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The separating topology for the Lorentz group L

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Some properties of the Lorentz group L are presented if it is endowed with a topology induced by one of the topologies for the Minkowski space M , proposed by E. C. Zeeman.

1. PRELIMINARIES

Let M denote Minkowski space, the four-dimensional real vector space $\mathbb{R}^4$, provided with the indefinite quadratic form

$$Q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2,$$

where $x = (x_0, x_1, x_2, x_3) \in M$. The vectors x of M are called *timelike* if $Q(x) > 0$, *lightlike* (or *isotropic*) if $Q(x) = 0$, and *spacelike* if $Q(x) < 0$.

L is the full Lorentz group (all linear maps leaving Q invariant). L' is the orthochronous Lorentz group that is the subgroup of L whose elements preserve the sign of the first coordinate. L'_+ is the subgroup of L' whose elements l have the property $\det l = +1$.

Using the canonical basis of $\mathbb{R}^4$ we introduce the *parity* p by

$$p = (p_{ij}), \quad 0 \leq i, j \leq 3,$$

$$p_{00} = 1, \quad p_{ii} = -1, \quad 1 \leq i \leq 3,$$

and $p_{ij} = 0$ for all $i \neq j$. We shall also use the *time reversal* $l = -p$.

Notice that $L/L'_+ \cong V_4$ where V_4 denotes Klein's four-group. By O_3^* we mean the centralizer of p in L'_+ that is to say the subgroup of L'_+ whose elements r have the property $prp^{-1} = r$. The elements of O_3^* are called *pure rotations*.

Z is the subgroup of L'_+ whose elements z have the form

$$z = \begin{pmatrix} \cosh \alpha & \sinh \alpha & 0 & 0 \\ \sinh \alpha & \cosh \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We introduce furthermore HS , being that subset of L'_+ whose elements h have the property $php^{-1} = h^{-1}$. Notice that $h = {}^t h$ (${}^t h$ is the transposed of h); h is called *hyperbolic screw*. L'_+ has no proper invariant subgroups, cf. Ref. 1.

Let $SL(2, \mathbb{C})$ be the group of unimodular 2×2 matrices over the complex numbers. As is known, there is a surjective homomorphism φ which induces an isomorphism

$$SL(2, \mathbb{C})/Z_2 \cong L'_+$$

where Z_2 is the set

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

The homomorphism φ can be described in the following way (cf. Ref. 2): Let $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ and let $\tilde{x}$ denote the Hermitian matrix

$$\begin{pmatrix} x_0 + x_1 & x_2 - ix_3 \\ x_2 + ix_3 & x_0 - x_1 \end{pmatrix} \in M(2, \mathbb{C}).$$

Consider the bijection $f: \mathbb{R}^4 \rightarrow M(2, \mathbb{C})$, given by $f(x) = \tilde{x}$. Let $l \in L'_+$ and $s = \varphi^{-1}(l) \in SL(2, \mathbb{C})$. We have the relation $l\tilde{x} = s\tilde{x}s^*$, where $\tilde{x} \rightarrow l\tilde{x}$ is a Hermitian map.

Using matrix language, we may write

$$\begin{pmatrix} y_0 + y_1 & y_2 - iy_3 \\ y_2 + iy_3 & y_0 - y_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x_0 + x_1 & x_2 - ix_3 \\ x_2 + ix_3 & x_0 - x_1 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

where $y = lx = (y_0, y_1, y_2, y_3)$ and $s = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

We shall also use the group G , that is the group generated by L , the group T of translations of M and the group of multiplications by a positive scalar of the vectors of M . G' is the subgroup of G that we obtain by considering L' instead of L .

There is a partial order $\ll$ on M given by $x \ll y$ if and only if $Q(y - x) > 0$ and $x_0 < y_0$. Another partial order $<$ on M is given by $x < y$ if and only if $Q(y - x) \geq 0$ and $x_0 \leq y_0$. We still need the relation $<^\circ$, given on M by $x <^\circ y$ if and only if $Q(y - x) = 0$ and $x_0 < y_0$. We introduce furthermore the sets:

$$C(x) = \{y \mid Q(y - x) = 0\},$$

$$S(x) = \{y \mid Q(y - x) < 0\},$$

$$I(x) = \{y \mid Q(y - x) > 0\},$$

$$I^*(x) = \{y \mid x \ll y\},$$

$$I^*(x) = \{y \mid y \ll x\}.$$

C is the group of bijections of M , preserving the relation $\ll$.

Zeeman³ proved that C and G' coincide. Zeeman's theorem has been generalized in several ways, cf. Refs. 4–9.

2. THE SEPARATING TOPOLOGY FOR MINKOWSKI SPACE M

Usually M is endowed with the Euclidean topology, but one can argue (Zeeman^{3,10}) that this is objectionable for physical reasons. On the other hand, it is impossible to define a topology for M by means of the indefinite quadratic form Q in a way similar to the Euclidean topology by means of the definite quadratic form. In

Ref. 10 Zeeman has proposed several non-Euclidean topologies for M related to the Lorentz group L . Nanda¹¹⁻¹³ investigated them and added some more of this kind of topologies. All these topologies have the property that the corresponding group of autohomeomorphisms of M coincides with G and for that reason they seem to be physically significant. Unfortunately, they are very complicated from a topological point of view; for instance, they fail to satisfy the normal property and hence they are not metrizable. In this section we shall deal with that one of the topologies, proposed by Zeeman, that seems to be the most suitable for physics, cf. Ref. 9. We call it the *separating topology*. Similar topologies are also proposed by Cole¹⁴ and Cel'nik.¹⁵

Let $d(x, y)$ denote the Euclidean metric

$$d(x, y) = \{(x_0 - y_0)^2 + (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2\}^{1/2}.$$

Given $x \in M$ and $\epsilon > 0$, let $N_E^\epsilon(x)$ denote the Euclidean ϵ -neighbourhood of x , given by

$$N_E^\epsilon(x) = \{y \mid d(x, y) < \epsilon\}.$$

We introduce

$$N_s^\epsilon(x) = N_E^\epsilon(x) \cap (C(x) \setminus \{x\})^*, \quad x \in M$$

(by V^* we mean the complement of a set V).

Definition: The separating topology for M is the topology, given by the basis of open sets $N_s^\epsilon(x)$, $x \in M$.

We use the notations M_s for M with the separating topology and M_E for M with the Euclidean topology.

Remark: It is also possible to define our topology by using only the relations $\ll$, $<$, and $<\cdot$. That offers the possibility of introducing the separating topology in more general causal spaces, cf. Refs. 9, 16.

Let $x, y, z \in M$; $y \ll x \ll z$ and let us write

$$O_x(y, z) = I^+(y) \cap I^-(z) \cap (C(x) \setminus \{x\})^*.$$

Clearly the topology for M with basic open sets $O_x(y, z)$ is equivalent with the topology with basic open sets $N_s^\epsilon(x)$. Notice that M_s is a Hausdorff space; it satisfies the first axiom of countability and it is a separable space but it does not have a countable basis. However M_s is locally connected and pathwise connected it is not locally compact. From a physical point of view it seems to be interesting that on lightlike lines the discrete topology is induced and that on timelike lines and spacelike hyperplanes the Euclidean topology is induced, cf. Ref. 10.

Comparing M_E and M_s we still note the following properties:

- (1) The set O is open in M_s and not in M_E if and only if for all $x \in O$ there is an $\epsilon > 0$ such that $N_s^\epsilon(x) \subset O$ and there is an $x \in O$ with the property $(C(x) \setminus \{x\}) \cap N_E^\epsilon(x) \cap O^* \neq \emptyset$ for all $\epsilon > 0$.
- (2) The subset X of M_s is compact in M_s if and only if X is compact in M_E and all $x \in X$ are isolated in $X \cap C(x)$ (with respect to M_E).
- (3) The group of autohomeomorphisms of M_s is G .

For details we refer to Ref. 9.

3. THE SEPARATING TOPOLOGY FOR LORENTZ GROUP L

This is the main part of our paper; we shall investigate the topology for L induced by the separating topology for M . As is to be expected, M_s induces a topology for L , deviating from the usual Lie group topology, such as we obtain by considering L as a six-dimensional manifold in $\mathbb{R}^9$. There are several ways to topologize a set of maps. In this section we shall deal with the topology of pointwise convergence. See e.g., Ref. 17.

A. Introduction

For each $x \in M_s$ and for every open set $O \subset M_s$ we define

$$(x, O) = \{l \in L_s^+ \mid lx \in O\}.$$

Let L_s denote L_s^+ , endowed with the topology that has the family of all sets (x, O) as a subbasis, and let L_E denote L_s^+ , endowed with the topology, defined in a similar way as for L_s , but coming from M_E instead of M_s . The family of intersections of sets of the form (x, O) is a basis for the topological space L_s , each number of this basis having the form $\bigcap_{i=1}^n (x_i, O_i)$, where $x_i \in M_s$ and O_i is open in M_s . Notice that L_s is finer than L_E , for M_s is finer than M_E . As we shall show below, L_s is strictly finer. Notice furthermore that L_s is a Hausdorff space, for M_s has that property.

It is also possible to describe our topology by means of convergence of nets (see, e.g., Ref. 17, p. 77). To that end one can define: The net of Lorentz transformations (l_ν) converges to l in L_s if and only if $(l_\nu x)$ converges to lx for all $x \in M_s$. We shall say that a set $O \subset L_s$ is open if and only if every net (l_ν) , converging to an element $l \in O$, is eventually in O . Remark that, if the net (l_ν) does not converge to l in L_E , it does not converge to l in L_s . As we shall show below, the converse is also true if we restrict ourselves to timelike vectors.

B. Properties of L_s

L_s is strictly finer than L_E . Example:

$$l_n = \begin{pmatrix} \cosh \frac{1}{n} & \sinh \frac{1}{n} & 0 & 0 \\ \sinh \frac{1}{n} & \cosh \frac{1}{n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

l is the unit element of L_s^+ . In L_E we find that (l_n) converges to l if $n \rightarrow \infty$, but $l_n x = e^{1/n} x$, and therefore (l_n) does not converge in L_s , for $l_n(x) \notin N_s^\epsilon(x)$, even for all n . Also in the case of spacelike vectors, there are nets converging in L_E , but not in L_s . The same sequence (l_n) as above, but applied to the spacelike vector $y = (1, 1, 0, 1)$, gives us $l_n y \notin N_s^\epsilon(y)$ for all n .

Theorem 1: L_s and L_E induce the same topology on the subgroup O_3^+ .

Proof: It suffices to prove that a net of pure rotations (r_ν) , converging in L_E , also converges in L_s (with the same limit). Suppose that (r_ν) converges to r in L_E . Then we have for all x that eventually $\{(r_\nu x)\} \subset N_E^\epsilon(rx)$.

On the other hand, we know that all $r_\nu x$ are situated in the same spacelike hyperplane through rx and therefore

$$\{(r_\nu x)\} \cap (C(rx) \setminus \{rx\}) = \emptyset$$

and, consequently,

$$\{(r_\nu x)\} \cap N_s^\varepsilon(rx) = \{(r_\nu x)\} \cap N_E^\varepsilon(rx).$$

This means that $(r_\nu x)$ eventually belongs to $N_s^\varepsilon(rx)$. In other words, (r_ν) converges to r in L_s . $\square$

Corollary: L_s induces the same topology as L_E on every compact subgroup of L_+^1 , because 0_+^1 is a maximal compact subgroup of L_E and consequently of L_s .

A *semitopological group* G is a topological space, provided with a group structure such that the product map $G \times G \rightarrow G$, given by $(a, b) \rightarrow ab$, $(a, b \in G)$, is separately continuous. See, e.g., Ref. 18.

Theorem 2: L_s is a semitopological group.

Proof: Suppose that (l_ν) converges to l , i.e., $(l_\nu x)$ converges to lx , $x \in M_s$. In particular, if we consider $l'x$ instead of x , then $(l_\nu l'x)$ converges to $ll'x$. Therefore, for all neighborhoods $O_{ll'}$ of $ll'x$ there is a neighborhood O_l of l such that $O_l \cdot l' \subset O_{ll'}$. On the other hand, we know that the elements of L_s are homeomorphisms of M_s and therefore it follows from $(l_\nu x)$ converges to lx that $(l' l_\nu x)$ converges to $l'l x$ for all $l' \in L$, i.e., for all $O_{ll'}$ there is a neighborhood O_l of l such that $l'O_l \subset O_{ll'}$. $\square$

C. The main theorem

The definition of L_s uses the action of L_+^1 on M and the topology of M_s . Now we want to give an intrinsic definition of L_s , by comparing it with L_E . In Sec. 1 we have seen that L_+^1 is very close to $SL(2, \mathbb{C})$.

Lemma 1: For timelike vectors x , $(l_\nu x)$ converges to lx in M_E if and only if $(l_\nu x)$ converges to lx in M_s .

Proof: Obviously, convergence in M_s implies convergence in M_E . To prove the converse, we remark that the nets, converging in L_E and not in L_s , are exactly those having the property that there is an x such that eventually

$$(l_\nu x - lx, l_\nu x - lx) = 0 \text{ and } l_\nu x \neq lx,$$

i.e.,

$$(l^{-1} l_\nu x - x, l^{-1} l_\nu x - x) = 0 \text{ and } l^{-1} l_\nu x \neq x.$$

It is sufficient to consider only one timelike vector. We choose $x' = (a, 0, 0, 0)$ and note that it is possible to transform all timelike vectors, situated on the same hypersurface $(x, x) = a^2$, into $(a, 0, 0, 0)$ by a suitable Lorentz transformation ($a \neq 0$). The intersection of $\{x | (x, x) = a^2\}$ and the light cone $C(x')$ consists only of the vertex x' of the cone. Therefore, the relations

$$(l^{-1} l_\nu x - x, l^{-1} l_\nu x - x) = 0$$

and $l^{-1} l_\nu x' \neq x'$ do not hold together. In other words, $(l_\nu x)$ converges to lx in M_E implies that $(l_\nu x)$ converges to lx in M_s . $\square$

Let φ denote the surjective homomorphism of $SL(2, \mathbb{C})$ onto L_+^1 (as introduced in Sec. 1) and let B denote the image under φ of the set of upper triangular matrices of the form $\begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix}$ with $|\alpha| \neq 1$.

Lemma 2: Let x be an isotropic vector and let (l_ν) be

a net of Lorentz transformations. Then $(l_\nu x)$ converges to lx in M_s if and only if

(i) $l_\nu x$ converges to lx in M_E ,

(ii) no $\bar{l} \in L$ exists such that eventually $\bar{l}^{-1} l^{-1} l_\nu \bar{l} \in B$.

Proof: Similarly, as in the proof of Lemma 1, it suffices to consider only one isotropic vector. We choose $x' = (1, 1, 0, 0)$ and (compare Sec. 1) the relation

$$\tilde{l}x = s\tilde{x}s^*,$$

written out and applied to our situation, becomes

$$\begin{pmatrix} y_0 + y_1 & y_2 - iy_3 \\ y_2 + iy_3 & y_0 - y_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix} = 2 \begin{pmatrix} |\alpha|^2 & \alpha\bar{\gamma} \\ \bar{\alpha}\gamma & |\gamma|^2 \end{pmatrix}.$$

Again, we have to exclude nets (l_ν) with the property that eventually $(l^{-1} l_\nu x - x, l^{-1} l_\nu x - x) = 0$ and $l^{-1} l_\nu x \neq x$. The intersection of $\{x | (x, x) = 0\}$ and the light cone $C(x')$ only consists of the line $\lambda(1, 1, 0, 0)$, $\lambda \in \mathbb{R}$, and therefore we must look for l_ν with $l^{-1} l_\nu x' = \lambda_\nu x'$ ($\lambda_\nu \neq 1$).

Let

$$l_\nu = \varphi \begin{pmatrix} \alpha_\nu & \beta_\nu \\ \gamma_\nu & \delta_\nu \end{pmatrix};$$

then for such l_ν we have

$$|\alpha_\nu|^2 \neq 1, \quad \alpha_\nu \bar{\gamma}_\nu = 0, \quad \bar{\alpha}_\nu \gamma_\nu = 0, \quad |\bar{\gamma}_\nu|^2 = 0;$$

in other words,

$$|\alpha_\nu|^2 \neq 1 \text{ and } \gamma_\nu = 0.$$

Consequently, the 2×2 matrices in question correspond with elements $b \in B$; i.e., $l^{-1} l_\nu = b$ or $l_\nu = lb$. The Lorentz transformations, leaving invariant the other one-dimensional isotropic subspaces, have the form $\bar{l}b\bar{l}^{-1}$, where $\bar{l}$ is a suitable Lorentz transformation. Summarizing, we have to exclude l_ν , eventually satisfying the relation $l^{-1} l_\nu = \bar{l}b\bar{l}^{-1}$ or $\bar{l}^{-1} l^{-1} l_\nu \bar{l} = b$. Now the proof is complete. $\square$

Corollary: L_s induces the discrete topology on the subgroup Z and on its conjugates.

Proof: As is known the elements of $\varphi^{-1}(z)$ have the form $\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$ with $t \in \mathbb{R}^*$, being a subset of B . $\square$

Notice that in the case of Z there are two isotropic eigenvectors, viz., $(1, \pm 1, 0, 0)$ but in the case of B there is only the isotropic eigenvector $(1, 1, 0, 0)$.

Let C be the image under φ of the matrices $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ of $SL(2, \mathbb{C})$ with properties:

$$(i) \quad |\alpha|^2 - |\beta|^2 - |\gamma|^2 + |\delta|^2 = 2,$$

$$(ii) \quad |\alpha|^2 - |\beta|^2 \neq 1.$$

Lemma 3: Let x be a spacelike vector and let (l_ν) be a net of Lorentz transformations. Then $(l_\nu x)$ converges to lx in M_s , if and only if:

(i) $(l_\nu x)$ converges to lx in M_E ,

(ii) no $\bar{l} \in L$ exists such that eventually $\bar{l}^{-1} l^{-1} l_\nu \bar{l} \in C$.

Proof: Again we only need one spacelike vector to start with and we choose $x' = (0, a, 0, 0)$, situated on the

hypersurface $(x, x) = -a^2$ ($a \neq 0$). Similarly, as for Lemma 2, we find

$$\begin{pmatrix} y_0 + y_1 & y_2 - iy_3 \\ y_2 + iy_3 & y_0 - y_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix} \\ = \begin{pmatrix} |\alpha|^2 - |\beta|^2 & \alpha\bar{\gamma} - \beta\bar{\delta} \\ \bar{\alpha}\gamma - \bar{\beta}\delta & |\gamma|^2 - |\delta|^2 \end{pmatrix}.$$

Now the intersection of $\{x|(x, x) = -a^2\}$ and $C(x')$ is situated in the hyperplane $x_1 = a$ and therefore we have to look for the elements of L_s , transforming $(0, a, 0, 0)$ into $(v, a, v \cos u, v \sin u)$, where $v \neq 0$.

This means that

$$\begin{pmatrix} v + a & v e^{-iu} \\ v e^{iu} & v - a \end{pmatrix} = a \begin{pmatrix} |\alpha|^2 - |\beta|^2 & \alpha\bar{\gamma} - \beta\bar{\delta} \\ \bar{\alpha}\gamma - \bar{\beta}\delta & |\gamma|^2 - |\delta|^2 \end{pmatrix}.$$

or

$$\begin{aligned} v + a &= a(|\alpha|^2 - |\beta|^2), \\ v - a &= a(|\gamma|^2 - |\delta|^2), \\ v e^{iu} &= a(\bar{\alpha}\gamma - \bar{\beta}\delta), \end{aligned}$$

and these relations are equivalent with the conditions:

- (i) $|\alpha|^2 - |\beta|^2 - |\gamma|^2 + |\delta|^2 = 2$,
- (ii) $|\alpha|^2 - |\beta|^2 \neq 1$ ($v \neq 0$),
- (iii) $(|\alpha|^2 - |\beta|^2 - 1)^2 = |\bar{\alpha}\gamma - \bar{\beta}\delta|^2$;

but condition (iii) is superfluous for it is implied by (i) and $\alpha\delta - \beta\gamma = 1$. Similarly, as for Lemma 2, it turns out that in this case we must exclude the nets (l_v) with the property that eventually $\bar{l}^{-1}l^{-1}l_v\bar{l} = c$, with $c \in C$. $\square$

Now we are able to state:

Theorem: Let (l_v) be a net of Lorentz transformations. Then (l_v) converges to l in L_s if and only if:

- (i) (l_v) converges to l in L_E ,
- (ii) no $\bar{l} \in L$ exists such that eventually $\bar{l}^{-1}l^{-1}l_v\bar{l} \in B \cup C$ (B and C as defined above).

Proof: The theorem follows immediately from the Lemmas 1, 2, and 3. $\square$

Remarks:

- 1. The theorem gives us an intrinsic definition of the topology of L_s by means of convergence of nets.
- 2. The topology of M_s can be recovered from L_s .

3. The Lorentz transformations that we have excluded for convergence, are exactly those leaving not only (x, x) invariant but also the intersections of the hypersurfaces $(x, x) = p$ and the light cones in the points of contact. In Lemma 1 this intersection only consists of one point and in Lemma 2 we found the one-dimensional subvarieties.

4. Probably the condition $|\alpha|^2 - |\beta|^2 - |\gamma|^2 + |\delta|^2 = 2$ has to do with the roots of the equation $s\tilde{x}s^* = \lambda\tilde{x}$.

5. The set $B \cup C$ has the property that $l \in B \cup C$ implies $l^{-1} \in B \cup C$ and therefore L_s is a T_1 -group (see Ref. 19, p. 27). As is known (cf. Ref. 1) L_s has no proper invariant subgroups and hence L_s is connected (see Ref. 19, p. 28).

6. Probably L_s has any representations that are not representations of L_E ; these representations might lead to new invariants of physics. I did not succeed in finding examples of these new representations until now.

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