p_i}{h_0 + h_i + p_i} \right] F_0^{(l_0)}(\tau(y_0^*) - 2) + \sum_{x=\tau(y_0^*)-1}^{\infty} \Pr\{D_i^{(l_i+1)} = \tilde{z}^*_i(y_0^* - 1 - x)\} \Pr\{D_0^{(l_0)} = x\} \quad \forall i \in J,$$

Proof: These inequalities follow directly from the properties that $\Delta G_{cyc}(y_0^*, \tilde{z}^*) \geq 0$ and $\Delta G_{cyc}(y_0^*-1, \tilde{z}^*) < 0$, and the result of Lemma 6. □

Note that $F_0^{(l_0)}(\tau(y_0) - 1)$ in (9) corresponds to the probability that retailers can reach inventory positions $y_i^*$ via shipments from the warehouse (i.e., there is no shortage at the warehouse). $F_i^{(l_i+1)}(y^*) - \frac{h_0 + p_i}{h_0 + h_i + p_i}$ is the overshoot from the target newsboy level for retailer $i$ due to discreteness. In case of continuous demand, there is no overshoot; moreover, the newsboy inequalities for the retailers and (9) can be satisfied with equality. Thus, (9) can be streamlined as $P_i(y_0^*, \tilde{z}^*) = \frac{p_i}{h_0 + h_i + p_i}$ for all $i \in J$ (cf. Diks and De Kok [1998]).

Corollary 2 $P_i(y_0^*, \tilde{z}^*) \geq \frac{p_i}{h_0 + h_i + p_i} \quad \forall i \in J, \forall y_0 \in Y_0^*$.

Proof: Result follows directly from (9) and the definition of $y_i^*$. □

The newsboy inequalities derived in Theorem 3 allow us to see the following direct relations between the holding cost parameters and the order-up-to levels under an optimal policy.

Corollary 3 If there exists a retailer $i \in J$ with $h_i = 0$ and an infinite support for its demand distribution $F_i^{(1)}$, then the warehouse becomes a cross-docking point under an optimal policy.

Proof: From Lemma 1, $y_i^* = \infty$. Thus, in each period, all available stock at the warehouse is allocated to the retailers under an optimal policy. □

Lemma 7.

(i) If $h_0 = 0$ then the inventory position of retailer $i$ can always be increased to at least $y_i^*$ for all $i \in J$ under an optimal policy $(y_0, \tilde{z}^*)$ with $y_0 \in Y_0^*$. 15
(ii) If \( h_0 = 0 \) and there is at least one retailer \( i \in J \) with an infinite support for its demand distribution \( F_i^{(1)} \), then \( y_0^* = \infty \) under an optimal policy \((y_0, \tilde{z}^*)\), \( y_0 \in Y_0^* = \{\infty\} \). Thus, infinite stock is kept at the warehouse.

Proof: See the Appendix.

For \( N = 1 \), the model reduces to a two-echelon serial system facing discrete demand. The newsboy inequalities discussed in this subsection hold for this system as well. In fact, following the model and notation of Chen and Zheng [1994], these newsboy inequalities can also be derived from their cost formulas. Moreover, using their formulas, these newsboy inequalities are easily worked out for \( N \)-echelon serial systems (see Doğru, De Kok, and Van Houtum [2004]).

### 3.6 Computational Issues

The results of the previous subsections are used to develop an efficient optimization scheme. The general line is a reminiscent of the Clark and Scarf [1960] approach developed for serial systems. First, \( y_i^* \forall i \in J \) are determined utilizing part (iv) of Lemma 1. Second, following the arguments in the proof of Theorem 1 and using parts (ii) and (iii) of Lemma 3, \( \tilde{z}^* \) is constructed. Finally, a simple search procedure can be run to find \( y_0^* \); details are as follows. Take a retailer \( i \in J \), preferably one with \( |Y_i^*| > 1 \). Start the search at \( y_0 \) for which \( P_i(y_0, \tilde{z}^*) \geq \frac{p_i}{h_0 + h_i + p_i} \) for the first time. Unless \( \Delta G_{cyc}(y_0, \tilde{z}^*) \geq 0 \), increase \( y_0 \) by a suitable step size (depending on the distribution of demand at retailer \( i \)) until \( \Delta G_{cyc}(y_0, \tilde{z}^*) \geq 0 \). Initiate a bisection procedure and terminate it when \( y_0^* \) is determined. Once \( y_0^* \) and \( \tilde{z}^* \) are obtained, the values are substituted into (7) and the optimal long-run average cost of the system under the balance assumption is obtained.

### 4. APPENDIX

Proof of Lemma 2:

(i) Observe that (5) consists of \( N \) independent components that are convex functions. In the absence of (6), the problem is separable and the minimization of each component solves the problem; i.e.,
\( z_i^*(x) \in Y_i^* \) for all \( i \in J \). In case \( x \geq \sum_{i \in J} y_i^* \), \( z_i^*(x) = y_i^* \) \( \forall i \in J \) constitutes an optimal solution, but any other \( z_i^*(x) \in Y_i^* \) is also possible as long as \( \sum_{i \in J} z_i^*(x) \leq x \).

(ii) For \( x \leq \sum_{i \in J} y_i^* \), consider a solution \( \{z_i(x)\}_{i \in J} \) such that \( \sum_{i \in J} z_i(x) < x \). Since \( \sum_{i \in J} z_i(x) < \sum_{i \in J} y_i^* \), there exists a retailer \( j \) with \( z_j(x) < y_j^* \). Now, allocate one unit extra to retailer \( j \). Since \( \Delta G_i(y_i) < 0 \) for \( y_i < y_i^* \), \( \forall i \in J \), the objective function improves. Thus, it is suboptimal to allocate less than \( x \) units when \( x \leq \sum_{i \in J} y_i^* \), independent of how the rationing is carried out. This makes (6) binding.

Proof of Lemma 3: Note that (6) is binding since we assume \( x < \sum_{i \in J} y_i^* \) (see Lemma 2).

(i) Gross [1956] considered a slightly different resource allocation problem and derived the same necessary and sufficient conditions. His proof applies to our problem and goes as follows. Let us start by proving the necessity of the condition by contradiction. Suppose \( \Delta G_k(z_k^*(x)) < \Delta G_m(z_m^*(x) - 1) \) for \( k, m \in J \) and \( k \neq m \). The solution \( z_k(x) = z_k^*(x) + 1, z_m(x) = z_m^*(x) - 1, z_j(x) = z_j^*(x) \forall j \in J \setminus \{k, m\} \) leads to a lower value for (5), which contradicts to the optimality of \( \{z_i^*(x)\}_{i \in J} \). The sufficiency follows from the convexity of the function \( G_i(\cdot) \) for \( i \in J \). For the details, see Gross [1956] or Saaty [1970], pp. 184-186. See also Fox [1966] for a more general study.

(ii)-(iii) Substitute the proposed solutions in parts (ii) and (iii) in the necessary and sufficient optimality condition given in part (i), and verify them. Parts (ii) and (iii) constitute the greedy steps in an incremental (marginal) allocation algorithm, which was first proposed by Gross [1956]. See Ibaraki and Katoh [1988] for an extensive discussion on incremental analysis. □

Proof of Lemma 4:

(i) Take \( x < \sum_{i \in J} y_i^* \), \( x \in \mathbb{Z} \) and a corresponding optimal solution \( \{z_i^*(x)\}_{i \in J} \) for (5)-(6). An optimal solution of (5)-(6) for \( x+1 \) can be constructed using part (ii) of Lemma 3. This is a solution \( z_k^*(x + 1) = z_k^*(x) + 1 \), and \( z_i^*(x + 1) = z_i^*(x) \forall i \in J \setminus \{k\} \) with \( z_k^*(x) < y_k^* \). Thus, \( \Delta H^*(x) = H^*(x + 1) - H^*(x) = \Delta G_k(z_k^*(x)) < 0 \).
(ii) This is a direct result that follows from part (i) of Lemma 2, and part (iv) of Lemma 1.

(iii) Distinguish between three regions: $x < \sum_{i \in J} y_i^* - 1$, $x = \sum_{i \in J} y_i^* - 1$, and $x \geq \sum_{i \in J} y_i^*$. Firstly, consider $\{z_i^*(x)\}_{i \in J}$ for $x < \sum_{i \in J} y_i^* - 1$, and $x \in \mathbb{Z}$. By implementing part (ii) of Lemma 3, optimal solutions for $x + 1$ and $x + 2$ can be determined, which are, $z_k^*(x + 1) = z_k^*(x) + 1$ for some $k \in J$ and no change $\forall i \in J \setminus \{k\}$, and $z_m^*(x + 2) = z_m^*(x + 1) + 1$ for some $m \in J$ and no change $\forall i \in J \setminus \{m\}$. Note that $k = m$ is possible. We find

$$\Delta^2 H^*(x) = [H^*(x + 2) - H^*(x + 1)] - [H^*(x + 1) - H^*(x)]$$

$$= \Delta G_m(z_m^*(x + 1) - \Delta G_k(z_k^*(x)) \geq 0.$$ 

The last step follows from part (i) of Lemma 3 in case $k \neq m$, and from the convexity of $G_k(\cdot)$ in case $k = m$. Secondly, for $x = \sum_{i \in J} y_i^* - 1$, by parts (i) and (ii), $\Delta^2 H^* = -\Delta H^*(\sum_{i \in J} y_i^* - 1) > 0$. Thirdly, for $x \geq \sum_{i \in J} y_i^*$, $\Delta H^*(x) = 0$ resulting in $\Delta^2 H^*(x) = 0$. □

Proof of Lemma 5:

(i) Follows directly from (7) and the definition of $H^*(\cdot)$.

(ii) For all $y_0 \in \mathbb{Z}$:

$$G_{cyc}(y_0, z^*) = G_0(y_0) + \sum_{x=0}^{\infty} H^*(y_0 - x) \Pr\{D_0^{(l_0)} = x\} \leq$$

$$G_0(y_0) + \sum_{x=0}^{\infty} \sum_{i \in J} G_i(\xi(y_0 - x)) \Pr\{D_0^{(l_0)} = x\} = G_{cyc}(y_0, z).$$

(iii) For all $y_0 \in \mathbb{Z}$:

$$\Delta G_{cyc}(y_0, z^*) = \Delta G_0(y_0) + \sum_{x=0}^{\infty} \Delta H^*(y_0 - x) \Pr\{D_0^{(l_0)} = x\} =$$

$$h_0 + \sum_{x=0}^{\infty} \Delta H^*(y_0 - x) \Pr\{D_0^{(l_0)} = x\}. \quad (10)$$

This proves part (iii) for $y_0 < \sum_{i \in J} y_i^*$. If $y_0 \geq \sum_{i \in J} y_i^*$ then the lower limit of the summation in (10) can be reduced to $y_0 - \sum_{i \in J} y_i^* + 1$ utilizing part (ii) of Lemma 4.

(iv) Since $G_0(x)$ and $H^*(x)$ are convex functions, so is $G_{cyc}(y_0, z^*)$. 

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(v) Due to the fact that $G_{cyc}(y_0, z^*)$ is convex with respect to $y_0$, the first minimizing value $(y_0^*)$ is obtained at the point where $\Delta G_{cyc}(y_0, z^*) \geq 0$ for the first time. If $\Delta G_{cyc}(y_0, z^*) = 0$, then there are multiple optimal values. The maximum of such points is the one where $\Delta G_{cyc}(y_0, z^*)$ turns positive. \qed

\textbf{Proof of Lemma 6:} Let $y_0 \in \mathbb{Z}$. Let us continue from part (iii) of Lemma 5:

\[
\Delta G_{cyc}(y_0, z^*) = h_0 + \sum_{x=\tau(y_0)}^{\infty} \Delta H^*(y_0-x)\Pr\{D^{(l_0)}_0 = x\} = h_0 + \sum_{x=\tau(y_0)}^{\infty} \left[\sum_{j \in J} G_j(\bar{z}_j^*(y_0+1-x)) - \sum_{j \in J} G_j(\bar{z}_j^*(y_0-x))\right] \Pr\{D^{(l_0)}_0 = x\} = h_0 + \sum_{x=\tau(y_0)}^{\infty} \min_{j \in J} \{\Delta G_j(\bar{z}_j^*(y_0-x))\} \Pr\{D^{(l_0)}_0 = x\}. \tag{11}\]

Note that for $x \geq \tau(y_0)$, $x \in \mathbb{Z}$:

\[
\Delta G_i(\bar{z}_i^*(y_0-x) - 1) \leq \min_{j \in J} \{\Delta G_j(\bar{z}_j^*(y_0-x))\} \leq \Delta G_i(\bar{z}_i^*(y_0-x)) \quad \forall i \in J. \tag{12}\]

While the upper bound in (12) is obvious, lower bound follows from part (i) of Lemma 3. Substituting (12) into (11) leads to, for all $i \in J$:

\[
h_0 + \sum_{x=\tau(y_0)}^{\infty} \Delta G_i(\bar{z}_i^*(y_0-x)-1)\Pr\{D^{(l_0)}_0 = x\} \leq \Delta G_{cyc}(y_0, z^*) \leq h_0 + \sum_{x=\tau(y_0)}^{\infty} \Delta G_i(\bar{z}_i^*(y_0-x))\Pr\{D^{(l_0)}_0 = x\}.\]

The lower bound may be rewritten in terms of $P_i(y_0, z^*)$, which is defined in (8):

\[
h_0 + \sum_{x=\tau(y_0)}^{\infty} \Delta G_i(\bar{z}_i^*(y_0-x)-1)\Pr\{D^{(l_0)}_0 = x\} = h_0 + \sum_{x=\tau(y_0)}^{\infty} \left[(h_0 + h_i + p_i)F^{(l_0)}_{i_1}(\bar{z}_i^*(y_0-x)-1) - (h_0 + p_i)\right] \Pr\{D^{(l_0)}_0 = x\} = (h_0 + p_i)F^{(l_0)}_{i_1}(\tau(y_0)-1) - p_i + \sum_{x=\tau(y_0)}^{\infty} \left[(h_0 + h_i + p_i)F^{(l_0+1)}_{i_1}(\bar{z}_i^*(y_0-x)-1)\right] \Pr\{D^{(l_0)}_0 = x\} = (h_0 + p_i)F^{(l_0)}_{i_1}(\tau(y_0)-1) - p_i + (h_0 + h_i + p_i) \sum_{x=\tau(y_0)}^{\infty} \left(F^{(l_0+1)}_{i_1}(\bar{z}_i^*(y_0-x))-\Pr\{D^{(l_0+1)}_i = \bar{z}_i^*(y_0-x)\}\right) \Pr\{D^{(l_0)}_0 = x\} = (h_0 + p_i)F^{(l_0)}_{i_1}(\tau(y_0)-1) - p_i + (h_0 + h_i + p_i) \sum_{x=0}^{\tau(y_0)-1} \left[F^{(l_0+1)}_{i_1}(\bar{z}_i^*(y_0-x))\right] \Pr\{D^{(l_0)}_0 = x\} - \sum_{x=\tau(y_0)}^{\infty} \Pr\{D^{(l_0+1)}_i = \bar{z}_i^*(y_0-x)\} \Pr\{D^{(l_0)}_0 = x\}.\]
Bethink from Lemma 1 that $F_{i}^{(l_i+1)}(y_i) = F_{i}^{(l_i+1)}(y_i^*)$ for $y_i \in Y_i^* \setminus \{y_i^*\}$. Hence, the expression
\[ \sum_{x=0}^{\tau(y_0)-1} F_{i}^{(l_i+1)}(\tilde{z}_i^*(y_0 - x)) \Pr\{D_0^{(l_0)} = x\} \] reduces to $F_{i}^{(l_i+1)}(y_i^*)F_0^{(l_0)}(\tau(y_0) - 1)$ and rearranging the terms leads to:
\[
\begin{align*}
    h_0 + \sum_{x=\tau(y_0)}^{\infty} \Delta G_i(\tilde{z}_i^*(y_0 - x)) & \Pr\{D_0^{(l_0)} = x\} = \left[(h_0 + p_i) - (h_0 + h_i + p_i)F_{i}^{(l_i+1)}(y_i^*) - p_i \right] F_0^{(l_0)}(\tau(y_0) - 1) - p_i + (h_0 + h_i + p_i) \Pr\{D_0^{(l_0)} = x\}.
\end{align*}
\]

Similarly, the upper bound can be expressed in terms of $P_i(y_0, \tilde{z}^*)$:
\[
\begin{align*}
    h_0 + \sum_{x=\tau(y_0)}^{\infty} \Delta G_i(\tilde{z}_i^*(y_0 - x)) & \Pr\{D_0^{(l_0)} = x\} = \left[(h_0 + p_i) - (h_0 + h_i + p_i)F_{i}^{(l_i+1)}(y_i^*) - p_i \right] F_0^{(l_0)}(\tau(y_0) - 1) - p_i + (h_0 + h_i + p_i) P_i(y_0, \tilde{z}^*). \quad \Box
\end{align*}
\]

Proof of Lemma 7:

(i) Take retailer $i \in J$ such that $\tilde{z}_i^*(\sum_{i \in J} y_i^* - 1) = y_i^* - 1$. From (8) and (9), for each $y_0 \in Y_0^*$:
\[
\begin{align*}
    P_i(y_0, \tilde{z}^*) & = \sum_{x=0}^{\tau(y_0)-1} F_{i}^{(l_i+1)}(\tilde{z}_i^*(y_0 - x)) \Pr\{D_0^{(l_0)} = x\} + \sum_{x=\tau(y_0)}^{\infty} F_{i}^{(l_i+1)}(\tilde{z}_i^*(y_0 - x)) \Pr\{D_0^{(l_0)} = x\} \geq \\
    & \frac{p_i}{h_i + p_i} + \left[F_{i}^{(l_i+1)}(y_i^*) - \frac{p_i}{h_i + p_i}\right] F_0^{(l_0)}(\tau(y_0) - 1),
\end{align*}
\]
which can be rewritten as
\[
\begin{align*}
    \sum_{x=\tau(y_0)}^{\infty} F_{i}^{(l_i+1)}(\tilde{z}_i^*(y_0 - x)) \Pr\{D_0^{(l_0)} = x\} \geq \frac{p_i}{h_i + p_i} \left(1 - F_0^{(l_0)}(\tau(y_0) - 1)\right) \quad (13)
\end{align*}
\]
using the property that $F_{i}^{(l_i+1)}(\tilde{z}_i^*(x)) = F_{i}^{(l_i+1)}(y_i^*)$ for $x \geq \sum_{i \in J} y_i^*$, $x \in \mathbb{Z}$. Further, the inequality in (13) may be rewritten as
\[
\begin{align*}
    \sum_{x=\tau(y_0)}^{\infty} F_{i}^{(l_i+1)}(\tilde{z}_i^*(y_0 - x)) \Pr\{D_0^{(l_0)} = x\} \geq \sum_{x=\tau(y_0)}^{\infty} \frac{p_i}{h_i + p_i} \Pr\{D_0^{(l_0)} = x\}. \quad (14)
\end{align*}
\]
From $\tilde{z}_i^*(\sum_{i \in J} y_i^* - 1) = y_i^* - 1$ and Lemma 1, $F_{i}^{(l_i+1)}(\tilde{z}_i^*(y_0 - x)) < \frac{p_i}{h_i + p_i}$ for $x \geq \tau(y_0)$. Thus, the inequality in (14) can only be satisfied if $\Pr\{D_0^{(l_0)} \geq \tau(y_0)\} = 0$, i.e., $F_0^{(l_0)}(\tau(y_0) - 1) = 1$. This implies that $y_0 \in Y_0^*$ is greater than or equal to any possible realization of $D_0^{(l_0)}$ plus $\sum_{i \in J} y_i^*$.

(ii) An infinite support for $F_i^{(1)}$, $i \in J$ implies that there is also an infinite support for $F_0^{(1)}$. From part (i), $F_0^{(l_0)}(\tau(y_0) - 1) = 1$ for $y_0 \in Y_0^*$ can only be attained when $y_0^* = \infty$. \quad \Box
REFERENCES


