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by

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Determining the Velocity as a Kinematic Boundary Condition in a Glass Pressing Problem

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Abstract

A typical stage in the manufacturing of container glass, such as bottles and jars, is the pressing phase. At this stage a gob of hot glass (1000 °C or higher) is pressed in a mechanical construction. Various aspects of this process can be investigated with respect to mathematical modeling and numerical simulation. This paper describes an important part of the process, the motion of the plunger. The problem is to determine the velocity of the latter which appears as a (kinematic) boundary condition of the glass flow problem. We show how to effectively uncouple these and to solve the resulting stiff differential equation numerically.

1 Introduction

Glass is an interesting material. It can be found in many forms [2], [3], [4], [9], [10], for which pressing, blowing, and casting is employed. The morphology is a complex physical process, in which heat exchange by convection, diffusion as well as radiation is still not fully understood cf. [1], [5], [6], [7]. In this paper we consider in particular the *pressing* of glass. This is a typical stage for many glass forming processes, which is being carried out when the glass is sufficiently hot. This pressing takes place in a mould with a moving part, called the *plunger*, powered by a piston. In a so called *press-blow* system we then obtain an intermediate form, called the *parison*, which is then blown into its final shape at the next stage. This paper is concerned with this pressing, and more specifically with the motion of the plunger. In Figure 1.1 we have sketched the various parts making up for the mould. The actual mould consists of the *baffle*, the *blank*, and the *neckring*. Initially the baffle part is removed and the mould is open from above. After the baffle has been closed the

plunger moves up gradually. The whole pressing process can be described by the following steps

Figure 1.1: Mechanical system, cross section.

1. The baffle is removed; the plunger is in its initial position.
2. A gob of glass falls down on top of the plunger into the blank.
3. The baffle closed.
4. The plunger starts moving up; the molten glass takes an appropriate form.
5. The baffle is removed; the mould is open; the parison is ready.

The plunger movement is a result of applying a certain pressure p_p from the bottom. It is of great importance to know when the parison is actually ready and the mould can be opened. Therefore the time elapse between steps 3 and 5, the so called *pressing time*, turns out to be a key parameter, indicating how long it will take to fill the mould. Depending on the specific process, this time is about 1-2 seconds. During this phase the pressure will increase and eventually will make the plunger stop. For a glass manufacturer it is important to distinguish between the actual motion and time duration after the plunger has been stopped. The first is called *pressing time* and the second *dwell time*. The first period should not take too long, while the

second is mainly used for cooling.

One of the characteristics of the production machinery unit is so called *cavity rate*, i.e. the number of parisons produced by a single machine per minute. In practice this can be regulated by a switch, which will adjust the machinery to a desired rate, thus producing a single parison in a longer or a shorter time. Obviously, one is interested to have this number as high as possible. At the same time, the cavity rate is directly linked to the pressing time, as described above. Setting this value too high can cause the mould to open while the *pressing* is still in progress. This will produce a parison which is unacceptable for further processing.

An essential ingredient in this clearly is the motion of the plunger. Since it is counteracted by (viscous) forces of the glass, there is a tight connection between the motion of this and that of the glass body, thus making a numerical simulation very complicated. We shall develop an alternative way, which is much cheaper. In this paper we shall model the motion of the glass, employing solutions of the Stokes equations. These equations require a velocity boundary condition, arising from the plunger. The latter in turn is to be found from the velocity field of the glass. We shall show how to virtually decouple these, thus enabling us to employ a marching procedure for the actual motion of both glass and plunger separately.

This paper is built up as follows. We first derive the equations which govern the motion of the glass fluid in Section 2. These equations determine the velocity and pressure of the gob, given the geometry of the glass body. Of great importance are the boundary conditions. We show that they are, roughly speaking, linearly depending on the velocity and the pressure. The latter fact turns out to be a crucial aspect in decoupling the motion of the plunger from that of the glass. In Section 3 we first derive the equation for this plunger motion. A seemingly simple approach for obtaining a new geometry for the Stokes equation at a subsequent time point is to employ an explicit integrator for the plunger motion. Although this decouples the two problems, it does not work in practice. Indeed, in Section 4 we show that the problem is stiff. The actual decoupling uses the aforementioned linear dependence on the velocity of the Stokes problem. In Section 5 we show how this is being done. This is illustrated by a numerical example. Finally we give a conclusion in Section 6.

2 Stokes Equations and the Boundary Conditions in Cylindrical Coordinates

Glass can be considered an incompressible Newtonian fluid. Let $\mathbf{v}$ denote the velocity of the fluid and p the pressure, then the stress tensor linearly depends on the pressure and can be expressed in the following form

$$\sigma(\mathbf{v}, p) = -pI + \eta(\nabla\mathbf{v} + (\nabla\mathbf{v})^T), \quad (2.1)$$

where I is the identity tensor. Note that σ is linear with respect to $\mathbf{v}$ and p .

The flow can be described by the Stokes equations which are a simplified form of the Navier-Stokes equations with small *Reynolds number*. For a more detailed description see [4]. Since the plunger and the mould are axisymmetric, it makes sense to model the flow as axisymmetric too, which will be done below. Let us denote by r and z the radial and axial coordinates respectively. The Stokes problem can then be formulated as follows: find a velocity field $\mathbf{v} := (u_r(r, z), u_z(r, z), 0)^T$ and a pressure field $p := p(r, z)$, which satisfy

$$\nabla \cdot \sigma(\mathbf{v}, p) = 0, \quad (2.2)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (2.3)$$

where $\sigma(\mathbf{v}, p)$ is the stress tensor, given by (2.1). Since all the derivatives must be taken in cylindrical coordinates, we can obtain a matrix representation of (2.1)

$$\sigma = \begin{pmatrix} -p + 2\eta \frac{\partial u_r}{\partial r} & \eta \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) & 0 \\ \eta \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) & -p + 2\eta \frac{\partial u_z}{\partial z} & 0 \\ 0 & 0 & -p + 2\eta \frac{u_r}{r} \end{pmatrix}. \quad (2.4)$$

Equations (2.2), (2.3) can then be written out in terms of cylindrical coordinates. As the third component of $\mathbf{div} \sigma(\mathbf{v}, p)$ is zero there are only two equations for (2.2). Together with the divergence they give the following system

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial}{\partial r}(\nabla \cdot \mathbf{v}) = \frac{1}{\eta} \frac{\partial p}{\partial r}, \quad (2.5)$$

$$\frac{\partial^2 u_z}{\partial r^2} + \frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{\partial}{\partial z}(\nabla \cdot \mathbf{v}) = \frac{1}{\eta} \frac{\partial p}{\partial z}, \quad (2.6)$$

$$\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} = 0. \quad (2.7)$$

Because of (2.3) the last terms on the left hand side of (2.5), (2.6) vanish.

Since we have an axisymmetric problem we obtain a glass domain Ω_t , as sketched in Figure 2.1. The boundary $\Gamma := \partial\Omega_t$ of the domain consists of four parts

$$\Gamma = \Gamma_s \cup \Gamma_m \cup \Gamma_p \cup \Gamma_f, \quad (2.8)$$

where the indices s, m, p, f represent the symmetric, mould, plunger and free

Figure 2.1: Problem domain.

boundaries respectively (see also Figure 1.1). Let $\mathbf{n}$, $\mathbf{t}$ be the normal and tangent unit vectors respectively for the boundary Γ in the directions as displayed in Figure 2.1. Then we find the following boundary conditions.

Because of symmetry, the boundary conditions on Γ_s are

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad (2.9)$$

$$\sigma \mathbf{n} \cdot \mathbf{t} = 0. \quad (2.10)$$

It is easy to see that

$$\mathbf{n} = (-1, 0)^T, \quad \mathbf{t} = (0, -1)^T, \quad \sigma \mathbf{n} = (-\sigma_{rr}, -\sigma_{rz})^T$$

on Γ_s . Using the expressions for the stress tensor components (2.4) we obtain

$$u_r = 0, \quad \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = 0.$$

Since $u_r \equiv 0$ on Γ_s , it follows that the derivative along Γ_s is also equal to zero, i.e., $\partial u_r / \partial z = 0$. As a result the boundary conditions on Γ_s can be written as

$$u_r = 0, \quad \frac{\partial u_z}{\partial r} = 0. \quad (2.11)$$

For the mould and the plunger we will allow both slip and no-slip boundary conditions and everything in between. A partial slip boundary condition for the mould means that the normal component of the velocity should be zero and the tangential component proportional to the tangential stress, i.e.

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad (2.12)$$

$$(\sigma \mathbf{n} + \beta_m \mathbf{v}) \cdot \mathbf{t} = 0, \quad (2.13)$$

where β_m is a friction coefficient. The first equation clearly represents a Dirichlet boundary condition, and the second a Robin boundary condition.

For the plunger which moves with velocity $\mathbf{v}_p$, we find

$$(\mathbf{v} - \mathbf{v}_p) \cdot \mathbf{n} = 0, \quad (2.14)$$

$$(\sigma \mathbf{n} + \beta_p (\mathbf{v} - \mathbf{v}_p)) \cdot \mathbf{t} = 0. \quad (2.15)$$

Note that $\mathbf{v}_p$ does not depend on r, z , and that β_p is again the friction coefficient. The physical meaning of these conditions is the same as for (2.12), (2.13), with the only difference that here we consider the velocity relative to $\mathbf{v}_p$, i.e., $\mathbf{v} - \mathbf{v}_p$. Also we are using the fact that $\sigma(\mathbf{v} - \mathbf{v}_p, p) = \sigma(\mathbf{v}, p)$. Let $V_p > 0$ be the absolute velocity of the plunger, then

$$\mathbf{v}_p = V_p \mathbf{e}_z := (0, V_p, 0)^T. \quad (2.16)$$

Actually, the velocity of the plunger V_p is an unknown function of time t , so we should write $V_p(t)$. Nevertheless, for the boundary conditions below and the Stokes problem as such, we view this as just a parameter. Hence, the boundary conditions read as follows

$$\mathbf{v} \cdot \mathbf{n} = V_p \mathbf{e}_z \cdot \mathbf{n}, \quad (2.17)$$

$$(\sigma \mathbf{n} + \beta_p \mathbf{v}) \cdot \mathbf{t} = \beta_p V_p \mathbf{e}_z \cdot \mathbf{t}. \quad (2.18)$$

Finally the boundary conditions at the free boundary Γ_f are defined as the vector relation

$$\sigma \mathbf{n} = -p_0 \mathbf{n}, \quad (2.19)$$

where p_0 is the external pressure. We can take the inner product of (2.19) with $\mathbf{n}$, $\mathbf{t}$ and obtain the boundary conditions in the form of two scalar equations

$$\sigma \mathbf{n} \cdot \mathbf{n} = -p_0, \quad (2.20)$$

$$\sigma \mathbf{n} \cdot \mathbf{t} = 0. \quad (2.21)$$

Note that the velocity field found from (2.2), (2.3) with the boundary conditions (2.9) – (2.21), is independent of the value of p_0 .

Once the boundary conditions are defined, (2.2), (2.3) can be solved numerically, using a finite element approach, for example (see [8]). As a result the velocity field $\mathbf{v}$ and the pressure p can be obtained. One can see that equations (2.2), (2.3) are time independent. In order to find the evolution of Ω_t we need to integrate the solution $\mathbf{v}$ found from the Stokes equation. In particular we have for $\mathbf{x} \in \Omega_t$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}). \quad (2.22)$$

Given Ω_{t^n} and with some extra care taken to restrict the motion of the glass within the boundaries of the mould, we can use a forward Euler step and thus obtain a numerical approximation for $\Omega_{t^{n+1}}$. This then is used to define the Stokes equations and boundary conditions at t^{n+1} . These steps can be continued until the glass will fill the physical domain, defined by the eventual mould and the plunger geometries. This procedure is not described in this paper, for more detail see [4].

It is easy to check that the velocity field $\mathbf{v}$ and the pressure field p are affine with respect to V_p . This is worked out in the following lemma.

Lemma 2.1 *Let $\mathbf{v}_1$, p_1 and $\mathbf{v}_2$, p_2 be the solutions of the Stokes equations (2.2), (2.3) with corresponding plunger velocities V_{p_1} and V_{p_2} respectively. Then $k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2$, $p_0 + k_1(p_1 - p_0) + k_2(p_2 - p_0)$ is also a solution of the same equations with $V_p = k_1 V_{p_1} + k_2 V_{p_2}$.*

Proof. Using the fact that $\nabla \cdot p_0 \mathbf{I} = 0$ it follows that

$$\begin{aligned} \nabla \cdot \sigma(k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2, p_0 + k_1(p_1 - p_0) + k_2(p_2 - p_0)) &= \\ k_1 \nabla \cdot \sigma(\mathbf{v}_1, p_1) + k_2 \nabla \cdot \sigma(\mathbf{v}_2, p_2) &= 0. \end{aligned}$$

Hence, (2.2) is satisfied. Clearly (2.3) holds as well, since

$$\nabla \cdot (k_0 \mathbf{v}_0 + k_1 \mathbf{v}_1) = k_0 \nabla \cdot \mathbf{v}_0 + k_1 \nabla \cdot \mathbf{v}_1 = 0.$$

In the same way this property can be shown for the boundary conditions. Considering the pressure field relative to p_0 , the boundary conditions (2.20), (2.21) are satisfied

$$\begin{aligned} \sigma(k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2, p_0 + k_1(p_1 - p_0) + k_2(p_2 - p_0)) \mathbf{n} &= \\ k_1(\sigma(\mathbf{v}_1, p_1) \mathbf{n} + p_0 \mathbf{n}) + k_2(\sigma(\mathbf{v}_2, p_2) \mathbf{n} + p_0 \mathbf{n}) - p_0 \mathbf{n} &= -p_0 \mathbf{n} \end{aligned}$$

This proves the lemma. □

Clearly, in order to define (2.17), (2.18) we need the velocity of the plunger $V_p(t)$ to be known at time t . On the other hand, $V_p(t)$ clearly depends on the solution of the Stokes equations. This will be discussed in detail in the next section.

3 Plunger motion

Consider now the velocity of the plunger $V_p(t)$, which was previously assumed to be a parameter in the boundary conditions. In practice, the plunger movement is the result of a certain pressure p_p applied to the bottom. Hence, $V_p(t)$ has to be computed during the process as well as the unknown $\mathbf{v}$ and p of the glass. Without further discussion we assume that Newton mechanics can be applied to a plunger body. Let $F(t)$ denote the total force on the plunger and m_p be the mass of the plunger. Then

$$\frac{dV_p(t)}{dt} = \frac{F(t)}{m_p}. \quad (3.1)$$

The total force is the sum of the force coming from the plunger F_p , and the force exerted by the glass F_g , so

$$F(t) = F_p + F_g(t). \quad (3.2)$$

Here F_p remains constant through the whole process and can be computed as

$$F_p = S_p p_p = \text{const},$$

where S_p is the area of the surface where pressure p_p is applied. The second term $F_g(t)$, can be expressed in terms of the stress tensor (2.1)

$$F_g(t) = \int_{S(t)} \sigma \mathbf{n} dS, \quad (3.3)$$

where $\sigma \equiv \sigma(t)$ is the stress tensor, and $S(t)$ is a part of the plunger surface which is in contact with the glass at time t . Obviously, F_g is tightly coupled with the motion of the glass.

Figure 3.1: Geometry of the plunger.

We can work this out as follows: consider Figure 3.1 which depicts one half of the plunger (see Figure 1.1) turned by 90 degrees. If z is the azimuthal variable and $R(z)$ denotes the form of the plunger we can derive

$$dS = 2\pi R_p(s) ds = 2\pi \sqrt{1 + R'_p(z)^2} R_p(z) dz,$$

where s represents the length over the plunger profile. The normal components n_r , n_z (see Figure 3.1) are computed as follows

$$(n_r, n_z)^T = -\frac{1}{\sqrt{1 + R'_p(z)^2}} (1, R'_p(z))^T.$$

Note that the right hand side in (3.3) is formally a vector. The formula requires integration of the second component of $\sigma \mathbf{n}$ only, as the first one will vanish due to such integration, because of the axisymmetrical nature of the problem. So, using the expressions (2.4) for the stress tensor components, (3.3) can be rewritten as follows

$$F_g(t) = 2\pi \int_{z_0}^{z_1} \left(\left(p - 2\eta \frac{\partial u_z}{\partial z} \right) R'_p(z) + \eta \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \right) R_p(z) dz. \quad (3.4)$$

Solving the Stokes equations completed with 3.4 is a cumbersome task. A simple idea to overcome this would be employing explicit data. This goes as follows: In

order to compute the velocity of the plunger $V_p(t)$ as a function of time, one should solve the ordinary differential equation

$$\frac{dV_p(t)}{dt} = \frac{F_g(t)}{m_p} + \frac{F_p}{m_p}, \quad (3.5)$$

$$V_p(0) = V_{p0}, \quad (3.6)$$

where V_{p0} is some initial velocity of the plunger. Note that we can compute $F_g(t)$, once u_r , u_z and p are known. The latter are obtained from the solution of the Stokes equations (2.2), (2.3). In order to solve the Stokes equations one needs some value for the plunger velocity V_p in (2.17), (2.18). So, at time $t = 0$ we use V_{p0} from (3.6) to solve the Stokes equations and find $F_g(0)$. We can thus perform an *explicit integration step* to solve (3.5). Suppose we use the Euler Forward scheme here, then

$$V_p^{n+1} = V_p^n + \Delta t(F_g^n + F_p)/m_p. \quad (3.7)$$

Having solved the Stokes equations, with the new velocity of the plunger V_p^{n+1} we can complete the boundary conditions for the Stokes problem at $t = t^{n+1}$. To this end the velocity of the plunger obtained from (3.7) is used.

Figure 3.2: Velocity of the plunger (numerical instabilities).

However, as illustrated in Figure 3.2, this algorithm turns out to be unstable. Looking more carefully at Figure 3.2 we detect a phenomenon that looks like stiffness. Of course, to overcome this we should take recourse to implicit methods. A fully implicit scheme is, however, practically impossible as we do not know the plunger

velocity at t^{n+1} . Indeed, we then would have to solve the complete, coupled problem at each time step.

4 Stiffness

In this section we shall investigate the stiffness of the ordinary differential equation (3.5). Clearly, we need to have a closer look at $F_g(t)$, as derived in (3.4). In general it is impossible to compute it exactly so we take recourse to a thin film approximation. Here we shall approach the problem analytically in order to point out the stiffness phenomenon detected in numerical simulation. For a more detailed discussion see [9]. We shall consider a simple, yet meaningful, geometry for the mould and the plunger, see Figure 4.1. Let each of them be defined by a parabola, say

$$R_m(z) = d_m \sqrt{z}, \quad R_p(z) = d_p \sqrt{z - z_0},$$

where coefficients d_m, d_p have positive values and z_0 is the position of the plunger.

Figure 4.1: Mould and plunger geometries defined by analytical function.

Let us define $\varepsilon := D/L$ to be the ratio between the length scales corresponding to the parison's wall thickness D and the height of the parison L . Since D is smaller than L , ε is a small parameter. The variables can be then scaled as follows

$$r = Dr^*, \quad z = Lz^*, \quad u_r = \varepsilon V u_r^*, \quad u_z = V u_z^*, \quad p = \frac{\eta V L}{D^2} p^*, \quad (4.1)$$

where V is the typical flow velocity. Using (4.1) we can make (3.4) dimensionless

$$F_g(t) := 2\pi\eta VL F_g^*(t). \quad (4.2)$$

Then (3.4) can be approximated by the following expression

$$\begin{aligned} F_g^*(t) &= \int_{z_0^*}^{z_1^*} \left(\left(p^* - 2\varepsilon^2 \frac{\partial u_z^*}{\partial z^*} \right) R_p^{*'}(z^*) + \left(\varepsilon^2 \frac{\partial u_r^*}{\partial z^*} + \frac{\partial u_z^*}{\partial r^*} \right) \right) R_p^*(z^*) dz \\ &\approx \int_{z_0^*}^{z_1^*} \left(p^* R_p^{*'}(z^*) + \frac{\partial u_z^*}{\partial r^*} \right) R_p^*(z^*) dz. \end{aligned}$$

Using (4.1) it is possible to find the exact solution of the Stokes equations (2.5), (2.6), (2.7) (see [9])

$$u_r^* = \frac{1}{r^*} \frac{d}{dz^*} \int_{r^*}^{R_m^*} r^* u_z^*(r^*, z^*) dr, \quad u_z^* = \frac{1}{4} r^{*2} \frac{dp^*}{dz^*} + A(z^*) \ln r^* + B(z^*), \quad (4.3)$$

where $A(z^*)$ and $B(z^*)$ can be obtained from the boundary conditions. Then, according to (4.2) we obtain

$$F_g(t) \approx 2\pi\eta VL V_p^*(t^*) \int_{z_0^*}^{z_1^*} \frac{c_m - c_p}{(b_m - b_p)^2 - (a_m - a_p)(c_m - c_p)} dz, \quad (4.4)$$

where $V_p^*(t^*)$ is the dimensionless velocity of the plunger scaled with V ; $a_m, a_p, b_m, b_p, c_m, c_p$ are defined as

$$a_m = \ln R_m^*(z^*) + s_m/R_m^*(z^*), \quad a_p = \ln R_p^*(z^*) + s_p/R_p^*(z^*),$$

$$b_m = R_m^{*2}(z^*)(1 + 2s_m/R_m^*(z^*)), \quad b_p = R_p^{*2}(z^*)(1 + 2s_p/R_p^*(z^*))$$

$$c_m = R_m^{*4}(z^*)(1 + 4s_m/R_m^*(z^*)), \quad c_p = R_p^{*4}(z^*)(1 + 4s_p/R_p^*(z^*)),$$

respectively. Here s_m, s_p are dimensionless parameters similar to the friction coefficients β_m, β_p as defined in Section 2. Note that all defined quantities are dimensionless.

The dimensionless integral in (4.4) can be computed numerically. The graph in Figure 4.2 shows the results of this integration as a function of upper bound z_1^* in (4.4). Using the same scaling (3.5) reads

$$\frac{dV_p^*}{dt^*} = V_p^* I(t) \frac{2\pi L^2 \eta}{V m_p} + \text{const}, \quad (4.5)$$

Figure 4.2: Dimensionless integral as a function of z_1^* .

where $t = t^* L/V$, $V_p = VV_p^*$, and $I(t)$ is the dimensionless integral from (4.4). The typical values for L and V are 10^{-1} m and 10^{-1} sec respectively. The mass of the plunger construction m_p is of order 1. The viscosity coefficient η for our problem is a large number

$$\eta \approx 10^4 \text{ kg/sec m.}$$

One can see that the coefficient of V_p^* in the right-hand side is a large. Indeed, taking $I(t) \approx 1$ (see Figure 4.2) we thus find

$$I(t) \frac{2\pi L^2 \eta}{Vm_p} \approx 10^4.$$

This, clearly, indicates that (4.5) is a stiff equation. One should note that η is the dominating quantity. This will also be the case for more complicated geometries; this then shows the inherent stiffness of the plunger motion equation.

5 Decoupling V_p

As stated in Section 2 the boundary conditions for the Stokes problem follow from the geometry and the (yet unknown) plunger velocity. We now consider the latter as a free parameter. From Lemma 2.1 it follows that we may then consider the velocity and pressure fields at some time t as affine functions of $V_p := V_p(t)$, so

$$\mathbf{v}(t; V_p) = V_p \mathbf{v}(t; 1),$$

$$p(t; V_p) = p_0 + V_p (p(t; 1) - p_0).$$

As a consequence we deduce from (3.4) that this then also holds for the glass force

$$F_g(t; V_p) = F_0(t) + V_p (F_g(t; 1) - F_0(t)), \quad (5.1)$$

where F_0 is the force on the glass due to normal air pressure

$$F_0(t) = 2\pi \int_{z_0}^{z_1} p_0 R_p'(z) R_p(z) dz.$$

Using (5.1) we can reformulate (3.5), (3.6) as follows

$$\begin{cases} \frac{dV_p(t)}{dt} = V_p(t) \frac{F_g(t; 1) - F_0(t)}{m_p} + \frac{F_p + F_0(t)}{m_p}, \\ V_p(0) = V_{p0}. \end{cases} \quad (5.2)$$

Note that one should use $V_p = 1$ for the boundary conditions (2.17), (2.18). By tracking the free boundary and defining the Stokes problem, the glass force $F_g(t; 1)$ can be computed for the changing domain Ω . As a consequence it makes sense to consider the force as a function of the plunger position, not the time. So we slightly change the notation

$$F_g := F_g(z; V_p), \quad V_p := V_p(z).$$

Equation (5.2) should be reformulated as follows

$$\begin{cases} \frac{1}{2} \frac{dV_p^2(z)}{dz} = V_p(z) \frac{F_g(z; 1) - F_0(z)}{m_p} + \frac{F_p + F_0(z)}{m_p}, \\ V_p^0 = V_{p0}. \end{cases} \quad (5.3)$$

Here we used the fact that

$$\frac{dV_p(t)}{dt} = \frac{dV_p(z)}{dz} V_p(z) = \frac{1}{2} \frac{dV_p^2(z)}{dz}.$$

In order to solve this equation one may e.g. use the Euler explicit method

$$\begin{cases} \frac{1}{2} \frac{V_p^{n+1} - V_p^n}{z^{n+1} - z^n} = V_p^n \frac{F_g(z^n; 1) - F_0(z^n)}{m_p} + \frac{F_p + F_0(z^n)}{m_p}, \\ V_p^0 = V_{p0}. \end{cases} \quad (5.4)$$

Clearly, this approach is identical to one in which plunger velocity for the boundary

conditions at the next time-step were obtained straight from the previous velocity field and pressure field

$$V_p(t + \Delta t) = V_p(t) + \Delta t(F_g(t) + F_p)/m_p. \quad (5.5)$$

The boundary conditions (2.17), (2.18) for the next stationary Stokes problem should use $V_p(t + \Delta t)$. We omit further discussion of (5.5).

Figure 5.1: Velocity of the plunger obtained using implicit scheme.

Now consider the implicit Euler method instead

$$\begin{cases} \frac{1}{2} \frac{V_p^{n+1,2} - V_p^{n,2}}{z^{n+1} - z^n} = V_p^{n+1} \frac{F_g(z^{n+1}; 1) - F_0(z^{n+1})}{m_p} + \frac{F_p + F_0(z^{n+1})}{m_p}, \\ V_p^0 = V_{p0}. \end{cases} \quad (5.6)$$

Although (5.6) is implicit, we just have a quadratic equation for V_p^{n+1} , which can be solved trivially. The result is depicted in Figure 5.1a. We clearly have a stable process now.

The velocity of the plunger in Figure 5.1a as it displayed is a function of z . In order to obtain the velocity as a function of t the following approximation can be used

$$\begin{cases} z^{n+1} = z^n + \Delta t^n V_p(z^n), \\ t^{n+1} = t^n + \Delta t^n, \end{cases} \quad (5.7)$$

where $t^0 = 0$. The final result is given in Figure 5.1b.

6 Conclusion

The computation of the plunger motion is intimately coupled with that of the glass. Moreover, the equation for the former turns out to be numerically stiff. This prevents us from using explicit methods which would virtually decouple the equation from the Stokes equations needed to compute the glass flow. The problems are overcome by noting that the equations for the glass flow are linear and the boundary conditions affine. This makes it possible to compute the flow variables independently of the actual plunger velocity in a way. The stiffness is overcome by switching from time to a space variable. The resulting ODE is only quadratic and can be solved numerically by an implicit method. The two ideas are combined in an efficient algorithm.

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