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The separating topology for the Lorentz group $L$

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Some properties of the Lorentz group $L$ are presented if it is endowed with a topology induced by one of the topologies for the Minkowski space $M$, proposed by E. C. Zeeman.

1. PRELIMINARIES

Let $M$ denote Minkowski space, the four-dimensional real vector space $R^4$, provided with the indefinite quadratic form

$$Q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2,$$

where $x = (x_0, x_1, x_2, x_3) \in M$. The vectors $x$ of $M$ are called timelike if $Q(x) > 0$, lightlike (or isotropic) if $Q(x) = 0$, and spacelike if $Q(x) < 0$.

$L$ is the full Lorentz group (all linear maps leaving $Q$ invariant). $L'$ is the orthochronous Lorentz group that is the subgroup of $L$ whose elements preserve the sign of the first coordinate. $L'$ is the subgroup of $L$ whose elements $t$ have the property $\det t = +1$.

Using the canonical basis of $R^4$ we introduce the parity $p$ by

$$p = (p_{ij}), \quad 0 \leq i, j \leq 3,$$

$$p_{00} = 1, \quad p_{ii} = -1, \quad 1 \leq i \leq 3,$$

and $p_{ij} = 0$ for all $i \neq j$. We shall also use the time reversal $t = -p$.

Notice that $L/L' \cong V_4$ where $V_4$ denotes Klein's four-group. By $O_2$ we mean the centralizer of $p$ in $L'$, that is to say the subgroup of $L'$ whose elements $r$ have the property $rpr^{-1} = r$. The elements of $O_2$ are called pure rotations.

$Z$ is the subgroup of $L'$ whose elements $z$ have the form

$$z = \begin{pmatrix}
\cosh \alpha & \sinh \alpha & 0 & 0 \\
\sinh \alpha & \cosh \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

We introduce furthermore $H$, being that subset of $L'$ whose elements $h$ have the property $php^{-1} = h^{-1}$. Notice that $h = h^t$ ($h$ is the transposed of $h$); $h$ is called hyperbolic screw. $L'$ has no proper invariant subgroups, cf. Ref. 1.

Let $SL(2, \mathbb{C})$ be the group of unimodular $2 \times 2$ matrices over the complex numbers. As is known, there is a surjective homomorphism $\varphi$ which induces an isomorphism

$$SL(2, \mathbb{C})/Z_2 \cong L',$$

where $Z_2$ is the set

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$
Ref. 10 Zeeman has proposed several non-Euclidean topologies for $M$ related to the Lorentz group $L$. Nanda\textsuperscript{11-13} investigated them and added some more of this kind of topologies. All these topologies have the property that the corresponding group of autohomomorphisms of $M$ coincides with $G$ and for that reason they seem to be physically significant. Unfortunately, they are very complicated from a topological point of view; for instance, they fail to satisfy the normal property and hence they are not metrizable. In this section we shall deal with that one of the topologies, proposed by Zeeman, that seems to be the most suitable for physics, cf. Ref. 9. We call it the separating topology. Similar topologies are also proposed by Cole\textsuperscript{14} and Cel'nik.\textsuperscript{15}

Let $d(x,y)$ denote the Euclidean metric

$$d(x,y) = \{(x_0-y_0)^2 + (x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2\}^{1/2}.$$ 

Given $x \in M$ and $\epsilon > 0$, let $N^\epsilon_2(x)$ denote the Euclidean $\epsilon$-neighbourhood of $x$, given by

$$N^\epsilon_2(x) = \{ y | d(x,y) < \epsilon \}.$$ 

We introduce

$$N^*_2(x) = N^\epsilon_2(x) \cap C(x) \setminus \{x\}^*, \quad x \in M \text{ (by } \setminus \text{ we mean the complement of a set V).}$$ 

Definition: The separating topology for $M$ is the topology, given by the basis of open sets $N^*_2(x)$, $x \in M$.

We use the notations $M_s$ for $M$ with the separating topology and $M_E$ for $M$ with the Euclidean topology.

Remark: It is also possible to define our topology by using only the relations $\prec, \prec_0, \preceq$. That offers the possibility of introducing the separating topology in more general causal spaces, cf. Refs. 9, 16.

Let $x, y, z \in M; y \prec x \prec_0 z$ and let us write

$$O_{1}(y,z) = I(y) \cap I(z) \cap C(x) \setminus \{x\}^*.$$ 

Clearly the topology for $M$ with basis open sets $O_{1}(y,z)$ is equivalent with the topology with basic open sets $N^*_2(x)$. Notice that $M_s$ is a Hausdorff space; it satisfies the first axiom of countability and it is a separable space but it does not have a countable basis. However $M_s$ is locally connected and pathwise connected it is not locally compact. From a physical point of view it seems to be interesting that on lightlike lines the discrete topology is induced and that on timelike lines and spacelike hyperplanes the Euclidean topology is induced, cf. Ref. 10.

Comparing $M_E$ and $M_s$ we still note the following properties:

1. The set $O$ is open in $M_s$ and not in $M_E$ if and only if for all $x \in O$ there is an $\epsilon > 0$ such that $N^\epsilon_2(x) \subset O$ and there is an $x \in O$ with the property $(C(x) \setminus \{x\}) \cap N^2_2(x) \cap \partial^* \neq \emptyset$ for all $\epsilon > 0$.

2. The subset $X$ of $M_s$ is compact in $M_E$ if and only if $X$ is compact in $M_s$ and all $x \in X$ are isolated in $X \cap C(x)$ (with respect to $M_E$).

3. The group of autohomomorphisms of $M_s$ is $G$.

For details we refer to Ref. 9.

3. THE SEPARATING TOPOLOGY FOR LORENTZ GROUP $L$

This is the main part of our paper; we shall investigate the topologies for $L$ induced by the separating topology for $M$. As is to be expected, $M_s$ induces a topology for $L_s$ deviating from the usual Lie group topology, such as we obtain by considering $L$ as a six-dimensional manifold in $\mathbb{R}^9$. There are several ways to topologize a set of maps. In this section we shall deal with the topology of pointwise convergence. See e. g., Ref. 17.

A. Introduction

For each $x \in M$ and for every open set $O \subset M$, we define

$$(x,O) = \{ t \in L_s | x \in O \}.$$ 

Let $L_s$ denote $L_s^*$, endowed with the topology that has the family of all sets $(x,O)$ as a subbasis, and let $L_E$ denote $L_E^*$, endowed with the topology, defined in a similar way as for $L_s$, but coming from $M_E$ instead of $M_s$. The family of intersections of sets of the form $(x,O)$ is a basis for the topological space $L_s$, each number of this basis having the form $\cap_{1}^{n} (x_i(O_i))$, where $x_i \in M$ and $O_i$ is open in $M_s$. Notice that $L_s$ is finer than $L_E$, for $M_s$ is finer than $M_E$. As we shall show below, $L_s$ is strictly finer. Notice furthermore that $L_s$ is a Hausdorff space, for $M_s$ has that property.

It is also possible to describe our topology by means of convergence of nets (see, e. g., Ref. 17, p. 77). To that end one can define: The net of Lorentz transformations $(l_n)$ converges to $l$ in $L_s$ if and only if $(l_n(x))$ converges to $l(x)$ for all $x \in M_s$. We shall say that a set $O \subset L_s$ is open if and only if every net $(l_n)$ converging to an element $l \in O_s$ is eventually in $O$. Remark that, if the net $(l_n)$ does not converge to $l$ in $L_s$, it does not converge to $l$ in $L_E$. As we shall show below, the converse is also true if we restrict ourselves to timelike vectors.

B. Properties of $L_s$

$L_s$ is strictly finer than $L_E$. Example.

Let $l_n = \left( \begin{array}{cccc} \cosh \frac{1}{n} & \sinh \frac{1}{n} & 0 & 0 \\ \sinh \frac{1}{n} & \cosh \frac{1}{n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \quad x = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right),$ 

$l$ is the unit element of $L_s$. In $L_E$ we find that $(l_n)$ converges to $l$ if $n \to \infty$, but $l_n(x) = e^{1/n}x$, and therefore $(l_n)$ does not converge in $L_s$, for $l_n(x) \notin N^*_2(x)$, even for all $n$. Also in the case of spacelike vectors, there are nets converging in $L_E$, but not in $L_s$. The same sequence $(l_n)$ as above, but applied to the spacelike vector $y = (1,1,0,1)$, gives us $l_n \notin N^*_2(y)$ for all $n$.

Theorem 1: $L_s$ and $L_E$ induce the same topology on the subgroup $O_C$.

Proof: It suffices to prove that a net of pure rotations $(r_n)$ converging in $L_E$, also converges in $L_s$ (with the same limit). Suppose that $(r_n)$ converges to $r$ in $L_E$. Then we have for all $x$ that eventually $[r_n(x)] \subset N^*_2(rx)$.
On the other hand, we know that all \( r, x \) are situated in the same spacelike hyperplane through \( r x \) and therefore
\[
\{(r, x) \in \mathbb{C}(r x) \setminus \{r x\} = \emptyset
\]
and, consequently,
\[
\{(r, x) \in N'_r(r x) = \{(r, x) \in N'_l(r x)\}
\]
This means that \((r, x)\) eventually belongs to \( N'_l(r x)\). In other words, \((r, x)\) converges to \( r \) in \( L_s \).

**Corollary:** \( L_s \) induces the same topology as \( L_s \) on every compact subgroup of \( L_s^* \), because \( O_l^* \) is a maximal compact subgroup of \( L_s \) and consequently of \( L_s \).

A semitopological group \( G \) is a topological space, provided with a group structure such that the product map \( G \times G \to G \), given by \((a, b) \to ab\), \((a, b) \in G\), is separately continuous. See, e.g., Ref. 18.

**Theorem 2:** \( L_s \) is a semitopological group.

**Proof:** Suppose that \((l_x)\) converges to \( l_y \), i.e., \((l_x)\) converges to \( l_x \), \( x \in M_s \). In particular, if we consider \( l'x \) instead of \( x \), then \((l_x)\) converges to \( l'x \). Therefore, for all neighborhoods \( O_y \) of \( ll' \) there is a neighborhood \( O_x \) of \( l \) such that \( O_x, l' \subset O_y \). On the other hand, we know that the elements of \( L_s \) are homeomorphisms of \( M_s \) and therefore it follows from \((l_x)\) converges to \( l'x \) for all \( l' \in L_s \), i.e., for all \( O_y \), there is a neighborhood \( O_y \) of \( l \) such that \( l' \in O_y \).

**C. The main theorem**

The definition of \( L_s \) uses the action of \( L_s^* \) on \( M_s \) and the topology of \( M_s \). Now we want to give an intrinsic definition of \( L_s \), by comparing it with \( L_s \). In Sec. 1 we have seen that \( L_s \) is very close to \( SL(2, \mathbb{C}) \).

**Lemma 1:** For timelike vectors \( x \), \((l_x)\) converges to \( l_x \), i.e., \((l_x)\) converges to \( l_x \) in \( M_s \) if and only if \((l_x)\) converges to \( l_x \) in \( M_s \).

**Proof:** Obviously, convergence in \( M_s \) implies convergence in \( M_s \). By the converse, we remark that the nets, converging in \( M_s \) and not in \( l_x \), are exactly those having the property that there is an \( x \) such that eventually
\[
(l_x - l_x, l_x - l_x) = 0 \quad \text{and} \quad l_x \neq l_x,
\]
i.e.,
\[
l^{-1}l_x - l_x, l^{-1}l_x - x = 0 \quad \text{and} \quad l^{-1}l_x x \neq x.
\]
It is sufficient to consider only one timelike vector. We choose \( x' = (a, 0, 0, 0) \) and note that it is possible to transform all timelike vectors, situated on the same hypersurface \( \{x \mid x = a \} \), into \( (a, 0, 0, 0) \) by a suitable Lorentz transformation \((a \neq 0)\). The intersection of \( \{x \mid x = a \} \) with the light cone \( C(x') \) consists only of the vertex \( x' \) of the cone. Therefore, the relations
\[
(l^{-1}l_x - x, l^{-1}l_x - x) = 0
\]
and \( l^{-1}l_x x \neq x \) do not hold together. In other words, \((l_x)\) converges to \( l_x \) in \( M_s \) implies that \((l_x)\) converges to \( l_x \) in \( M_s \).

Let \( \varphi \) denote the surjective homomorphism of \( SL(2, \mathbb{C}) \) onto \( L \) (as introduced in Sec. 1) and let \( B \) denote the image under \( \varphi \) of the set of upper triangular matrices of the form \([a \ b; 0 \ c]\) with \( |a| \neq 1 \).

**Lemma 2:** Let \( x \) be an isotropic vector and let \((l_x)\) be a net of Lorentz transformations. Then \((l_x)\) converges to \( l_x \) in \( M_s \) if and only if
\begin{enumerate}
\item \( l_x \) converges to \( l_x \) in \( G_s \),
\item no \( \tilde{l} \in L \) exists such that eventually \( \tilde{l}^{-1}l_x \tilde{l} \in B \).
\end{enumerate}

**Proof:** Similarly, as in the proof of Lemma 1, it suffices to consider only one isotropic vector. We choose \( x' = (0, 1, 0, 0) \) and (compare Sec. 1) the relation
\[
l_x = s x, x \ast
\]
written out and applied to our situation, becomes
\[
\left( y_0 + y_1, y_2 - y_3 \right) = \left( \alpha \beta \gamma \delta \right) \left( y_0 \ 0 \ 0 \ \gamma \right) = 2 \left( \alpha \beta \gamma \delta \right) = 2 \left( \alpha \beta \gamma \delta \right).
\]
Again, we have to exclude nets \((l_x)\) with the property that eventually \( l^{-1}l_x x = x, l^{-1}l_x x = x \) and \( l^{-1}l_x x \neq x \). The intersection of \( \{x \mid x = 0\} \) and the light cone \( C(x') \) only consists of the line \( \lambda_1 = (1, 0, 0, 0), \lambda \in \mathbb{R}, \) and therefore we must look for \( l_x \) with \( l^{-1}l_x x = \lambda, x' (\lambda \neq 1) \).

Let
\[
l_x = \varphi \left( \alpha \beta \gamma \delta \right);
\]
then for such \( l_x \) we have
\[
\left| \alpha \right|^2 = 1, \quad \left| \alpha \beta \gamma \delta \right| = 0, \quad \left| \alpha \beta \gamma \delta \right| = 0;
\]
in other words,
\[
\left| \alpha \beta \gamma \delta \right| = 1 \quad \text{and} \quad \left| \alpha \beta \gamma \delta \right| = 0.
\]
Consequently, the \( 2 \times 2 \) matrices in question correspond with elements \( b \in B \); i.e., \( l^{-1}l_x = b \) or \( l^{-1}l_x = b \). The Lorentz transformations, leaving invariant the other one-dimensional isotropic subspaces, have the form \( \tilde{l}l \tilde{b}, \tilde{l}l \tilde{b}^{-1} \), where \( \tilde{l} \) is a suitable Lorentz transformation.

Summarizing, we have to exclude \( l_x \) eventually satisfying the relation \( l^{-1}l_x = \tilde{l}l \tilde{b}^{-1} \) or \( l^{-1}l_x \tilde{l} = b \). Now the proof is complete.

**Corollary:** \( L_s \) induces the discrete topology on the subgroup \( Z \) and on its conjugates.

**Proof:** As is known the elements of \( \varphi^{-1}(z) \) have the form \([y \ z; 0 \ 0]\) with \( l \in \mathbb{R}' \), being a subset of \( B \).

Notice that in the case of \( Z \) there are two isotropic eigenvectors, viz., \((1, 0, 0, 0, 0) \) but in the case of \( B \) there is only the isotropic eigenvector \((1, 1, 0, 0) \).

Let \( C \) be the image under \( \varphi \) of the matrices \([y \ z; 0 \ 0]\) of \( SL(2, \mathbb{C}) \) with properties:
\begin{enumerate}
\item \( \left| \alpha \right|^2 + \left| \beta \right|^2 + \left| \gamma \right|^2 + \left| \delta \right|^2 = 2,
\item \( \left| \alpha \beta \gamma \delta \right| = 0.
\end{enumerate}

**Lemma 3:** Let \( x \) be a spacelike vector and let \((l_x)\) be a net of Lorentz transformations. Then \((l_x)\) converges to \( l_x \) in \( M_s \) if and only if
\begin{enumerate}
\item \( l_x \) converges to \( l_x \) in \( M_s \),
\item no \( \tilde{l} \in L \) exists such that eventually \( \tilde{l}^{-1}l_x \tilde{l} \in C \).
\end{enumerate}

**Proof:** Again we only need one spacelike vector to start with and we choose \( x' = (0, a, 0, 0) \) and \( [a \ b; 0 \ c]\) with \( |a| \neq 1 \).

\[ \text{1212 J. Math. Phys., Vol. 16, No. 6, June 1975} \]

P.G. Vroegindeweij
hypsersurface \( (x, x) = -a^2 \) (\( a \neq 0 \)). Similarly, as for Lemma 2, we find
\[
\begin{align*}
(\gamma + i y, y - iy) &= (\alpha, \beta) (0, 0, \sqrt{\alpha}, \sqrt{\beta}) \\
(\gamma + i y, y - y) &= (\alpha, \beta) (0, 0, \sqrt{a}, \sqrt{b}) \\
(\alpha, \beta) &= (\alpha^2 - \beta^2, \alpha \gamma - \beta b).
\end{align*}
\]

Now the intersection of \( \{x \mid (x, x) = -a^2 \} \) and \( C(x) \) is situated in the hyperplane \( x_1 = a \) and therefore we have to look for the elements of \( L'_s \), transforming \( (0, a, 0, 0) \) into \((v, a, \nu \cos u, v \sin u)\), where \( v \neq 0 \).

This means that
\[
\begin{align*}
v + a &\equiv \nu e^{i \alpha} \\
v - a &= \nu e^{i \beta},
\end{align*}
\]

or
\[
\begin{align*}
v + a &= e(\alpha^2 - \beta^2), \\
v - a &= e(\gamma^2 - \delta^2), \\
v e^{i \varphi} &= e(\nu - \beta b),
\end{align*}
\]

and these relations are equivalent with the conditions:
(i) \( \alpha^2 - \beta^2 - \gamma^2 + \delta^2 = 2 \),
(ii) \( \alpha^2 - \beta^2 = 1 \ (v \neq 0) \),
(iii) \( (\alpha^2 - \beta^2 - 1)^2 = (\alpha \gamma - \beta b)^2 \),

but condition (iii) is superfluous for it is implied by (i) and \( \alpha \beta = \beta \gamma = 1 \). Similarly, as for Lemma 2, it turns out that in this case we must exclude the nets \( (L) \) with the property that eventually \( T^{-1} \cdot T \cdot \ell = c \), with \( c \in C \).

Now we are able to state:

**Theorem:** Let \((L)\) be a net of Lorentz transformations. Then \((L)\) converges to \( l \) in \( L_s \) if and only if:
(i) \( (l) \) converges to \( l \) in \( L_x \),
(ii) \( l \) is a point in \( L_s \) and each such point eventually \( T^{-1} \cdot T \cdot \ell \in B \cup C \) (\( B \) and \( C \) as defined above).

**Proof:** The theorem follows immediately from the Lemmas 1, 2, and 3.

**Remarks:**
1. The theorem gives us an intrinsic definition of the topology of \( L_s \) by means of convergence of nets.
2. The topology of \( M_s \) can be recovered from \( L_s \).

3. The Lorentz transformations that we have excluded for convergence, are exactly those leaving not only \( (x, x) \) invariant but also the intersections of the hypersurfaces \( (x, x) = p \) and the light cones in the points of contact. In Lemma 1 this intersection only consists of one point and in Lemma 2 we found the one-dimensional subvarieties.

4. Probably the condition \( |\alpha| - |\beta| + |\gamma| - |\delta| = 2 \) has to do with the roots of the equation \( x^2 = c \).

5. The set \( B \cup C \) has the property that \( l \in B \cup C \) implies \( T^{-1} \cdot T \cdot \ell \in B \cup C \) and therefore \( L_s \) is a \( T_1 \)-group (see Ref. 19, p. 27). As is known (cf. Ref. 1) \( L_s \) has no proper invariant subgroups and hence \( L_s \) is connected (see Ref. 19, p. 28).

6. Probably \( L_s \) has any representations that are not representations of \( L \); these representations might lead to new invariants of physics. I did not succeed in finding examples of these new representations until now.