The lambda-cube with classes of terms modulo conversion

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The $\lambda$-cube with classes of terms modulo conversion

by

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94/46
On II-conversion in Type Theory

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1 Introduction

Type theory has almost always been studied without II-conversion (which is the analogue of \(\beta\)-conversion on product type level). That is, \((\lambda_{x:A}.b)C \rightarrow_{\beta} b[x := C]\) is always assumed but not \((\Pi_{x:A}.B)C \rightarrow_{\Pi} B[x := C]\). The exception for this are some Automath languages in \([\text{de Bruijn 74}]\) and the current work of \([\text{KN 94}]\) and \([\text{KN 9x}]\).

Moreover, the conversion rule of the usual typing relation of the systems of the \(\lambda\)-Cube \(\vdash_{\beta}\), is given in terms of \(\vdash_{\beta}\) rather than \(\vdash_{\beta\Pi}\) and the application axiom is as follows:

\[
\begin{align*}
\Gamma \vdash_{\beta} F : \Pi_{x:A}.B & \quad \Gamma \vdash_{\beta} a : A \\
\hline
\Gamma \vdash_{\beta} Fa : B[x := a]
\end{align*}
\]

The fact that the type of \(Fa\) is taken to be \(B[x := a]\) rather than \((\Pi_{x:A}.B)a\), implies the loss of the compatibility property for the typing of application. So whereas in the \(\lambda\)-Cube one has compatibility for the typing of abstraction (i.e. \(A : B \Rightarrow \lambda_{x:C}.A : \Pi_{x:C}.B\)), one does not have \(A : B \Rightarrow AC : BC\).

The extended metatheory of the systems of the \(\lambda\)-Cube with II-conversion was first studied in detail in \([\text{KN 94}]\) and \([\text{KN 9x}]\). The aim of that study was to create a canonical typing operator which can easily find the type of a term, and to make the judgement \(\Gamma \vdash A\) typable in \(\Gamma\) independent of the type of \(A\). That is, the aim was to split \(\Gamma \vdash A : B\) into \(\Gamma \vdash A\) and \(\tau(\Gamma, A) =_{\Pi} B\) where \(\tau(A)\) may indeed contain II-redexes. In fact, the \(\Gamma\)-canonical type of a term \(A\), \(\tau(\Gamma, A)\), was almost an exact copy of \(A\) where the heart of \(A\) is replaced by its type in \(\Gamma\), the \(\Pi\)'s are removed and the \(\lambda\)-redexes are changed to II-redexes. For example:

\[
\tau(\Pi_{x:.}(\lambda_{y:.}(\lambda_{z:.}x)y)(\Pi_{y:.}(\lambda_{x:.}y))y) = (\Pi_{y:.}(\Pi_{x:.}.)(\lambda_{z:.}x)y)(\Pi_{x:.}.)(\lambda_{z:.}x)y)
\]
In [KN 9x], II-reduction was needed as a relation between the canonical type of a pseudo-term and any other of its types. It was shown however, that including II-reduction in the typing relation of the Cube is rather redundant for the splitting of \( \Gamma \vdash A : B \) in two parts. In fact, if \( \vdash_{\beta} \) is the relation which is defined as \( \vdash_{\beta} \) but where the conversion rule uses \( \equiv_{\beta} \) rather than \( \equiv_{\beta} \), then it was shown that \( \vdash_{\beta} A : B \iff \vdash_{\beta} A \wedge \pi(\Gamma, A) =_{\beta} B \wedge B \) is \( \vdash_{\beta} \)-legal type. Hence, really \( \vdash_{\beta} \) is not needed yet \( \equiv_{\beta} \) is as it compares the canonical type of \( A \) (which may contain many II-redexes) with the \( \vdash_{\beta} \)-type of \( A \).

A further result was shown in [KN 9x]. That is: if \( \vdash_{\beta} A : B \) then \( \vdash_{\beta} A \) contains no II-redexes and \( B \) is either II-redex free or is itself the unique II-redex in \( B \). This means that \( \vdash_{\beta} \) has almost the same legal terms as \( \vdash_{\beta} \). We do however, think that the small change in legal types which allows them now to be themselves II-redexes, and the elegance that results from writing \( \vdash_{\beta} A : (\Pi x : A. B)a \) rather than \( B[x := a] \) in the application rule are worth studying.

Now we come to the heart of this paper. In [KN 9x], it was shown that Subject Reduction (SR) fails for \( \vdash_{\beta} \) (that is: \( \Gamma \vdash_{\beta} A : B \wedge A' \not\equiv_{\beta} \Gamma \vdash_{\beta} A' : B \) yet holds if \( B \) is \( \vdash_{\beta} \)-legal. The failure of SR was a result of the failure of type correctness. In fact, one may have \( \Gamma \vdash_{\beta} A : B \) without having \( B \equiv_{\beta} \emptyset \) or \( \Gamma \vdash_{\beta} B : S \) for some \( S \). We find this failure of type correctness annoying and once we establish it, we can re-establish SR. In this paper, we shall show that adding definitions to the Cube re-establishes type correctness and full Subject Reduction for \( \vdash_{\beta} \). This is quite intriguing. In fact, most implementations of Pure Type Systems such as Coq ([Dow 91]), Lego ([LP 92]) and HOL ([GM 93]) include definitions. Furthermore, our work on generalising reduction in the Cube in [BKN 9y] showed that definitions are necessary. Now, again, definitions play a role in II-reduction.

Following the above observations, we divide the paper as follows:

1. In section 2, we introduce the formal machinery of the cube extended with definitions and introduce \( \vdash_{\beta} \) as in [KN 9x] observing that correctness of types and subject reduction do not hold for \( \vdash_{\beta} \).

2. In section 3, we introduce \( \vdash_{\beta} \) extended with definitions and show that now, correctness of types and subject reduction hold for \( \vdash_{\beta} \). We show further that all other properties of the cube remain valid for \( \vdash_{\beta} \).

2 The Cube with II-reduction

The systems of the Cube (see [Barendregt 92]), are based on a set of pseudo-expressions or terms \( T \) defined by the following abstract syntax:

\[
T = \star \mid \square \mid V \mid TT \mid \pi V T T
\]

where \( \pi \) ranges over \( \Pi \) and \( \lambda, V \) is an infinite collection of variables over which \( \alpha, \beta, x, y, z, \ldots \) range. \( \star \) and \( \square \) are called sorts over which \( S, S_1, S_2, \ldots \) are used to range. We take \( A, B, a, b \ldots \) to range over \( T \).

Bound and free variables and substitution are defined as usual. We write \( BV(A) \) and \( FV(A) \) to represent the bound and free variables of \( A \) respectively. We write \( A[x := B] \) to denote the term where all the free occurrences of \( x \) in \( A \) have been replaced by \( B \). Furthermore,
we take terms to be equivalent up to variable renaming. For example, we take \( \lambda x:A.x \equiv \lambda y:A.y \) where \( \equiv \) is used to denote syntactical equality of terms. We assume moreover, the Barendregt variable convention which is formally stated as follows:

**Convention 2.1 (BC: Barendregt’s Convention)**
Names of bound variables will always be chosen such that they differ from the free ones in a term. Moreover, different \( \lambda \)'s have different variables as subscript. Hence, we will not have \((\lambda x:A.x)x\), but \((\lambda y:A.y)x\) instead.

Terms can be related via a reduction relation \( \rightarrow_r \). We assume the usual definition of the compatibility of a reduction relation, and define \( \rightarrow_r \) to be its reflexive transitive closure and \( =_r \) to be its equivalence closure. We use in this paper two reduction relations: \( \rightarrow_\beta \) generated by the axiom \((\lambda x:A.B)C \rightarrow_\beta B[x:=C]\) and \( \rightarrow_{\beta\Pi} \) generated by the axiom \((\pi x:A.B)C \rightarrow_{\beta\Pi} B[x:=C]\) (remember that \( \pi \) ranges over both \( \lambda \) and \( \Pi \)).

**Definition 2.2 (declarations, definitions, pseudocontexts, \( \subseteq' \))**

1. A declaration \( d \) is of the form \( \lambda x:A \). We define \( \text{subj}(d) \) and \( \text{pred}(d) \) to be \( x \) and \( A \) respectively.
2. A definition \( d \) is of the form \((\pi x:A.-)B\) and defines \( x \) of type \( A \) to be \( B \). We define \( \text{subj}(d) \), \( \text{pred}(d) \) and \( \text{def}(d) \) to be \( x \), \( A \), and \( B \) respectively.
3. We use \( d, d_1, d_2, \ldots \) to range over declarations and definitions.
4. A pseudocontext \( \Gamma \) is a (possibly empty) concatenation of declarations and definitions \( d_1, d_2, \ldots, d_n \) such that if \( i \neq j \), then \( \text{subj}(d_i) \neq \text{subj}(d_j) \). We use \( \Gamma, \Delta, \Gamma', \Gamma_1, \Gamma_2, \ldots \) to range over pseudocontexts.
5. We define \( \text{dom}(\Gamma) = \{ \text{subj}(d) \mid d \in \Gamma \} \), \( \Gamma\text{-decl} = \{ d \in \Gamma \mid d \text{ is a declaration} \} \) and \( \Gamma\text{-def} = \{ d \in \Gamma \mid d \text{ is a definition} \} \) for any pseudocontext \( \Gamma \). Note that \( \text{dom}(\Gamma) = \{ \text{subj}(d) \mid d \in \Gamma\text{-decl} \cup \Gamma\text{-def} \} \).
6. Define \( \subseteq' \) between pseudocontexts as the least reflexive transitive relation satisfying:
   - \( \Gamma, \Delta \subseteq' \Gamma, d, \Delta \) for \( d \) a declaration or a definition.
   - \( \Gamma, \lambda x:A, \Delta \subseteq' \Gamma, (\lambda x:A,-)B, \Delta \)

**Remark 2.3** We only consider definitions when the reduction relation is \( \rightarrow_{\beta\Pi} \). Then \( \lambda \)-redexes and \( \Pi \)-redexes have—intuitively—equal meanings, therefore we allow definitions to be \( \Pi \)-redexes as well as \( \lambda \)-redexes.

**Definition 2.4 (statements, judgements, \( \vdash \))**

1. A statement is of the form \( A : B \), \( A \) and \( B \) are called the subject and the predicate of the statement respectively.
2. When \( \Gamma \) is a pseudocontext and \( A : B \) is a statement, we call \( \Gamma \vdash A : B \) a judgement, and write \( \Gamma \vdash A : B : C \) to mean \( \Gamma \vdash A : B \land \Gamma \vdash B : C \).
3. For \( \Gamma \) a pseudocontext and \( d \in \Gamma\text{-def} \cup \Gamma\text{-decl} \), we say \( \Gamma \) invites \( d \), notation \( \Gamma \prec d \), iff
Definition 2.5 (Definitional r-equality) Let $\Gamma$ be one of the relations $=_{\beta}$, $=_{\beta 1}$. For all pseudocontexts $\Gamma$ we define the binary relation $\Gamma \vdash \cdot =_{\text{def}} \cdot$ to be the equivalence relation generated by

- if $A =_{\cdot} B$ then $\Gamma \vdash A =_{\text{def}} B$
- if $d \in \Gamma$-def and $A, B \in T$ such that $B$ arises from $A$ by substituting one particular occurrence of $\text{subj}(d)$ in $A$ by $\text{def}(d)$, then $\Gamma \vdash A =_{\text{def}} B$.

Remark 2.6 If no definitions are present in $\Gamma$ then $\Gamma \vdash A =_{\text{def}} B$ is the same as $A =_{\cdot} B$.

Definition 2.7 Let $\Gamma$ be a pseudocontext.

1. Let $d, d_1, \ldots, d_n$ be declarations and definitions. We define $\Gamma \vdash d$ and $\Gamma \vdash d_1 \cdots d_n$ simultaneously as follows:

- If $d$ is a declaration: $\Gamma \vdash d$ iff $\Gamma \vdash \text{subj}(d) : \text{pred}(d)$.
- If $d$ is a definition: $\Gamma \vdash d$ iff $\Gamma \vdash \text{subj}(d) : \text{pred}(d) \land \Gamma \vdash \text{def}(d) : \text{pred}(d) \land \Gamma \vdash \text{def}(d) : \text{subj}(d) =_{\text{def}} \text{def}(d)$.
- $\Gamma \vdash d_1 \cdots d_n$ iff $\Gamma \vdash d_i$ for all $1 \leq i \leq n$.

2. $\Gamma$ is called legal if $\exists P, Q \in T$ such that $\Gamma \vdash P : Q$.

3. $A \in T$ is called a $\Gamma$-term if $\exists B \in T[\Gamma \vdash A : B \lor \Gamma \vdash B : A]$.

We take $\Gamma$-terms $= \{A \in T \mid \exists B \in T[\Gamma \vdash A : B \lor \Gamma \vdash B : A]\}$.

$A \in T$ is called legal if $\exists \Gamma[A \in \Gamma$-terms$]$.

In the Cube as presented in [Barendregt 92], the only declarations allowed are of the form $\lambda x : A$. Hence there are no definitions. Therefore, $\Gamma \vdash d$ is of the form $\Gamma \vdash \lambda x : A$ and means that $\Gamma \vdash A : S$ for some $S$ and that $x$ is fresh in $\Gamma, A$. Moreover, for any $d \equiv \lambda x : A$, remember that $\text{subj}(d) \equiv x$ and $\text{pred}(d) \equiv A$. Moreover, $\Pi$-reduction is not allowed. Hence, in the following definition, $d$ is a meta-variable for declarations only and $=_{\text{def}}$ is the same as $=_{\beta}$ (which is independent of $\vdash$).

Definition 2.8 (Axioms and rules of the Cube: $d$ is a declaration, $=_{\text{def}}$ is $=_{\beta}$)

(axiom) $<> \vdash_{\beta} * : \Box$

(start rule) $\frac{\Gamma \vdash d}{\Gamma, d \vdash_{\beta} \text{subj}(d) : \text{pred}(d)}$

(weakening rule) $\frac{\Gamma \vdash d \quad \Gamma \vdash_{\beta} D : E}{\Gamma, d \vdash_{\beta} D : E}$
(application rule) \[
\frac{\Gamma \vdash_\beta F : \Pi_{x:A}. B \quad \Gamma \vdash_\beta a : A}{\Gamma \vdash_\beta F a : B[x := a]}
\]

(abstraction rule) \[
\frac{\Gamma, \lambda x:A. b : B}{\Gamma \vdash_\beta \lambda x:A. b : \Pi_{x:A}. B}
\]

(conversion rule) \[
\frac{\Gamma \vdash_\beta A : B \quad \Gamma \vdash_\beta B' : S}{\Gamma \vdash_\beta A : B'}
\]

(formation rule) \[
\frac{\Gamma \vdash_\beta A : S_1 \quad \Gamma, \lambda x:A. \vdash_\beta B : S_2}{\Gamma \vdash_\beta \Pi_{x:A}. B : S_2}
\]

Each of the eight systems of the Cube is obtained by taking the \((S_1, S_2)\) rules allowed from a subset of \{(*, *), (*, \Box), (\Box, *), (\Box, \Box)\}. The basic system is the one where \((S_1, S_2) = (\ast, \ast)\) is the only possible choice. All other systems have this version of the formation rules, plus one or more other combinations of \((\ast, \Box), (\Box, \ast)\) and \((\Box, \Box)\) for \((S_1, S_2)\). Here is the table which presents the eight systems of the Cube:

<table>
<thead>
<tr>
<th>System</th>
<th>Set of specific rules</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda)</td>
<td>((\ast, \ast))</td>
</tr>
<tr>
<td>(\lambda 2)</td>
<td>((\ast, \ast)), ((\Box, \ast))</td>
</tr>
<tr>
<td>(\lambda P)</td>
<td>((\ast, \ast)), ((\ast, \Box))</td>
</tr>
<tr>
<td>(\lambda P2)</td>
<td>((\ast, \ast)), ((\Box, *), (\ast, \Box))</td>
</tr>
<tr>
<td>(\lambda w)</td>
<td>((\ast, \ast)), ((\ast, \Box)), ((\Box, \ast))</td>
</tr>
<tr>
<td>(\lambda Pw)</td>
<td>((\ast, \ast)), ((\ast, \Box)), ((\Box, \ast))</td>
</tr>
<tr>
<td>(\lambda Pw = \lambda C)</td>
<td>((\ast, \ast)), ((\ast, \Box)), ((\Box, \ast))</td>
</tr>
</tbody>
</table>

[KN 9x] extended the above Cube by changing \(\rightarrow_\beta\) to \(\rightarrow_{\beta\Pi}\) and by changing \(\vdash_\beta\) to \(\vdash_{\beta\Pi}\):

**Definition 2.9** (\(\vdash_{\beta\Pi}\)) We define \(\vdash_{\beta\Pi}\) as \(\vdash_\beta\) with the difference that the application and conversion rules change as follows:

(new application rule) \[
\frac{\Gamma \vdash_{\beta\Pi} F : \Pi_{x:A}. B \quad \Gamma \vdash_{\beta\Pi} a : A}{\Gamma \vdash_{\beta\Pi} F a : (\Pi_{x:A}. B)a}
\]

(new conversion rule) \[
\frac{\Gamma \vdash_{\beta\Pi} A : B \quad \Gamma \vdash_{\beta\Pi} B' : S}{\Gamma \vdash_{\beta\Pi} A : B' \quad \Gamma \vdash_{\beta\Pi} B =_{\text{det}} B'}
\]

Note that \(\Gamma \vdash_{\beta\Pi} B =_{\text{det}} B'\) is the same as \(B =_{\beta\Pi} B'\), as no definitions are allowed in the context.

Now we list the properties of \(\vdash_{\beta\Pi}\) without proofs (see [KN 9x]).

**Theorem 2.10** (The Church Rosser Theorem CR, for \(\rightarrow_{\beta\Pi}\))
If \(A \rightarrow_{\beta\Pi} B\) and \(A \rightarrow_{\beta\Pi} C\) then there exists \(D\) such that \(B \rightarrow_{\beta\Pi} D\) and \(C \rightarrow_{\beta\Pi} D\) \(\Box\)

**Lemma 2.11** (Free variable lemma for \(\vdash_{\beta\Pi}\))
Let \(\Gamma\) be a \(\vdash_{\beta\Pi}\)-legal context such that \(\Gamma \vdash_{\beta\Pi} B : C\). Then we have:

1. For all \(d, d' \in \Gamma\text{-decl}\), \(\text{subj}(d) \neq \text{subj}(d')\).
2. \(\text{FV}(B), \text{FV}(C) \subseteq \text{dom}(\Gamma)\).
3. If \( \Gamma \equiv \Gamma_1 \cdot \Gamma_2 \) then \( FV(d) \subseteq \text{dom}(\Gamma_1) \).

**Lemma 2.12** (Start Lemma for \( \vdash_{\beta_{\Pi}} \))

Let \( \Gamma \) be a \( \vdash_{\beta_{\Pi}} \)-legal context. Then \( \Gamma \vdash_{\beta_{\Pi}} * : \Box \) and \( \forall d \in \Gamma[\Gamma \vdash_{\beta_{\Pi}} d] \).

**Lemma 2.13** (Transitivity Lemma for \( \vdash_{\beta_{\Pi}} \))

Let \( \Gamma \) and \( \Delta \) be \( \vdash_{\beta_{\Pi}} \)-legal contexts. Then:\[ \Delta \vdash_{\beta_{\Pi}} A \land A \vdash_{\beta_{\Pi}} B \Rightarrow \Delta \vdash_{\beta_{\Pi}} A : B \]

**Lemma 2.14** (Substitution Lemma for \( \vdash_{\beta_{\Pi}} \))

If \( \Gamma \vdash_{\beta_{\Pi}} B : C \) and \( \Gamma \vdash_{\beta_{\Pi}} D : A \) then \( \Gamma.(\Delta[x := D]) \vdash_{\beta_{\Pi}} B[x := D] : C[x := D] \).

**Lemma 2.15** (Thinning Lemma for \( \vdash_{\beta_{\Pi}} \))

Let \( \Gamma \) and \( \Delta \) be \( \vdash_{\beta_{\Pi}} \)-legal contexts such that \( \Gamma \vdash_{\beta_{\Pi}} A \land A \vdash_{\beta_{\Pi}} B \Rightarrow \Delta \vdash_{\beta_{\Pi}} A : B \).

**Lemma 2.16** (Generation Lemma for \( \vdash_{\beta_{\Pi}} \))

1. \( \Gamma \vdash_{\beta_{\Pi}} S : C \Rightarrow S \equiv * , C =_{\beta_{\Pi}} \Box \) and, if \( C \neq \Box \) then \( \Gamma \vdash_{\beta_{\Pi}} C : S' \) for some sort \( S' \).
2. \( \Gamma \vdash_{\beta_{\Pi}} x : C \Rightarrow \exists B =_{\beta_{\Pi}} C[\lambda_x : B] \in \Gamma \land \text{if } C \neq B \text{ then } \Gamma \vdash_{\beta_{\Pi}} C : S \) for some sort \( S \).
3. \( \Gamma \vdash_{\beta_{\Pi}} \Pi_{x : A} . B : C \Rightarrow \exists (S_1, S_2)[\Gamma \vdash_{\beta_{\Pi}} A : S_1 \land \Gamma \vdash_{\beta_{\Pi}} B : S_2 \land \Pi_{x : A} : B =_{\beta_{\Pi}} \Pi_{x : A} : C] \)
4. \( \Gamma \vdash_{\beta_{\Pi}} \lambda_{x : A} . B : C \Rightarrow \exists (S, B)[\Gamma \vdash_{\beta_{\Pi}} B \land \Gamma \vdash_{\beta_{\Pi}} B : C =_{\beta_{\Pi}} \Pi_{x : A} : B \land [C \neq \Pi_{x : A} : B \Rightarrow \exists S[\Gamma \vdash_{\beta_{\Pi}} C : S]]]
5. \( \Gamma \vdash_{\beta_{\Pi}} F a : C \Rightarrow \exists A, B, x[\Gamma \vdash_{\beta_{\Pi}} F : \Pi_{x : A} : B \land \Gamma \vdash_{\beta_{\Pi}} a : A \land C =_{\beta_{\Pi}} \Pi_{x : A} : B[a \neq C \Rightarrow \exists S[\Gamma \vdash_{\beta_{\Pi}} C : S]]].

**Remark 2.17** (Correctness of types does not hold for \( \vdash_{\beta_{\Pi}} \))

The new legal terms of the form \( \Pi_{x : B} : C \) imply the failure of type correctness for \( \vdash_{\beta_{\Pi}} \). That is, even in \( \lambda_{x :} \), \( \Gamma \vdash_{\beta_{\Pi}} A : B \neq (B \equiv \Box \lor \Gamma \vdash_{\beta_{\Pi}} B : S \text{ for some sort } S) \). For example, if \( \Gamma \equiv \lambda_{x :} \lambda_{x :} \lambda_{x :} \vdash_{\beta_{\Pi}} (\Pi_{y : z} : z) \vdash_{\beta_{\Pi}} (\Pi_{y : z} : z) : S \), but \( \Gamma \not\vdash_{\beta_{\Pi}} (\Pi_{y : z} : z) : S \) from Lemma 2.19. Type correctness of course holds for \( \vdash_{\beta} \).

Failure of correctness of types implies failure of Subject Reduction even in \( \lambda_{x :} \):

**Example 2.18** In \( \lambda_{x :} \), \( \lambda_{x :} \lambda_{x :} \vdash_{\beta_{\Pi}} (\Pi_{y : z} : z) \vdash_{\beta_{\Pi}} (\Pi_{y : z} : z) : S \), which is absurd by Lemma 2.19. Yet in \( \lambda_{x :} \), \( \lambda_{x :} \lambda_{x :} \vdash_{\beta_{\Pi}} (\Pi_{y : z} : z) \vdash_{\beta_{\Pi}} (\Pi_{y : z} : z) : S \) from Lemma 2.19.

We do however, a weak subject reduction which we will prove after we show the relationship between \( \vdash_{\beta_{\Pi}} \) and \( \vdash_{\beta} \).

**Lemma 2.19** For any \( A, B, C, S, \Gamma : \Gamma \vdash_{\beta_{\Pi}} (\Pi_{x : A} : B) C : S \).

We do have the following lemma which is a sort of weak generation corollary:

**Lemma 2.20** \( \Gamma \vdash_{\beta_{\Pi}} A : B \land B \) is not a \( \Pi \)-redex \( \Rightarrow \Gamma \vdash_{\beta_{\Pi}} B : S \).

**Lemma 2.21** (Legal terms and contexts for \( \vdash_{\beta_{\Pi}} \) and \( \vdash_{\beta_{\Pi}} \))

1. If \( \Gamma \vdash_{\beta_{\Pi}} A : B \) then \( A \) and \( \Gamma \) are free of \( \Pi \)-redexes, and either \( B \) contains no \( \Pi \)-redexes or \( B \) is the only \( \Pi \)-redex in \( B \).
2. If $\Pi(x,D,E)B$ is $\beta_{\Pi}$-legal, then $E[x := B]$ contains no $\Pi$-redexes.

To relate $\beta$ and $\beta_{\Pi}$, we introduce a notation which removes the unique $\Pi$-redex in a $\beta_{\Pi}$-legal term (if it exists):

**Definition 2.22** For $A \vdash_{\beta_{\Pi}}$-legal, let $\hat{A}$ be $C[x := D]$ if $A \equiv (\Pi_{x,B}.C)D$ and $A$ otherwise.

**Lemma 2.23**

1. If $\Gamma \vdash_{\beta} A : B$ then $\Gamma \vdash_{\beta} A : \hat{B}$.

2. If $\Gamma \vdash_{\beta} A : B$ then $\Gamma \vdash_{\beta_{\Pi}} A : B$.

**Lemma 2.24** (Weak Subject Reduction for $\vdash_{\beta_{\Pi}}$ and $\rightarrow_{\beta_{\Pi}}$)

$\Gamma \vdash_{\beta_{\Pi}} A : B \land A \rightarrow_{\beta_{\Pi}} A' \Rightarrow \Gamma \vdash_{\beta_{\Pi}} A' : \hat{B}$

**Corollary 2.25** (WSR Corollary for $\vdash_{\beta_{\Pi}}$ and $\rightarrow_{\beta_{\Pi}}$)

If $\Gamma \vdash_{\beta_{\Pi}} A : B_1$ and $B_1 \rightarrow_{\beta_{\Pi}} B_2$ then $\Gamma \vdash_{\beta_{\Pi}} A : \hat{B}_2$.

**Remark 2.26** We cannot replace Corollary 2.25 by: If $\Gamma \vdash_{\beta_{\Pi}} A : B$ and $B \rightarrow_{\beta_{\Pi}} B'$ then $\Gamma \vdash_{\beta_{\Pi}} A : B'$. For example, take $\Gamma \equiv \lambda_{x : \alpha} \lambda_{y : \alpha} A \equiv (\lambda_{x : \alpha} z)((\lambda_{x : \alpha} x)y), B \equiv (\Pi_{x : \alpha} \alpha)((\lambda_{x : \alpha} z)y)$ and $B' \equiv (\Pi_{x : \alpha} \alpha)y$. Then, $\Gamma \vdash_{\beta_{\Pi}} A : B$ but $\Gamma \not\vdash_{\beta_{\Pi}} A : B'$ because if otherwise, we get by generation, $\Gamma \vdash_{\beta_{\Pi}} (\Pi_{x : \alpha} \alpha)y : S$, absurd by Lemma 2.19.

**Lemma 2.27** (Unicity of Types for $\vdash_{\beta_{\Pi}}$ and $\rightarrow_{\beta_{\Pi}}$)

$\Gamma \vdash_{\beta_{\Pi}} A : B_1 \land \Gamma \vdash_{\beta_{\Pi}} A : B_2 \Rightarrow B_1 =_{\beta_{\Pi}} B_2$

**Theorem 2.28** (Strong Normalisation with respect to $\vdash_{\beta_{\Pi}}$ and $\rightarrow_{\beta_{\Pi}}$)

If $A$ is $\vdash_{\beta_{\Pi}}$-legal then $\text{SN}_{\rightarrow_{\beta_{\Pi}}}(A)$; i.e. $A$ is strongly normalising with respect to $\rightarrow_{\beta_{\Pi}}$.

3 Extending $\vdash_{\beta_{\Pi}}$ with definitions

We shall extend the derivation rules of $\vdash_{\beta_{\Pi}}$ so that we can use definitions in the context. The rules remain unchanged except for the following points:

- One rule, the (def rule), is added.
- The use of $\Gamma \vdash B =_{\text{def}} B'$ in the conversion rule really has an effect now, rather than simply postulating $B =_{\beta_{\Pi}} B'$.
- Not only declarations but also definitions are allowed in contexts.

**Definition 3.1** (Axioms and rules of the Cube extended with definitions; $d$ ranges over declarations and definitions)

We extend the relation $\vdash_{\beta_{\Pi}}$ to $\vdash_{\beta_{\Pi}e}$ by adding the following definition rule:

(def rule) 

$\frac{\Gamma, (\pi_{x:A} \cdot \cdot \cdot) B \vdash_{\beta_{\Pi}e} C : D}{\Gamma \vdash_{\beta_{\Pi}e} (\pi_{x:A}C)B : D[x := B]}$
The (def rule) says that if $C : D$ can be deduced using a definition $d \equiv (\pi_{x:A.-})B$, then $(\pi_{x:A.C})B$ will be of type $D$ where $d$ has been unfolded in $D$. We do not get type $(\pi_{x:A.D})B$ in order to avoid things like $(\pi_{x:A.-})B$. Note that the (def rule) does global substitution in the predicate of all the occurrences of subj$(d)$. The reason is that $d$ no longer remains in the context. In the conversion rule however, substitution is local as $\Gamma$ keeps all its information (see Definition 2.5).

Remark 3.2 Our approach to definitions is slightly different from that of [SP 93], for details we refer to [BKN 9y].

Lemma 3.3 (Free variable lemma for $\vdash_{\beta \eta e}$)
Let $\Gamma$ be a legal context such that $\Gamma \vdash_{\beta \eta e} B : C$. Then the following holds:

1. If $d$ and $d'$ are two different elements of $\Gamma$-decl $\cup$ $\Gamma$-def, then subj$(d)$ $\neq$ subj$(d')$.
2. FV$(B)$, FV$(C)$ $\subseteq$ dom$(\Gamma)$.
3. If $\Gamma \equiv \Gamma_1.d.\Gamma_2$ then FV$(d)$ $\subseteq$ dom$(\Gamma_1)$.

Proof: All by induction on the derivation of $\Gamma \vdash_{\beta \eta e} B : C$. $\square$

Lemma 3.4 (Start Lemma for $\vdash_{\beta \eta e}$)
Let $\Gamma$ be a legal context. Then $\Gamma \vdash_{\beta \eta e} * : \bot$ and \forall $d \in \Gamma[\Gamma \vdash_{\beta \eta e} d]$.

Proof: $\Gamma$ legal $\Rightarrow \exists B, C[\Gamma \vdash_{\beta \eta e} B : C]$; now use induction on $\Gamma \vdash_{\beta \eta e} B : C$. $\square$

Lemma 3.5 (Transitivity Lemma for $\vdash_{\beta \eta e}$)
Let $\Gamma$ and $\Delta$ be legal contexts. Then: $[\Gamma \vdash_{\beta \eta e} \Delta \land \Delta \vdash_{\beta \eta e} A : B] \Rightarrow \Gamma \vdash_{\beta \eta e} A : B$.

Proof: Induction on the derivation $\Delta \vdash_{\beta \eta e} A : B$. $\square$

Lemma 3.6 (Thinning for $\vdash_{\beta \eta e}$)

1. If $\Gamma_1.\Gamma_2 \vdash_{\beta \eta e} A =_{\text{def}} B$, $\Gamma_1.\Delta.\Gamma_2 \vdash_{\beta \eta e} A =_{\text{def}} B$.
2. If $\Gamma$ and $\Delta$ are legal contexts such that $\Gamma \subseteq' \Delta$ and if $\Gamma \vdash_{\beta \eta e} A : B$, then $\Delta \vdash_{\beta \eta e} A : B$.

Proof: 1. is by induction on the derivation $\Gamma_1.\Gamma_2 \vdash_{\beta \eta e} A =_{\text{def}} B$. 2. is as follows:

• If $\Gamma.\Delta \vdash_{\beta \eta e} A : B$, $\Gamma \vdash_{\beta \eta e} C : S$, $x$ is fresh, then also $\Gamma.(\lambda_{x:C}.\Delta) \vdash_{\beta \eta e} A : B$. We show this by induction on the derivation $\Gamma.\Delta \vdash_{\beta \eta e} A : B$ using 1. for conversion.

• If $\Gamma.\Delta \vdash_{\beta \eta e} A : B$, $\Gamma \vdash_{\beta \eta e} C : D : S$, $x$ is fresh, then also $\Gamma.(\pi_{x:D.-})C.\Delta \vdash_{\beta \eta e} A : B$. We show this by induction on the derivation $\Gamma.\Delta \vdash_{\beta \eta e} A : B$.

• If $\Gamma.\lambda_{x:A}.\Delta \vdash_{\beta \eta e} B : C$, $\Gamma \vdash_{\beta \eta e} D : A$, then $\Gamma.(\lambda_{x:A.-})D.\Delta \vdash_{\beta \eta e} B : C$ is shown by induction on the derivation $\Gamma.\lambda_{x:A}.\Delta \vdash_{\beta \eta e} B : C$ (for conversion, use 1.). $\square$

Lemma 3.7 (Substitution lemma for $\vdash_{\beta \eta e}$)

1. If $\Gamma.(\pi_{x:C.-})D.\Delta \vdash_{\beta \eta e} A =_{\text{def}} B$, $A$ and $B$ are $\Gamma.(\pi_{x:C.-})D.\Delta$-legal, then $\Gamma.\Delta[x := D] \vdash_{\beta \eta e} A[x := D] =_{\text{def}} B[x := D]$.

2. If $B$ is a $\Gamma.(\pi_{x:C.-})D$-legal term, then $\Gamma.(\pi_{x:C.-})D \vdash_{\beta \eta e} B =_{\text{def}} B[x := D]$. 8
3. If $\Gamma.(\pi_{x:A}.)B.\Delta \vdash_{\beta\Pi_{\mathrm{e}}} C : D$, then $\Gamma.\Delta[x := B] \vdash_{\beta\Pi_{\mathrm{e}}} C[x := B] : D[x := B]$.

4. If $\Gamma.\lambda_{x:A}.\Delta \vdash_{\beta\Pi_{\mathrm{e}}} C : D$, $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} B : A$, then $\Gamma.\Delta[x := B] \vdash_{\beta\Pi_{\mathrm{e}}} C[x := B] : D[x := B]$.

**Proof:**

1. Induction on the derivation rules of $=_{\mathrm{def}}$.

2. Induction on the structure of $B$.

3. Induction on the derivation rules, using 1., 2. and thinning.

4. Idem.

**Lemma 3.8** (Generation Lemma for $\vdash_{\beta\Pi_{\mathrm{e}}}$)

1. If $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} x : A$ then for some $d \in \Gamma$, $x \equiv \text{subj}(d)$, $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} A =_{\mathrm{def}} \text{pred}(d)$ and $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} A : S$ for some sort $S$.

2. If $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} \lambda_{x:A}.B : C$ then for some $D$ and sort $S$: $\Gamma.\lambda_{x:A} \vdash_{\beta\Pi_{\mathrm{e}}} B : D$, $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} \Pi_{x:A}.D : S$, $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} \Pi_{x:A}.D =_{\mathrm{def}} C$ and if $\Pi_{x:A}.D \neq C$ then $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} C : S'$ for some sort $S'$.

3. If $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} \Pi_{x:A}.B : C$ then for some sorts $S_1, S_2$: $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} A : S_1$, $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} B : S_2$, $(S_1, S_2)$ is a rule, $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} C =_{\mathrm{def}} S_2$ and if $S_2 \neq C$ then $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} C : S$ for some $S$.

4. If $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} Fa : C$, $F \neq \lambda_{x:A}.B$, then for some $D, E$: $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} a : D$, $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} F : \Pi_{x:D}.E$, $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} (\Pi_{x:D}.E)a =_{\mathrm{def}} C$ and if $(\Pi_{x:D}.E)a \neq C$ then $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} C : S$ for some $S$.

5. If $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} (\pi_{x:A}.B)D : C$, then $\Gamma.(\pi_{x:A}.)B \vdash_{\beta\Pi_{\mathrm{e}}} D : C$.

**Proof:** 1., 2., 3. and 4. follow by a tedious but straightforward induction on the derivations (use the thinning lemma). As to 5., an easy induction to the derivation rules shows that one of the following two cases is applicable:

- $\Gamma.(\pi_{x:A}.)B \vdash_{\beta\Pi_{\mathrm{e}}} \Pi_{x:F}.G$, $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} C =_{\mathrm{def}} (\Pi_{y:F}.G)B$ and if $(\Pi_{y:F}.G)B \neq C$ then $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} C : S$ for some sort $S$.

In the first case we know by thinning that $\Gamma.(\pi_{x:A}.)B \vdash_{\beta\Pi_{\mathrm{e}}} C'[x := B] =_{\mathrm{def}} C$ and also $\Gamma.(\pi_{x:A}.)B \vdash_{\beta\Pi_{\mathrm{e}}} C'[x := B]$, hence by conversion $\Gamma.(\pi_{x:A}.)B \vdash_{\beta\Pi_{\mathrm{e}}} D : C$.

In the second case 2. tells us $\Gamma.\lambda_{x:A} \vdash_{\beta\Pi_{\mathrm{e}}} D : H$, $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} \Pi_{x:A}.H : S$, $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} \Pi_{x:A}.H =_{\mathrm{def}} \Pi_{y:F}.G$ and if $\Pi_{x:A}.H \neq \Pi_{y:F}.G$ then $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} \Pi_{y:F}.G : S'$ for some sort $S'$.

This means that $x \equiv y$, $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} A =_{\mathrm{def}} F$ and $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} H =_{\mathrm{def}} G$. Out of $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} \Pi_{x:A}.H : S$ we get by 3. that $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} A : S''$ for some sort $S''$, hence by conversion $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} B : A$.

Now by thinning we get $\Gamma.(\lambda_{x:A}.)B \vdash_{\beta\Pi_{\mathrm{e}}} D : H$. As we know $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} C =_{\mathrm{def}} (\Pi_{x:F}.G)B$, $\Gamma \vdash_{\beta\Pi_{\mathrm{e}}} G =_{\mathrm{def}} H$ and $\Gamma.(\lambda_{x:A}.)B \vdash_{\beta\Pi_{\mathrm{e}}} H =_{\mathrm{def}} H[x := B]$, we get (use thinning) $\Gamma.(\lambda_{x:A}.)B \vdash_{\beta\Pi_{\mathrm{e}}} H =_{\mathrm{def}} C$ so by conversion $\Gamma.(\lambda_{x:A}.)B \vdash_{\beta\Pi_{\mathrm{e}}} D : C$.  \(\Box\)
Lemma 3.9 (\(\Lambda\Pi\)-exchanging)
\[ \Gamma, \lambda x:A.\Delta \vdash_{\beta} \Gamma, \Pi x:A.\Delta \vdash_{\beta} C : D \quad \text{and} \quad \Gamma, (\lambda x:A.\Delta) B \vdash_{\beta} C : D \iff \Gamma, (\Pi x:A.\Delta) B \vdash_{\beta} C : D \]

Proof: induction on the derivation rules. We treat one case of the start rule:
\[ \Gamma, (\lambda x:A.\Delta) B \vdash_{\beta} x : A \text{ as a consequence of } \Gamma \vdash (\lambda x:A.\Delta) B. \text{ Then also } \Gamma \vdash (\Pi x:A.\Delta) B, \text{ so } \Gamma, (\Pi x:A.\Delta) B \vdash_{\beta} C : D. \]

\[ \square \]

Corollary 3.10 (Correctness of Types)
If \( \Gamma \vdash_{\beta} A : B \) then \( B \equiv \Box \) or \( \Gamma \vdash_{\beta} B : S \) for some sort \( S \).
Proof: By induction to the derivation rules. The interesting cases are the definition and application rules.

- In case \( \Gamma \vdash_{\beta} (\pi x:A. D) B : C[x := B] \) as a consequence of \( \Gamma \vdash_{\beta} \pi x:A. D : C \), then by IH \( C \equiv \Box \) or \( \Gamma \vdash_{\beta} \pi x:A. D : C : S \) for some sort \( S \). In the first case also \( C[x := B] \equiv \Box \), in the second case by the Substitution Lemma \( \Gamma \vdash_{\beta} C[x := B] : S[x := B] \equiv S \).

- In case \( \Gamma \vdash_{\beta} \lambda a : (\Pi x:A. B) a : (\Pi x:A. B) a : (\Pi x:A. B) \vdash_{\beta} C[x := B] \) as a consequence of \( \Gamma \vdash_{\beta} \lambda a : (\Pi x:A. B) a : (\Pi x:A. B) a : (\Pi x:A. B) \vdash_{\beta} C[x := B] \), then by IH \( C \equiv \Box \) or \( \Gamma \vdash_{\beta} \lambda a : (\Pi x:A. B) a : (\Pi x:A. B) a : (\Pi x:A. B) \vdash_{\beta} C[x := B] : S \) and by the definition rule \( \Gamma \vdash_{\beta} (\Pi x:A. B) \vdash_{\beta} C[x := B] : S[x := B] \equiv S \).

\[ \square \]

Theorem 3.11 (Subject Reduction for \(\vdash_{\beta} \) and \(\rightarrow_{\beta} \))
If \( \Gamma \vdash_{\beta} A : B \) and \( A \rightarrow_{\beta} A' \) then \( \Gamma \vdash_{\beta} A' : B \).
Proof: We prove by simultaneous induction on the derivation rules:

1. If \( \Gamma \vdash_{\beta} A : B \) and \( \Gamma' \) results from contracting one of the terms in the declarations and definitions of \( \Gamma \) by a one step \(\Pi\)-reduction, then \( \Gamma' \vdash_{\beta} A : B \)

2. If \( \Gamma \vdash_{\beta} A : B \) and \( A \rightarrow_{\beta} A' \) then \( \Gamma \vdash_{\beta} A' : B \)

- (axiom): nothing to prove
- (start rule): We consider the case \( A \equiv (\lambda x:A.\Delta) B \), \( A \rightarrow_{\beta} A' \). The other cases are similar or easy.

  We have: \( \Gamma, (\lambda x:A.\Delta) B \vdash_{\beta} x : A \text{ as a consequence of } \Gamma \vdash (\lambda x:A.\Delta) B, \text{ i.e. } \Gamma \vdash_{\beta} B : A : S \). By the induction hypothesis \( \Gamma \vdash_{\beta} A' : S \) and by the induction hypothesis and conversion \( \Gamma \vdash_{\beta} B : A' \). Hence \( \Gamma \vdash_{\beta} (\lambda x:A.\Delta) B \vdash_{\beta} x : A' \) and again by conversion \( \Gamma \vdash_{\beta} (\lambda x:A.\Delta) B \vdash_{\beta} x : A \).

- (weak), (formation), (conversion): use the induction hypothesis.
- (abstraction): use the induction hypothesis and conversion.
- (definition): \( \Gamma \vdash_{\beta} (\pi x:A. D) B : C[x := B] \) as a consequence of \( \Gamma \vdash_{\beta} \pi x:A. D : C \). Now \( \Gamma' \vdash_{\beta} (\pi x:A. D) B : C[x := B] \), \( \Gamma \vdash_{\beta} (\pi x:A. D) B : C[x := B] \) and \( \Gamma \vdash_{\beta} (\pi x:A. D) B : C[x := B] \) by the induction hypothesis.

  Furthermore, if \( B \rightarrow_{\beta} B' \) then \( C[x := B] \equiv_{\beta} C[x := B'] \) and by the induction hypothesis and definition rule we get \( \Gamma \vdash_{\beta} (\pi x:A. D) B' : C[x := B'] \). Now by Lemma
3.10, $C = \square$ or $\Gamma, \pi x:A.e \vdash_{\beta n} C : S$ for some sort $S$. In the first case, $C[x := B] = C \equiv C[x := B]$ and we are done, in the second case by the Substitution Lemma $\Gamma \vdash_{\beta n} C[x := B] : S[x := B] \equiv S$, so by conversion $\Gamma \vdash_{\beta n} (\pi x:A.D)B' : C[x := B]$.

For the last possibility, $(\pi x:A.D)B \to_{\beta n} D[x := B]$, we remark that by the Substitution Lemma we get out of $\Gamma, \pi x:A.e \vdash_{\beta n} D[x := B]$ that $\Gamma \vdash_{\beta n} D[x := B] : C[x := B]$.

- (application): $\Gamma \vdash_{\beta n} F a : (\Pi x:A.B)a$ as a consequence of $\Gamma \vdash_{\beta n} F : \Pi x:A.B$ and $\Gamma \vdash_{\beta n} a : A$. Then $\Gamma' \vdash_{\beta n} F a : (\Pi x:A.B)a$ and $\Gamma \vdash_{\beta n} F' a : (\Pi x:A.B)a$ by the induction hypothesis, and $\Gamma \vdash_{\beta n} F a' : (\Pi x:A.B)a'$ because by the induction hypothesis $\Gamma \vdash_{\beta n} F a' : (\Pi x:A.B)a'$, by Lemma 3.10 $\Gamma \vdash_{\beta n} (\Pi x:A.B)a : S$ for some sort $S$, so by conversion $\Gamma \vdash_{\beta n} F a' : (\Pi x:A.B)a$.

Now the crucial case: $F \equiv (\pi y:C.D), F a \to_{\beta n} D[y := a]$. Then $\Gamma \vdash_{\beta n} (\pi y:C.D) a : (\Pi x:A.B)a$ so by the Generation Lemma $\Gamma, (\pi y:C.\_\_) \vdash_{\beta n} D : (\Pi x:A.B)a$, now by the Substitution Lemma $\Gamma \vdash_{\beta n} D[y := a] = ((\Pi x:A.B)a)[y := a]$, but by the Barendregt convention $((\Pi x:A.B)a)[y := a] \equiv (\Pi x:A.B)a$ so we are done. \hfill $\Box$

The proof of Strong Normalisation is based on Strong Normalisation of the $\lambda$-cube extended with definitions as in [BKN 9y]. First we change $\Pi$-redexes into $\lambda$-redexes.

**Definition 3.12**

- For all pseudo-expressions $A$ we define $\tilde{A}$ to be the term $A$ where all partnered $\Pi$-symbols have been changed into $\lambda$-symbols.

- For a context $\Gamma \equiv d_1, \ldots, d_n$ we define $\tilde{\Gamma}$ to be $\tilde{d}_1, \ldots, \tilde{d}_n$, where $\pi x:A \equiv \lambda x:\tilde{A}$ and $(\pi x:A.\_\_)B \equiv (\lambda x:\tilde{A}.\_\_B)$.

**Lemma 3.13** If $\Gamma \vdash_{\beta n} A : B$ then $\tilde{\Gamma} \vdash_{e} \tilde{A} : \tilde{B}$, where $\vdash_{e}$ is the typing relation of systems of the $\lambda$-cube extended with definitions (no $\Pi$-definitions are allowed in the context, the reduction relation is $\beta$-reduction only, not $\beta\Pi$-reduction, and the abstraction rule has the old format, i.e. $\vdash_{\beta n} \Gamma \vdash_{\beta n} F : \Pi x:A.B \quad \Gamma \vdash_{\beta n} a : A \quad \text{see [BKN 9y].}$

**Proof**: Induction on the derivation rules of $\vdash_{\beta n}$. All rules except (application rule) are trivial since they are also rules in $\vdash_e$.

Now suppose $\Gamma \vdash_{\beta n} F a : (\Pi x:A.B)a$ as a consequence of $\Gamma \vdash_{\beta n} F : \Pi x:A.B$ and $\Gamma \vdash_{\beta n} a : A$. Then by the induction hypothesis $\tilde{\Gamma} \vdash_{e} \tilde{F} : \Pi x:\tilde{A}.\tilde{B}$ and $\tilde{\Gamma} \vdash_{e} \tilde{a} : \tilde{A}$, so by the application rule of $\vdash_{e}$, $\tilde{\Gamma} \vdash_{e} \tilde{F} \tilde{a} : \tilde{B}[x := \tilde{a}]$.

As a consequence of $\tilde{\Gamma} \vdash_{e} \tilde{F} : \Pi x:\tilde{A}.\tilde{B}$ we also get $\tilde{\Gamma}.\lambda x:\tilde{A} \vdash_{e} \tilde{B} : S$ and hence by thinning and definition rule for $\vdash_{e}$, $\tilde{\Gamma} \vdash_{e} (\lambda x:\tilde{A}.\tilde{B})\tilde{a} : S$, so by conversion $\tilde{\Gamma} \vdash_{e} \tilde{F} \tilde{a} : (\lambda x:\tilde{A}.\tilde{B})\tilde{a}$.

But $F$ cannot contain a $\Pi$-symbol which will mix with a in $Fa$ to form a $\Pi$-redex. Otherwise, one can show by the generation lemma that $\Pi x:A.B =_{def} S$ for some $S$. But this is impossible, hence $\tilde{Fa} \equiv \tilde{Fa}$. \hfill $\Box$

**Theorem 3.14** (Strong Normalisation for the Cube with respect to $\vdash_{\beta n}$ and $\rightarrow_{\beta n}$)

If $A$ is a $\vdash_{\beta n}$-legal term then $A$ is strongly normalising with respect to $\rightarrow_{\beta n}$.

**Proof**: If $A$ is $\vdash_{\beta n}$-legal then $\tilde{A}$ is $\vdash_{e}$-legal by Lemma 3.13 and hence $\tilde{A}$ is strongly normalising with respect to $\rightarrow_{\beta}$ (see [BKN 9y]). But then also $A$ is strongly normalising with respect to $\rightarrow_{\beta n}$. \hfill $\Box$
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