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Observer based 2-D filter design

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Abstract

In this paper it is shown that recursive local state estimators (related to Roesser models) can be constructed using global state observers (related to column to column propagation models). These observers can be designed based on a result (pole assignment) of algebraic 2-D systems theory.
Introduction

In this paper an observer based filter design algorithm is described for a 2-D system. The method is an application of algebraic 2-D systems theory (the 2-D system is considered a 1-D system over a ring. See [3], [5]).

Let a 2-D system be given by a Roesser model (see [8]). Such a model consists of the following set of partial difference equations.

\[
\begin{bmatrix}
R_{k+1,h} \\
S_{k,h+1}
\end{bmatrix}
= \begin{bmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{bmatrix}
\begin{bmatrix}
R_{kh} \\
S_{kh}
\end{bmatrix}
+ \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u_{kh} + \begin{bmatrix}
C_1 \\
C_2
\end{bmatrix} v_{kh},
\]

\[
y_{kh} = \begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
\begin{bmatrix}
R_{kh} \\
S_{kh}
\end{bmatrix} + D w_{kh}.
\]

Here the vectors \(R_{kh}\) and \(S_{kh}\) denote the local state variables. The input \(u_{kh}\) is assumed to be known and \(v_{kh}, w_{kh}\) denote independent Gaussian white noise processes.

The problem is to estimate \(R_{kh}\) and \(S_{kh}\) for all \(k\) and \(h\) given the output (and the input) \(y_{k_1,h_1}\) for \(k_1 \leq k\) and \(h_1 \leq h\). In [1] it is shown that a least squares estimate of \(R_{kh}\) and \(S_{kh}\) cannot be constructed, given the output and the input on the above index set, using a straightforward generalization of the 1-D concept of "observer". If this were possible one would have the 2-D analogue of the Kalman filter (optimal observer). Of course, one might use a Wiener filter here but this does not allow a recursive solution of the problem. Recursibility is very important here.
because of the huge amount of data involved in 2-D signal processing.

One might view the Roesser model as a 1-D (column to column propagation) model. An advantage would be its 1-D character but a disadvantage definitely is its high (infinite) dimensionality. This 1-D approach does not allow a recursive (in both directions) solution of the least squares estimation problem. The solution would be a very high order Kalman filter which is not very attractive. Even if one might be able to compute the gains of such a filter the implementation of this filter might give rise to serious problems because generally the recursibility in both directions would be lost. See [9]. Of course, one might be content with approximations and then one can use the fact that the operators describing the Kalman filter can be reasonably well approximated by Toeplitz operators (see [9]). These Toeplitz operators can be implemented using FFT algorithms.

In [1] one chooses a different approach. A suboptimal observer is constructed, using a straightforward generalization of the structure of 1-D observers. The selection of gains is based on some characteristics of the error covariance. Because of the structure chosen in [1] one has to solve a feedback stabilization problem for a 2-D system given by the Roesser model. However, up to now this stabilization problem has not been solved (2-D pole assignment analogue).

In the next we will describe a filter design method which preserves recursibility in both directions and on the other hand offers the possibility to obtain arbitrary dynamics for the error equation of the estimator. The parameters of the estimator (filter) may be adjusted using error covariance information. It will be shown that a large number of observers (estimators) can be parameterized by a relatively small number of gain factors.
Results

The Roesser model (1) can also be written as (column to column propagation)

\[
\begin{bmatrix}
R_{k+1,0} \\
R_{k+1,1} \\
R_{k+1,2} \\
\vdots
\end{bmatrix} =
\begin{bmatrix}
A_0 & 0 & 0 \\
A_1 & A_0 & 0 \\
A_2 & A_1 & A_0 \\
\vdots & \vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
R_{k,0} \\
R_{k,1} \\
R_{k,2} \\
\vdots
\end{bmatrix} +
\begin{bmatrix}
B_0 & 0 & 0 \\
B_1 & B_0 & 0 \\
B_2 & B_1 & B_0 \\
\vdots & \vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
u_{k,0} \\
u_{k,1} \\
u_{k,2} \\
\vdots
\end{bmatrix}
\]

(2)

\[
\begin{bmatrix}
y_{k,0} \\
y_{k,1} \\
y_{k,2} \\
\vdots
\end{bmatrix} =
\begin{bmatrix}
C_0 & 0 & 0 \\
C_1 & C_0 & 0 \\
C_2 & C_1 & C_0 \\
\vdots & \vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
R_{k,0} \\
R_{k,1} \\
R_{k,2} \\
\vdots
\end{bmatrix} +
\begin{bmatrix}
D & 0 & 0 \\
0 & D & 0 \\
0 & 0 & D \\
\vdots & \vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
w_{k,0} \\
w_{k,1} \\
w_{k,2} \\
\vdots
\end{bmatrix}
\]

This is an infinite dimensional 1-D system where the operators are all lower (block) Toeplitz operators (same matrices along the diagonals). We will suppose the initial conditions to be zero. This is just for convenience, it is not really a restriction. The matrices \(A_i, B_i, G_i, C_i, D\) are determined by the matrices \(A_1, A_2, A_3, A_4, B_1, B_2, G_1, G_2, C_1, C_2, D\) in the Roesser model (1). This kind of system description is closely
related to [6]. An equivalent way of describing the above infinite dimensional 1-D system is the following.

\[
R_{k+1}(s) = A(s)R_k(s) + B(s)u_k(s) + G(s)v_k(s),
\]
(3)

\[
y_k(s) = C(s)R_k(s) + Dw_k(s).
\]

Here

\[
R_k(s) = \sum_{h=0}^{\infty} R_{kh} s^{-h}
\]
\[
A(s) = \sum_{i=0}^{\infty} A_i s^{-i}
\]

and the other variables are defined analogously. The infinite summations are just formal power series. There is no convergence involved yet. See [3], [5]. We can now show how the parameters in (2) relate to the parameters in (1).

In fact we have

\[
A(s) = A_1 + A_2 \left[ sI - A_4 \right]^{-1} A_3
\]
\[
B(s) = B_1 + A_2 \left[ sI - A_4 \right]^{-1} B_2
\]
\[
G(s) = G_1 + A_2 \left[ sI - A_4 \right]^{-1} G_2
\]
\[
C(s) = C_1 + C_2 \left[ sI - A_4 \right]^{-1} A_3
\]

Therefore we have

\[
A_0 = A_1, A_1 = A_2 A_4^{i-1} A_3 \quad i = 1, 2, 3, ...
\]

and analogously for the other matrices involved. We will suppose \( A_4 \) to have eigenvalues strictly in the unit circle.
Now we have obtained a 1-D system over the ring of 1-D transfer functions (in the variable s) as an equivalent description of (1) and (2). It can be shown that (3) is a global state space model for the 2-D system given by the local state space model (1), see [3], [5]. The problem of state estimation is now: Find an estimate \( \hat{R}_k(s) \) for \( R_k(s) \) based on the inputs \( u_{k_1}(s) \) for \( k_1 \leq k \) and the outputs \( y_{k_1}(s) \) for \( k_1 \leq k \). Furthermore this has to be done in such a way that the estimator for \( R_{kh} \) only uses \( u_{k_1,h_1} \) and \( y_{k_1,h_1} \) for \( k_1 \leq h \) (and \( k_1 \leq k \)). (This is not crucial for recursibility because weakly causal systems could be used (see [4]). We have chosen to do so because our results can directly be interpreted in terms of Roesser models. To this end we may proceed as in the 1-D case where the analogue of this problem is solved by means of an observer.

The structure of an observer for (3), meeting all requirements described above, is completely analogous to the 1-D case. We have (formally)

\[
\hat{R}_{k+1}(s) = A(s)\hat{R}_k(s) + B(s)u_k(s) + K(s) [y_k(s) - C(s)\hat{R}_k(s)].
\]

Here \( \hat{R}_k(s) \) denotes the estimate for \( R_k(s) \). Observe that the input and the output of the original system are inputs to the observer. Let us define the error

\[
e_k(s) = \hat{R}_k(s) - R_k(s).
\]

Then we have

\[
e_{k+1}(s) = [A(s) - K(s)C(s)]e_k(s) + K(s)Dw_k(s) - G(s)v_k(s).
\]
Next we define the expectation operator $E$ for formal power series

$$E \{ e_k(s) \} = \sum_{h=0}^{\infty} E e_{kh} s^{-h}.$$ 

Thus we obtain

$$E \{ e_{k+1}(s) \} = [A(s) - K(s) C(s)] E(e_k(s)) .$$

We have

$$E \{ K(s) Dw_k(s) \} = 0 ,$$

$$E \{ G(s) v_k(s) \} = 0 ,$$

because $w_{kh}$ and $v_{kh}$ are white noise processes.

As in the case of 1-D systems (over the real numbers) we want the error to tend to zero (for $k \to \infty$). Thus we have to choose $K(s)$ such that (5) represents an asymptotically stable system. Therefore we should be able to choose $K(s)$ such that

$$[zI - A(s) + K(s) C(s)]^{-1}$$

is a stable 2-D transfer matrix.

The denominator of this transfer matrix is

$$\det[zI - A(s) + K(s) C(s)]$$

multiplied by some polynomial in $s$ due to the fact that $A(s)$, $C(s)$, $K(s)$ are (rational) transfer matrices themselves. Stability of (6) is obtained if $A(s)$, $C(s)$, $K(s)$ are stable 1-D transfer matrices and if
\[
\det[zI - A(s) + K(s) C(s)] \neq 0 \ ; \ |z| \geq 1, |s| \geq 1.
\]

It is possible to construct \( K(s) \) such that (6) is stable if the pair \((C(s),A(s))\) satisfies an observability condition. The right notion of observability is here \((A^T(s),C^T(s))\) is a reachable pair which means that

\[
[C^T(s), A^T(s) C^T(s), \ldots, (A^T(s)^{R-1} C^T(s)]
\]

has a right inverse over the ring \( R \) of stable proper transfer functions (in the variable \( s \)), see [5], [3].

(A rational function \( \frac{p(s)}{q(s)} \) is called stable if \( q(s) \neq 0, |s| \geq 1 \) and it is called proper if the degree of \( q(s) \) is not less than the degree of \( p(s) \)).

In fact we have the following theorem.

Theorem (pole assignment).

Let \( A(s) \) be an \( n \times n \)-matrix over \( R \) and let \( B(s) \) be an \( n \times m \)-matrix over \( R \) such that \((A(s),B(s))\) is a reachable pair. Let \( \lambda_1(s), \ldots, \lambda_n(s) \) be elements in \( R \). Then there exists an \( m \times n \)-matrix \( K(s) \) over \( R \) such that

\[
\det[zI - A(s) + B(s) K(s)] = (z - \lambda_1(s)) \ldots (z - \lambda_n(s)).
\]

Proof. The proof in [6] for the case of matrices over a polynomial ring also holds for matrices over a principal domain. Therefore the case considered above (\( R \) is a principal ideal domain) is proved.

This theorem shows that we can obtain arbitrary dynamics for the error equation (5).
In the theorem we may choose \( \lambda_1(s), \ldots, \lambda_n(s) \) to be \( s \)-independent. Thus we may choose \( \lambda_1, \ldots, \lambda_n \) such that \( |\lambda_i| < 1 \) for \( i = 1, \ldots, n \). This gives a separable model for the error. (\( \det[zI - A(s) + K(s)C(s)] \) is a polynomial in \( z \).) We will now describe the construction of \( K(s) \) for the case of a single output system \( y_{kh} \) is a scalar for all \( k, h \) in more detail.

Now we have a row vector \( C(s) \) and an \( n \times n \)-matrix \( A(s) \). The observability condition becomes:

\[
[C^T(s), A^T(s)C^T(s), \ldots, (A^T(s)^{n-1}C^T(s))] \text{ has a right inverse over } \mathbb{R}.
\]

This can also be stated as

\[
\det \begin{bmatrix}
C(s) \\
C(s)A(s) \\
\vdots \\
C(s)A(s)^{n-1}
\end{bmatrix} = \frac{p(s)}{q(s)}
\]

where \( \frac{p(s)}{q(s)} \) is invertible in \( \mathbb{R} \). This means that \( \deg p(s) = \deg q(s) \) and \( p(s) \neq 0 \) for \( |s| \geq 1 \). (\( q(s) \) is also a stable polynomial because \( C(s) \) and \( A(s) \) are supposed to be stable 1-D transfer matrices).

If we have observability in the above sense then we can construct a matrix \( T(s) \) such that

\[
C(s)T(s) = [0, \ldots, 0, 1]
\]

and

\[
T(s)^{-1}A(s)T(s) = \begin{bmatrix}
0 & \ldots & 0 & -\alpha_0(s) \\
1 & \ddots & & \\
& \ddots & \ddots & \\
& & 0 & \ddots \\
& & & \ddots & 0 \\
0 & \ldots & 0 & 1 & -\alpha_{n-1}(s)
\end{bmatrix}
\]
Here $\alpha_0(s) + \alpha_1(s)z + \ldots + \alpha_{n-1}(s)z^{n-1} + z^n$ is the characteristic polynomial of $A(s)$. The construction of $T(s)$ is completely analogous to the case of a 1-D system over the real numbers. See [2].

Next, let $\lambda_1(s), \ldots, \lambda_n(s)$ be the desired poles of the error equation (5). Define $\beta_0(s), \ldots, \beta_{n-1}(s)$ as follows:

$$\beta_0(s) + \beta_1(s)z + \ldots + \beta_{n-1}(s)z^{n-1} + z^n = (z - \lambda_1(s)) \ldots (z - \lambda_n(s)).$$

The construction of the gain matrix $K(s)$ proceeds as follows.

Define $F(s)$ as

$$F(s) = \begin{bmatrix} \beta_0(s) & -\alpha_0(s) \\ \vdots & \vdots \\ \beta_{n-1}(s) & -\alpha_{n-1}(s) \end{bmatrix}$$

and let $K(s)$ be defined by

$$K(s) = T(s)F(s).$$

Then we have

$$A(s) - K(s)C(s) = T(s) - \begin{bmatrix} \begin{bmatrix} 0 & -\alpha_0(s) \\ 1 & \vdots \end{bmatrix} & \begin{bmatrix} 0 & \ldots & \beta_0(s) & -\alpha_0(s) \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \beta_{n-1}(s) & -\alpha_{n-1}(s) \end{bmatrix} \\ \end{bmatrix}T(s)^{-1}.$$
Thus

\[ A(s) - K(s)C(s) = T(s) \begin{bmatrix} 0 & -s_0(s) \\ 1 & \vdots \\ \vdots & \vdots \\ 1 & -s_{n-1}(s) \end{bmatrix} T(s)^{-1}. \]

This shows that \( A(s) - K(s)C(s) \) has \( \lambda_1(s), \ldots, \lambda_n(s) \) as its eigenvalues. Observe that we do not only have pole assignability but even coefficient assignability (coefficients of the characteristic polynomial). In the multi-output case we generally do not have coefficient assignability.

It will be clear that this restricts the class of 2-D polynomials which can be obtained as denominator polynomial for the transfer matrix of the error equation.

Remark. If \((C(s), A(s))\) does not satisfy the observability condition above in the sense that \( \deg p(s) < \deg q(s) \) we can still construct an observer with arbitrary dynamics for the error equation. In this case \( K(s) \) is not proper any more. This does not prevent a recursive implementation of the observer because in this case weakly causal systems can be used. We will not go into this any further and we refer to [4] for the details.
The estimator (4) which has been built based on the matrix K(s), constructed above, can be implemented as a Roesser model again and then we have obtained an estimator for $R_{kh}$ in the model (1).

Because

$$S_{k+1} = A_3 R_{kh} + A_4 S_{kh} + B_2 u_{kh} + C_2 v_{kh},$$

we can also construct an estimator for $S_{kh}$.

$$\hat{S}_{k+1} = A_3 \hat{R}_{kh} + A_4 \hat{S}_{kh} + B_2 u_{kh} + L(y - C_1 \hat{R}_{kh} - C_2 \hat{S}_{kh}).$$

The dynamics of the associated error equation may be improved by choosing L such that $A_4 - LC_2$ has a "better" spectrum than $A_4$.

Combining the error equation (5) and the error equation associated with $(\hat{S}_{kh} - S_{kh})$ it is not difficult to prove that the resulting error equation for $(\hat{R}_{kh} - R_{kh})$ and $(\hat{S}_{kh} - S_{kh})$ is stable.

Of course one has to choose K(s), in A(s) - K(s), based on some criterion (which may be related to the covariance of $(\hat{R}_{kh} - R_{kh})$). Also for the selection of L, in $A_4 - LC_2$, one has to use some criterion.

One way to do this (suboptimally) is by parameterizing K(s) and then one might select a satisfactory one by means of an optimization technique. A special case is where K(s) is chosen a priori such that $\det[zI - A(s) + K(s)C(s)]$ is a polynomial independent of s. Then all such K(s) can be parameterized by the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A(s) - K(s)C(s)$. In this case the error equation (5) is a separable 2-D system. This means that a Roesser model for this system can be chosen such that the corresponding $A_3$-matrix or $A_2$-matrix is zero.
This property may be advantageous with respect to the computation of co-variance matrices for $\hat{R}_{kh} - R_{kh}$ as can be seen in [1] (the structure of the equation is considerably simplified). This is because the error (in this case) is a sum of independent random variables as can easily be seen.
Conclusion

A new method of construction of 2-D filters has been described. The filter is given as a local state estimator for a Roesser model. The actual construction of the filter is based on the design of an observer for the global state in a state space model (column to column propagation) which is equivalent to the (local state space) Roesser model.
References


