On approximation of (slowly growing) analytic functions by special polynomials

de Graaf, J.

Published: 01/01/1990

Document Version
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the author’s version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):
ON APPROXIMATION OF
(SLOWLY GROWING)
ANALYTIC FUNCTIONS
BY SPECIAL POLYNOMIALS
by
J. de Graaf
ON APPROXIMATION OF
(SLOWLY GROWING) ANALYTIC FUNCTIONS
BY SPECIAL POLYNOMIALS

J. de Graaf

Eindhoven University of Technology
Department of Mathematics, P.O. Box 513,

0. Introduction and Summary

Let $K = (K_{mn})_{m,n=0}^\infty$ be an infinite matrix with $K_{mn} = 0$ if $m > n$.
We denote $K \in UTM$, i.e. the class of upper triangular matrices. If we suppose that $K_{ii} \neq K_{jj}$ if $i \neq j$ there exists a unique $S \in UTM$ with all $S_{ii} = 1$ such that $S^{-1}KS = \Delta_k = \text{diag}(K_{00}, K_{11}, \ldots)$. If $K$ has suitable growth properties we construct classes of diagonal matrices $M = \text{diag}(\mu_0, \mu_1, \ldots)$ such that $MS^{-1}$ and $M^{-1}S^{-1}$ are $L_2$-bounded matrices. Of course, the growth properties of $M$ are related to those of $K$.

This general result enables us to construct a great variety of polynomial Riesz bases for weighted $L_2$-spaces of analytic functions. Thus we obtain very detailed results on approximation of slowly growing entire functions by special polynomials (Jacobi, Hermite, Laguerre, etc.).

It also leads to both a refinement and an extension of Szegö’s classical theorem on Jacobi expansions of analytic functions on an ellips.

1. Basic concepts

We start with a separable Hilbert space $X$ and a selected orthonormal basis $(T_n)_{n=0}^\infty \subset X$.

Associated with a sequence $\mu = (\mu_n)_{n=0}^\infty$, $\mu_n > 0$, $\mu_n \to \infty$ as $n \to \infty$, we introduce a Hilbert space $X_\mu$ which is dense in $X$,

$$X_\mu = \{ f \in X : \sum_{n=0}^\infty \mu_n^2 |(f, T_n)|^2 < \infty \}.$$

Note that $f = \sum_{n=0}^\infty a_n T_n \in X_\mu$ iff $\sum_{n=0}^\infty \mu_n^2 |a_n|^2 < \infty$. Note also that $(\mu^{-1}_n T_n)_{n=0}^\infty$ is an orthonormal basis in $X_\mu$. By means of a transition matrix $S \in UTM$ we define a sequence of vectors $(R_n)_{n=0}^\infty \subset X$ by $R_n = \sum_{j=0}^n S_{jn}^{-1} T_j$. We take all $S_{nn} = 1$ so that $S$ has an inverse $S^{-1} \in UTM$ of the same type and $T_n = \sum_{j=0}^n S_{jn}^{-1} R_j$. 
Our first question is: When is the sequence \((\mu_n^{-1} R_n)_{n=0}^{\infty}\) a Riesz basis in \(X_\mu\)?

Recall that a sequence \((\psi_n)_{n=0}^{\infty} \subset X\) is a Riesz basis iff for each \(f \in X\) there is a unique expansion

\[
 f = \sum_{n=0}^{\infty} \alpha_n \psi_n \quad \text{with} \quad (\alpha_n)_{n=0}^{\infty} \subset l_2. \quad \text{See e.g. [N].}
\]

**THEOREM 1.1.**

Let \(M \in UTM\) be defined by \(M = \text{diag}(\mu_0, \mu_1, \cdots)\). The sequence \((\mu_n^{-1} R_n)_{n=0}^{\infty}\) is Riesz basis in \(X_\mu\) iff both matrices \(M S M^{-1}\) and \(MS^{-1} M^{-1}\) are \(l_2\)-bounded. This means that they can be regarded as bounded operators in \(l_2\).

**Proof.**

On \(\text{span}(T_n)\) define the operators \(M\) and \(S\) by \(M T_n = \mu_n T_n\) and \(S T_n = R_n\), followed by linear extension. On \(\text{span}(T_n)\) the inverse \(S^{-1}\) exists and \(S^{-1} R_n = T_n\). Observe that \((\mu_n^{-1} R_n)_{n=0}^{\infty}\) is a Riesz basis in \(X_\mu\) iff \(S\) extends to a continuous bijection on \(X_\mu\) iff both \(S\) and \(S^{-1}\) extend to continuous mappings on \(X_\mu\) iff both matrices \(M S M^{-1}\) and \(M S^{-1} M^{-1}\) are \(l_2\)-bounded. The latter equivalence follows from

\[
 \|S f\|_\mu = \|M f\| = \|(M S M^{-1}) M f\|. \]

Let \(K\) be a densely defined operator in \(X\) which acts invariantly on \(\text{span}(T_n)\). Suppose that, with respect to the basis \((T_n)_{n=0}^{\infty}\), the operator \(K\) is represented by a matrix \(K \in UTM\) with mutually distinct entries on its diagonal. Then there exists a unique \(S \in UTM\), with all \(S_{ii} = 1\), such that \(S^{-1} K S = \Delta_K = \text{diag}(K_{00}, K_{11}, \cdots)\). Consequently, there exists a unique sequence \((T^K_n)_{n=0}^{\infty} \subset \text{span}(T_n)\) of eigenvectors of \(K\).

Note that \(T^K_n = \sum_{j=0}^{n} S_{jn} T_j\) and \(K T^K_n = K_{nn} T^K_n\). Now we turn \(\text{span}(T_n)\) into a pre-Hilbert space by declaring the sequence \((\mu_n^{-1} T^K_n)_{n=0}^{\infty}\) to be an orthonormal sequence: The Hilbert space completion is then denoted by \(X^K_\mu\). Note in particular that \((\mu_n^{-1} T^K_n)_{n=0}^{\infty}\) is a Riesz basis in \(X_\mu\) iff \(X_\mu = X^K_\mu\) as topological vector spaces. Note that \(X_\mu = X^K_\mu\). Two problems can now be posed:

* The classification problem.
  
  For which pairs \((K, \mu)\), \((K', \mu')\) do we have \(X^K_\mu = X^{K'}_{\mu'}\) as topological vector spaces. Or, more generally, when a second Hilbert space \(Y\) with a similar construction is involved. When do we have \(X^K_\mu = Y^L_v\)?

* The characterization problem.
  
  If \(X\) is a function space, describe the elements of \(X^K_\mu\) in classical analytic terms.

When dealing with the classification problem we want to apply Theorem 1.1. Then we must relate growth properties of \(K\) to boundedness properties of \(M S M^{-1}\) and \(M S^{-1} M^{-1}\). This is the subject of the next section.
2. Some estimations on infinite diagonalizing matrices.

In this section we solve problem of the following type: Let there be given an infinite matrix \( K \in UT M \) with distinct diagonal elements. Let \( S \in UT M \) with all \( S_{ii} = 1 \) be the diagonalizer of \( K \), so \( S^{-1}KS = \Delta_K = \text{diag}(K_{11}, K_{22}, \cdots) \). Find diagonal matrices \( M = \text{diag}(u_0, u_1, \cdots) \) with the property \( MSM^{-1} \) and \( MS^{-1}M^{-1} \) are \( l_2 \)-bounded. We consider matrices \( M \) of the form \( M = e^{tA} \) with \( \Re t > 0 \) and \( \Lambda = \text{diag}(\lambda_0, \lambda_1, \cdots) \). The conditions we impose on \( K \) and the conditions on \( \Lambda \) that we look for are growth conditions.

**THEOREM 2.1.**

Let \( K \in UT M \). Take \( \mu > 1 \), fixed. Put \( C_n = \max_{1 \leq i < j \leq n} |K_{ij}| \). Suppose

(i) \[ \theta = \sup_{i \neq j} |K_{ii} - K_{jj}|^{-1} < \infty \]

(ii) \[ \forall \epsilon > 0 \sup_{n \geq 0} \left( e^{-\epsilon n - 1} C_n \right) < \infty. \]

Let \( \Lambda = \text{diag}(\lambda_0, \lambda_1, \cdots) \) with

(iii) \[ \Re \lambda_n = n^\mu (1 + \epsilon_1(n)), \quad \epsilon_1(n) \to 0 \text{ as } n \to \infty \]

\[ \Im \lambda_n = \epsilon_2(n) \quad \epsilon_2(n) \to 0 \text{ as } n \to \infty \]

\[ \Re (\lambda_n - \lambda_m) \geq (n - m) n^{\mu - 1} (1 + \epsilon_3(n, m)), \quad n > m, \]

with \( \epsilon_3(n, m) \to 0 \text{ as } \min(n, m) \to \infty \)

\[ \Im (\lambda_n - \lambda_m) = \epsilon_4(n, m) \Re (\lambda_n - \lambda_m) \]

with \( \epsilon_4(n, m) \to 0 \text{ as } \min(n, m) \to \infty \).

Let \( S \in UT M \) be the diagonalizer of \( K \) with \( S_{ii} = 1, 0 \leq i < \infty \). Then for all \( t \in \mathbb{C}, \Re t > 0 \), the matrices

\[ e^{tA}S e^{-tA} \text{ and } e^{tA}S^{-1} e^{-tA} \] are \( l_2 \)-bounded.

Their strict upper diagonal parts are Hilbert-Schmidt.

**Proof.**

The proof consists of parts (a)-(e).

(a) Since \( KS = S \Delta_K S^{-1} \) and \( S^{-1}K = \Delta_K S^{-1} \) we have the following recurrence relations for the entries of \( S \) and \( S^{-1} \):

\[ S_{n-q, n} = (K_{n-q, n} - K_{nn})^{-1} \sum_{k=0}^{q-1} K_{n-q, n-k} S_{n-k, n}, \]

\[ S_{nn} = 1, \quad 1 \leq q \leq n - 1. \]
\[ S^{-1}_{m,m+q} = (K_{mm} - K_{m+q,m+q})^{-1} \sum_{k=0}^{q-1} K_{m+k,m+q} S^{-1}_{m,m+k}, \]
\[ S^{-1}_{mm} = 1, \quad q \geq 1. \]

We estimate
\[ |S_{n-q,n}| \leq C_n \sum_{k=0}^{q-1} |S_{n-k,n}|, \quad 1 \leq q \leq n - 1 \]
\[ |S_{m,m+q}| \leq C_{m+q} \sum_{k=0}^{q-1} |S^{-1}_{m,m+k}|, \quad q \geq 1. \]

By induction it follows that
\[ \frac{|S_{m,n}|}{|S^{-1}_{m,n}|} \leq (1 + C_n)^{n-m}. \]

(b) Let \( t \in \mathbb{C}, \text{Re} t > 0 \), be fixed. Put \( t = t_1 + i t_2, t_1, t_2 \in \mathbb{R}, t_1 > 0 \). Using condition (ii) define
\[ M_n = \sup_{n \geq 0} [(1 + C_n) \exp(-\frac{1}{4} t_1 n^{\mu-1})]. \]

Put
\[ \sigma'_mn = (e^{i\Lambda} S e^{-i\Lambda})_{mn} = S_{mn} e^{-i(n_m - \lambda_m)}. \]

With the estimate of part (a) we find
\[ |\sigma'_mn| \leq \exp\{(n-m) [\log M_n + \frac{1}{4} t_1 n^{\mu-1}] - t_1 \text{Re}(\lambda_n - \lambda_m) + t_2 \text{Im}(\lambda_n - \lambda_m)\}. \]

(c) We now show that each row in \( \sigma' \) is an \( l_2 \)-sequence. So let \( m \) be fixed
\[ |\sigma'_mn| \leq \exp\{-m \log M_n + \frac{1}{4} t_1 n^{\mu-1} + t_1 \text{Re} \lambda_m - t_2 \text{Im} \lambda_m\} \cdot \exp\{n \log M_n + \frac{1}{4} t_1 n^{\mu} - t_1 [n^{\mu}(1+\varepsilon_1(n)) - \frac{t_2}{t_1} \varepsilon_2(n)]\}. \]

The first factor is bounded. For large \( n \), the second factor is smaller than \( \exp - \frac{1}{2} t_1 n^{\mu} \). This proves the assertion.

(d) We want to show that for \( M \) sufficiently large \( \sum_{n > m \geq M} |\sigma'_mn|^2 < \infty \). So, starting from the estimate obtained in (b)
\[ |\sigma'_mn| \leq \exp\{(n-m) [\log M_n + \frac{1}{4} t_1 n^{\mu-1}] - t_1 [(n-m) n^{\mu-1}(1+\varepsilon_3(n,m)) (1-\frac{t_2}{t_1} \varepsilon_4(n,m))]\}. \]

For \( m = \min(m,n) \) sufficiently large, larger than \( M \) say, this is smaller than \( \exp - \frac{1}{4} t_1 (n-m) n^{\mu-1} \). So, below the \( M^{th} \) row the strict upper diagonal part of \( \sigma' \) is
Hilbert-Schmidt.

(e) According to (c) the first $M$ rows of $\sigma'$ establish a Hilbert-Schmidt matrix. So together with (d) we get the result.
For $e^{tA} S^{-1} e^{-tA}$ the proof is exactly the same.

In the next theorem we want to deal with matrices $M$ of growth about $e^{in}$ as $n \to \infty$. For results in this direction we have to put conditions of greater subtlety on $K$.

**THEOREM 2.2.**

Let $K \in UTM$. Put $C_{n} = \max_{1 \leq i < j \leq n} |K_{ij}|$ and $\overline{C}_{n} = n \max_{1 \leq i \leq n} \left( \frac{1}{i} C_{i} \right)$. Suppose

(i) $\exists D > 0 \ \forall n > m \ |K_{mn} - K_{nn}|^{-1} \leq \frac{D}{n(n-m)}$

(ii) $\sup_{n \geq 0} \left( \frac{1}{n} \overline{C}_{n} \right) = \delta < \infty$

Let $\Lambda = \text{diag}(\lambda_0, \lambda_1, \cdots)$, with

(iii) $\text{Re} \lambda_n = n(1 + \varepsilon_1(n))$, $\varepsilon_1(n) \to 0$ as $n \to \infty$

$\text{Im} \lambda_n = \varepsilon_2(n) \text{Re} \lambda_n$, $\varepsilon_2(n) \to 0$ as $n \to \infty$

$\text{Re}(\lambda_n - \lambda_m) \geq (n-m) (1 + \varepsilon_3(n,m))$, $n > m$

with $\varepsilon_3(n,m) \to 0$ as $\min(n,m) \to \infty$

$\text{Im}(\lambda_n - \lambda_m) = \varepsilon_4(n,m) \text{Re}(\lambda_n - \lambda_m)$

with $\varepsilon_4(n,m) \to 0$ as $\min(n,m) \to \infty$.

Let $S \in UTM$ be the diagonalizer of $K$ with $S_{ii} = 1$, $0 \leq i < \infty$. Then for all $t \in \mathbb{C}$, $\text{Re} t > 0$, the matrices $e^{tA} S e^{-tA}$ and $e^{tA} S^{-1} e^{-tA}$ are $l_2$-bounded.

**Proof.**

The proof consists of parts (a)-(f).

(a) We show that

$$\left\{ \begin{array}{c} |S_{m,n}|^{-1} \\ |S_{m,n}|^{-1} \end{array} \right\} \leq \exp 2 \left\{ \frac{D(n-m)}{n} \overline{C}_{n} \right\}^{\frac{1}{2}}.$$

From the recurrence relations for the entries of $S$ we estimate
\[ |S_{n-q,n}| \leq \frac{DC_n}{nq} \sum_{k=0}^{q-1} |S_{n-k,n}|, \quad 1 \leq q \leq n, \quad S_{nn} = 1. \]

By induction
\[ |S_{n-q,n}| \leq \frac{DC_n}{nq} \prod_{l=1}^{q-1} \left( 1 + \frac{1}{l} \frac{DC_n}{n} \right), \quad 1 \leq q \leq n. \]

Because of \( \prod_{l=1}^{q} \left( 1 + \frac{x}{l} \right) \leq e^{xq} \) and \( C_n \leq \overline{C}_n \) we arrive at the first inequality.

For the second one
\[ |S_{m+q,n}| \leq \frac{DC_{m+q}}{2(m+q)} \sum_{k=0}^{q-1} |S_{m+q-k}|, \quad q = 1, 2, \ldots, S_{m+q} = 1. \]

By induction
\[ |S_{m+q,n}| \leq \frac{DC_{m+q}}{q(m+q)} \prod_{l=1}^{q} \left( 1 + \frac{1}{l} \frac{DC_{m+l}}{m+l} \right), \quad q = 1, 2, \ldots \]
\[ \leq \prod_{l=1}^{q} \left( 1 + \frac{1}{l} \frac{DC_{m+q}}{m+q} \right) \leq \exp \left\{ q \frac{DC_{m+q}}{m+q} \right\}. \]

(b) Put \( t = t_1 + it_2 \) and \( \sigma_{m+n} \) as in the proof of the preceding theorem. With the estimate of part (a) we find
\[ |\sigma_{m+n}| \leq \exp \left\{ 2 |D(n-m)| n \frac{\overline{C}_n}{n^2} \right\} - t_1 \Re(\lambda_n - \overline{\lambda}_m) + t_2 \Im(\lambda_n - \overline{\lambda}_m) \right\}. \]

(c) We now show that each row in \( \sigma' \) in an \( l_2 \)-sequence. So let \( m \) be fixed
\[ |\sigma_{m+n}| \leq \exp \{ t_1 \Re(\lambda_m + t_2 \Im(\lambda_m)) \cdot \exp \left[ 2n \left( \frac{D\overline{C}_n}{n^2} \right) \right] - t_1 n(1 + \epsilon_1(n) + \frac{t_2}{t_1} \epsilon_2(n)) \right\}. \]

Take \( N \) so large that for \( n > N \) we have
\[ 2 \left( \frac{D\overline{C}_n}{n^2} \right)^{\frac{1}{2}} \leq \frac{1}{4} \quad \text{and} \quad \left( 1 + \epsilon_1(n) + \frac{t_2}{t_1} \epsilon_2(n) \right) > \frac{1}{2}. \]

For \( n > N \) the second factor is smaller than \( e^{-\frac{1}{4}t_1 n} \). Hence the result.

(d) Starting from the inequality in (b) we find, applying condition (ii)
\[ |\sigma_{m+n}| \leq \exp \{ 2(D \delta(n-m))^{\frac{1}{2}} - t_1 (n-m) (1 + \epsilon_3)(1 - \frac{t_2}{t_1} \epsilon_3) \right\}. \]

Below the \( M^{th} \) row, \( M \) sufficiently large, we have
\[ |\sigma'_{mn}| \leq \exp\{2(D \delta(n-m))^{\frac{1}{2}} - \frac{1}{2} t_1(n-m)\} . \]

Therefore each codiagonal is a bounded sequence. Finally for \( m \geq M \) and \( (n-m) \geq \frac{1}{t_1} 8(D \delta)^{\frac{1}{2}} = L \)

\[ |\sigma'_{mn}| \leq e^{-\frac{1}{2} t_1(n-m)} . \]

(e) We now split the matrix \( \sigma' \) in 2 parts:

- The first \( M \) rows. They represent a bounded \( l_2 \)-operator which is even Hilbert Schmidt, cf. (c).
- The part below the \( M^{\text{th}} \) row also represents a bounded \( l_2 \)-operator because of the "codiagonal estimate", use (d),

\[ \sum_{k=0}^{\infty} (\sup_{j \in k} |K_{ij}|) < \infty . \]

(f) For \( e^{tA}S^{-1}e^{-tA} \) the parts (b)-(e) of the proof apply in exactly the same way.

**THEOREM 2.3.**

(a) The conditions on \( A \) in Theorems 2.1 and 2.2 are satisfied if \( \lambda_n = n^\mu (1 + \rho(n)), \mu \geq 1, \) and

\[ \lim_{n \to \infty} \rho(n) = 0, \quad \lim_{m \to \infty} \sup_{n \geq m+1} \left| \frac{\rho(n) - \rho(m)}{n-m} \right| = 0 . \]

(b) The conditions of (a) are satisfied if \( \rho \) is a (complex valued) differentiable function on \((0,\infty)\) with \( \lim_{\xi \to 0} \rho(\xi) = 0 \) and \( \lim_{\xi \to \infty} \xi \rho'(\xi) = 0 \).

(b) If \( A = N^\alpha = \text{diag} (\cdots, (n(n+a))^{\frac{1}{2}} \mu, \cdots), a \in \mathbb{C} \), then condition (b) is satisfied.

**Proof.**

(a) We compute

\[ \lambda_n - \lambda_m = (n-m) n^{\mu-1} \left( 1 + \left\{ \frac{1 - \left( \frac{m}{n} \right)^\mu}{\frac{1}{n}} - 1 \right\} + \left\{ \frac{n \rho(n) - m \rho(m)}{n-m} \right\} + \left\{ \frac{m}{n} \frac{(1 - \frac{m}{n})^{\mu-1}}{1 - \frac{m}{n}} \rho(m) \right\} \right) . \]

On the interval \([0,1)\) one has
For $e_3(n, m)$ take the real parts of the 3rd and 4th terms between { }. We omit the simple verification of the other conditions.

(b) Follows by application of the mean value estimation on $(n - m)^{-1} (p(n) n - p(m) m)$.

(c) We have $p(x) = \left(1 + \frac{a_1}{x} \right)^2 - 1$ which satisfies (b).

3. Application to classical polynomials.

Take $X = L_2([-1, 1], (1 - x^2)^{-\frac{1}{2}} \, dx)$ with the orthonormal basis of normalized Chebyshev polynomials $(T_n)_{n=0}^{\infty}$. Recall that $T_0(\cos \theta) = \pi^{-\frac{1}{2}}$, $T_n(\cos \theta) = \left(\frac{2}{\pi}\right)^\frac{1}{2} \cos n \theta$, $n = 1, 2, \ldots$. Note that $\text{span}(T_n) = \{\text{all polynomials}\}$.

For $K$ we take

$$K = \left( A_1 x^2 + A_2 x + A_3 \right) \frac{d^2}{dx^2} + (A_4 x + A_5) \frac{d}{dx},$$

with fixed $A_i \in \mathbb{C}$. If $A_1 = 0$ then $A_4 \neq 0$, if $A_1 \neq 0$ then $1 - \frac{A_4}{A_1} \notin \mathbb{N}$.

The matrix $K$ of $K$ with respect to $(T_n)_{n=0}^{\infty}$ belongs to $UTM$. From a straightforward calculation, using the transformation $x = \cos \theta$, it follows that

$$K_{nn} = A_1 n^2 - (A_1 - A_4) n \quad \text{and} \quad \exists D > 0 \mid \left| K_{mn} \right| \leq D n^2.$$

So the entries of $K$ satisfy the conditions (i) and (ii) of Theorem 2.1 for all $\mu > 1$. Let us look at the polynomials $T_n^K$ for some special cases.

- **Generalized Laguerre polynomials $L_n^{(\alpha)}$.**
  $A_2 = 1, A_4 = -1, A_5 = \alpha + 1, A_1 = A_3 = 0, \alpha \in \mathbb{C}$.

  We normalize, cf. [AS],

  $$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} x^n + \cdots = \frac{(-1)^n (2\pi)^{\frac{1}{2}}}{2^n n!} T_n(x) + \cdots T_{n-1}(x) + \cdots.$$  

  In this case $T_n^K = (-1)^n 2^n n! (2\pi)^{\frac{1}{2}} L_n^{(\alpha)}$.

- **Hermite polynomials $H_n$.**
  $A_3 = 1, A_4 = -2, A_1 = A_2 = A_5 = 0$. We normalize

  $$H_n(x) = 2^n x^n + \cdots = (2\pi)^{\frac{1}{2}} T_n(x) + \cdots.$$  

  In this case $T_n^K = (2\pi)^{-\frac{1}{2}} H_n$. 

Monomials $x^n$.

$A_1 = A_4 = 1$, $A_2 = A_3 = A_5 = 0$. We normalize

$$x^n = \frac{(2\pi)^{\frac{1}{2}}}{2^n} T_n + \cdots T_{n-1} + \cdots.$$ 

In this case $K_n = \frac{1}{2^n} x^n$.

Jacobi polynomials $T_n^{\alpha \beta}$.

They are eigenfunctions of the differential operators

$$K = A_{\alpha \beta} = -(1-x^2) \frac{d^2}{dx^2} + (\alpha + \beta + 2)x \frac{d}{dx} - (\beta - \alpha) \frac{d}{dx}, \alpha, \beta \in \mathbb{C}, -(1+\alpha+\beta) \in \mathbb{N}.$$ 

In this case the entries of $K$ satisfy

$$|K_{mn}| \leq (2|\beta - \alpha| + |\alpha + \beta + 1|) n.$$

Therefore in this case the conditions of Theorem 2.2 are satisfied! We normalize

$$T_n^{\alpha \beta}(x) = T_n^{\alpha \beta}(x) = T_n(x) + \cdots.$$ 

Note that $T_n^{\alpha \beta} = T_n$. In the special case that $\alpha > -1$, $\beta > -1$, there exist constants $\kappa_{n}^{\alpha \beta}$, uniformly bounded and bounded away from zero, such that $T_n^{\alpha \beta} = \kappa_{n}^{\alpha \beta} R_n^{\alpha \beta}$. Here the $R_n^{\alpha \beta}$ are the normalized Jacobi polynomials which establish an orthonormal basis in $L_2([-1,1], (1-x)^\alpha (1+x)^\beta dx)$. For the details see [GE].

In the next two theorems a fixed sequence $(\rho(n))_{n=0}^\infty$ occurs which is, on some interval $(M, \infty)$, the restriction of a differentiable function $\rho$ with $\lim_{\xi \to \infty} \rho(\xi) = 0$ and $\lim_{\xi \to \infty} \xi \rho'(\xi) = 0$.

Cf. Theorem 2.3(b). For example, a sequence like

$$(\log n)^A n^B \exp(t n^A + Cn))_{n=0}^\infty, A,B,C \in \mathbb{R}, t > 0, \mu \geq 1,$$

is of type $\exp(t n^B (1+\rho(n)))$.

The following result is a consequence of Theorem 2.1.

**THEOREM 3.1.**

Let $\mu > 1$, $t > 0$. Let $\rho(n)$ be as above and fixed. Define $\theta = (\theta_n)_{n=0}^\infty$ by $\theta_n = \exp t n^\mu (1+\rho(n))$.

Let $\theta$ be an entire analytic function. Let $\alpha, \beta, \gamma \in \mathbb{C}, -(1+\alpha+\beta) \in \mathbb{N}$. Consider the expansions

$$f(z) = \sum_{n=0}^\infty a_n \{\theta_n^{-1} 2^n z^n\} = \sum_{n=0}^\infty b_n \{\theta_n^{-1} T_n(z)\} = \sum_{n=0}^\infty c_n^{\alpha \beta} \{\theta_n^{-1} T_n^{\alpha \beta}(z)\} = \sum_{n=0}^\infty h_n \{\theta_n^{-1} H_n(z)\} = \sum_{n=0}^\infty d_n \{\theta_n^{-1} 2^n n! L_n^\mu(z)\}.$$ 

Either the coefficient sequences $(a_n)_{n=0}^\infty$, etc., are all $l_2$-sequence or none of them is $l_2$. In other words, the restriction to $[-1,1]$ of each of the function systems $\{\cdots\}$ is a Riesz basis in $X_\theta$. 

REMARKS 3.2.

- As a topological vector space, $X_\theta$ is equal to $Y_\theta$ with $Y$ any of the Hilbert spaces $L_2([-1,1], (1-x)^\alpha (1+x)^\beta \, dx)$, $\alpha > -1$, $\beta > -1$, with the $(R_n^{\alpha \beta})_{n=0}^\infty$ as their selected bases. Cf. [GE].

- In [GE] for certain sequences $\theta$ the space $X_\theta$ has been characterized as a weighted $L_2$-space of entire functions:
  If $\theta_n = n^\gamma e^{i\theta n}$, then $f \in X_\theta$ iff
  \[ \int_E |f(x+iy)|^2 \, w(x+iy) \, dx \, dy < \infty, \]
  with
  \[ w(z) = |z|^{-2} \left( \log |z| \left| z \right| \frac{1}{\log 2} \right), \]
  \[ \gamma = \frac{2 \nu + \nu - \frac{1}{2}}{2(1-\nu)}, \quad t = (1-\nu) (\nu)^{\nu(1-\nu)}, \quad \nu = \frac{1}{1-\nu}. \]

The final result of this paper is a consequence of Theorem 2.2.

THEOREM 3.3.
Let $t > 0$. Let $\rho(n)$ be a above and fixed. Define the sequence $\mu = (\mu_n)_{n=0}^\infty$ by $\mu_n = \exp t n(1+\rho(n))$. Let $f$ be an analytic function on $[-1,1]$. In the expansions

\[ f(z) = \sum_{n=0}^\infty b_n \{ \mu_n^{-1} T_n(z) \} = \sum_{n=0}^\infty C_n^{\alpha \beta} \{ \mu_n^{-1} T_n^{\alpha \beta}(z) \}. \]

Either all coefficient sequences $(b_n)_{n=0}^\infty$ and $(c_n^{\alpha \beta})_{n=0}^\infty$ are $l_2$-sequences or none of them is $l_2$.
Hence, the restriction to $[-1,1]$ of each of the function systems \{\ldots\} is a Riesz bases in $X_\mu$. \hfill \Box

REMARKS 3.4.

- The first of the Remarks 3.2 also applies here.
- In [G] we show that the $X_\mu$ can be characterized as weighted Sobolev spaces on an ellips in the complex plane. Thus refining Szegö's result [Sz].

REFERENCES


