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A PRIORI RESULTS IN
LINEAR-QUADRATIC OPTIMAL
CONTROL THEORY
(preliminary draft)

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Eindhoven, January 1989
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A PRIORI RESULTS IN LINEAR–QUADRATIC OPTIMAL CONTROL THEORY

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Let us consider the linear, time-invariant, finite-dimensional system

\[ \Sigma: \dot{x} = Ax + Bu, \quad x(0) = x_0, \quad u(\cdot) \text{ locally square integrable over } \mathbb{R}^+, \]

together with the quadratic cost criterion

\[ J(x_0, u) = \int_0^\infty w(x, u) dt, \quad w(x, u) = x^TQx + 2u'Sx + u'Ru, \quad Q = Q', \quad R = R'. \]

If \( T \) denotes a given subspace, then we define the linear–quadratic control problem with stability modulo \( T \) (LQCP) as follows: For all \( x_0 \), determine

\[ J_T(x_0) := \inf \{ J(x_0, u) \mid u \in U(x_0) \text{ such that } (x(x_0, u))/T(\infty) = 0 \}. \]

Here \( U(x_0) \) stands for the set of locally square integrable inputs for which \( J(x_0, u) \) is either real, \( +\infty \) or \( -\infty \). If we assume that \( (A, B) \) is stabilizable, then, for all \( x_0 \), \( U(x_0) \) is not empty. Also, it is easily seen that \( J_T(x_0) \neq -\infty \) only if \( R \geq 0 \). Now we introduce the dissipation inequality (1)

\[ F(K) = \begin{bmatrix} Q + A'K + KA & KB + S' \\ B'K + S & R \end{bmatrix} \geq 0, \]

with \( K \) a real, symmetric matrix. Let \( \Gamma := \{ K = K' \mid F(K) \geq 0 \} \) and \( \Gamma_{\min} := \{ K \mid K \in \Gamma, \text{ rank } (F(K)) = \inf_{K \in \Gamma} \text{ rank } (F(K)) \}. \)
Standing assumption

(A, B) is stabilizable and \( \exists K_0 \in \Gamma : K_0 \preceq 0 \).

**Remark:** If \( w \geq 0 \), then \( K_0 = 0 \in \Gamma \).

**Theorem**

For every \( T \) and \( x_0 \), \( J_T(x_0) \) is real, and it equals \( x_0'K_Tx_0 \) with \( K_T \) a real, symmetric matrix. It holds that \( \rho := \inf_{K \in \Gamma} \text{rank } (F(K)) \) is fixed and, since \( \Gamma_{\text{min}} \) is not empty, \( \rho = \min_{K \in \Gamma} \text{rank } (F(K)) \). Moreover, \( K_T \in \Gamma_{\text{min}} \).

Hence the set \( \Gamma_{\text{min}} \) is of intrinsic importance since it contains all possible candidates for representing optimal costs for linear–quadratic control problems. In a final paper we would like to provide a structural method for computing \( \Gamma_{\text{min}} \). This technique is based upon work done in [2].

Next, we observe that for all \( T > 0 \) and for all \( u ([1]) \),

\[
\int_0^T w(x, u) \, dt + ((x(T))'K_Tx(T)) = \int_0^T y_{K_T}'y_{K_T} \, dt + x_0'K_Tx_0,
\]

where \( y_{K_T} = C_{K_T}x + D_{K_T}u \), the latter two matrices determined by the factorization \( F(K_T) = [C_{K_T} \quad D_{K_T}][C_{K_T} \quad D_{K_T}] \).

Then, we state the following
Theorem

Let \( u \in U(x_0) \) be such that \( J(x_0, u) \) is real and \( (x(x_0, u)/T)(\infty) = 0 \).

i) It holds that \( J(x_0, u) \geq \int_0^\infty y_{K_T}^T y_{K_T} \, dt + x_0^T K_T x_0 \).

ii) If \( \lim_{T \to \infty} ((x(\cdot))' K_T x(\cdot))_\infty := 1 \) and \( ((x(T))' K_T x(\cdot))_\infty \), then this limit exists and it is \( \leq 0 \).

iii) \( J(x_0, u) = x_0^T K_T x_0 \) if and only if \( \{x(x(\cdot))' K_T x(\cdot)\}_\infty = 0 \) and \( y_{K_T} = 0 \).

iv) \( \inf \{ \int_0^\infty y_{K_T}^T y_{K_T} \, dt | u \text{ such that } (x(x_0, u)/T)(\infty) = 0 \} = 0 \).

The theorem given above turns out to be a powerful instrument to obtain a priori results on LQCP's, apart from the fact that it contains such results itself. A few examples:

a) Let \( R > 0 \) (the regular problems). Then it is directly found that optimal inputs are state feedback laws. Furthermore, the optimal cost can be found with no difficulty.

b) Assume that \( w \succeq 0 \) (the non-negative definite problems). Then always \( K_T \succeq 0 \) and hence (ii) \( (x(x_0, u)/\ker(K_T))(\infty) = 0 \). It follows that \( J_T(x_0) = J_{\ker(K_T)}(x_0) \).

c) Assume that \( R \) is merely positive semi-definite. Then (measurable) optimal inputs need not exist ([3]). One way out of this problem is, to introduce distributions of the kind presented in [3] with respect to the system \( \Sigma_{K_T} = (A, B, C_{K_T}, D_{K_T}) \). This makes sense, since we can prove that the set of distributions that yield regular outputs \( y_{K_T} \) is independent of \( K_T \). Moreover, the space of strongly reachable states \( W(\Sigma_{K_T}) =: W_{K_T} \) ([3], [2]) is \( K_T \)-independent and it is computed together with \( \Gamma_{\min} \).
Finally, we mention the next

Conjecture

Let $\Gamma^0_{\min} := \{K \in \Gamma_{\min} | J_{\ker(K)}(x_0) = x_0'Kx_0\}$, then $K_T \in \Gamma^0_{\min}$.

If this conjecture would be true in general, then $K_{\|Rn}$ is the smallest element of $\Gamma^0_{\min}$. Note that always $K^+ := K_0 \in \Gamma^0_{\min}$. Also, the conjecture is true for $w \geq 0$ and/or $R > 0$!

References

