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The separating topology for the Lorentz group $L$

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Some properties of the Lorentz group $L$ are presented if it is endowed with a topology induced by one of the topologies for the Minkowski space $M$, proposed by E. C. Zeeman.

1. PRELIMINARIES

Let $M$ denote Minkowski space, the four-dimensional real vector space $\mathbb{R}^4$, provided with the indefinite quadratic form

$$Q(x) = x_1^2 - x_2^2 - x_3^2 - x_4^2,$$

where $x = (x_0, x_1, x_2, x_3) \in M$. The vectors $x$ of $M$ are called timelike if $Q(x) > 0$, lightlike (or isotropic) if $Q(x) = 0$, and spacelike if $Q(x) < 0$.

$L$ is the full Lorentz group (all linear maps leaving $Q$ invariant). $L'$ is the orthochronous Lorentz group that is the subgroup of $L$ whose elements preserve the sign of the first coordinate. $L''$ is the subgroup of $L'$ whose elements $l$ have the property $\det l = +1$.

Using the canonical basis of $\mathbb{R}^4$ we introduce the parity $p$ by

$$p = (p_{ij}), \quad 0 \leq i, j \leq 3,$$

$$p_{00} = 1, \quad p_{ii} = -1, \quad 1 \leq i \leq 3,$$

and $p_{ij} = 0$ for all $i \neq j$. We shall also use the time reversal $l = -p$.

Notice that $L/L', \cong V_4$, where $V_4$ denotes Klein's four-group. By $G'$ we mean the centralizer of $p$ in $L'$, that is to say the subgroup of $L'$ whose elements $r$ have the property $pp^{-1} = r$. The elements of $G'$ are called pure rotations.

$Z$ is the subgroup of $L'$, whose elements $z$ have the form

$$z = \begin{pmatrix}
\cosh \alpha & \sinh \alpha & 0 & 0 \\
\sinh \alpha & \cosh \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

We introduce furthermore $HS$, being that subset of $L'$ whose elements $h$ have the property $h p h^{-1} = p$. Notice that $h = h^t$ ($h^t$ is the transposed of $h$); $h$ is called hyperbolic screw. $L'$ has no proper invariant subgroups, cf. Ref. 1.

Let $SL(2, \mathbb{C})$ be the group of unimodular $2 \times 2$ matrices over the complex numbers. As is known, there is a surjective homomorphism $\varphi$ which induces an isomorphism

$$SL(2, \mathbb{C})/Z_2 \cong L',$$

where $Z_2$ is the set

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

The homomorphism $\varphi$ can be described in the following way (cf. Ref. 2): Let $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ and let $\bar{x}$ denote the Hermitian matrix

$$\begin{pmatrix}
x_0 + x_1 & x_2 - i x_3 \\
x_2 + i x_3 & x_0 - x_1
\end{pmatrix} \in M(2, \mathbb{C}).
$$

Consider the bijection $f : \mathbb{R}^4 \rightarrow M(2, \mathbb{C})$, given by $f(x) = \bar{x}$. Let $l \in L'$ and $s = \varphi^{-1}(l) \in SL(2, \mathbb{C})$. We have the relation $s \bar{x} s^* = \bar{x}$.

Using matrix language, we may write

$$\begin{pmatrix}
y_0 + y_1 & y_2 - i y_3 \\
y_2 + i y_3 & y_0 - y_1
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta \\
\bar{\beta} & \bar{\alpha}
\end{pmatrix}\begin{pmatrix}
x_0 + x_1 & x_2 - i x_3 \\
x_2 + i x_3 & x_0 - x_1
\end{pmatrix}\begin{pmatrix}
\alpha & \beta \\
\bar{\beta} & \bar{\alpha}
\end{pmatrix},$$

where $\gamma = ix = (y_0, y_1, y_2, y_3)$ and $s = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$.

We shall also use the group $G$, that is the group generated by $L$, the group $T$ of translations of $M$ and the group of multiplications by a positive scalar of the vectors of $M$. $G'$ is the subgroup of $G$ that we obtain by considering $L'$ instead of $L$.

There is a partial order $\ll$ on $M$ given by $x \ll y$ if and only if $Q(y-x) > 0$ and $x_0 < y_0$. Another partial order $<$ on $M$ is given by $x < y$ if and only if $Q(y-x) > 0$ and $x_0 < y_0$. We still need the relation $\ll$, given on $M$ by $x < y$ if and only if $Q(y-x) = 0$ and $x_0 < y_0$. We introduce furthermore the sets:

$$C(x) = \{ y | Q(y-x) = 0 \},$$

$$S(x) = \{ y | Q(y-x) < 0 \},$$

$$I(x) = \{ y | Q(y-x) > 0 \},$$

$$I'(x) = \{ y | x \ll y \},$$

$$I''(x) = \{ y | y \ll x \}.$$

$C$ is the group of bijections of $M$, preserving the relation $\ll$.

Zeeman proved that $C$ and $G'$ coincide. Zeeman's theorem has been generalized in several ways, cf. Refs. 4–9.

2. THE SEPARATING TOPOLOGY FOR MINKOWSKI SPACE $M$

Usually $M$ is endowed with the Euclidean topology, but one can argue (Zeeman, Ref. 10) that this is objectionable for physical reasons. On the other hand, it is impossible to define a topology for $M$ by means of the indefinite quadratic form $Q$ in a way similar to the Euclidean topology by means of the definite quadratic form. In
is locally connected and path wise connected.

From a physical point of view it seems

that on timelike lines and spacelike

lines the discrete topology is induced and that on lightlike lines the

discrete topology is also possible to describe our topology by means

Remark: It is also possible to define our topology by using only the relations \( \leq, <, > \). That offers the possibility of introducing the separating topology in more general causal spaces, cf. Refs. 9, 16.

Let \( x, y, z \in M \); \( y < x < z \) and let us write

\[
O_{1}(y, z) = \{ r \in \mathbb{R}^{n} | r \in (x, y) \} \cap \mathbb{R}^{n}/(x, z) \cap (\mathbb{R}^{n} \setminus (y, z)).
\]

Clearly the topology for \( M \) with basic open sets \( O_{1}(x, z) \) is equivalent with the topology with basic open sets \( N_{1}(x) \). Notice that \( M_{s} \) is a Hausdorff space; it satisfies the first axiom of countability and it is a separable space but it does not have a countable basis. However \( M_{s} \) is locally connected and pathwise connected, it is not locally compact. From a physical point of view it seems to be interesting that on lightlike lines the discrete topology is induced and that on timelike lines and spacelike hyperplanes the Euclidean topology is induced, cf. Ref. 10.

Comparing \( M_{s} \) and \( M_{s} \), we still note the following properties:

1. The set \( O \) is open in \( M_{s} \) and not in \( M_{s} \) if and only if for all \( x \in O \) there is an \( \epsilon > 0 \) such that

\[
N_{2}(x) \subset O
\]

and there is an \( x \in O \) with the property \( (C(x) \setminus \{ x \}) \cap O \neq \emptyset \) for all \( \epsilon > 0 \). If \( L \) is a six-dimensional manifold in \( \mathbb{R}^{n} \). There are several ways to topologize a set of maps. In this section we shall deal with the topology of pointwise convergence. See e.g., Ref. 17.

A. Introduction

For each \( x \in M \) and for every open set \( O \subset M \), we define

\[
\{ (x, O) | x \in L \} \subset L_{E} \setminus O
\]

Let \( L \), denote \( L_{s} \), endowed with the topology that has the family of all sets \( (x, O) \) as a subbasis, and let \( L_{s} \) denote \( L_{s} \), endowed with the topology, defined in a similar way as for \( L_{s} \), but coming from \( L_{s} \) instead of \( M_{s} \). The family of intersections of sets of the form \( (x, O) \) is a basis for the topological space \( L_{s} \), each number of this basis having the form \( \cap_{i=1}^{n} (x, O_{i}) \), where \( x \in M \) and \( O_{i} \) is open in \( M \). Notice that \( L_{s} \) is finer than \( L_{s} \), for \( M_{s} \) is finer than \( M_{s} \). As we shall show below, \( L_{s} \) is strictly finer. Notice furthermore that \( L_{s} \) is a Hausdorff space, for \( M_{s} \) has that property.

It is also possible to describe our topology by means of convergence of nets (see, e.g., Ref. 17, p. 77). To that end one can define: The net of Lorentz transformations \( \{ l_{n} \} \) converges to \( l \) in \( L_{s} \), if and only if \( (l_{n}, x) \) converges to \( l x \) for all \( x \in M \). We shall say that a set \( O \subset L_{s} \) is open if and only if every net \( \{ l_{n} \} \), converging to an element \( l \in O \), is eventually in \( O \). Remark that, if the net \( \{ l_{n} \} \) does not converge to \( l \) in \( L_{s} \), it does not converge to \( l \) in \( L_{s} \). As we shall show below, the converse is also true if we restrict ourselves to timelike vectors.

B. Properties of \( L_{s} \)

\( L_{s} \) is strictly finer than \( L_{s} \). Example.

\[
L_{s} = \begin{pmatrix}
\cosh \frac{1}{n} & \sinh \frac{1}{n} & 0 & 0 & 1 \\
\sinh \frac{1}{n} & \cosh \frac{1}{n} & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\( l \) is the unit element of \( L_{s} \). In \( L_{s} \) we find that \( \{ l_{n} \} \) converges to \( l \) if \( n \to \infty \), but \( l_{n} x = l_{n}^{1/n} x \), and therefore \( \{ l_{n} \} \) does not converge in \( L_{s} \), for \( l_{n} x \notin N_{2}(x) \), even for all \( n \). Also in the case of spacelike vectors, there are nets converging in \( L_{s} \), but not in \( L_{s} \). The same sequence \( \{ l_{n} \} \) as above, but applied to the spacelike vector \( y = (1, 1, 0, 1) \), gives us \( l_{n} y \notin N_{2}(y) \) for all \( n \).

Theorem 1: \( L_{s} \) and \( L_{s} \) induce the same topology on the subgroup \( O_{s} \).

Proof: It suffices to prove that a net of pure rotations \( \{ r_{n} \} \), converging in \( L_{s} \), also converges in \( L_{s} \) (with the same limit). Suppose that \( \{ r_{n} \} \) converges to \( r \) in \( L_{s} \). Then we have for all \( x \) that eventually \( \{ r_{n} x \} \subset N_{2}(r x) \).
On the other hand, we know that all \( r, x \) are situated in the same spacelike hyperplane through \( r x \) and therefore
\[
\{(r x)\} \cap (C(r x) \setminus \{r x\}) = \emptyset
\]
and, consequently,
\[
\{(r x)\} \cap N'_r (r x) = \{(r x)\} \cap N'_r (r x).
\]
This means that \( (r x) \) eventually belongs to \( N'_r (r x) \). In other words, \( (r x) \) converges to \( r \) in \( L' \).

**Corollary:** \( L' \) induces the same topology as \( L_g \) on every compact subgroup of \( L'_r \) because \( G'_r \) is a maximal compact subgroup of \( L_g \) and consequently of \( L'_r \).

A semitopological group \( G \) is a topological space, provided with a group structure such that the product map \( G \times G \rightarrow G \), given by \( (a, b) \rightarrow ab \), \((a, b) \in G \), is separately continuous. See, e.g., Ref. 18.

**Theorem 2:** \( L_r \) is a semitopological group.

**Proof:** Suppose that \( (l_r) \) converges to \( l_y \), i.e., \( (l_r x) \) converges to \( l_x, x \in M_g \). In particular, if we consider \( l' x \) instead of \( x \), then \( (l'_r x) \) converges to \( l' x \). Therefore, for all neighborhoods \( O_{1 l'} \) of \( l' \), there is a neighborhood \( O_1 \) of \( l \) such that \( O_1 \cup l' \subset O_{1 l'} \). On the other hand, we know that the elements of \( L_g \) are homeomorphisms of \( M_g \) and therefore it follows from \( (l_r x) \) converges to \( l_x \) that \( (l'_r x) \) converges to \( l' x \) for all \( l' \in L \), i.e., for all \( O_{1 l'} \) there is a neighborhood \( O_1 \) of \( l \) such that \( O_1 \cup l' \subset O_{1 l'} \).

**C. The main theorem**

The definition of \( L_g \) uses the action of \( L'_r \) on \( M \) and the topology of \( M_g \). Now we want to give an intrinsic definition of \( L_g \), by considering \( L_g \) in \( SL(2, C) \).

**Lemma 1:** For timelike vectors \( x \), \((l_r x)\) converges to \( lx \) in \( M_g \) if and only if \( (l_r x) \) converges to \( lx \) in \( M_g \). Proof: Obviously, convergence in \( M_g \) implies convergence in \( M_g \). To prove the converse, we remark that the nets, converging in \( M_g \) and not in \( L_g \), are exactly those having the property that there is an \( x \) such that eventually
\[
(l_r x - lx, l_x - lx) = 0 \quad \text{and} \quad l_r x \neq l_x,
\]
i.e.,
\[
(l^{-1} l_r x - x, l^{-1} l_r x - x) = 0 \quad \text{and} \quad l^{-1} l_r x \neq x.
\]
It is sufficient to consider only one timelike vector. We choose \( x' = (a_0, 0, 0) \) and note that it is possible to transform all timelike vectors, situated on the same hypersurface \( \{x, x = a_0\} \), into \( (a_0, 0, 0) \) by a suitable Lorentz transformation \((a \neq 0)\). The intersection of \( \{x, x = a_0\} \) and the light cone \( C(x') \) consists only of the vertex \( x' \) of the cone. Therefore, the relations
\[
(l^{-1} l_r x - x, l^{-1} l_r x - x) = 0
\]
and \( l^{-1} l_r x \neq x' \) do not hold together. In other words, \((l_r x)\) converges to \( lx \) in \( M_g \) implies that \((l_r x)\) converges to \( lx \) in \( M_g \).

Let \( \varphi \) denote the surjective homomorphism of \( SL(2, C) \) onto \( L' \); as introduced in Sec. 1) and let \( B \) denote the image under \( \varphi \) of the set of upper triangular matrices of the form \( \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \) with \( |a| \neq 1 \).

**Lemma 2:** Let \( x \) be an isotropic vector and let \((l_r)\) be a net of Lorentz transformations. Then \((l_r x)\) converges to \( lx \) in \( M_g \) if and only if
(i) \( l_r x \) converges to \( lx \) in \( M_g \),
(ii) no \( \tilde{t} \in L \) exists such that eventually \( \tilde{t}^{-1} l_r \tilde{t} \in B \).

**Proof:** Similarly, as in the proof of Lemma 1, it suffices to consider only one isotropic vector. We choose \( x' = (1, 1, 0, 0) \) and (compare Sec. 1) the relation \( \tilde{t} x = s \tilde{t} x * \), written out and applied to our situation, becomes
\[
\begin{pmatrix} y_0 + y_1 & y_2 - i y_3 \\ y_2 + i y_3 & y_0 - y_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & -i \beta \\ i \beta & \alpha \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\alpha|^2 \end{pmatrix}.
\]
Again, we have to exclude nets \((l_r)\) with the property that eventually \( (l^{-1} l_r x - x, l^{-1} l_r x - x) = 0 \) and \( l^{-1} l_r x \neq x \).

The intersection of \( \{x, (x, x) = 0\} \) and the light cone \( C(x') \) only consists of the line \( (1, 1, 0, 0) \), \( x \in R \), and therefore we must look for \( l_r \) with \( l^{-1} l_r x = \lambda_r x' \) \((\lambda_r \neq 1)\).

Let
\[
l_r = \varphi, ((0, a, 0, 1)),
\]
then for such \( l_r \) we have
\[
|\alpha|^2 = 1 \quad \alpha \beta = 0, \quad \alpha \gamma = 0, \quad |\gamma|^2 = 0;
\]
in other words,
\[
|\alpha|^2 = 1 \quad \text{and} \quad \gamma = 0.
\]
Consequently, the \( 2 \times 2 \) matrices in question correspond with elements \( b \in B \); i.e., \( l^{-1} l_r = b \) or \( l_r = b l \). The Lorentz transformations, leaving invariant the other one-dimensional isotropic subspaces, have the form \( bb^{-1} \), where \( b \) is a suitable Lorentz transformation.

Summarizing, we have to exclude \( l_r \), eventually satisfying the relation \( l^{-1} l_r = bb^{-1} \) or \( \tilde{t}^{-1} l_r \tilde{t} = b \). Now the proof is complete.

**Corollary:** \( L_g \) induces the discrete topology on the subgroup \( Z \) and on its conjugates.

**Proof:** As is known the elements of \( \varphi^{-1}(x) \) have the form \( \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \) with \( i \in R' \), being a subset of \( B \).

Notice that in the case of \( Z \) there are two isotropic eigenvectors, viz., \((1, 1, 0, 0) \) but in the case of \( B \) there is only the isotropic eigenvector \((1, 1, 0, 0) \).

Let \( C \) be the image under \( \varphi \) of the matrices \( \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \) of \( SL(2, C) \) with properties:
(i) \( |\alpha|^2 - |\beta|^2 - |\gamma|^2 + |\delta|^2 = 2 \),
(ii) \( |\alpha|^2 - |\beta|^2 = 1 \).

**Lemma 3:** Let \( x \) be a spacelike vector and let \((l_r)\) be a net of Lorentz transformations. Then \((l_r x)\) converges to \( lx \) in \( M_g \) if and only if
(i) \( l_r x \) converges to \( lx \) in \( M_g \),
(ii) no \( \tilde{t} \in L \) exists such that eventually \( \tilde{t}^{-1} l_r \tilde{t} \in C \).

**Proof:** Again we only need one spacelike vector to start with and we choose \( x' = (0, a, 0, 0) \), situated on the
hypersurface \((x, x) = -a^2\) \((a \neq 0)\). Similarly, as for Lemma 2, we find
\[
\begin{pmatrix}
y_0 + y_1 & y_2 - iy_3 \\
y_2 + iy_3 & y_0 - y_1
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \begin{pmatrix}
\bar{a} & 0 \\
0 & \bar{a}
\end{pmatrix} \begin{pmatrix}
y \bar{\gamma} \\
\bar{\gamma}
\end{pmatrix}
\]
\[
= \begin{pmatrix}
|\alpha|^2 - |\beta|^2 & \alpha \bar{\gamma} - \beta \bar{\delta} \\
\bar{\alpha} \gamma - \bar{\beta} \delta & |\gamma|^2 - |\delta|^2
\end{pmatrix}.
\]
Now the intersection of \(\{x \mid (x, x) = -a^2\}\) and \(C(x')\) is situated in the hyperplane \(x_1 = a\) and therefore we have to look for the elements of \(L_1\), transforming \((0, a, 0, 0)\) into \((v, a, v \cos u, v \sin u)\), where \(v \neq 0\).

This means that
\[
\begin{pmatrix}
v + a & ve^{iu} \\
ve^{iu} & v - a
\end{pmatrix} = \begin{pmatrix}
|\alpha|^2 - |\beta|^2 & \alpha \bar{\gamma} - \beta \bar{\delta} \\
\bar{\alpha} \gamma - \bar{\beta} \delta & |\gamma|^2 - |\delta|^2
\end{pmatrix},
\]
or
\[
v + a = a(\alpha^2 - |\beta|^2),
v - a = a(|\gamma|^2 - |\delta|^2),
ve^{iu} = a(\bar{\alpha} \gamma - \bar{\beta} \delta),
\]
and these relations are equivalent with the conditions:
(i) \(|\alpha|^2 - |\beta|^2 - |\gamma|^2 + |\delta|^2 = 2\),
(ii) \(|\alpha|^2 - |\beta|^2 \neq 1 \quad (v \neq 0)\),
(iii) \((|\alpha|^2 - |\beta|^2 - 1)^2 = |\bar{\alpha} \gamma - \bar{\beta} \delta|^2\);
but condition (iii) is superfluous for it is implied by (i) and \(a \delta - \beta r = 1\). Similarly, as for Lemma 2, it turns out that in this case we must exclude the nets \((l)\) with the property that eventually \(\tilde{t}^{-1} l^{-1} l = c\), with \(c \in C\).

Now we are able to state:

Theorem: Let \((l_u)\) be a net of Lorentz transformations. Then \((l_u)\) converges to \(l\) in \(L_1\) if and only if:
(i) \((l_u)\) converges to \(l\) in \(L_1\),
(ii) no \(l \in E \) exists such that eventually \(\tilde{t}^{-1} l^{-1} l_u \in B \cup C\) \((B \text{ and } C \text{ as defined above})\).

Proof: The theorem follows immediately from the Lemmas 1, 2, and 3.

Remarks:
1. The theorem gives us an intrinsic definition of the topology of \(L_1\) by means of convergence of nets.
2. The topology of \(M_1\) can be recovered from \(L_1\).