On approximation of (slowly growing) analytic functions by special polynomials

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0. Introduction and Summary

Let \( K = (K_{mn})_{m,n=0}^\infty \) be an infinite matrix with \( K_{mn} = 0 \) if \( m > n \).
We denote \( K \in \text{UTM} \), i.e. the class of upper triangular matrices. If we suppose that \( K_{ii} \neq K_{jj} \) if \( i \neq j \) there exists a unique \( S \in \text{UTM} \) with all \( S_{ii} = 1 \) such that \( S^{-1} K S = \Delta = \text{diag}(K_{00}, K_{11}, \ldots) \).
If \( K \) has suitable growth properties we construct classes of diagonal matrices \( M = \text{diag}(\mu_0, \mu_1, \ldots) \) such that \( M S M^{-1} \) and \( MS^{-1}M^{-1} \) are \( L_2 \)-bounded matrices. Of course, the growth properties of \( M \) are related to those of \( K \).

This general result enables us to construct a great variety of polynomial Riesz bases for weighted \( L_2 \)-spaces of analytic functions. Thus we obtain very detailed results on approximation of slowly growing entire functions by special polynomials (Jacobi, Hermite, Laguerre, etc.).
It also leads to both a refinement and an extension of Szegö's classical theorem on Jacobi expansions of analytic functions on an ellips.

1. Basic concepts

We start with a separable Hilbert space \( X \) and a selected orthonormal basis \( (T_n)_{n=0}^\infty \subset X \).
Associated with a sequence \( \mu = (\mu_n)_{n=0}^\infty \), \( \mu_n > 0 \), \( \mu_n \to \infty \) as \( n \to \infty \), we introduce a Hilbert space \( X_\mu \) which is dense in \( X \),
\[
X_\mu = \{ f \mid f \in X, \sum_{n=0}^\infty \mu_n^2 |(f,T_n)|^2 < \infty \}.
\]
Note that \( f = \sum_{n=0}^\infty a_n T_n \in X_\mu \) iff \( \sum_{n=0}^\infty \mu_n^2 |a_n|^2 < \infty \). Note also that \( (\mu_n^{-1} T_n)_{n=0}^\infty \) is an orthonormal basis in \( X_\mu \). By means of a transition matrix \( S \in \text{UTM} \) we define a sequence of vectors \( (R_n)_{n=0}^\infty \subset X \) by \( R_n = \sum_{j=0}^n S_{jn} T_j \). We take all \( S_{nn} = 1 \) so that \( S \) has an inverse \( S^{-1} \in \text{UTM} \) of the same type and \( T_n = \sum_{j=0}^n S_{jn}^{-1} R_j \).
Our first question is: When is the sequence $(\mu_n^{-1} R_n)_{n=0}^\infty$ a Riesz basis in $X_\mu$?
Recall that a sequence $(\psi_n)_{n=0}^\infty \subset X$ is a Riesz basis iff for each $f \in X$ there is a unique expansion
\[ f = \sum_{n=0}^\infty \alpha_n \psi_n \text{ with } (\alpha_n)_{n=0}^\infty \subset l_2. \]
See e.g. [N].

THEOREM 1.1.
Let $M \in UTM$ be defined by $M = \text{diag}(\mu_0, \mu_1, \cdots)$. The sequence $(\mu_n^{-1} R_n)_{n=0}^\infty$ is a Riesz basis in $X_\mu$ iff both matrices $MSG^{-1}$ and $MS^{-1}M^{-1}$ are $l_2$-bounded. This means that they can be regarded as bounded operators in $l_2$.

Proof.
On span$(T_n)$ define the operators $M$ and $S$ by $MT_n = \mu_n T_n$ and $ST_n = R_n$, followed by linear extension. On span$(T_n)$ the inverse $S^{-1}$ exists and $S^{-1} R_n = T_n$. Observe that $(\mu_n^{-1} R_n)_{n=0}^\infty$ is a Riesz basis in $X_\mu$ iff $S$ extends to a continuous bijection on $X_\mu$ iff both $S$ and $S^{-1}$ extend to continuous mappings on $X_\mu$ iff both matrices $MSG^{-1}$ and $MS^{-1}M^{-1}$ are $l_2$-bounded. The latter equivalence follows from $\|Sf\|_\mu = \|Mf\| = \|(MSG^{-1})Mf\|$. 

Let $K$ be a densely defined operator in $X$ which acts invariantly on span$(T_n)$. Suppose that, with respect to the basis $(T_n)_{n=0}^\infty$, the operator $K$ is represented by a matrix $K \in UTM$ with mutually distinct entries on its diagonal. Then there exists a unique $S \in UTM$, with all $S_{ii} = 1$, such that $S^{-1} K S = \Delta_K = \text{diag}(K_{00}, K_{11}, \cdots)$. Consequently, there exists a unique sequence $(T_n^K)_{n=0}^\infty \subset \text{span}(T_n)$ of eigenvectors of $K$.

Note that $T_n^K = \sum_{j=0}^n S_{jn} T_j$ and $K T_n^K = K_{nn} T_n^K$. Now we turn span$(T_n)$ into a pre-Hilbert space by declaring the sequence $(\mu_n^{-1} T_n^K)_{n=0}^\infty$ to be an orthonormal sequence: The Hilbert space completion is then denoted by $X_\mu^K$. Note in particular that $(\mu_n^{-1} T_n^K)_{n=0}^\infty$ is a Riesz basis in $X_\mu$ iff $X_\mu = X_\mu^K$ as topological vector spaces. Note that $X_\mu = X_\mu^M$. Two problems can now be posed:

* The classification problem.
  For which pairs $(K, \mu), (K', \mu')$ do we have $X_\mu^K = X_\mu^{K'}$ as topological vector spaces. Or, more generally, when a second Hilbert space $Y$ with a similar construction is involved. When do we have $X_\mu^K = Y_\nu^L$?

* The characterization problem.
  If $X$ is a function space, describe the elements of $X_\mu^K$ in classical analytic terms.

When dealing with the classification problem we want to apply Theorem 1.1. Then we must relate growth properties of $K$ to boundedness properties of $MSG^{-1}$ and $MS^{-1}M^{-1}$. This is the subject of the next section.
2. Some estimations on infinite diagonalizing matrices.

In this section we solve problem of the following type: Let there be given an infinite matrix \( K \in UTM \) with distinct diagonal elements. Let \( S \in UTM \) with all \( S_{ii} = 1 \) be the diagonalizer of \( K \), so \( S^{-1}KS = \Delta_K = \text{diag}(K_{00}, K_{11}, \ldots) \). Find diagonal matrices \( M = \text{diag}(u_0, u_1, \ldots) \) with the property \( MSM^{-1} \) and \( MS^{-1}M^{-1} \) are \( l_2 \)-bounded. We consider matrices \( M \) of the form \( M = e^{tA} \) with \( \text{Re } t > 0 \) and \( \Lambda = \text{diag}(\lambda_0, \lambda_1, \ldots) \). The conditions we impose on \( K \) and the conditions on \( \Lambda \) that we look for are growth conditions.

**THEOREM 2.1.**

Let \( K \in UTM \). Take \( \mu > 1 \), fixed. Put \( C_n = \max_{1 \leq i < j \leq n} |K_{ij}| \). Suppose

(i) \( \theta = \sup_{i \neq j} |K_{ii} - K_{jj}|^{-1} < \infty \)

(ii) \( \forall \epsilon > 0 \sup_{n \geq 0} (e^{-\epsilon n} - C_n) < \infty \).

Let \( \Lambda = \text{diag}(\lambda_0, \lambda_1, \ldots) \) with

(iii) \( \text{Re } \lambda_n = n^\mu (1 + \epsilon_1(n)), \quad \epsilon_1(n) \to 0 \) as \( n \to \infty \)

\( \text{Im } \lambda_n = \epsilon_2(n) \text{Re } \lambda_n, \quad \epsilon_2(n) \to 0 \) as \( n \to \infty \)

\( \text{Re } (\lambda_n - \lambda_m) \geq (n - m) n^{\mu - 1} (1 + \epsilon_3(n,m)), \quad n > m, \)

with \( \epsilon_3(n,m) \to 0 \) as \( \min(n,m) \to \infty \)

\( \text{Im } (\lambda_n - \lambda_m) = \epsilon_4(n,m) \text{Re } (\lambda_n - \lambda_m) \)

with \( \epsilon_4(n,m) \to 0 \) as \( \min(n,m) \to \infty \).

Let \( S \in UTM \) be the diagonalizer of \( K \) with \( S_{ii} = 1 \), \( 0 \leq i < \infty \). Then for all \( t \in \mathbb{C}, \text{Re } t > 0 \), the matrices

\[ e^{tA} S e^{-tA} \quad \text{and} \quad e^{tA} S^{-1} e^{-tA} \]

are \( l_2 \)-bounded.

Their strict upper diagonal parts are Hilbert-Schmidt.

**Proof.**

The proof consists of parts (a)-(e).

(a) Since \( KS = S \Delta_K \) and \( S^{-1}K = \Delta_K S^{-1} \) we have the following recurrence relations for the entries of \( S \) and \( S^{-1} \):

\[ S_{n-q,n} = (K_{n-q,n} - K_{nn})^{-1} \sum_{k=0}^{q-1} K_{n-q,n-k} S_{n-k,n}, \]

\[ S_{nn} = 1, \quad 1 \leq q \leq n - 1. \]
\[
S_{-1}^{-1} = (K_{mn} - K_{m+q,m+q})^{-1} \sum_{k=0}^{q-1} K_{m+k,m+q} S_{m,m+k}
\]

\[
S_{mn}^{-1} = 1, \quad q \geq 1.
\]

We estimate

\[
|S_{-q,n}| \leq \theta C_n \sum_{k=0}^{q-1} |S_{n-k,n}|, \quad 1 \leq q \leq n - 1
\]

\[
|S_{m,m+q}| \leq \theta C_{m+q} \sum_{k=0}^{q-1} |S_{m,m+k}|, \quad q \geq 1.
\]

By induction it follows that

\[
\begin{aligned}
|S_{m,n}| &\leq (1 + \theta C_n)^{n-m} \\
|S_{m,n}^{-1}| &\leq (1 + \theta C_n)^{n-m}.
\end{aligned}
\]

(b) Let \( t \in \mathbb{C}, \Re t > 0 \), be fixed. Put \( t = t_1 + i t_2, \ t_1, t_2 \in \mathbb{R}, \ t_1 > 0 \). Using condition (ii) define

\[
M_t = \sup_{n \geq 0} \left[ (1 + \theta C_n) \exp(-\frac{1}{4} t_1 n^{w-1}) \right].
\]

Put

\[
\sigma_{nn}' = (e^{i\Lambda} S e^{-i\Lambda})_{mn} = S_{mn} e^{-i(\omega_\Lambda - \omega_m)}.
\]

With the estimate of part (a) we find

\[
|\sigma_{mn}'| \leq \exp\left( (n-m) \left[ \log M_t + \frac{1}{4} t_1 n^{w-1} \right] - t_1 \Re(\lambda_n - \lambda_m) + t_2 \Im(\lambda_n - \lambda_m) \right).
\]

(c) We now show that each row in \( \sigma' \) is an \( l_2 \)-sequence. So let \( m \) be fixed

\[
|\sigma_{mn}'| \leq \exp\left( -m \log M_t - \frac{1}{4} t_1 m n^{w-1} + t_1 \Re \lambda_m - t_2 \Im \lambda_m \right) \cdot \\
\cdot \exp\left[ n \log M_t + \frac{1}{4} t_1 n^{w} - t_1 \left( n^{w}(1 + e_1(n)) - \frac{t_2}{t_1} e_2(n) \right) \right].
\]

The first factor is bounded. For large \( n \), the second factor is smaller than \( \exp - \frac{1}{2} t_1 n^{w} \). This proves the assertion.

(d) We want to show that for \( M \) sufficiently large \( \sum_{n > m \geq M} |\sigma_{mn}'|^2 < \infty \). So, starting from the estimate obtained in (b)

\[
|\sigma_{mn}'| \leq \exp\left( (n-m) \left[ \log M_t + \frac{1}{4} t_1 n^{w-1} \right] - t_1 [(n-m) n^{w-1}(1 + e_3(n,m)) (1 - \frac{t_2}{t_1} e_4(n,m))] \right).
\]

For \( m = \min(m,n) \) sufficiently large, larger than \( M \) say, this is smaller than \( \exp - \frac{1}{4} t_1 (n-m) n^{w-1} \). So, below the \( M^{th} \) row the strict upper diagonal part of \( \sigma' \) is
Hilbert-Schmidt.

(e) According to (c) the first $M$ rows of $\sigma'$ establish a Hilbert-Schmidt matrix. So together with (d) we get the result.

For $e^{tA} S^{-1} e^{-tA}$ the proof is exactly the same.

In the next theorem we want to deal with matrices $M$ of growth about $e^{tn}$ as $n \to \infty$. For results in this direction we have to put conditions of greater subtlety on $K$.

THEOREM 2.2.

Let $K \in UTM$. Put $C_n = \max_{1 \leq i < j \leq n} |K_{ij}|$ and $\bar{C}_n = n \max_{1 \leq i \leq n} \frac{1}{n} C_i$. Suppose

(i) $\exists D > 0 \ \forall n > m \ |K_{nm} - K_{nn}|^{-1} \leq \frac{D}{n(n-m)}$

(ii) $\sup_{n \geq 0} \left( \frac{1}{n} \bar{C}_n \right) = \delta < \infty$.

Let $\Lambda = \text{diag}(\lambda_0, \lambda_1, \cdots)$, with

(iii) $\text{Re} \lambda_n = n(1 + \epsilon_1(n))$, $\epsilon_1(n) \to 0$ as $n \to \infty$

$\text{Im} \lambda_n = \epsilon_2(n) \text{Re} \lambda_n$, $\epsilon_2(n) \to 0$ as $n \to \infty$

$\text{Re}(\lambda_n - \lambda_m) \geq (n-m)(1 + \epsilon_3(n,m))$, $n > m$

with $\epsilon_3(n,m) \to 0$ as $\min(n,m) \to \infty$

$\text{Im}(\lambda_n - \lambda_m) = \epsilon_4(n,m) \text{Re}(\lambda_n - \lambda_m)$

with $\epsilon_4(n,m) \to 0$ as $\min(n,m) \to \infty$.

Let $S \in UTM$ be the diagonalizer of $K$ with $S_{ii} = 1$, $0 \leq i < \infty$. Then for all $t \in \mathbb{C}$, $\text{Re} t > 0$, the matrices

$e^{tA} S e^{-tA}$ and $e^{tA} S^{-1} e^{-tA}$ are $l_2$-bounded.

Proof.

The proof consists of parts (a)-(f).

(a) We show that

$$\left\{ \frac{|S_{m,n}|}{|S_{m,n}^{-1}|} \right\} \leq \exp \left\{ 2 \left( \frac{D(n-m)}{n} \bar{C}_n \right)^{1/2} \right\}.$$

From the recurrence relations for the entries of $S$ we estimate
\[ |S_{n-q,n}| \leq \frac{DC_n}{n} \sum_{k=0}^{q-1} |S_{n-k,n}|, \quad 1 \leq q \leq n, \quad S_{nn} = 1. \]

By induction
\[ |S_{n-q,n}| \leq \frac{DC_n}{n} \prod_{l=1}^{q-1} \left( 1 + \frac{1}{l} \frac{DC_n}{n} \right), \quad 1 \leq q \leq n. \]

Because of \( \prod_{l=1}^{q} \left( 1 + \frac{x}{l} \right) \leq e^{2\sqrt{x}} \) and \( C_n \leq C_n \) we arrive at the first inequality.

For the second one
\[ |S_{m,m+q}^{-1}| \leq \frac{DC_{m+q}}{2(m+q)} \sum_{k=0}^{q-1} |S_{m,m+k}^{-1}|, \quad q = 1, 2, \ldots, S_{mm} = 1. \]

By induction
\[ |S_{m,m+q}^{-1}| \leq \frac{DC_{m+q}}{q} \prod_{l=1}^{q-1} \left( 1 + \frac{D}{l} \frac{C_{m+1}}{m+1} \right), \quad q = 1, 2, \ldots \]
\[ \leq \prod_{l=1}^{q} \left( 1 + \frac{1}{l} \frac{DC_{m+q}}{m+q} \right) \leq \exp 2 \left\{ q D \frac{C_{m+q}}{m+q} \right\}. \]

(b) Put \( t = t_1 + i t_2 \) and \( \sigma'_{mn} \) as in the proof of the preceding theorem. With the estimate of part (a) we find
\[ |\sigma'_{mn}| \leq \exp \left\{ 2 \left| D(n-m) \frac{C_n}{n^2} \right|^\frac{1}{2} - t_1 \Re(\lambda_n - \lambda_m) + t_2 \Im(\lambda_n - \lambda_m) \right\}. \]

(c) We now show that each row in \( \sigma' \) in an \( l_2 \)-sequence. So let \( m \) be fixed
\[ |\sigma'_{mn}| \leq \exp \left\{ t_1 \Re(\lambda_m) + t_2 \Im(\lambda_m) \right\} \cdot \exp \left\{ 2n \left[ \frac{DC_n}{n^2} \right]^\frac{1}{2} - t_1 n (1 + e_1(n) + t_2 \frac{e_2(n)}{t_1}) \right\}. \]

Take \( N \) so large that for \( n > N \) we have
\[ 2 \left[ \frac{DC_n}{n^2} \right]^\frac{1}{2} \leq \frac{1}{4} t_1 \quad \text{and} \quad (1 + e_1(n) + t_2 \frac{e_2(n)}{t_1}) > \frac{1}{2}. \]

For \( n > N \) the second factor is smaller than \( e^{-\frac{1}{2}t_1n} \). Hence the result.

(d) Starting from the inequality in (b) we find, applying condition (ii)
\[ |\sigma'_{mn}| \leq \exp \left\{ 2 \left( D \delta(n-m) \right)^\frac{1}{2} - t_1 (n-m) \left( 1 + e_3(n) \right) \left( 1 - \frac{t_2}{t_1} e_4(n) \right) \right\}. \]

Below the \( M^{th} \) row, \( M \) sufficiently large, we have
\[ |\sigma_{m,n}^l| \leq \exp\left(2(D \delta(n-m))^{\frac{1}{2}} - \frac{1}{2} t_1(n-m)\right). \]

Therefore each codiagonal is a bounded sequence. Finally for \( m \geq M \) and

\[(n-m) \geq \frac{1}{t_1} 8(D \delta)^{\frac{1}{2}} = L \]

\[ |\sigma_{m,n}^l| \leq e^{-\frac{1}{2} t_1(n-m)}. \]

(e) We now split the matrix \( \sigma^l \) in 2 parts:

- The first \( M \) rows. They represent a bounded \( l_2 \)-operator which is even Hilbert Schmidt, cf. (c).
- The part below the \( M^{th} \) row also represents a bounded \( l_2 \)-operator because of the "codiagonal estimate", use (d),

\[ \sum_{k=0}^{\infty} (\sup_{j-i=k} |K_{ij}|) < \infty. \]

(f) For \( e^{iA} S^{-1} e^{-iA} \) the parts (b)-(e) of the proof apply in exactly the same way.

\[ \square \]

**THEOREM 2.3.**

(a) The conditions on \( A \) in Theorems 2.1 and 2.2 are satisfied if \( \lambda_n = n^\mu(1+p(n)), \mu \geq 1, \) and

\[ \lim_{n \to \infty} p(n) = 0, \lim\sup_{m \to \infty} \frac{p(n) n - p(m) m}{n - m} = 0. \]

(b) The conditions of (a) are satisfied if \( p \) is a (complex valued) differentiable function on \((0,\infty)\) with \( \lim_{\xi \to \infty} p(\xi) = 0 \) and \( \lim_{\xi \to \infty} \xi p'(\xi) = 0. \)

(b) If \( A = N_n^a = \text{diag} \left( \cdots, (n(n+a))^{\frac{1}{2}}, \cdots \right), a \in \mathbb{C}, \) then condition (b) is satisfied.

**Proof.**

(a) We compute

\[ \lambda_n - \lambda_m = (n-m) n^{\mu-1} \left[ 1 + \left\{ 1 - \left( \frac{m}{n} \right)^\mu \frac{n}{1 - \frac{m}{n}} - 1 \right\} + \right. \]

\[ + \left\{ \frac{n p(n) - m p(m)}{n - m} \right\} \left. \right\} \frac{m}{n} \left(1 - \frac{m}{n}\right)^{\mu-1} p(m) \right\}. \]

On the interval \([0,1)\) one has
\[
\frac{1 - x^k}{1 - x} \geq 1 \quad \text{and} \quad \frac{1 - x^{k-1}}{1 - x} \leq \max(1, \mu - 1) \leq \mu.
\]

For \( e_3(n,m) \) take the real parts of the 3\textsuperscript{rd} and 4\textsuperscript{th} terms between \{ \}. We omit the simple verification of the other conditions.

(b) Follows by application of the mean value estimation on \((n - m)^{-1} (p(n) n - p(m) m)\).

(c) We have \( p(x) = (1 + \frac{A}{x})^2 - 1 \) which satisfies (b).

3. Application to classical polynomials.

Take \( X = L_2([-1,1], (1 - x^2)^{-\frac{1}{2}} \, dx) \) with the orthonormal basis of normalized Chebyshev polynomials \((T_n)_{n=0}^{\infty}\). Recall that \( T_0(\cos \theta) = \frac{1}{2} \), \( T_n(\cos \theta) = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \cos n \theta \), \( n = 1, 2, \cdots \). Note that \( \text{span}(T_n) = \{ \text{all polynomials} \} \).

For \( K \) we take

\[
K = (A_1 x^2 + A_2 x + A_3) \frac{d^2}{dx^2} + (A_4 x^2 + A_5) \frac{d}{dx},
\]

with fixed \( A_i \in \mathbb{C} \). If \( A_1 = 0 \) then \( A_4 \neq 0 \), if \( A_1 \neq 0 \) then \( 1 - \frac{A_4}{A_1} \notin \mathbb{N} \).

The matrix \( K \) of \( K \) with respect to \((T_n)_{n=0}^{\infty}\) belongs to \( UTM \). From a straightforward calculation, using the transformation \( x = \cos \theta \), if follows that

\[
K_{nn} = A_1 n^2 - (A_1 - A_4) n \quad \text{and} \quad \exists D > 0 \ |K_{mn}| \leq D n^2.
\]

So the entries of \( K \) satisfy the conditions (i) and (ii) of Theorem 2.1 for all \( \mu > 1 \). Let us look at the polynomials \( T_n^K \) for some special cases.

- **Generalized Laguerre polynomials** \( L_n^{(\alpha)} \).

\( A_2 = 1, A_4 = -1, A_5 = \alpha + 1, A_1 = A_3 = 0, \alpha \in \mathbb{C} \).

We normalize, cf. \[AS\],

\[
L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} x^n + \cdots = \frac{(-1)^n (2\pi)^{\frac{1}{2}}}{2^n n!} T_n(x) + \cdots T_{n-1}(x) + \cdots.
\]

In this case \( T_n^K = (-1)^n 2^n n ! (2\pi)^{\frac{1}{2}} L_n^{(\alpha)} \).

- **Hermite polynomials** \( H_n \).

\( A_3 = 1, A_4 = -2, A_1 = A_2 = A_5 = 0 \). We normalize

\[
H_n(x) = 2^n x^n + \cdots = (2\pi)^{\frac{1}{2}} T_n(x) + \cdots.
\]

In this case \( T_n^K = (2\pi)^{\frac{1}{2}} H_n \).
• Monomials $x^n$.

$A_1 = A_4 = 1$, $A_2 = A_3 = A_5 = 0$. We normalize

$$x^n = \frac{(2\pi)^\frac{1}{2}}{2^n} T_n + \cdots T_{n-1} + \cdots.$$  

In this case $T_n^K = (2\pi)^{-\frac{1}{2}} 2^n x^n$.

• Jacobi polynomials $T_n^{\alpha\beta}$.

They are eigenfunctions of the differential operators

$$K = A_{ab} = -(1-x^2) \frac{d^2}{dx^2} + (\alpha + \beta + 2)x \frac{d}{dx} - (\beta - \alpha) \frac{d}{dx}, \alpha, \beta \in \mathbb{C}, -(1+\alpha+\beta) \notin \mathbb{N}.$$

In this case the entries of $K$ satisfy

$$K_{nn} = n(n + \alpha + \beta + 1) \text{ and } |K_{nn}| \leq (2|\beta - \alpha| + |\alpha + \beta + 1|) n.$$  

Therefore in this case the conditions of Theorem 2.2 are satisfied. We normalize

$$T_n^K (x) = T_n^{\alpha\beta} (x) = T_n(x) + \cdots.$$  

Note that $T_n^K = T_n$. In the special case that $\alpha > -1$, $\beta > -1$, there exist constants $\kappa_n^{\alpha\beta}$, uniformly bounded and bounded away from zero, such that $T_n^{\alpha\beta} = \kappa_n^{\alpha\beta} R_n^{\alpha\beta}$. Here the $R_n^{\alpha\beta}$ are the normalized Jacobi polynomials which establish an orthonormal basis in $L_2([-1,1], (1-x)^\alpha (1+x)^\beta \, dx)$. For the details see [GE].

In the next two theorems a fixed sequence $(\rho(n))_{n=0}^\infty$ occurs which is, on some interval $(M, \infty)$, the restriction of a differentiable function $\rho$ with $\lim_{\xi \to \infty} \rho(\xi) = 0$ and $\lim_{\xi \to \infty} \xi \rho'(\xi) = 0$.

Cf. Theorem 2.3(b). For example, a sequence like

$$(\log n)^\lambda n^\beta \exp(t n^\mu + Cn))_{n=0}^\infty, A,B,C \in \mathbb{R}, t > 0, \mu \geq 1,$$

is of type $\exp(t n^\mu(1+\rho(n)))$.

The following result is a consequence of Theorem 2.1.

**THEOREM 3.1.**

Let $\mu > 1$, $t > 0$. Let $\rho(n)$ be as above and fixed. Define $\theta = (\theta_n)_{n=0}^\infty$ by $\theta_n = \exp t n^\mu(1+\rho(n))$.

Let $f$ be an entire analytic function. Let $\alpha, \beta, \gamma \in \mathbb{C}, -(1+\alpha+\beta) \notin \mathbb{N}$. Consider the expansions

$$f(z) = \sum_{n=0}^\infty a_n \{\theta_n^{-1} 2^n z^n\} = \sum_{n=0}^\infty b_n \{\theta_n^{-1} T_n(z)\} = \sum_{n=0}^\infty c_n^{\alpha\beta} \{\theta_n^{-1} T_n^{\alpha\beta}(z)\}$$

$$= \sum_{n=0}^\infty d_n \{\theta_n^{-1} H_n(z)\} = \sum_{n=0}^\infty e_n \{\theta_n^{-1} 2^n n! L_n(z)\}.$$  

Either the coefficient sequences $(a_n)_{n=0}^\infty$, etc., are all $l_2$-sequence or none of them is $l_2$. In other words, the restriction to $[-1,1]$ of each of the function systems $\{\cdots\}$ is a Riesz basis in $X_\theta$. 

REMARKS 3.2.

- As a topological vector space, $X_\theta$ is equal to $Y_\theta$ with $Y$ any of the Hilbert spaces $L_2([-1,1], (1-x)^\alpha (1+x)^\beta \, dx)$, $\alpha > -1$, $\beta > -1$, with the $(R_n^\theta)_{n=0}^\infty$ as their selected bases. Cf. [GE].

- In [GE] for certain sequences $\theta$ the space $X_\theta$ has been characterized as a weighted $L_2$-space of entire functions:
  If $\theta_n = n^\gamma e^{nt}$, then $f \in X_\theta$ iff
  \[ \iint |f(x+iy)|^2 w(x+iy) \, dx \, dy < \infty, \]
  with
  \[ w(z) = |z|^{-2} (\log |z|)^2 \exp\left(-2 \frac{1}{\lambda} |\log w| 1 \right)^{1/2}, \]
  \[ \gamma = \frac{2k+2 \mu - 1}{2(1-\nu)} , \quad t = (1-\nu) (\lambda \nu)^{\nu/(1-\nu)} , \quad \mu = \frac{1}{1-\nu}. \]

The final result of this paper is a consequence of Theorem 2.2.

THEOREM 3.3.

Let $t > 0$. Let $\rho(n)$ be a above and fixed.
Define the sequence $\mu = (\mu_n)_{n=0}^\infty$ by $\mu_n = \exp t n (1+\rho(n))$. Let $f$ be an analytic function on $[-1,1]$. In the expansions
\[ f(z) = \sum_{n=0}^\infty b_n \{ \mu_n^{-1} T_n(z) \} = \sum_{n=0}^\infty C_n^{\phi} \{ \mu_n^{-1} T_n^{\phi}(z) \}. \]

Either all coefficient sequences $(b_n)_{n=0}^\infty$ and $(c_n^{\phi})_{n=0}^\infty$ are $l_2$-sequences or none of them is $l_2$. Hence, the restriction to $[-1,1]$ of each of the function systems $\cdots$ is a Riesz bases in $X_\mu$. \( \Box \)

REMARKS 3.4.

- The first of the Remarks 3.2 also applies here.

- In [G] we show that the $X_\mu$ can be characterized as weighted Sobolev spaces on an ellips in the complex plane. Thus refining Szegö's result [Sz].

REFERENCES


