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Wilms, R.J.G.

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Properties of Fourier-Stieltjes sequences of distributions with support in \([0,1)\)

R.J.G. Wilms

Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
5600 MB Eindhoven
The Netherlands

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The Netherlands
PROPERTIES OF FOURIER-STIELTJES SEQUENCES OF DISTRIBUTIONS
WITH SUPPORT IN [0,1).

R.J.G. Wilms

Abstract
In the literature, the relation between characteristic functions and distribution functions is described by the uniqueness theorem, the inversion theorem, the convolution theorem and the continuity theorem. In this paper we consider the Fourier-Stieltjes sequence of distribution functions that only have probability mass in the half-open interval [0,1). We study the connection between properties of this Fourier-Stieltjes sequence and properties of the distribution function.

1. INTRODUCTION.

The fundamental properties of characteristic functions (ch.f.'s) and their relation to distribution functions (d.f.'s) can be found in e.g. Lukacs (1970). The best-known of these properties are the uniqueness theorem, the inversion theorem, the convolution theorem and the continuity theorem. Considering distributions on the circle, Mardia (1972) proves that the ch.f. determines the distribution uniquely. Mardia deals also with an inversion formula, the continuity theorem and central limit theorems on the circle.

A discrete analogue of the ch.f.'s is provided by the Fourier-Stieltjes sequence (F.S.S.; see e.g. Kawata (1972)). He states a uniqueness theorem showing that there is a unique correspondence between the d.f.'s and their F.S.S.'s if the d.f. is continuous at 0 or at 1, and further he gives an inversion formula. Feller (1971) shows that a distribution on the circle determines uniquely its F.S.S. In addition, he proves that a sequence represents the F.S.S. of a distribution on the circle if, and only if, it is positive definite (a sequence \( (\varphi_p)_{p=-\infty}^{\infty} \) is said to be positive definite if for all \( m \in \mathbb{N} \) and all \( z_1', \ldots, z_m \in \mathbb{C} \).

1 Postal adress: Eindhoven University of Technology, Department of Mathematics and Computing Science, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.
In this paper, we review the connection between d.f.'s with support in the half-open interval \([0,1)\) and the corresponding F.S.S.'s. First we give some definitions.

**Definition 1.1.** A function \(F\) defined on \(\mathbb{R}\) is said to be a distribution function if it satisfies

(a) \(F\) is non-decreasing on \(\mathbb{R}\)

(b) \(F\) is right-continuous

(c) \(\lim_{x \to -\infty} F(x) = 0, \lim_{x \to \infty} F(x) = 1.\)

We denote by \(\mathcal{F}(0,1)\) the class of d.f.'s with support in the half-open interval \([0,1)\). Furthermore, we define the F.S.S. of such d.f.'s as follows:

**Definition 1.2.** Let \(F \in \mathcal{F}(0,1)\). The Fourier-Stieltjes sequence \(c = (c(k))_{k=-\infty}^{\infty}\) of \(F\) is defined by

\[
c(k) = \int_{0,1} e^{2\pi ikx} \, dF(x) \quad \text{for } k \in \mathbb{Z}. \tag{1.1}\]

It is quite obvious that \(c(0) = 1, \left| c(k) \right| \leq 1 \) (for \(k \in \mathbb{Z}\)), and \(c(-k) = \overline{c(k)} \) (for \(k \geq 0\)). Clearly, the F.S.S. \(c = (c(k))_{k=-\infty}^{\infty}\) is determined by the \(c(k)\) for \(k \geq 0\). Therefore, we shall often define a F.S.S. in terms of \(c(k)\) for \(k \geq 0\) only. Moreover, considering a random variable (r.v.) \(X\) with corresponding d.f. \(F \in \mathcal{F}(0,1)\), we denote the F.S.S. \(C_X\) of \(F\) as a ch.f. \(\varphi_X\) of \(X\):

\[
\varphi_X(2\pi k) = \mathbb{E}e^{2\pi ikX} = c_X(k) \quad \text{for } k \in \mathbb{Z}. \tag{1.2}\]

Now we give three simple examples of F.S.S.'s.

**Example 1.3.** (i) Let \(U\) be the uniform distribution on \([0,1)\), then the corresponding F.S.S. \(C_U\) is given by

\[
c_U(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{1.3}\]

(ii) Let \(F \in \mathcal{F}(0,1)\) be a purely discrete d.f. with discontinuity points
\[ \{ \xi_m \}_{m=1}^{\infty} \] and let \( p_m \) be the jump of \( F \) at \( \xi_m \). A straightforward computation leads to the corresponding F.S.S.

\[ c_F(k) = \sum_{m=1}^{\infty} p_m e^{2\pi i k \xi_m} \quad \text{for } k \in \mathbb{Z}. \quad (1.4) \]

(iii) \( c \) is the F.S.S. of a lattice distribution (i.e. with \( \xi_m = a + mh \)) if, and only if, \( |c(k)| = 1 \) for some \( k \neq 0 \) (see e.g. Mardia (1972)).

We proceed by indicating the contents of the subsequent sections. In section 2 we give the uniqueness theorem, which describes the unique correspondence between d.f.'s on \([0,1)\) and F.S.S.'s. In section 3 we give the inversion formula, which determines the d.f. in terms of the F.S.S. In addition, we express the jump at a point \( x \) of the d.f. in terms of the F.S.S., and we give a sufficient condition on the F.S.S. for the corresponding d.f. to be absolutely continuous. In section 4 we give the convolution theorem, which describes the unique correspondence between the convolution (modulo 1) of the d.f.'s on \([0,1)\) and the multiplication of their F.S.S.'s. In addition, using properties of purely discrete d.f.'s, we derive another sufficient condition on the F.S.S. for a d.f. to be continuous. In section 5 we give a proof of the continuity theorem, which states that the mapping from the set of d.f.'s on \([0,1)\) to the set of F.S.S.'s is bi-continuous in the topology of weak convergence adapted slightly to d.f.'s on \([0,1)\). Finally, in section 6, we give applications of these results to the fractional parts of random variables.

2. THE UNIQUENESS THEOREM.

The following theorem is an analogue of the uniqueness theorem for F.S.S.'s (see e.g. Kawata (1972)). Here we assume that the d.f.'s have support in \([0,1)\). We give now the unique correspondence between d.f.'s and F.S.S.'s.

Theorem 2.1 (the uniqueness theorem). Let \( F, G \in \mathcal{F}[0,1) \). If \( c_F = c_G \), then \( F(x) = G(x) \) (for \( x \in \mathbb{R} \)).

Proof: Let \( H(x) = F(x) - G(x) \) for \( x \in [0,1] \). Then \( H \) is of bounded variation and from (2.1) we find...
\[ \int_{[0,1)} e^{2\pi i k x} \, dH(x) = 0 \quad \text{for } k \geq 0. \quad (2.2) \]

Furthermore, we deduce for all \( k \in \mathbb{Z} \):

\[ \int_{[0,1)} e^{2\pi i k x} \, dH(x) = -\int_{[0,1)} (1 - e^{2\pi i k x}) \, dH(x) + \int_{[0,1)} dH(x) \]

\[ = 2\pi i k \int_{[0,1)} e^{2\pi i k y} \left\{ \int_{[0,1)} e^{2\pi i k y} \, dy \right\} dH(x) + H(1-0) - H(0-0) \]

\[ = 2\pi i k \int_{[0,1)} e^{2\pi i k y} \left\{ \int_{[y,1)} dH(x) \right\} dy. \]

Hence we obtain

\[ \int_{[0,1)} e^{2\pi i k x} \, dH(x) = -2\pi i k \int_{[0,1)} H(x) \, e^{2\pi i k x} \, dx. \quad (2.3) \]

From (2.2) and (2.3) it follows immediately

\[ \int_{[0,1)} H(x) \, e^{2\pi i k x} \, dx = 0 \quad \text{for } k \geq 1. \]

Let

\[ A = \int_{[0,1)} H(x) \, dx \]

then

\[ \int_{[0,1)} (H(x) - A) \, e^{2\pi i k x} \, dx = 0 \quad \text{for } k \geq 0. \]

Hence by the completeness of the sequence \( e^{2\pi i k x} \) \( k \geq 0 \) in the class \( L^1([0,1]) \) (see e.g. Kawata (1972)), we find

\[ H(x) = A \quad (\text{a.e.}) \quad \text{for } x \in (0,1). \]

Now we take a sequence \( (x_m) \) \( m=1 \) with \( x_m \uparrow 1 \) through the set where

\[ A = \lim_{m \to \infty} H(x_m) = H(1-0) = 0. \]

Since \( H(x) \) is right-continuous, it follows that \( H(x) = 0 \) on \([0,1]\), i.e. that \( F(x) = G(x) \) on \([0,1]\), and hence on \( \mathbb{R} \).
3. THE INVERSION THEOREM.

In the preceding section we showed that a d.f. with support in \([0,1)\) is uniquely determined by its F.S.S. In this section we give explicit formulas for the d.f. in terms of its F.S.S. We further give a sufficient condition on the d.f. to be absolutely continuous. The proofs of the theorems stated here are very similar to the proofs of the analogous theorems for ch.f.'s such as given in Kawata (1972) or Lukacs (1970). Before proving the inversion theorem, we give a well-known result for Fourier series.

**Lemma 3.1.** For all \(x \in \mathbb{R}\) and for all \(m \neq 0\) we have

\[
\frac{x}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \begin{cases} 
 1 & 1 \leq m < x \leq (2m+1)\pi \\
 \frac{1}{2} + m & 2m\pi < x \leq (2m+1)\pi \\
 \frac{1}{2} - m & -(2m+1)\pi \leq x < -2m\pi \\
 -m & x = -2m\pi.
\end{cases}
\]

**Proof:** See e.g. Kawata (1972).

The following corollary is a consequence of this lemma.

**Corollary 3.2.** For all \(v \in (-1,1)\) we have

\[
v + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi nv)}{n} = \begin{cases} 
 \frac{1}{2} & 0 < v < 1 \\
 0 & v = 0 \\
 -\frac{1}{2} & -1 < v < 0.
\end{cases}
\]

**Theorem 3.3 (the inversion theorem).** Let \(c\) be the F.S.S. of \(F \in \mathcal{F}[0,1)\). Then

\[
F(z) - F(y) = \lim_{K \to \infty} \sum_{k=-K}^{K} \frac{e^{i2\pi k z} - e^{i2\pi k y}}{2\pi i k} c(k)
\]

(3.1)

for all continuity points \(y, z \in [0,1)\) of \(F\); here the term

\[
\frac{e^{i2\pi k z} - e^{i2\pi k y}}{2\pi i k}
\]

is interpreted to be \(z - y\) when \(k = 0\).

**Proof:** Let \(y, z \in [0,1)\) be continuity points of \(F\), and put
\[ S_K(y,z) = \sum_{k=-K}^{K} \frac{e^{2\pi iky} - e^{2\pi ikz}}{2\pi ik} c(k) \quad \text{for } y, z \in [0,1). \]

Because of the symmetry of the F.S.S. c, we can reverse the summation order to obtain

\[ S_K(y,z) = \sum_{k=-K}^{K} \frac{e^{-2\pi ikz} - e^{-2\pi iky}}{2\pi ik} c(k) \quad \text{for } y, z \in [0,1), \]

or equivalently

\[
S_K(y,z) = \sum_{k=-K}^{K} \left[ \frac{e^{-2\pi ikz} - e^{-2\pi iky}}{2\pi ik} \int_{[0,1)} e^{2\pi iku} dF(u) \right] + (y-z)
\]

\[
= \int_{[0,1)} \left[ \sum_{k=1}^{K} \frac{\sin(2\pi k(u-z)) - \sin(2\pi k(u-y))}{\pi k} \right] dF(u) + (y-z)
\]

\[
= \int_{[0,1)} \left[ (u-z) + (u-y) + \sum_{k=1}^{K} \frac{\sin(2\pi k(u-z)) - \sin(2\pi k(u-y))}{\pi k} \right] dF(u).
\]

By corollary 3.2 the integrand of this integral is uniformly bounded, and applying the bounded convergence theorem, we find

\[
\lim_{K \to \infty} S_K(y,z) = \frac{1}{2} \int_{[0,z)} dF(u) - \frac{1}{2} \int_{(z,1)} dF(u) - \frac{1}{2} \int_{[0,y)} dF(u) + \frac{1}{2} \int_{(y,1)} dF(u)
\]

\[ = \frac{1}{2} F(z) + \frac{1}{2} F(z) - \frac{1}{2} F(y) + \frac{1}{2} F(y). \]

Thus the theorem is proved.

It is quite obvious that the inversion theorem above implies the uniqueness theorem. When F is discrete, then the following theorem determines F by giving its jumps.

**Theorem 3.4.** Let c be the F.S.S. of \( F \in \mathcal{F}(0,1) \). Then the limit

\[
P_x = \lim_{K \to \infty} \frac{1}{2K} \sum_{k=-K}^{K} c(k) e^{-2\pi ikx}
\]
exists for all \( x \in (0,1) \), and is equal to the jump \( F(x) - F(x-0) \).

**Proof:** First we compute

\[
\sum_{k=-K}^{K} c(k) e^{-2\pi ikx} = \sum_{k=-K}^{K} \left[ e^{-2\pi ikx} \int_{[0,1]} e^{2\pi iku} \, dF(u) \right]
\]

\[
= \int_{[0,1]} \left[ \sum_{k=-K}^{K} e^{2\pi ik(u-x)} \right] \, dF(u)
\]

\[
= 2 \int_{[0,1]} \left[ \frac{\sin((K+\frac{1}{2})(2\pi(u-x)))}{2\sin(\pi(u-x))} \right] \, dF(u) \quad (3.2)
\]

where the integrand is called the Dirichlet kernel, defined to equal \( K + \frac{1}{2} \) for \( (u-x) \in \mathbb{Z} \). Let

\[ h_K(x) = \frac{\sin((K+\frac{1}{2})x)}{2\sin\frac{1}{2}x} \quad \text{for} \ x \in \mathbb{R}. \]

Clearly

\[ h(x) = \lim_{K \to \infty} \frac{1}{K} h_K(2\pi x) = \begin{cases} 1 & x \in \mathbb{Z} \\ 0 & x \in \mathbb{R}\setminus\mathbb{Z}. \end{cases} \quad (3.3) \]

Applying the bounded convergence theorem, we obtain therefore from formulas (3.2) and (3.3)

\[
\lim_{K \to \infty} \frac{1}{2K} \sum_{k=-K}^{K} c(k) e^{-2\pi ikx} = \int_{[0,1]} h(2\pi(u-x)) \, dF(u) = F(x) - F(x-0). \]

\[
\blacksquare
\]

Next, we show that the corresponding d.f. \( F \) is absolutely continuous if its F.S.S. satisfies the condition

\[
\sum_{k=\infty}^{\infty} |c(k)| < \infty. \quad (3.4)
\]

We summarize this in the following theorem.

**Theorem 3.5.** Let \( c \) be the F.S.S. of \( F \in \mathcal{F}[0,1] \). If (3.4) holds, then \( F \) is absolutely continuous and the formula

\[
f(x) = F'(x) = \sum_{k=-\infty}^{\infty} c(k) e^{2\pi ikx}
\]
expresses its probability density function $f$ as an ordinary Fourier series. Moreover, the function $f$ is continuous on $(0,1)$.

**Proof:** Let $x+h$ and $x-h$ be continuity points of $F$. The inversion formula (3.1) then gives

$$F(x+h) - F(x-h) = \lim_{K \to \infty} \sum_{k=-K}^{K} c(k) \frac{2\pi k x}{2\pi kh} \left( e^{2\pi kh} - 1 \right)$$

$$= 1 + \sum_{k=0}^{\infty} c(k) \frac{2\pi k x}{2\pi kh} \sin(2\pi kh). \quad (3.5)$$

In view of (3.4), the dominated convergence theorem implies that formula (3.5) converges to

$$\sum_{k=-\infty}^{\infty} c(k) e^{2\pi ikx}$$

as $h$ tends to zero.

Now we have to prove the assertion that $f$ is continuous. We obtain straightforwardly

$$|f(x+h) - f(x)| = \left| \sum_{k=-\infty}^{\infty} c(k) e^{-2\pi ikx} \left( e^{-2\pi ikh} - 1 \right) \right|$$

$$\leq \sum_{|k| < A} |c(k)| |e^{-2\pi ikh} - 1| + \sum_{|k| \geq A} |c(k)| |e^{-2\pi ikh} - 1|. \quad (3.6)$$

Under the condition (3.4), the second term in (3.6) can be made arbitrarily small for all $h$ by choosing $A$ sufficiently large. Next, taking $h$ sufficiently small makes the first term of (3.6) as small as we wish. This proves the theorem.

$$\blacksquare$$

**4. THE CONVOLUTION THEOREM.**

The convolution theorem for ch.f.'s is well known (see e.g. Lukacs
First we define the convolution of two d.f.'s.

**Definition 4.1.** Let \( F_1, F_2 \in \mathcal{F}(0,1) \). The function

\[
F(z) = \int_{0}^{z} F_1(z-x) \, dF_2(x) = \int_{0}^{z} F_2(z-x) \, dF_1(x) \quad \text{for } z \in [0,2) \tag{4.1}
\]

is said to be the convolution of \( F_1 \) and \( F_2 \), and is denoted by \( F = F_1 * F_2 \). It is obvious that \( F(z) \) is a d.f.

In section 6 we point out the application of the results of this paper to the fractional parts of random variables. Therefore, we now define the convolution \( F \) of two d.f.'s in \([0,1)\) such that \( F \in \mathcal{F}(0,1) \), which can be interpreted as a convolution modulo one.

**Definition 4.2.** Let \( G, H \in \mathcal{F}(0,1) \). The function

\[
F(z) = G*H(z) + G*H(1+z) - G*H(1-0) \quad \text{for } z \in [0,1) \tag{4.2}
\]

is said to be the convolution modulo one of \( G \) and \( H \), and is denoted by \( F = G \oplus H \). It is obvious that \( F \in \mathcal{F}(0,1) \).

We remark that if we consider two r.v.'s \( X_1 \) and \( X_2 \) distributed in \([0,1)\), we can then interpret (4.2) (cf. (1.2)) as the d.f. of the fractional part \( \{X_1 + X_2\} \), which is denoted by

\[
F_{\{X_1 + X_2\}} = F_{X_1} \oplus F_{X_2} \tag{4.3}
\]

or similarly,

\[
\{X_1 + X_2\} = X_1 \cdot c_{X_2} \tag{4.4}
\]

We shall use this interpretation for the proof of the convolution theorem, which describes the unique correspondence between the convolution of d.f.'s on \([0,1)\) in the sense of definition (4.2) and the product of the corresponding F.S.S.'s.

**Theorem 4.3 (the convolution theorem).** Let \( F_1, F_2, F \in \mathcal{F}(0,1) \), and let \( c_1, c_2 \) and \( c \) be the corresponding F.S.S.'s, then

\[
F = F_1 \oplus F_2 \text{ if, and only if, } c = c_1 \cdot c_2. \tag{4.5}
\]

**Proof:** Let \( X_1 \) and \( X_2 \) be the r.v.'s corresponding to \( F_1 \) and \( F_2 \) respecti-
vely. Because of (4.3) it suffices to note the following relation
\[
\begin{align*}
\frac{2\pi ikx_1}{\epsilon} + \frac{2\pi ikx_2}{\epsilon} &= \frac{2\pi ik(x_1 + x_2)}{\epsilon} \\
\frac{c_1(k)}{e^{-\epsilon k}} + \frac{c_2(k)}{e^{-\epsilon k}} &= \frac{c(k)}{e^{-\epsilon k}} \quad \text{for } k \in \mathbb{Z},
\end{align*}
\]
where \(x_1 + x_2\) is the fractional part of \(x_1 + x_2\).

This theorem leads to the following corollary.

**Corollary 4.4.** If \(c_1\) and \(c_2\) are the F.S.S.'s of two d.f.'s in \(F[0,1]\), then \(c_1 \cdot c_2\) is the F.S.S. of a d.f. in \(F[0,1]\).

Before deriving another criterion to decide whether a d.f. is continuous, we define the conjugate d.f. as follows:

Let \(X\) be a r.v. with d.f. \(F \in F[0,1]\). Then the conjugate d.f. \(\tilde{F}\) of \(F\) is defined by

\[
\tilde{F}(x) = \begin{cases} 
0 & x < 0 \\
1 + F(0) - F(1 - x) & x \in [0,1) \\
1 & x \geq 1,
\end{cases}
\]

and moreover, the corresponding r.v. \(\tilde{X}\) is denoted by \(\tilde{X} = (1 - X)\), where \((1 - X)\) is the fractional part of \(1 - X\). Evidently \(\tilde{F} \in F[0,1]\), and by an elementary computation similar to (2.3), we find

\[
\int_{[0,1)} e^{2\pi i kx} \, dF(x) = c(-k) = \overline{c(k)} \quad \text{for } k \geq 0.
\]

We proceed by stating two theorems. The proofs of these theorems are very similar to the proofs of the analogous theorems for ch.f.'s (see e.g. Lukacs (1970)).

**Theorem 4.5.** Let \(F_1, F_2 \in F[0,1]\) be two purely discrete d.f.'s with discontinuity points \((x_\alpha)_{\alpha=1}^\infty \) and \((y_\beta)_{\beta=1}^\infty \) respectively. Then \(F = F_1 \oplus F_2\) is also purely discrete and the discontinuity points of \(F\) are \((x_\alpha + y_\beta)_{\alpha=1, \beta=1}^\infty\), where \((x_\alpha + y_\beta)\) means \((x_\alpha + y_\beta) \mod 1\).

In addition, let \(a_\alpha\) be the jump of \(F_1\) at \(x_\alpha\) and \(b_\beta\) be the jump of \(F_2\) at \(y_\beta\) and suppose that \(\xi\) is a discontinuity point of \(F\). The jump of \(F\) at \(\xi\)
is then
\[ \sum_{\xi=\{x^\alpha y^\beta\}} a^\alpha b^\beta \]

**Proof:** From (1.4) we have
\[ c_1(k) = \sum_{\alpha=1}^{\infty} a^\alpha e^{2\pi ikx^\alpha} \quad \text{for } k \in \mathbb{Z}, \]
\[ c_2(k) = \sum_{\beta=1}^{\infty} b^\beta e^{2\pi iky^\beta} \quad \text{for } k \in \mathbb{Z}. \]

According to the convolution theorem, the F.S.S. \( c \) of \( F \) is
\[ c(k) = c_1(k) c_2(k) = \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} a^\alpha b^\beta e^{2\pi ik(x^\alpha + y^\beta)} \quad \text{for } k \in \mathbb{Z}. \]

This completes the proof. \( \square \)

**Theorem 4.6.** Let \( c \) be the F.S.S. of \( F \in \mathbb{F}(0,1) \), and let \( p_j \) (for \( j \geq 1 \)) be an enumeration of the jumps of \( F \), then
\[ \lim_{K \to \infty} \frac{1}{2K} \sum_{k=-K}^{K} |c(k)|^2 = \sum_{j=1}^{\infty} p_j^2. \]

**Proof:** If \( F \) has a discontinuity point \( x_j \in (0,1) \) with jump \( p_j \), then the conjugate d.f. \( \tilde{F} \) of \( F \) has a discontinuity at \((1-x_j) \mod 1\) with the same jump \( p_j \). Then theorem 4.5 implies that \( \Phi \tilde{F} \) has a jump of size \( \sum_{j=1}^{\infty} p_j^2 \) at 0. On the other hand, by theorem 3.4 we find that the jump at 0 is given by
\[ \lim_{K \to \infty} \frac{1}{2K} \sum_{k=-K}^{K} |c(k)|^2, \]
which completes the proof. \( \square \)

**5. The Continuity Theorem.**

To obtain the continuity theorem, we need a lemma. In order to formulate this lemma we denote the F.S.S. of a d.f. \( F_n \in \mathbb{F}(0,1) \) (for \( n \geq 1 \)) by \( c_n \). We proceed by giving the lemma.
Lemma 5.1. Let \((F_n)_{n=1}^{\infty}\) be a sequence of d.f.'s in \(\mathcal{F}[0,1]\), which converges weakly to a d.f. \(F \in \mathcal{F}[0,1]\). Then

\[
\lim_{n \to \infty} c_n(k) = c(k) \quad \text{for } k \geq 0.
\]

Proof: From Helly's Second Theorem (see e.g. Lukacs (1970)) we know that

\[
\lim_{n \to \infty} \int_{-1}^{1} e^{2\pi i k x} dF_n(x) = \int_{-1}^{1} e^{2\pi i k x} dF(x) \quad \text{for } k \geq 0.
\]

Since over the half-open interval \([-1,0)\) \(F_n(x) = F(x) = 0\) (for \(n \geq 1\)), and \(F\) is continuous at \(-1\) and \(1\), this proves the lemma.

In the foregoing we considered d.f.'s \(F \in \mathcal{F}[0,1]\), which are continuous at 1. In the following theorem we interpret such d.f.'s as d.f.'s modulo 1, i.e., possible probability mass at 1 is transformed to 0. Naturally, we can also interpret such d.f.'s as d.f.'s on the unit circle by identifying the endpoints 0 and 1 (see e.g. Feller (1971)). We state now the continuity theorem.

Theorem 5.2 (the continuity theorem). Let \((F_n)_{n=1}^{\infty}\) be a sequence of d.f.'s in \(\mathcal{F}[0,1]\) and let \((c_n)_{n=1}^{\infty}\) be the corresponding sequence of F.S.S.'s. If

\[
\lim_{n \to \infty} c_n(k) = c(k)
\]

exists for all \(k \geq 0\), then the sequence \((F_n)_{n=1}^{\infty}\) converges weakly to \(F \in \mathcal{F}[0,1]\). This sequence \(c\) is the F.S.S. of \(F\).

Proof: Considering the sequence \((F_n)_{n=1}^{\infty}\) as a sequence of d.f.'s on \([0,1]\), from Helly's First Theorem (see e.g. Lukacs (1970)) we can select a subsequence \((F_{n_i})_{i=1}^{\infty}\) of \((F_n)_{n=1}^{\infty}\) such that this sequence \((F_{n_i})_{i=1}^{\infty}\) converges weakly to a d.f. \(F \in \mathcal{F}[0,1]\), which we interpret modulo 1. As a result, from lemma 5.1 it follows immediately that the sequence \((c_{n_i})_{i=1}^{\infty}\) converges to the F.S.S. of \(F\). Then the uniqueness theorem implies that the sequence \(c\) is the F.S.S. of \(F\).
6. APPLICATIONS.

In the foregoing sections we described the relation between properties of F.S.S.'s and properties of d.f.'s. As indicated in section 4, we apply these results to the fractional parts of r.v.'s distributed on \( \mathbb{R} \). Let \( X \) be a r.v. distributed on \( \mathbb{R} \). Denote by \([X]\) the integer part of \( X \), the largest integer not exceeding \( X \), and let \((X) = X - [X]\) be the fractional part. Moreover, let \( \varphi_X(t) \) be its ch.f. The following useful identity was pointed out to us by Prof. Dr. P. Groeneboom (Delft University of Technology).

\[ \varphi_X(2\pi k) = E e^{2\pi i k X} = E e^{2\pi i k (X)} = c_{(X)}(k) \quad \text{for } k \in \mathbb{Z}. \quad (6.1) \]

To indicate how we are going to apply this interpretation, we give the following example (see Holewijn (1969)).

Example 6.1. (i) Let \( X_1, X_2, \ldots, X_n \) be r.v.'s, independent and identically distributed (i.i.d.) on \( \mathbb{R} \) with d.f. \( F \) and let \( c \) be the corresponding F.S.S. Furthermore, denote \( S_n = X_1 + \ldots + X_n \) (for \( n \geq 1 \)). From the convolution theorem we conclude

\[ \varphi_n(2\pi k) = E e^{2\pi i k S_n} = E e^{2\pi i k (S_n)} = (c(k))^n \quad \text{for } n \geq 1, \ k \geq 0. \]

As a consequence, if we assume that \(|c(k)| < 1\) for all \( k > 0 \), we find

\[ \lim_{n \to \infty} \varphi_n(2\pi k) = c_U(k) \quad \text{for } k \geq 0, \]

where \( c_U \) is the F.S.S. of the uniform distribution on \([0,1)\) (see (1.3)). Then the continuity theorem implies that the distribution of the fractional part \((X_1 + \ldots + X_n)\) converges to the uniform distribution on \([0,1)\) when \( n \) tends to infinity.

On the other hand, if we assume that \( F \in \mathcal{F}[0,1) \) has a lattice d.f. on \( M \) points with zero as a discontinuity point, then the distribution of \( \{S_n\} \) converges to a discrete uniform distribution, i.e. a lattice d.f. on \( M \) points with equal probability \( \frac{1}{M} \) at \( x = \frac{m}{M} \) for all \( m = 0,1,\ldots,M-1 \) (see (1.4); e.g. Mardia (1972)).

Obviously, if \(|c(k)| = 1\) for all \( k \in \mathbb{Z} \), then \( F \) is concentrated at some point \( x \in [0,1) \), i.e. \( F \) is the degenerate d.f.

(ii) Let \( X_1 \) be uniformly distributed on \( \mathbb{R} \) and let \( X_2, \ldots, X_n \) have an
arbitrary distribution, and let further $X_1, \ldots, X_n$ be independent. It follows immediately that $c_{\{S_n\}}(k) = 0$ for all $k \neq 0$, and hence $\{S_n\}$ is uniformly distributed on $[0,1)$.

In future research we are interested in other similar problems, e.g., when does the sequence $\{\max(X_1, \ldots, X_n)\}$ converge in distribution; does this sequence ever converge in distribution?

We note that in problem 247 in Statistica Neerlandica (1990) it is shown that this sequence does not converge in distribution when $X_1, X_2, \ldots$ are i.i.d. and exponentially distributed.

In addition, we shall be interested in other properties of the distribution of fractional parts, for instance, is there a sufficient or necessary condition on the fractional part for the distribution to be infinitely divisible, self-decomposable or stable?

7. REFERENCES.


JAGERS, A.A. (1990), Solution of Problem 247, Statistica Neerlandica, 44, nr.3.


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