Impossibility results for the equational theory of timed CCS

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Impossibility Results for the Equational Theory of Timed CCS*

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Abstract. We study the equational theory of Timed CCS as proposed by Wang Yi in CONCUR'90. Common to Wang Yi's paper, we particularly focus on a class of linearly-ordered time domains exemplified by the positive real or rational numbers. We show that, even when the set of basic actions is a singleton, there are parallel Timed CCS processes that do not have any sequential equivalent and thus improve on the Gap Theorem for Timed CCS presented by Godskesen and Larsen in FSTTCS'92. Furthermore, we show that timed bisimilarity is not finitely based both for single-sorted and two-sorted presentations of Timed CCS. We further strengthen this result by showing that, unlike in some other process algebras, adding the untimed or the timed left-merge operator to the syntax and semantics of Timed CCS does not solve the axiomatizability problem.

1 Introduction

In [12], Wang Yi proposed Timed CCS (TCCS) as a possible timed extension of Milner's CCS [7]. (See [10] for another timed extension of CCS.) There, he gave syntax and operational semantics of the calculus as well as a number of equational laws, including a form of *expansion law* that allows one to resolve parallelism and transform parallel processes into a nondeterministic composition of sequential processes.

However, it turned out that the expansion law of [12] is not sound with respect to timed bisimilarity [5,13]. In [13], Wang Yi put forward an alternative correct version of the expansion law of [12]. However, the correction involved the introduction of the so-called *time variables* and a substantially more complicated calculus. A natural question was then whether there is a sound expansion theorem for the simple calculus of [12] and whether the calculus of [12] affords a finite complete (respectively, ω-complete) axiomatization.

The former question was answered negatively in [5] by showing that for all $n > 0$ there are expressions with $n + 1$ parallel components for which there are

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no bisimilar terms with n parallel components or less (Gap Theorem). In other words, parallel composition cannot in general be eliminated from TCCS terms.

The latter question has been addressed partially in [14] and [1], which present complete axiomatizations for the finite [14] and regular [1] fragments of TCCS, respectively. However, it has remained an open question whether the full calculus, including parallel composition, affords a finite (ω-)complete axiomatization or not.

The aim of this paper is to re-visit this question. We show that different presentations of TCCS cannot be finitely axiomatized modulo timed bisimilarity. We further strengthen this result by showing that, unlike in some other process algebras, adding the left-merge operator (timed or un-timed) to the theory of TCCS does not solve the axiomatizability problem. We also present an improved version of the gap theorem and show that even in the presence of a single action, parallel composition cannot be resolved in TCCS.

The rest of this paper is organized as follows. In Section 2, we present the basic definitions concerning TCCS, timed bisimilarity and equational logic. Section 3 is devoted to the single-sorted presentation of TCCS and shows that it cannot be finitely axiomatized modulo timed bisimilarity. In Section 4, we study the two-sorted presentation of TCCS and show, first of all, that, even in the presence of just one action, parallelism cannot be resolved in TCCS (Gap Theorem). We also prove that the theory of the two-sorted presentation of TCCS cannot be finitely axiomatized either. Section 5 studies the addition of the untimed as well as the timed lef-merge operator to TCCS and shows that adding neither of these operators solves the axiomatizability problem. Section 6 concludes the paper and presents directions of ongoing and future research.

2 Preliminaries

2.1 TCCS: Syntax and Semantics

Following [6], we define a monoid $(X, +, 0)$ to be:

- *left-cancellative* iff $(x + y = x + z) \Rightarrow (y = z)$, and
- *anti-symmetric* iff $(x + y = 0) \Rightarrow (x = y = 0)$.

We define a partial order on X as $x \leq y$ iff $x + z = y$ for some $z \in X$. A *time domain* is a left-cancellative anti-symmetric monoid $(D, +, 0)$ such that $\leq$ is a total order. If $d_0 \leq d_1$, then we write $d_1 - d_0$ for the unique d such that $d_1 = d_0 + d$. A time domain is *non-trivial* if D contains at least two elements. Note that every non-trivial time domain does not have a largest element. A time domain has 0 as *cluster point* iff for each $d \in D$ such that $d \neq 0$ there is a $d' \in D$ such that $0 < d' < d$. In the remainder of the paper, we assume that our time domain, denoted henceforth by D, is non-trivial and has 0 as a cluster point.

The syntax of TCCS *processes* (closed terms) is given by the following grammar.

$$t ::= 0 \mid \mu.t \mid \epsilon(d).t \mid s + t \mid s \,||\, t$$

In the grammar given above, 0 stands for the deadlocking process (not to be confused with 0 in the time domain), $\mu.\_$ represents action-prefix operators for $\mu \in A$ where A is the set of (delayable) actions. Given a time domain D, $\epsilon(d).\_$ is an operator for each $d \in D$, and represents a time delay of length at least d before proceeding with the remaining process. For the sake of simplicity, we assume that all delays are non-zero; $\epsilon(0).t$ can be interpreted as a syntactic sugar for t, but we avoid zero delays altogether throughout the rest of this paper. Nondeterministic choice is denoted by $+$ and parallel composition is denoted by $||$.

We write μ for $\mu.0$ and $\mu(d)$ for $\epsilon(d).\mu$. Our proofs, in the remainder of this paper, remain sound even when the set A of actions is a singleton $\{a\}$. We write a^n to stand for 0 if $n = 0$, and $a.a^{n-1}$, otherwise.

TCCS (open) *terms* are constructed inductively using the operators of the syntax and a countably infinite set of variables V, with typical members x, x_0, y, y_0, ... The *size* of a term is its length in symbols. The set of variables appearing in term t is denoted by $vars(t)$. A substitution σ is a function from variables to TCCS terms. The range of a *closing substitution* is the set of TCCS processes. The domain of a substitution is lifted naturally from variables to terms.

We take two different approaches to formalizing the syntax of TCCS in a term algebra.

1. The first approach is to use a single-sorted algebra with the only available sort representing processes. Then, both $a.\_$ and $\epsilon(d).\_$ are sets of unary operators for each $a \in A$ and $d \in D$.
2. The other approach is to take two different sorts, one for time and one for processes, denoted by $\mathbb{T}$ and $\mathbb{P}$, respectively. Then, $\epsilon(\_)$ is a single function symbol with arity $\mathbb{T} \times \mathbb{P} \to \mathbb{P}$. When using this approach, we use d, d', d_0, ... as variables of sort $\mathbb{T}$ and closing substitutions map variables of sort $\mathbb{T}$ to elements of the time domain D.

The Plotkin-style rules defining the operational semantics of TCCS are given below.

$$\frac{}{0 \stackrel{\epsilon(d)}{\to} 0} \qquad \frac{}{\mu.x \stackrel{\mu}{\to} x} \qquad \frac{}{\mu.x \stackrel{\epsilon(d)}{\to} \mu.x}$$

$$\frac{}{\epsilon(d).x \stackrel{\epsilon(d)}{\to} x} \qquad \frac{}{\epsilon(d+e).x \stackrel{\epsilon(d)}{\to} \epsilon(e).x} \qquad \frac{x \stackrel{\epsilon(e)}{\to} y}{\epsilon(d).x \stackrel{\epsilon(d+e)}{\to} y}$$

$$\frac{x_0 \stackrel{\mu}{\to} y}{x_0 + x_1 \stackrel{\mu}{\to} y} \qquad \frac{x_1 \stackrel{\mu}{\to} y}{x_0 + x_1 \stackrel{\mu}{\to} y} \qquad \frac{x_0 \stackrel{\epsilon(d)}{\to} y_0 \quad x_1 \stackrel{\epsilon(d)}{\to} y_1}{x_0 + x_1 \stackrel{\epsilon(d)}{\to} y_0 + y_1}$$

$$\frac{x_0 \stackrel{\mu}{\to} y_0}{x_0 \,||\, x_1 \stackrel{\mu}{\to} y_0 \,||\, x_1} \qquad \frac{x_1 \stackrel{\mu}{\to} y_1}{x_0 \,||\, x_1 \stackrel{\mu}{\to} x_0 \,||\, y_1} \qquad \frac{x_0 \stackrel{\epsilon(d)}{\to} y_0 \quad x_1 \stackrel{\epsilon(d)}{\to} y_1}{x_0 \,||\, x_1 \stackrel{\epsilon(d)}{\to} y_0 \,||\, y_1}$$

The rules above, define two types of transition relations: $\stackrel{a}{\to}$, where $a \in A$, for action transitions and $\stackrel{\epsilon(d)}{\to}$, where $d \in D$, for time-delay transitions. We use $\stackrel{a^n}{\to}$ to denote n consecutive a-transitions (whose intermediate processes are irrelevant). The following lemma lists some interesting properties of the semantics of TCCS.

Lemma 1. *The following statements hold for each process p and time delay d.*

1. *There exists a unique process p_d such that $p \stackrel{\epsilon(d)}{\rightarrow} p_d$.*
2. *If p does not contain parallel composition and $p \stackrel{a}{\rightarrow} p'$, then $p_d \stackrel{a}{\rightarrow} p'$, where p_d is defined above.*

The notion of equivalence over TCCS that we are interested in is the following notion of timed bisimilarity.

Definition 2. *A symmetric relation R on TCCS processes is a timed bisimulation relation when for all $(p, q) \in R$,*

1. *for all actions a and processes p', if $p \stackrel{a}{\rightarrow} p'$ then there exists a process q' such that $q \stackrel{a}{\rightarrow} q'$ and $(p', q') \in R$;*
2. *for all time delays d and processes p', if $p \stackrel{\epsilon(d)}{\rightarrow} p'$ then there exists a process q' such that $q \stackrel{\epsilon(d)}{\rightarrow} q'$ and $(p', q') \in R$.*

Two processes p and q are timed bisimilar, or just bisimilar, denoted by $p \,\underline{\leftrightarrow}\, q$ when there exists a timed bisimulation relation R such that $(p, q) \in R$.

The notion of bisimilarity generalizes naturally to open terms: s and t are bisimilar, when $\sigma(s) \,\underline{\leftrightarrow}\, \sigma(t)$ for each closing substitution σ.

It is well-known that timed bisimilarity is a congruence over TCCS [12].

We define the following notion of time insensitive processes and show that a TCCS process is time insensitive if and only it is bisimilar to a CCS process.

Definition 3. *A TCCS process p is* initially time insensitive *when for all d, if $p \stackrel{\epsilon(d)}{\rightarrow} p_d$, then $p_d \,\underline{\leftrightarrow}\, p$.*

The set of time insensitive *processes is the largest set P_{TI} of TCCS processes such that, whenever $p \in P_{TI}$, (i) p is initially time-insensitive, and (ii) if $p \stackrel{a}{\rightarrow} p_a$ then $p_a \in P_{TI}$ is time insensitive for each $a \in A$ and each process p_a.*

For example, $a(d)$ is not (initially) time insensitive, $a.a(d)$ is initially time insensitive but not time insensitive, and $a.\epsilon(d).0$ is (initially) time insensitive.

Theorem 4. *A TCCS process p is time insensitive if and only if there exists a process q such that $p \,\underline{\leftrightarrow}\, q$ and q does not contain time-delay prefixing operators, i.e., q is a CCS process.*

2.2 Equational Theory

Given a signature Σ, a set E of equations $t = t'$, where t and t' are terms (of the same sort), is called an *axiom system*.

We write $E \vdash t = t'$ when $t = t'$ is derivable from E by the following set of deduction rules. Deduction rule is a rule schema for each operator f in the signature.

$$\frac{}{E \vdash t = t} \quad \frac{E \vdash t = t'}{E \vdash t' = t} \quad \frac{E \vdash t_0 = t_1 \quad E \vdash t_1 = t_2}{E \vdash t_0 = t_2}$$

$$\frac{E \vdash t_0 = t'_0 \quad \ldots \quad E \vdash t_n = t'_n}{E \vdash f(t_0, \ldots, t_n) = f(t'_0, \ldots, t'_n)} \quad \frac{t = t' \in E}{E \vdash \sigma(t) = \sigma(t')}$$

Without loss of generality, we assume that E is closed under symmetry, i.e., $t = t' \in E$ if and only if $t' = t \in E$, so that need not be considered in proofs. It is well-known that if an equation relating two closed terms can be proven from an axiom system E, then there is a closed proof for it.

An equation $t = t'$ is *sound* (modulo timed bisimilarity) if the terms t and t' are timed bisimilar. An axiom system is sound if each of its equations is sound. An example of a collection of equations from [12] that are sound with respect to timed bisimilarity is given below. The axioms A4, M1 and D1 (used from left to right) are enough to establish that each TCCS term that is bisimilar to 0 is also provably equal to 0. Thus, in the technical developments from Section 4 onwards, we shall assume, without loss of generality, that each axiom system we consider includes the equations given below. This assumption means, in particular, that our axiom systems allows us to identify each term that is bisimilar to 0 with 0.

A1	$x + y = y + x$	A2	$(x + y) + z = x + (y + z)$
A3	$x + x = x$	A4	$x + 0 = x$
M1	$0 \,\|\|\, x = 0$	M2	$x \,\|\|\, 0 = x$
D1	$\epsilon(d).0 = 0$	D2	$\epsilon(d).(x + y) = \epsilon(d).x + \epsilon(d).y$
D3	$\epsilon(d).(x \,\|\|\, y) = \epsilon(d).x \,\|\|\, \epsilon(d).y$	D4	$\epsilon(d).\epsilon(d').x = \epsilon(d + d').x$
P	$a.x = a.x + \epsilon(d).a.x$		

Henceforth, process terms are considered modulo associativity and commutativity of $+$ and $||$. We use a *summation* and *a product*, denoted by $\sum_{i \in \{1,\ldots,k\}} s_i$ and $\prod_{j \in \{1,\ldots,k'\}} t_j$, to stand for $s_1 + \cdots + s_k$ and $t_1 \,||\, \cdots \,||\, t_{k'}$, respectively, where the empty sum and product represent 0. We say that a term t has a 0 *factor* if it contains a subterm of the form $\prod_{j \in \{1,\ldots,k'\}} t_j$, where some t_j is bisimilar to 0. It is easy to see that, modulo the equations given above, every TCCS term s can be written as $\sum_{i \in I} s_i$, for some finite index set I, and terms s_i $(i \in I)$ that are not 0 and do not have themselves the form $s' + s''$, for some terms s' and s''. The terms s_i $(i \in I)$ will be referred to as the *summands* of t. Again modulo the equations given above, each s_i can be assumed to have no 0 factors.

3 Single-Sorted TCCS

In this section, as a warm up for the more complex results to follow, we show that single-sorted TCCS has no finite basis provided that the time domain is infinite. (Note that each time domain D that we consider in this paper does not have a largest element and is therefore infinite.)

Theorem 5. *If time domain D is infinite, then bisimilarity over single-sorted TCCS has no finite basis.*

We start with proving the following lemma which implies the above theorem.

Lemma 6. *Assume that E is a finite axiom system that is sound modulo bisimilarity. Let d be greater than the maximal delay prefixing mentioned in terms in E. For all provable equations $t = u$ such that either t or u contain $\epsilon(d)$, then both t and u contain $\epsilon(d)$.*

Proof. To prove Lemma 6, we proceed by an induction on the derivation structure for $E \vdash t = u$ and make a case distinction based on the last deduction rule applied to derive $E \vdash t = u$. The cases for , and are either trivial or follow immediately from the induction hypothesis. The most involved case is when the last deduction rule is .

For a TCCS process p, we define the action depth of p, denoted by $adepth(p)$, as the length of the maximal action trace that p affords (by omitting the time-delay transitions in between). It then follows that, for any two TCCS terms s and t, if $s \underline{\leftrightarrow} t$ then $adepth(\sigma(s)) = adepth(\sigma(t))$ for all closing substitutions σ.

Lemma 7. *Let t, u be bisimilar TCCS terms. Then $vars(t) = vars(u)$.*

Proof. Assume $x \in vars(t) \setminus vars(u)$. Construct a substitution σ that maps x to a^n, for some n larger than the sizes of both t and u, and all other variables to 0. Then, $adepth(\sigma(t)) \geq n > adepth(\sigma(u))$ and hence t and u are not bisimilar.

We are now ready to complete the proof of Lemma 6. Assume that $t' = u' \in E$, $t = \sigma(t')$, $u = \sigma(u')$ and $\epsilon(d)$ occurs in t. Since d is greater than the largest constant appearing in E, neither t' nor u' contain occurrences of $\epsilon(d)$. Thus, there exists a variable $x \in vars(t')$ such that $\sigma(x)$ has an occurrence of $\epsilon(d)$. By Lemma 7 and the soundness of the equation $t' = u'$ modulo bisimilarity, $x \in vars(u')$. Thus, $\sigma(u')$ also contains $\sigma(x)$ as a subterm, which in turn has an occurrence of $\epsilon(d)$. ⊠

Proof of Theorem 5. Assume that single-sorted TCCS affords a finite complete axiomatization E and d is greater than the largest delay appearing in E. (If no element of D appears in terms in E then let d be an arbitrary element of D.) Axiom D1 is sound. However, it follows from Lemma 6 that the instance of D1 for $d \in D$ is not derivable from E and thus Theorem 5 follows. ⊠

The lesson to be drawn from the above result is that, in the presence of an infinite time domain, when studying the equational theory of TCCS, it is much more natural to consider a two-sorted presentation of the calculus. For this reason, the rest of this paper is devoted to the study of the equational theory of two-sorted TCCS.

4 Two-Sorted TCCS

4.1 Gap Theorem

In this section, we present and prove the so-called *gap theorem* for TCCS, originally offered in [5], which shows that parallel composition cannot be eliminated in general from TCCS terms. Our presentation and the proof of this theorem improves on that of [5] in two ways; first, our version of the gap theorem holds even in the presence of a single action while the gap theorem of [5] requires the presence of countably many different actions. Secondly, our proof is purely process algebraic in nature while the proof of [5] goes through a translation of TCCS to timed automata [2] and the argument is based on the number of clocks in the translated timed automata.

Theorem 8. *Define* p *as* $a(d_0) \,||\, \prod_{i \in \{1,\ldots,n\}} a.a(d_i)$*, for some action* $a \in A$*, positive integer* n *and delays* $d_0, d_1, \ldots, d_n \in D$*. There exists no* q *such that* $p \underline{\leftrightarrow} q$ *and* $q = \sum_{j \in J} \prod_{i \in \{1,\ldots,n_j\}} q_{ij}$ *where* $n_j \leq n$ *and* q_{ij} *does not contain parallel composition.*

Informally, the above theorem states that for all $n > 0$, there are TCCS processes with $n+1$ parallel components which do not have any bisimilar counterpart with (summands comprising) n or fewer parallel components. *Proof.* Assume, towards a contradiction, that $p \underline{\leftrightarrow} q$ and $q \equiv \sum_{j \in J} \prod_{i \in \{1,\ldots,n_j\}} q_{ij}$. By the definition of p, we have that $p \stackrel{a^n}{\to} p' \equiv \prod_{i \in \{0,\ldots,n\}} a(d_i)$. Hence there should exist a $j \in J$ such that $q_j \equiv \prod_{i \in \{1,\ldots,n_j\}} q_{ij} \stackrel{a^n}{\to} q'$ for some q' such that $q' \underline{\leftrightarrow} \prod_{i \in \{0,\ldots,n\}} a(d_i)$. Then, either all parallel components of q_j contribute exactly one action to the trace a^n or there exists a component in q_j that contributes more than one action to a^n. Next, we analyze these two possibilities and show that both lead to a contradiction.

1. Assume that all parallel components of q_j contribute exactly one action to the trace a^n, i.e., $n_j = n$, $q' = \prod_{i \in \{1,\ldots n\}} q'_{ij}$, for some q'_{ij} such that for all $i \leq n$, $q_{ij} \stackrel{a}{\to} q'_{ij}$, and $\prod_{i \in \{0,\ldots n\}} a(d_i) \underline{\leftrightarrow} \prod_{i \in \{1,\ldots,n\}} q'_{ij}$.
   Since D has 0 as a cluster point, there is a $d' \in D$ such that $0 < d' < d_0$. It follows from Lemma 1.(2) that $q_{ij} \stackrel{\epsilon(d')}{\to} q''_{ij} \stackrel{a}{\to} q'_{ij}$; thus, $q \stackrel{\epsilon(d')}{\to} q'' \stackrel{a^n}{\to} q'$. Furthermore, $p \stackrel{\epsilon(d')}{\to} p'' \equiv a(d_0 - d') \,||\, \prod_{i \in \{1,\ldots n\}} a.a(d_i)$ and it should hold that $p'' \underline{\leftrightarrow} q''$. However, $p'' \stackrel{a^n}{\to} a(d_0 - d') \,||\, \prod_{i \in \{1,\ldots n\}} a(d_i)$ (as $d' < d_0$, this is the only a^n-derivative of p''), which is clearly not bisimilar to $\prod_{i \in \{0,\ldots n\}} a(d_i)$, and hence, not bisimilar to q'.
2. Assume that there is a component in q_j that contributes more than one action to a^n, i.e., there exists an $l \in \{1,\ldots,n_j\}$ such that $q_{lj} \stackrel{a^{k-2}}{\to} q_{a^{k-2}lj} \stackrel{a}{\to} q_{a^{k-1}lj} \stackrel{a}{\to} q_{a^k lj}$ for some $k > 1$ and for some $q_{a^{k-2}lj}$, $q_{a^{k-1}lj}$ and $q_{a^k lj}$. For notational convenience, we assume that $k = 2$ but the proof technique can easily be adapted for $k > 2$. Note that by Lemma 1.(2) $q_{lj} \stackrel{a}{\to} q_{alj} \stackrel{\epsilon(d')}{\to} q_{ad'lj} \stackrel{a}{\to} q_{ad'alj} \equiv q_{a^2lj}$ for an arbitrary d'.
   It follows from the semantics of parallel composition and nondeterministic choice that $q \stackrel{a^{n-1}}{\to} q_{a^{n-1}} \equiv q_{alj} \,||\, \prod_{i \in \{1,\ldots,n_j\} \setminus \{l\}} q'_{ij}$ and $q_{a^{n-1}} \underline{\leftrightarrow}\, a.a(d_m) \,||\, \prod_{i \in \{0,\ldots,n\} \setminus \{m\}} a(d_i)$ for some $0 < m \leq n$. From the above bisimilarity, we have that, for any d' such that $d' > d_i$, for each $i \leq n$,

$q_{a^{n-1}} \stackrel{\epsilon(d')}{\rightarrow} q_{a^{n-1}d'}$ for some $q_{a^{n-1}d'} \equiv q_{ad'lj} \,||\, \prod_{i\in\{1,\ldots,n_j\}\setminus\{l\}} q'_{d'ij}$ such that $q_{a^{n-1}d'} \leftrightarrow a.a(d_m) \,||\, \prod_{i\in\{0,\ldots,n-1\}} a$. Furthermore, $q_{a^{n-1}d'}$ can make one further a-transition, due to $q_{ad'lj}$ resulting in some $q_{a^n d'}$ such that $q_{a^n d'} \leftrightarrow a(d_m) \,||\, \prod_{i\in\{0,\ldots,n-1\}} a$ or $q_{a^n d'} \leftrightarrow a.a(d_m) \,||\, \prod_{i\in\{0,\ldots,n-2\}} a$. It follows from the aforementioned bisimilarities that $q_{a^n d'} \stackrel{a^n}{\rightarrow} q' \leftrightarrow a(d_m)$.
Either all a-transitions in the latter a^n-trace are due to $q_{d'ij}$ with $i \neq l$, or some of them are performed by $q_{ad'alj} \equiv q_{a^2lj}$.
In the former case, then $q_{a^{n-2}} \equiv q_{lj} \,||\, \prod_{i\in\{1,\ldots,n_j\}\setminus\{l\}} q'_{ij} \stackrel{\epsilon(d')}{\rightarrow} \stackrel{a^{n+2}}{\rightarrow} \overline{q}$ for some $\overline{q} \leftrightarrow a(d_m)$ since, first, $\prod_{i\in\{1,\ldots,n_j\}\setminus\{l\}} q'_{d'ij}$ can make n consecutive a-transitions, and $q_{d'lj}$ can make two a-transition afterwards. However, p cannot mimic this behavior, i.e., $\stackrel{a^{n-1}}{\rightarrow} \stackrel{\epsilon(d')}{\rightarrow} \stackrel{a^{n+2}}{\rightarrow}$, for it has only $n+1$ a-transitions enabled after an initial a^{n-2} trace and a time delay of d', which results in some process bisimilar to $a(d_m) \,||\, a(d'_m)$, for some $m' \neq m \in \{1, \ldots n\}$.
In the latter case, i.e., $q_{ad'alj} \equiv q_{a^2lj}$ contributes to some of the a-transitions, in a^n, say some u a-transitions such that $u > 0$, then $q_{lj} \stackrel{a^2}{\rightarrow} q_{a^2lj} \stackrel{a^u}{\rightarrow} q'$ for some q' and hence $q \stackrel{a^{n+u}}{\rightarrow} q''$ for some q''. But p can initially make at most n consecutive a-transitions, hence a contradiction follows. ⊠

As a corollary to the above theorem, one can conclude that TCCS affords no expansion theorem, i.e., parallel composition cannot be resolved in TCCS.

Corollary 9. *Two-sorted dense-time TCCS has no expansion theorem.*

4.2 Axiomatizability

Our next milestone in this section is to prove a theorem witnessing that TCCS, does not have a finite basis modulo timed bisimilarity. The problem underscored by the proof of this result is the inability of any finite and sound axiom system E to "expand" the initial behavior of terms of the form $p \,||\, q$ when either p or q have sufficiently many summands (namely, larger than the size of terms in the equations in E). All of the impossibility results presented henceforth also hold for conditional equations of the form $P \Rightarrow t = u$ where P is an arbitrary predicate over the time domain.

Theorem 10. *Timed bisimilarity over two-sorted dense-time TCCS has no finite basis.*

The above result dates back to [11] (for the case of CCS without time) and our proof follows the same structure as that of [11]; namely, we prove that for each finite and sound set of axioms E for TCCS modulo bisimilarity and for sufficiently large n, with respect to the size of the largest term appearing in E, the following sound equation is not provable

$$E \vdash a \,||\, \Phi_n = a.\Phi_n + \sum_{i\in\{1,\ldots,n\}} a.(a \,||\, \phi_i) \ ,$$

where $\Phi_n = \sum_{i\in\{1,\ldots,n\}} a.\phi_i$ and $\phi_n = \sum_{i\in\{1,\ldots,n\}} a^i$.

As we show in the next section, unlike in the setting of Milner's CCS, even adding two variations on the left-merge operator does not improve the situation with respect to axiomatizability.

5 Two-Sorted TCCS with Left-Merge

Classical Left-Merge. Bergstra and Klop suggested to add an auxiliary left-merge operator, denoted by $\lfloor\!\lfloor$, which would allow for a finite axiomatization of parallel composition in CCS. The semantics of the left-merge operator is captured by the following deduction rule.

$$\frac{x_0 \xrightarrow{a} y_0}{x_0 \lfloor\!\lfloor x_1 \xrightarrow{a} y_0 \,||\, x_1}$$

However, adding the left-merge operator with the above semantics does not result in a finitely axiomatizable theory.

Theorem 11. *Timed bisimilarity over two-sorted dense-time TCCS extended with the untimed left-merge operator has no finite basis.*

Timed Left-Merge. Following the tradition of Bergstra and Klop, the left-merge operator was given a timed semantics as follows [4].

$$\frac{x_0 \xrightarrow{a} y_0}{x_0 \lfloor\!\lfloor x_1 \xrightarrow{a} y_0 \,||\, x_1} \qquad \frac{x_0 \xrightarrow{\epsilon(d)} y_0 \quad x_1 \xrightarrow{\epsilon(d)} y_1}{x_0 \lfloor\!\lfloor x_1 \xrightarrow{\epsilon(d)} y_0 \lfloor\!\lfloor y_1}$$

This operator enjoys most of the axioms for the classic left-merge operator that lead to a finite axiomatization of bisimilarity [3]. The following lemma lists the most important properties that this timed left-merge operator possesses. Note that Lemma 1 also remains valid over TCCS extended with the above left-merge operator.

Lemma 12. *For the left-merge operator with the semantics given above, the following axioms are sound:*

$$\begin{array}{ll} x \,||\, y = (x \lfloor\!\lfloor y) + (y \lfloor\!\lfloor x) & \quad 0 \lfloor\!\lfloor x = 0 \\ (x + y) \lfloor\!\lfloor z = (x \lfloor\!\lfloor z) + (y \lfloor\!\lfloor z) & \quad x \lfloor\!\lfloor 0 = x. \end{array}$$

Thanks to axioms and , one can show that terms bisimilar to 0 can be removed as arguments of the left-merge. Henceforth, when we write p does not contain 0-factors, we mean that it does not contain a parallel composition *or a left-merge* with an argument bisimilar to 0.

However, the new left-merge operator does not help in giving TCCS a finite basis either, as we prove in the remainder of this section. The reason is that the axiom $(a.x) \lfloor\!\lfloor y = a.(x \,||\, y)$, which is a sound axiom in the untimed setting, is in general unsound over TCCS. For example, consider the process $a \lfloor\!\lfloor \epsilon(d).a$; after making a time delay of length d, it results in $a \lfloor\!\lfloor a$, which is capable of performing two consecutive a-transitions. However, $a.(0 \,||\, \epsilon(d).a)$ after a time delay of length d remains the same process and can only perform one a-transition since the second a-transition still has to wait for some time, i.e., d, before becoming enabled.

However, axiom is sound for the class of TCCS processes that are initially time insensitive; see Definition 3. Indeed if q is a process such that $q \stackrel{\epsilon(d)}{\to} q_d$ implies $q \underline{\leftrightarrow} q_d$ for each delay d, then it holds that $(a.p) \lfloor\!\lfloor q \underline{\leftrightarrow} a.(p \,||\, q)$. For instance, $a \lfloor\!\lfloor \Phi_n \underline{\leftrightarrow} a.\Phi_n$ for each $n \geq 0$, where the process Φ_n is defined as in Section 4.2. Unfortunately, the class of initially time insensitive processes cannot be characterized by a finite (head-)normal form and this constitutes the key idea in our non-finite axiomatizability proof, given below.

Theorem 13. *Two-sorted TCCS extended with the timed left-merge operator affords no finite axiomatization modulo timed bisimilarity.*

Proof. Towards a contradiction, we assume that TCCS with left-merge does have a finite axiomatization E. We prove the theorem by showing that the following lemma holds. (In the remainder of this proof, we assume that all terms appearing in equations do not contain parallel composition since, by axiom , parallel composition is a derived operator.)

Lemma 14. *Consider the equality* $a \lfloor\!\lfloor \Phi_n = a.\Phi_n$*. Let* n_0 *be the size of the biggest term* t *or* u*, appearing in equations* $(t = u) \in E$*. The above equation is not derivable from* E *for* $n > max(n_0, 2)$*.*

Once the above lemma is proven the theorem follows since the above equality is sound yet not derivable from E for $n > \max(n_0, 2)$. Lemma 14 is a consequence of the following result that establishes a property of equations that are derivable from E but that is not afforded by the equation $a \lfloor\!\lfloor \Phi_n = a.\Phi_n$ for suitably large values of n.

Lemma 15. *If* $E \vdash p = q$*,*

1. *p and q do not contain 0 summands or factors,*
2. *$p \underline{\leftrightarrow} a \lfloor\!\lfloor \Phi_n$ for $n > max(n_0, 2)$, and*
3. *p has a summand of the form $p_0 \lfloor\!\lfloor p_1$ where $p_0 \underline{\leftrightarrow} a$ and $p_1 \underline{\leftrightarrow} \Phi_n$,*

then q *has a summand of the form* $q_0 \lfloor\!\lfloor q_1$ *where* $q_0 \underline{\leftrightarrow} a$ *and* $q_1 \underline{\leftrightarrow} \Phi_n$*.*

If we prove the above lemma then it follows that $a \lfloor\!\lfloor \Phi_n = a.\Phi_n$ for $n > \max(2, n_0)$ is not provable from E because the left-hand side satisfies the requirements of the statement but the right-hand side does not contain any summand of the form $q_0 \lfloor\!\lfloor q_1$.

In the proof of Lemma 15, we shall have some use for the following definition and the subsequent lemma [8,11].

Definition 16. *Process* p *is irreducible when for all* p_0 *and* p_1*, if* $p \underline{\leftrightarrow} p_0 \,||\, p_1$ *then* $p_0 \underline{\leftrightarrow} 0$ *or* $p_1 \underline{\leftrightarrow} 0$*. We say that* p *is prime when it is irreducible and is not bisimilar to* 0*.*

Lemma 17. *The following processes are prime:*

1. ϕ_i, *for an arbitrary* $i \geq 1$*;*
2. Φ_i, *for all* $i > 1$*;*
3. $a.\Phi_i$, *for all* $i > 1$*.*

The proof of the above lemma is standard and is omitted for brevity. Note that neither item 2 nor item 3 in the above lemma hold for $i = 1$ since $\Phi_1 \equiv a.\phi_1 \equiv a.a \underline{\leftrightarrow} a \,||\, a$.

To prove Lemma 15, we use an induction on the derivation structure for $p = q$ and distinguish the following cases based on the last deduction rule applied in the derivation. (Since p and q have neither 0 summands nor factors, reasoning as in [9], we may assume that none of the terms mentioned in the proof of $p = q$ has 0 summands or factors.) The statement is trivial if $E \vdash p = q$ is due to . If $E \vdash p = q$ is due to , then there exists a term r such that $E \vdash p = r$ and $E \vdash r = q$ and the lemma follows by applying the induction hypothesis first on $E \vdash p = r$ and then on $E \vdash r = q$. If the last applied deduction rule is , then we distinguish the following cases based on the head operator of p and q.

1. $p \equiv a.p'$ and $q \equiv a.q'$; this case is vacuous since p should contain at least one summand which is of the form $p_1 \lfloor\lfloor\, p_2$;
2. $p \equiv \epsilon(d).p'$ and $q \equiv \epsilon(d).q'$; impossible, see above.
3. $p \equiv p_0 + p_1$, $q \equiv q_0 + q_1$, $E \vdash p_0 = q_0$ and $E \vdash p_1 = q_1$; without loss of generality, we assume that p_0 contains a summand of the form $p'_0 \lfloor\lfloor\, p'_1$ where $p'_0 \underline{\leftrightarrow} a$ and $p'_1 \underline{\leftrightarrow} \Phi_n$. It is not hard to see that $p_0 \underline{\leftrightarrow} a \lfloor\lfloor\, \Phi_n$ because $p \underline{\leftrightarrow} a \lfloor\lfloor\, \Phi_n$. It follows from the induction hypothesis that q_0 contains a summand that is of the form $q'_0 \lfloor\lfloor\, q'_1$ such that $q'_0 \underline{\leftrightarrow} a$, $q'_1 \underline{\leftrightarrow} \Phi_n$ and hence, so does q.
4. $p \equiv p_0 \lfloor\lfloor\, p_1$, $q \equiv q_0 \lfloor\lfloor\, q_1$, $E \vdash p_0 = q_0$ and $E \vdash p_1 = q_1$; it follows from the hypothesis of the lemma that $p_0 \underline{\leftrightarrow} a$ and $p_1 \underline{\leftrightarrow} \Phi_n$. By the soundness of E, we have that $p_0 \underline{\leftrightarrow} q_0 \underline{\leftrightarrow} a$ and $p_1 \underline{\leftrightarrow} q_1 \underline{\leftrightarrow} \Phi_n$ and thus the lemma follows.

It remains to consider the case where the last deduction rule applied is a closed instantiation of an axiom $(t = u) \in E$. In this case, there exists a substitution σ such that $\sigma(t) = p$ and $\sigma(u) = q$. Assume that $t \equiv \sum_{i \in I} t_i$ and $u \equiv \sum_{j \in J} u_j$ such that the t_i's and u_j's are not bisimilar to 0 and do not have $+$ as their head operator. Let t_i be a summand of t such that $\sigma(t_i)$ has a summand of the form $p_0 \lfloor\lfloor\, p_1$ and $p_0 \underline{\leftrightarrow} a$ and $p_1 \underline{\leftrightarrow} \Phi_n$. We analyze the following cases based on the structure of t_i.

- $t_i \equiv x$ It is not difficult to prove that, since $t = u$ is sound modulo bisimilarity, there exists $j \in J$ such that $u_j \equiv x$. Then the lemma follows since $\sigma(x)$, and hence $\sigma(u)$, contains a summand of the form $p_0 \lfloor\lfloor\, p_1$ and $p_0 \underline{\leftrightarrow} a$ and $p_1 \underline{\leftrightarrow} \Phi_n$.
- $t_i \equiv a.t'_i$ Impossible since $\sigma(t_i)$ must have a summand of the form $p_0 \lfloor\lfloor\, p_1$.
- $t_i \equiv \epsilon(d).t'_i$ Impossible since $\sigma(t_i)$ must have a summand of the form $p_0 \lfloor\lfloor\, p_1$.
- $t_i \equiv t'_i \lfloor\lfloor\, t''_i$ Then, it is not hard to see that $\sigma(t'_i) \underline{\leftrightarrow} a$ and $\sigma(t''_i) \underline{\leftrightarrow} \Phi_n$. Write $t''_i = \sum_{k \in K} v_k$ where no v_k is bisimilar to 0 or has $+$ as head operator. Since $\Phi_n \xrightarrow{a} \phi_i$, for each $0 < i \leq n$, the term $\sigma(t''_i)$ should mimic these

transitions, and because $2|K| < n$, there exists a $k \in K$ such that $\sigma(v_k) \xrightarrow{a} p'_i \underline{\leftrightarrow} \phi_i$ for at least three different i's. By a case distinction on the structure of v_k, we argue that v_k can only be a variable:

(a) $v_k \equiv a.v'_k$: This leads to a contradiction. Indeed, then $\sigma(v'_k) \underline{\leftrightarrow} \phi_i \underline{\leftrightarrow} \phi_j$ for different i and j.

(b) $v_k \equiv \epsilon(d).v'_k$: Then $\sigma(v_k)$ cannot make an a-transition, which is a contradiction.

(c) $v_k \equiv v'_k \| v''_k$: Recall that $\sigma(v_k)$ can make an a-transition to ϕ_i and ϕ_j for some $i \neq j$. Hence $\sigma(v_k) \equiv \sigma(v'_k) \| \sigma(v''_k) \xrightarrow{a} p_i \,||\, \sigma(v''_k) \underline{\leftrightarrow} \phi_i$ for some p_i. Since ϕ_i is prime (Lemma 17) $p_i \underline{\leftrightarrow} 0$ and $\sigma(v''_k) \underline{\leftrightarrow} \phi_i$. Similarly, $\sigma(v_k) \equiv \sigma(v'_k) \| \sigma(v''_k) \xrightarrow{a} p_j \,||\, \sigma(v''_k) \underline{\leftrightarrow} \phi_j$ for some p_j and thus, $\sigma(v''_k) \underline{\leftrightarrow} \phi_j$. Concluding, $\phi_i \underline{\leftrightarrow} \sigma(v''_k) \underline{\leftrightarrow} \phi_j$ for $i \neq j$, which is a contradiction.

Therefore $v_k \equiv x$ for some variable x and $\sigma(x)$ can make a-transitions to ϕ_i, ϕ_j and ϕ_k for different i, j and k. Then, x is not a summand of t, for this would contradict our assumption that $p \underline{\leftrightarrow} a \| \Phi_n$. Indeed, the action depth of $\sigma(x)$ after an a-transition is at most n (the action depth of ϕ_n) while the action depth of $a \| \Phi_n$ after an a-transition is $n+1$ (the action depth of Φ_n). Furthermore, $x \notin vars(t'_i)$ or otherwise it would not hold that $\sigma(t'_i) \underline{\leftrightarrow} a$ since $\sigma(t'_i)$ would have an action depth larger than 1. Also, x can only appear in the summands of t''_i that are of the form x or $\epsilon(d).x$. Indeed, if x occurred in summands that have any form other than x or $\epsilon(d).x$, then $\sigma(t''_i) \underline{\leftrightarrow} \Phi_n$ would not be sound since $\sigma(t''_i)$ could then make two or more a-transitions (possibly interleaved with time delays) resulting in ϕ_i for some $i > 1$, which cannot be mimicked by Φ_n. Hence, we conclude that $t''_i = x + t'' + \sum_{i' \in I'} \epsilon(d_{i'}).x$ for some term t'' such that $x \notin vars(t'')$.

Consider the new substitution σ' defined to map x to $\epsilon(d).a.\Phi_n$, where d is *smaller than* each delay occurring in p or q, and to agree with σ on all other variables. (Such d exists since D has 0 as a cluster point.)

We have that $\sigma'(t_i) \xrightarrow{\epsilon(d)} p'_d \equiv \sigma(t'_i)_d \| (a.\Phi_n + \sigma(t'')_d + \sum_{i' \in I'} \epsilon(d_{i'} - d).\sigma'(x))$ where $\sigma(t'_i) \xrightarrow{\epsilon(d)} \sigma(t'_i)_d$ and $\sigma(t'') \xrightarrow{\epsilon(d)} \sigma(t'')_d$. Furthermore, as $\sigma(t'_i) \underline{\leftrightarrow} a$ and axiom P is sound, $p'_d \xrightarrow{a} p'_{da} \underline{\leftrightarrow} a.\Phi_n + \sigma(t'')_d$. Observe that the action depth of $\sigma(t'')_d$ can at most be $n+1$. It follows from $(t = u) \in E$ and the soundness of E that $\sigma'(t) \underline{\leftrightarrow} \sigma'(u)$ and hence $\sigma'(u) \xrightarrow{\epsilon(d)} \sigma'(u)_d \xrightarrow{a} q'_{da} \underline{\leftrightarrow} p'_{da}$ for some q'_{da}. Thus, there exists a summand u_j of u (for some $j \in J$) such that $\sigma'(u_j) \xrightarrow{\epsilon(d)} q'_d \xrightarrow{a} q'_{da}$. It holds that $x \in vars(u'_j)$ since otherwise, $\sigma(u_j) \equiv \sigma'(u_j)$ and q'_{da} would have action depth of at most $n+1$ (since $\sigma(u) \underline{\leftrightarrow} a \| \Phi_n$). We distinguish the following cases based on the structure of u_j.

$u_j \equiv x$ Impossible since then $q'_{da} \equiv \Phi_n$ which is not bisimilar to $a.\Phi_n + \sigma(t'')_d \underline{\leftrightarrow} p'_{da}$.

$u_j \equiv \epsilon(e).u'_j$ Impossible since d is smaller than each delay in p and q, which means that $d < e$ and thus, $\sigma(u_j)$ cannot perform an action after a time delay of length d.

$u_j \equiv a.u'_j$ We argue that this case leads to a contradiction. To this end, observe that, first of all, variable x can appear only in summands of u'_j which are of the form x or $\epsilon(d').x$. Otherwise, if u'_j has an action prefixing or left-merge operator with an argument containing x among its variables, the action depth of $\sigma'(u'_j)$ would be at least $n+3$, which is larger than the action depth of $p'_{da} \underline{\leftrightarrow}\, a.\Phi_n + \sigma(t'')_d$. Hence, $u'_j \,\underline{\leftrightarrow}\, x + u'' + \sum_{j' \in J'} \epsilon(d_{j'}).x$ for some term u'' such that $x \notin \mathit{vars}(u'')$. Thus, $q'_d \,\underline{\leftrightarrow}\, a.\ (\epsilon(d).a.\Phi_n + \sigma'(u'') + \sum_{j' \in J'} \epsilon((d_{j'} + d)).\sigma'(x))$ and $q'_{ad} \equiv \epsilon(d).a.\Phi_n + \sigma'(u'') + \sum_{j' \in J'} \epsilon(d_{j'} + d).\sigma'(x)$. It should hold that $q'_{ad} \,\underline{\leftrightarrow}\, a.\Phi_n + \sigma(t'')_d$; but $a.\Phi_n + \sigma(t'')_d \xrightarrow{a} \Phi_n$ and a matching a-transition of q'_{ad} can only be due to $\sigma'(u'')$, which does not contain x and thus is the same as $\sigma(u'')$. It holds that $\mathit{adepth}(\sigma(u'')) < \mathit{adepth}(\sigma(u_j)) \leq \mathit{adepth}(\sigma(u)) \leq n + 2 = \mathit{adepth}(a.\Phi_n + \sigma(t'')_d)$. Therefore, any a-derivative of q'_{da} will have action depth of at most n. Hence, it cannot hold that $q'_{da} \,\underline{\leftrightarrow}\, a.\Phi_n + \sigma(t'')_d$, contradicting our assumption.

$u_j \equiv u'_j \,⌊\!⌊\, u''_j$ By one of our assumptions $\sigma(u'_j)$ and $\sigma(u''_j)$ are not bisimilar to 0. Therefore, $\sigma'(u'_j)$ and $\sigma'(u''_j)$ are not bisimilar to 0, either. By our assumption, $\sigma'(u'_j) ⌊\!⌊ \sigma'(u''_j) \xrightarrow{\epsilon(d)} \xrightarrow{a} \sigma'(u'_j)_{da} \,||\, \sigma'(u''_j)_d \,\underline{\leftrightarrow}\, p'_{da}$. Recall that $p'_{da} \underline{\leftrightarrow} a.\Phi_n + \sigma(t'')_d$ where $\sigma(t'') \xrightarrow{d} \sigma(t'')_d$ and $\sigma(t'') + \Phi_n \underline{\leftrightarrow} \Phi_n$. It follows from the latter bisimilarity that $\sigma(t'')_d + \Phi_n \,\underline{\leftrightarrow}\, \Phi_n$. We claim that $a.\Phi_n + \sigma(t'')_d$ is prime and hence, $\sigma'(u'_j)_{da} \,\underline{\leftrightarrow}\, 0$ and $\sigma'(u''_j) \,\underline{\leftrightarrow}\, p'_{da}$.

To prove the above claim assume towards a contradiction that $r \,||\, s \,\underline{\leftrightarrow}\, a.\Phi_n + \sigma(t'')_d$ for r and s not bisimilar to 0. We distinguish the following cases based on the behavior of $\sigma(t'')_d$.

i. Assume that $\sigma(t'')_d \,\underline{\leftrightarrow}\, 0$. It follows that $r \,||\, s \,\underline{\leftrightarrow}\, a.\Phi_n$. However, this is impossible since $a.\Phi_n$ is prime (Lemma 17, item 3).

ii. Assume that $\sigma(t'')_d \xrightarrow{\epsilon(e)} \sigma(t'')_{d+e} \xrightarrow{a} \sigma(t'')_{(d+e)a} \,\underline{\leftrightarrow}\, \phi_i$ for some $i \leq n$. Then, without loss of generality, $r \,||\, s \xrightarrow{\epsilon(e)} r_e \,||\, s_e \xrightarrow{a} r' \,||\, s_e \,\underline{\leftrightarrow}\, \phi_i$ for some r' such that $r \xrightarrow{\epsilon(e)} r_e \xrightarrow{a} r'$. It follows from primality of ϕ_i that $r' \,\underline{\leftrightarrow}\, 0$ and $s_e \,\underline{\leftrightarrow}\, \phi_i$. It also holds that $a.\Phi_n + \sigma(t'')_d \xrightarrow{a} \Phi_n$. Thus, $r \,||\, s$ should be able to mimic this transition; the transition cannot be due to r because then $s \,\underline{\leftrightarrow}\, \Phi_n$ (since Φ_n is also prime), which contradicts $s_e \,\underline{\leftrightarrow}\, \phi_i$. Hence, $r \,||\, s \xrightarrow{a} r \,||\, s' \,\underline{\leftrightarrow}\, \Phi_n$. It follows from primality of Φ_n that $r \,\underline{\leftrightarrow}\, \Phi_n$ and $r_e \,\underline{\leftrightarrow}\, \Phi_n$. Thus, using congruence of $\underline{\leftrightarrow}$ with respect to $||$, we have that $r_e \,||\, s_e \,\underline{\leftrightarrow}\, \Phi_n \,||\, \phi_i$. Since the action depth of $a.\Phi_n + \sigma(t'')_{d+e}$ is $n + 2$, we infer that $\phi_i \,\underline{\leftrightarrow}\, a$ and $i = 1$. But even then, $r_e \,||\, s_e \xrightarrow{a} v \,\underline{\leftrightarrow}\, \phi_n \,||\, a$, which cannot be mimicked by

$a.\Phi_n + \sigma(t'')_{d+e}$ (since $\sigma(t'')_{d+e}$ does not have the sufficient action depth and the only a-transition afforded by $a.\Phi_n$ results in Φ_n).

Thus, we conclude that $a.\Phi_n+\sigma(t'')_d$ is prime and hence, $\sigma'(u'_j)_{da} \leftrightarrow 0$ and $\sigma'(u''_j) \leftrightarrow a.\Phi_n + \sigma(t'')_d$. We claim that since d is smaller than all delays mentioned in p and q and $\sigma'(u'_j) \stackrel{\epsilon(d)}{\to} \stackrel{a}{\to} \sigma'(u'_j)_{da} \leftrightarrow 0$, then $\sigma(u'_j) \stackrel{a}{\to} \sigma(u'_j)_a$ for some $\sigma(u'_j) \leftrightarrow 0$. From this claim (whose proof is given next), it follows that $\sigma(u_j) \equiv \sigma(u'_j) \lfloor\lfloor \sigma(u''_j) \stackrel{a}{\to} \sigma(u'_j)_a \,||\, \sigma(u''_j) \leftrightarrow \sigma(u''_j)$. On the other hand, $q \equiv \sigma(u) \leftrightarrow a \lfloor\lfloor \Phi_n$ and thus, $\sigma(u''_j) \leftrightarrow \Phi_n$. Hence, the action depth of $\sigma(u'_j)$ is 1 and therefore $\sigma(u'_j) \leftrightarrow a$. To summarize, we have proved then that $\sigma(u_j) \equiv \sigma(u'_j) \lfloor\lfloor \sigma(u''_j)$, $\sigma(u'_j) \leftrightarrow a$ and $\sigma(u''_j) \leftrightarrow \Phi_n$, which was to be shown. Thus, it only remains to prove the following lemma.

Lemma 18. *Assume that d is smaller than each delay in $\sigma(u)$ and let r be a process that is not bisimilar to 0. Define σ' to map x to $\epsilon(d).a.r$ and all other variables y to $\sigma(y)$. Assume that $\sigma'(u) \stackrel{\epsilon(d)}{\to} \sigma'(u)_d \stackrel{a}{\to} \sigma'(u)_{da} \leftrightarrow 0$. Then, $\sigma(u) \stackrel{a}{\to} \sigma(u)_a$ for some $\sigma(u)_a \leftrightarrow 0$.*

Proof. By an induction on the structure of u. For brevity, we only give the proof for the case $u \equiv y$; the proofs for other cases are similar.

Assume that $u \equiv y$. First of all observe that y cannot be the same as x. Indeed $\sigma(x) \equiv \epsilon(d).a.r \stackrel{\epsilon(d)}{\to} a.r \stackrel{a}{\to} r$ and it does not hold that $r \leftrightarrow 0$ by one of the provisos in the lemma. Thus, $\sigma'(y) \equiv \sigma(y) \stackrel{\epsilon(d)}{\to} \stackrel{a}{\to} q'$ and $q' \leftrightarrow 0$. We proceed with an induction on the structure of $\sigma'(y) \equiv \sigma(y)$.

$\sigma(y) \equiv a.q'$ Then the lemma follows since $\sigma(y) \stackrel{a}{\to} q'$.

$\sigma(y) \equiv \epsilon(e).q'$ Impossible since then $\sigma(y)$ would not afford an a-transition after a time delay of length d because $d < e$.

$\sigma(y) \equiv q_0 + q_1$ Assume without loss of generality that $q_0 \stackrel{\epsilon(d)}{\to} \stackrel{a}{\to} q'$. Time delay d is smaller than the delays mentioned in $\sigma(y)$ and, hence, in q_0. It follows from the induction hypothesis that $q_0 \stackrel{a}{\to} q''$ for some $q'' \leftrightarrow 0$ and therefore $\sigma(y) \equiv q_0 + q_1 \stackrel{a}{\to} q'' \leftrightarrow 0$.

$\sigma(y) \equiv q_0 \lfloor\lfloor q_1$ Then, $q_0 \stackrel{\epsilon(d)}{\to} \stackrel{a}{\to} q'_0 \leftrightarrow 0$ and $q' \equiv q'_0 \,||\, q_{1d} \leftrightarrow 0$ where $q_1 \stackrel{\epsilon(d)}{\to} q_{1d}$; hence, $q_{1d} \leftrightarrow q_1 \leftrightarrow 0$. It follows from the induction hypothesis that $q_0 \stackrel{a}{\to} q''_0$ for some $q''_0 \leftrightarrow 0$ and thus, $\sigma(y) \equiv q_0 \lfloor\lfloor q_1 \stackrel{a}{\to} q''_0 \,||\, q_1 \leftrightarrow 0$.

6 Conclusions

In this paper, we studied the equational theory of TCCS as proposed by Wang Yi in [12]. We improved upon the Gap Theorem of [5] and proved that, even in the presence of a single basic action, parallelism in TCCS cannot be resolved in general. Furthermore we showed that TCCS, in its single- and two-sorted presentations, as well as its extensions with the untimed or the timed left-merge operator, does not afford a finite axiomatization.

It is an open question whether there exists a binary operator that, when added to TCCS, can give timed bisimilarity a finite basis. (A similar question is still open for untimed process algebras, i.e., whether there exists a single binary operator that can axiomatize communication and concurrency; the answer in both cases is expected to be negative.) Towards achieving this goal, in the extended version of this paper, we prove that adding two different variants of the timed left-merge operator $\lfloor\!\lfloor_0$ and $\lfloor\!\lfloor_1$ with the following semantics does not lead to a finite axiomatization for bisimilarity. (The leftmost rule below applies to both $\lfloor\!\lfloor_0$ and $\lfloor\!\lfloor_1$.)

$$\frac{x_0 \xrightarrow{a} y_0}{x_0 \lfloor\!\lfloor_{0,1} x_1 \xrightarrow{a} y_0 \,||\, x_1} \qquad \frac{x_0 \xrightarrow{\epsilon(d)} y_0}{x_0 \lfloor\!\lfloor_0 x_1 \xrightarrow{\epsilon(d)} y_0 \,||\, x_1} \qquad \frac{x_0 \xrightarrow{\epsilon(d)} y_0 \quad x_1 \xrightarrow{\epsilon(d)} y_1}{x_0 \lfloor\!\lfloor_1 x_1 \xrightarrow{\epsilon(d)} y_0 \,||\, y_1}$$

In the case of two-sorted TCCS, our proofs make use of the fact that the time domain has 0 as a cluster point. It remains open whether discrete-time TCCS (or its extension with (timed) left-merge) is finitely axiomatizable modulo bisimilarity.

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