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van der Laan, P.; Eeden, van, C.

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On using a loss function in selecting the best of two gamma populations in terms of their scale parameters

P. van der Laan
C. van Eeden

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The Netherlands
ON USING A LOSS FUNCTION IN SELECTING THE BEST OF TWO GAMMA POPULATIONS IN TERMS OF THEIR SCALE PARAMETERS

Paul van der Laan
Eindhoven University of Technology
The Netherlands

Constance van Eeden
University of British Columbia
Université du Québec à Montréal
Canada

SUMMARY

This paper continues the study of the subset selection procedure proposed by van der Laan and van Eeden (1993). In this 1993 paper the authors consider a location problem and base their procedure on a continuous loss function. This loss function takes into account the "distance", in parameter values, between the populations under consideration and the best one among the ones in the selected subset. In defining this "distance", they incorporate the notion of "ε-best" studied by, e.g., Desu (1970), Lam (1986), van der Laan (1992), Gill and Sharma (1993) and Gill, Sharma and Misra (1993). As an example of their results, van der Laan and van Eeden (1993) consider the case of two normal populations with equal known variances.

The present paper develops a similar procedure for scale parameters. The case of two gamma populations with equal known shape parameters is studied in detail.

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1 Introduction

In van der Laan and van Eeden (1993) a subset selection procedure based on a continuous loss function is proposed and studied. The problem these authors consider is a location problem and their loss function takes into account the "distance", in parameter values, between the $k$ populations under study and the best population in the selected subset. In defining this "distance", the authors use the notion of "$\varepsilon$-best". This "$\varepsilon$-best" notion has been applied in selection procedures by, e.g., Desu (1970), Lam (1986), van der Laan (1992), Gill and Sharma (1993) and by Gill, Sharma and Misra (1993). But, in contrast to van der Laan and van Eeden (1993), each of these other papers uses a 0-1 loss function.

In the present paper the approach of van der Laan and van Eeden (1993) is applied to the scale parameter problem. The case of two gamma populations with equal known shape parameters is studied in detail.

A description of the selection goals is given in Section 2. Section 3 treats the special case of two gamma populations and Section 4 contains the proofs of some of our results. Tables of the risk function and of the expected subset size are given in Section 5 for the two-sample gamma-population case with a quadratic loss function.

2 Description of the selection goals

Let, for $i = 1, \ldots, k$, $X_{i,1}, \ldots, X_{i,n}$ be independent, identically distributed random variables with distribution function $F(x/\theta_i)$, where $\theta_i > 0$ is unknown and $F$ is known.

The goal of a selection procedure is to "find" the best one, in the sense of smallest scale parameter, among these $k$ populations. In most selection procedures discussed in the literature, the procedure is chosen so as to give a prescribed minimum probability of correct selection, i.e. minimum probability that the selected subset contains the best population.

In the present paper we follow the ideas of van der Laan and van Eeden (1993) and use a continuous loss function. This loss function is defined as follows. Let $d \subset K = \{1, \ldots, k\}$ and let $D = \{\theta_i | i \in d\}$ be the selected subset. Further, let $\varepsilon \geq 0$ be a given number and let $\theta_{[1]} \leq \ldots \leq \theta_{[k]}$ be the ordered scale parameters. Then the loss function is given by

$$L(\theta, d, \varepsilon) = \begin{cases} 0 & \text{if } \min_{i \in d} \theta_i \leq (1 + \varepsilon)\theta_{[1]} \\ h \left( \frac{\min_{i \in d} \theta_i}{(1 + \varepsilon)\theta_{[1]}} \right) & \text{if not}, \end{cases}$$

(2.1)
where $h$ is a nondecreasing function defined on $R^+$ with $h(1) = 0$. Using such a loss function means that, whenever the best population in $D$ is "$\varepsilon$-best" among all populations in the sense that $\min_{i \in d} \theta_i / \theta_{[1]} \leq 1 + \varepsilon$, then the loss is zero. If $D$ does not contain any of the $\varepsilon$-best populations then the loss is a nondecreasing function of $\min_{i \in d} \theta_i / \theta_{[1]}$.

Now let $\delta$ be the decision function, i.e. a function defined on the sample space taking values in the set of nonempty subsets of $K$. Further let $S$ be the size of the selected subset and let $X = (X_{i,1}, \ldots, X_{i,n}; i = 1, \ldots, k)$. Then our selection goals are expressed in terms of an upper bound on $E_{\theta}L(\theta, \delta(X), \varepsilon)$ and (or) $E_{\theta}S$, where $\theta = (\theta_1, \ldots, \theta_k)$.

In the next section the case of two gamma populations with equal known shape parameters and $h(x) = (x - 1)^p, p > 0$, is studied in detail.

3 The case of two gamma populations

In this section the case where $k = 2, n = 1$, $h(x) = (x - 1)^p, p > 0$ and the $X_i/\theta_i, i = 1, 2$, have density

$$f(x; \alpha) = \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}I(x > 0)$$  \hspace{1cm} (3.1)

is considered. Note that, by sufficiency, this covers the case where $n > 1$. It also covers the case of two independent samples $Y_{i,1}, \ldots, Y_{i,n}$ from normal populations with variances $\theta_i, i = 1, 2$. In that case one can take, for $i = 1, 2$,

$$2X_i = \sum_{j=1}^{n} Y_{i,j}^2$$

and $\alpha = n/2$ when $EY_{i,j} = 0$, and $2X_i = \sum_{j=1}^{n} (Y_{i,j} - \bar{Y}_i)^2$, with $n\bar{Y}_i = \sum_{j=1}^{n} Y_{i,j}$ and $\alpha = (n/2) - 1$ when $EY_{i,j}$ is estimated by $\bar{Y}_i$.

The decision rules, $\delta_c$, are of the form

$$\delta_c(X) = \begin{cases} 
  d_1 = \{1\} & \text{if } X_1 < \frac{1}{c}X_2 \\
  d_2 = \{2\} & \text{if } X_1 > \frac{1}{c}X_2 \\
  d_{1,2} = \{1, 2\} & \text{if } cX_2 \leq X_1 \leq \frac{1}{c}X_2,
\end{cases}$$  \hspace{1cm} (3.2)

where $X = (X_1, X_2)$ and $c \in (0, 1]$.

Two characteristics of a selection procedure are its risk function, $E_{\theta}L(\theta, \delta_c, \varepsilon)$, and its expected subset size, $E_{\theta}S_c$. We first obtain, in the next two theorems, expressions for each of these as functions of $\theta, c$ and $\varepsilon$.

**Theorem 3.1** The risk function of the rule $\delta_c$ is given by

$$R(\rho, \delta_c, \varepsilon) = E_{\theta}L(\theta, \delta_c, \varepsilon) = P_\theta(T > \rho/c) \left(\frac{\rho}{1 + \varepsilon} - 1\right)^p I(\rho > 1 + \varepsilon),$$

where $T = \theta_2 X_1 / (\theta_1 X_2)$ and $\rho = \theta_{[2]}/\theta_{[1]}$. 

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Proof. Without loss of generality assume that $\theta_1 < \theta_2$. Then

$$\mathcal{E}_d L(\theta, \delta_c, \varepsilon) = P_\theta (d = d_2) \left( \frac{\theta_2}{(1 + \varepsilon) \theta_1} - 1 \right)^p \mathbb{I} \left( \frac{\theta_2}{\theta_1} > 1 + \varepsilon \right)$$

$$= P_\theta \left( \frac{X_1}{X_2} > \frac{1}{c} \right) \left( \frac{\theta_2}{(1 + \varepsilon) \theta_1} - 1 \right)^p \mathbb{I} \left( \frac{\theta_2}{\theta_1} > 1 + \varepsilon \right)$$

$$= P_\theta \left( T > \frac{\rho}{c} \right) \left( \frac{\rho}{1 + \varepsilon} - 1 \right)^p \mathbb{I} (\rho > 1 + \varepsilon).$$

Note that the random variable $T = (X_1/\theta_1)/(X_2/\theta_2)$, where the $X_i/\theta_i, i = 1, 2$, have density given by (3.1). Thus, $T$ has the distribution of the ratio of two independent variables, each with density (3.1) and its density is given by (see e.g. Stuart and Ord (1987; p. 378, exercise 11.10))

$$g(t; \alpha) = C(\alpha) \frac{t^{\alpha-1}}{(1 + t)^{2\alpha}} \mathbb{I} (t > 0),$$

where $C(\alpha) = \Gamma(2\alpha)/\Gamma^2(\alpha)$. So, $R(\rho, \delta_c, \varepsilon)$ can be written as

$$R(\rho, \delta_c, \varepsilon) = \int_{\rho/c}^\infty g(t; \alpha) dt \left( \frac{\rho}{1 + \varepsilon} - 1 \right)^p \mathbb{I} (\rho > 1 + \varepsilon).$$

**Theorem 3.2** The expected subset size is given by

$$\mathcal{E}_d S_c = 1 + C(\alpha) \int_{\rho/c}^\infty \frac{t^{\alpha-1}}{(1 + t)^{2\alpha}} dt.$$ 

*Proof.* From the selection rule (3.2) it follows that

$$\mathcal{E}_d S_c = P_\theta \left( \frac{X_1}{X_2} < \frac{1}{c} \right) + P_\theta \left( \frac{X_1}{X_2} > \frac{1}{c} \right) + 2P_\theta \left( \frac{X_1}{X_2} \leq \frac{1}{c} \right)$$

$$= 1 + P_\theta \left( \rho c \leq T \leq \frac{\rho}{c} \right) = 1 + C(\alpha) \int_{\rho/c}^\infty \frac{t^{\alpha-1}}{(1 + t)^{2\alpha}} dt.$$ 

The influence of $\alpha$ on the risk function, $R(\rho, \delta_c, \varepsilon)$, is given in the next theorem and two corollaries.

**Theorem 3.3** The risk function $R(\rho, \delta_c, \varepsilon)$ is, for each $\rho > 1 + \varepsilon$, each $c \in (0, 1]$, each $\varepsilon \geq 0$ and each $\rho > 0$, strictly decreasing in $\alpha$. 

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Proof. Because \( p > 1 + \varepsilon \), \( c \in (0,1] \) and \( \varepsilon \geq 0 \) it needs to be shown that

\[
G(x; \alpha) = C(\alpha) \int_0^x \frac{t^{\alpha-1}}{(1 + t)^{2\alpha}} dt
\]

is, for \( x > 1 \), strictly increasing in \( \alpha \). In order to prove this, first note that \( G(1; \alpha) = 1/2 \) for all \( \alpha > 0 \). This follows from the fact that \( X_1/\theta_1 \) and \( X_2/\theta_2 \) are independent and identically distributed and thus \( P_\theta(T > 1) = P_\theta(T < 1) = 1/2 \) for all \( \theta \) and all \( \alpha > 0 \). Further, for \( \alpha' > \alpha \) and \( x > 0 \),

\[
\left. \frac{d}{dx} (G(x; \alpha') - G(x; \alpha)) \right|_{x = 1} = C(\alpha') \frac{x^{\alpha'-1}}{(1 + x)^{2\alpha'}} - C(\alpha) \frac{x^{\alpha-1}}{(1 + x)^{2\alpha}} \begin{cases} > 0 \\ < 0 \end{cases}
\]

where \( A = (C(\alpha)/C(\alpha'))^{1/(\alpha'-\alpha)} \). Using the fact that \( G(x; \alpha) \), \( x > 0 \), is, for each \( \alpha > 0 \), a distribution function, it then follows that \(-Ax^2 + (1 - 2A)x - A = 0\) must have two positive roots, which implies that \( A < 1/4 \). The product of these two roots equals 1, so exactly one of them, \( x_o \) say, is larger than 1. From (3.4) and \( A < 1/4 \) it then follows that

\[
\left. \frac{d}{dx} (G(x; \alpha') - G(x; \alpha)) \right|_{x = 1} = C(\alpha') \frac{x^{\alpha'-1}}{(1 + x)^{2\alpha'}} (1 - (4A)^{\alpha'-\alpha}) > 0,
\]

which implies that \( G(x; \alpha') - G(x; \alpha) \) is increasing on \((1, x_o)\) and decreasing on \((x_o, \infty)\). The result then follows from the fact that \( G(1; \alpha') - G(1; \alpha) = G(\infty; \alpha') - G(\infty; \alpha) = 0 \). \( \square \)

The following corollary is an immediate consequence of Theorem 3.3 and Lemma 4.1, part iv).

**Corollary 3.1** For each \( c \in (0,1] \), each \( \varepsilon \geq 0 \) and each \( p > 0 \),

\[
\max\{R(p, \delta_c, \varepsilon) | p \geq 1 + \varepsilon\}
\]

is, for \( \alpha > p \), strictly decreasing in \( \alpha \).

Now let \( R_o \in (0, \max\{R(p, \delta_c, \varepsilon) | p \geq 1 + \varepsilon\}) \). Then, by the continuity of \( R(p, \delta_c, \varepsilon) \) in \( p \) and by Lemma 4.1, part iv), there exists a unique interval \( I(R_o, c, \varepsilon) = (a(R_o, c, \varepsilon), b(R_o, c, \varepsilon)) \) of \( p \)-values such that

\[
R(p, \delta_c, \varepsilon) = \begin{cases} > R_o \text{ if } p \in I(R_o, c, \varepsilon) \\ \leq R_o \text{ if not.} \end{cases}
\]

Further, \( b(R_o, c, \varepsilon) < \infty \) if and only if \( \alpha > p \).

The next corollary follows immediately from Theorem 3.3.
Corollary 3.2 For each $c \in (0,1]$, each $\varepsilon \geq 0$ and each $p > 0$, $a(R_o, c, \varepsilon)$ is strictly increasing in $\alpha$ and, for $\alpha > p$, $b(R_o, c, \varepsilon)$ is strictly decreasing in $\alpha$.

Some properties of $R(\rho, \delta_c, \varepsilon)$ and $E_\theta S_c$, as functions of $\rho, c$ and $\varepsilon$ are proved in Section 4. These properties can serve as a guide to the choice of $c$, and thus of the decision rule $\delta_c$. Of course, one would like a small loss, as well as a small subset size. But, as can be seen from Lemma 4.1, part ii), and Lemma 4.2, part ii), the expected loss is increasing in $c$ while the expected subset size is decreasing in $c$. So, a compromise must be made. This can be done, e.g., by putting an upper bound on the "more important" one of these characteristics of the procedure. And this upper bound can be required to hold for all $\rho \geq 1$ or for a subset of $\rho$-values.

First look at the case where $\alpha \geq p$. Then, for each $\varepsilon \geq 0$, $R_m(c, \varepsilon) = \max\{R(\rho, \delta_c, \varepsilon)|\rho \geq 1 + \varepsilon\}$ is, by Lemma 4.1, part v), strictly increasing and continuous in $c$ with $R_m(c, \varepsilon) \rightarrow 0$ when $c \rightarrow 0$ and $\rightarrow R^*$, with $0 < R^* < \infty$, when $c \rightarrow 1$. So, for each $\varepsilon \geq 0$ and each $R_o \in (0, R^*)$, there exists a $c(R_o, \varepsilon)$ such that

$$R_m(c, \varepsilon) \begin{cases} > R_o & \iff c > c(R_o, \varepsilon) \\ < & \end{cases}$$

Further, $E_\theta S_c$ is, by Theorem 3.2, independent of $\varepsilon$ and, by Lemma 4.2, part ii), strictly decreasing and continuous in $c$ for each fixed $\rho \geq 1$. So, if one requires of the procedure that $R_m(c, \varepsilon) \leq R_o$ with $\max\{E_\theta S_c|\rho \geq 1\}$ as small as possible, then $\delta_c(R_o, \varepsilon)$ is the unique rule satisfying this requirement.

One could also start with an upper bound $1 + \eta_o$ on $S_m(c) = \max\{E_\theta S_c|\rho \geq 1\}$. By Lemma 4.2, parts iii) and iv), this maximum is attained for $\rho = 1$ and $S_m(c)$ is strictly decreasing and continuous in $c$ with $S_m(0) = 2$ and $S_m(1) = 1$. So, for each $\eta_o \in (0,1)$, there exists a $c(\eta_o) \in (0,1)$ such that

$$S_m(c) \begin{cases} > 1 + \eta_o & \iff c > c(\eta_o) \\ < & \end{cases}$$

Then, because $R_m(c, \varepsilon)$ is strictly increasing and continuous in $c$, the unique procedure for which $S_m(c) \leq 1 + \eta_o$ with $R_m(c, \varepsilon)$ as small as possible is given by $\delta_c(\eta_o)$.

Requiring of the procedure that the bound $R_o$ on $R(\rho, \delta_c, \varepsilon)$ holds only for $\rho \notin$ an interval of the form $(a(R_o, c, \varepsilon), b(R_o, c, \varepsilon))$, leads to a larger value of $c$ and thus to a smaller value of $S_m(c)$. In the same way, requiring that $E_\theta S_c \leq 1 + \eta_o$ holds only for $\rho \geq \rho_o$ for some $\rho_o > 1$, decreases $c$ and thus decreases $R_m(c, \varepsilon)$.

When $\alpha < p$, the situation is somewhat different, because in that case (see Lemma 4.1, parts iii) and iv)) $R(\rho, \delta_c, \varepsilon)$ is strictly increasing in $\rho$ and $\rightarrow \infty$.
as \( p \to \infty \), and this for all \( c \in (0,1] \) and all \( \varepsilon \geq 0 \). So, here one could require that \( R(p, \delta_c, \varepsilon) \leq R_o \) on an interval of the form \([1 + \varepsilon, \rho_o] \) for some \( R_o > 0 \) and \( \rho_o > 1 + \varepsilon \). This gives an upper bound, \( c(R_o, \rho_o) \) say, for \( c \) and \( \delta_c(R_o, \rho_o) \) is then the unique procedure satisfying the requirement with \( S_m(c) \) as small as possible. Putting an upper bound, \( 1 + \eta_o \) say, on \( E_p S_c \) (either for all \( p \geq 1 \) or only for \( p \geq \rho_o \) for some \( \rho_o > 1 \)) leads, in this case where \( \alpha < p \), to a lower bound, \( c(\eta_o, \rho_o) \) say, on \( c \). Then \( \delta_c(\eta_o, \rho_o) \) is the unique rule satisfying the requirement with \( R_m(c, \varepsilon) \) as small as possible.

As for as the choice of \( \varepsilon \geq 0 \), \( R(p, c, \varepsilon) \) is, by Lemma 4.3, strictly decreasing in \( \varepsilon \) for each \( p > 1 + \varepsilon \) and \( c \in (0,1] \). So, \( \varepsilon \) should be chosen as large as is possible taking into account that the two populations are judged "equal" when \( \theta(\alpha) / \theta(\beta) \leq 1 + \varepsilon \).

Finally, by Theorem 3.3, \( R(p, c, \varepsilon) \) is, for each \( p > 1 + \varepsilon \), each \( c \in (0,1] \), each \( \varepsilon \geq 0 \) and each \( p > 0 \), strictly decreasing in \( \alpha \). So, for fixed \( c \) and \( \varepsilon \), the larger \( \alpha \) is, the smaller is the risk function, uniformly in \( p \).

### 4 Some lemmas

In this section some properties of the risk function \( R(p, \delta_c, \varepsilon) \) and the expected subset size \( E_p S_c \) as functions of \( p, c \) and \( \varepsilon \) are stated and proved.

**Lemma 4.1** For each \( p > 0 \), the risk function \( R(p, \delta_c, \varepsilon) \) satisfies

i) for each \( \varepsilon \geq 0 \), \( R(p, \delta_c, \varepsilon) = 0 \) for all \( c \in (0,1] \) when \( 1 \leq p \leq 1 + \varepsilon \),

ii) for each \( \varepsilon \geq 0 \) and each \( p > 1 + \varepsilon \), \( R(p, \delta_c, \varepsilon) \) is strictly increasing and continuous in \( c \) for \( c \in (0,1] \) and \( \to 0 \) as \( c \to 0 \),

iii) for each \( c \in (0,1] \) and each \( \varepsilon \geq 0 \)

\[
\lim_{\rho \to \infty} R(p, \delta_c, \varepsilon) = \begin{cases} 
+\infty & \text{if } \alpha < p \\
0 & \text{if } \alpha > p \\
\frac{c(p)}{p} \left( \frac{c}{1 + \varepsilon} \right)^p & \text{if } \alpha = p,
\end{cases}
\]

iv) for each \( c \in (0,1] \) and each \( \varepsilon \geq 0 \), \( R(p, \delta_c, \varepsilon) \) is, for \( p > 1 + \varepsilon \), strictly increasing in \( p \) when \( \alpha \leq p \). When \( \alpha > p \), \( R(p, \delta_c, \varepsilon) \) has, for \( p > 1 + \varepsilon \), a unique maximum at \( p = \rho_m(c, \varepsilon) \), where \( \rho_m(c, \varepsilon) \) is the unique solution, in \( p \), to

\[
p \in P_\theta \left(T > \frac{\rho}{c} \right) = g \left( \frac{\rho}{c} \right) \left( \rho - (1 + \varepsilon) \right),
\]

\[
\rho > 1 + \varepsilon,
\]
Further, $R(p, \delta, \varepsilon)$ is strictly increasing in $p$ on $(1 + \varepsilon, \rho_m(c, \varepsilon))$ and strictly decreasing on $(\rho_m(c, \varepsilon), \infty)$.

vi) for each $\alpha > p$ and $\varepsilon > 0$, $R_m(c, \varepsilon) = \max\{R(p, \delta, \varepsilon) | p \geq 1 + \varepsilon\}$ is strictly increasing and continuous in $c$ for $c \in (0, 1]$. This maximum converges to 0 when $c \to 0$ and to a positive finite value when $c \to 1$.

Proof. Parts i) and ii) follow immediately from Theorem 3.1 and part vi) follows immediately from part ii). For part iii), note that l'Hopital's rule gives

$$
\lim_{\rho \to \infty} R(p, \delta, \varepsilon) = C(\alpha) \lim_{\rho \to \infty} \frac{1}{(1 + \varepsilon)^{-p}} \int_0^\infty \frac{t^{\alpha-1}}{(1 + t)^{2\alpha}} dt
$$

$$
= C(\alpha) \lim_{\rho \to \infty} \frac{c \rho^{\alpha-1}}{(1 + \rho c)^{2\alpha} \rho (1 + \varepsilon - 1)^{-p-1} \frac{1}{1 + \varepsilon}}
$$

$$
= C(\alpha) \frac{c^\alpha (1 + \varepsilon)}{\rho} \lim_{\rho \to \infty} \frac{\rho^{\alpha-1} (\rho c - 1)^{p+1}}{(1 + \rho c)^{2\alpha}},
$$

from which the result follows immediately.

For iv), first note that, for $\rho > 1 + \varepsilon$,

$$
\frac{d}{d\rho} R(p, \delta, \varepsilon) = \left(\frac{\rho}{1 + \varepsilon} - 1\right)^{p-1} h(\rho, c),
$$

where

$$
h(\rho, c) = \frac{\rho}{1 + \varepsilon} P_\theta \left(T > \frac{\rho}{c}\right) - \frac{1}{c} g \left(\frac{\rho}{c}\right) \left(\frac{\rho}{1 + \varepsilon - 1}\right).
$$

Further, for each $c \in (0, 1]$,

$$
\lim_{\rho \to \infty} h(\rho, c) = 0 \quad \text{and} \quad h(1 + \varepsilon, c) > 0
$$

and

$$
\frac{d}{d\rho} h(\rho, c) = -\frac{\rho}{(1 + \varepsilon)c} g \left(\frac{\rho}{c}\right) - \frac{1}{(1 + \varepsilon)c} g \left(\frac{\rho}{c}\right) - \frac{1}{c} \left(\frac{\rho}{1 + \varepsilon - 1}\right) \frac{d}{d\rho} g \left(\frac{\rho}{c}\right),
$$

with

$$
\frac{d}{dt} g(t) = C(\alpha) \frac{\alpha - 1 - t(\alpha + 1)}{(1 + t)^{2\alpha+1}} t^{\alpha-2} = g(t) \frac{\alpha - 1 - t(\alpha + 1)}{t(1 + t)}. \quad (4.3)
$$
This gives, putting \( c^* = c/(1 + \varepsilon) \) and \( \rho^* = \rho/(1 + \varepsilon) \),

\[
\frac{d}{dp} h(\rho, c) =
\]

\[
g \left( \frac{\rho}{c} \right) \frac{1}{c} \left\{ \frac{p + 1}{1 + \varepsilon} - \left( \frac{\rho}{1 + \varepsilon} - 1 \right) \left( \frac{\alpha - 1)c - \rho(\alpha + 1)}{\rho(\rho + c)} \right\} = (4.4)
\]

\[
g \left( \frac{\rho}{c} \right) \frac{H(\rho, c)}{c\rho^*(\rho + c)}.
\]

where

\[
H(\rho, c) = \rho^*2(\alpha - p) - \rho^*(c^*(\alpha + p) + (\alpha + 1)) + (\alpha - 1)c^*. \quad (4.5)
\]

First consider the case where \( \alpha = p \). Then, for \( \rho > 1 + \varepsilon \),

\[
H(\rho, c) = -\rho^*(2pc^* + p + 1) + (p - 1)c^* < -(p + 1)(c^* + 1) < 0. \quad (4.6)
\]

From (4.2), (4.4) and (4.6) it then follows that \( h(\rho, c) > 0 \) for \( \rho > 1 + \varepsilon \), which implies (by (4.1)) that \( R(\rho, \delta_c, \varepsilon) \) is strictly increasing in \( \rho \) for \( \rho > 1 + \varepsilon \).

Now consider the case where \( \alpha < p \). Then it follows from (4.5) that \( H(\rho, c) \) attains, for fixed \( c \in (0, 1] \), its maximum value at

\[
\rho = \frac{(\alpha + p)c + (\alpha + 1)(1 + \varepsilon)}{2(p - \alpha)} < 0. \quad (4.7)
\]

Further, for all \( \alpha > 0 \),

\[
H(\rho, c)|_{\rho = 1 + \varepsilon} = -\frac{(p + 1)(1 + \varepsilon + c)}{(1 + \varepsilon)} < 0. \quad (4.8)
\]

From (4.4), (4.7) and (4.8) and the fact that \( \alpha < p \) it then follows that \( \frac{d}{dp} h(\rho, c) < 0 \) for all \( \rho > 1 + \varepsilon \), which, in the same way as for the case where \( \alpha = p \), implies that \( R(\rho, \delta_c, \varepsilon) \) is strictly increasing in \( \rho \).

Finally, let \( \alpha > p \). From (4.4) and (4.8) it then follows that there exists a unique solution, \( \rho_o \) say, to

\[
\frac{d}{d\rho} h(\rho, c) = 0, \; \rho > 1 + \varepsilon
\]

and that, for \( \rho > 1 + \varepsilon \),

\[
\frac{d}{d\rho} h(\rho, c) \begin{cases} > 0 & \rho > \rho_o \\ < 0 & \rho < \rho_o \end{cases} \quad (4.9)
\]
But this implies, by (4.2), that there exists a unique solution to
\[ h(\rho, c) = 0, \rho > 1 + \varepsilon \]
and the result then follows from (4.1).
For \( v \), the result follows from part iii) for the case where \( \alpha = p \). For \( \alpha > p \), the continuity in \( c \) follows from the fact that, by Theorem 3.1, \( R(\rho, \delta, \varepsilon) \) is continuous in \((\rho, c)\) on \([1 + \varepsilon, \infty) \times (0, 1]\) and that, by part iv) of the present lemma, \( \rho_m(c, \varepsilon) \) is continuous in \( c \). That \( R_m(c, \varepsilon) \) is strictly increasing in \( c \) follows from part ii) of the present lemma. The continuity of \( R_m(c, \varepsilon) \) in \( c \) for \( c \in (0, 1] \), implies that \( R_m(c, \varepsilon) \to R_m(1, \varepsilon) < \infty \) as \( c \to 1 \). Finally, in order to prove that \( R_m(c, \varepsilon) \to 0 \) as \( c \to 0 \), first note that \( 1 + \varepsilon \leq \rho_m(c, \varepsilon) \leq \rho_\alpha \), where \( \rho_\alpha \) is the solution to \( \{ H(\rho, c) = 0, \rho > 1 + \varepsilon \} \). Further, it is easily seen from (4.5) that \( \rho_\alpha \leq \rho'_\alpha \), where \( \rho'_\alpha \) is the solution to
\[ \rho'^2(\alpha - p) - \rho(2\alpha + p + 1) - |\alpha - 1|(1 + \varepsilon) = 0. \]
This \( \rho'_\alpha \) is independent of \( c \) and the result then follows from the fact that, for each \( \rho \in [1 + \varepsilon, \rho'_\alpha] \),
\[ R(\rho, \delta, \varepsilon) \leq P_\theta \left( T > \frac{1 + \varepsilon}{c} \right) \left( \frac{\rho'_\alpha}{1 + \varepsilon} - 1 \right)^p \to 0 \text{ as } c \to 0. \]

\[ \square \]

**Lemma 4.2** *The expected subset size, \( E_\theta S_c \), satisfies*

i) for each \( \rho \geq 1 \), \( E_\theta S_c = 1 \) when \( c = 1 \),

ii) for each \( \rho \geq 1 \), \( E_\theta S_c \) is strictly decreasing and continuous in \( c \) for \( c \in (0, 1) \) and converges to 2 as \( c \to 0 \),

iii) for each \( c \in (0, 1) \), \( E_\theta S_c \) is strictly decreasing and continuous in \( \rho \) for \( \rho \geq 1 \) and converges to 1 as \( \rho \to \infty \),

iv) for \( c \in (0, 1) \), \( S_m(c) = \max\{E_\theta S_c | \rho \geq 1\} \) is strictly decreasing and continuous in \( c \) with \( \lim_{c \to 0} S_m(c) = 2 \) and \( S_m(1) = 1 \).

**Proof.** Parts i) and ii) follow immediately from Theorem 3.2. For part iii) note that, for \( c \in (0, 1) \),
\[ \frac{d}{d\rho} E_\theta S_c = C(\alpha)\rho^{\alpha-1}c^\alpha \left( \frac{1}{(c + \rho)^2} - \frac{1}{(1 + \rho c)^2} \right) < 0 \text{ for } \rho > 1, \]
because \( (0 < c < 1, \rho > 1) \) implies that \( c + \rho > 1 + \rho c \).

For part iv), note that, by part ii), \( E_\theta S_c \) is strictly decreasing and continuous
in c for any fixed $\rho \geq 1$. The result then follows from the fact that, by part iii), $S_c$ attains its maximum at $\rho = 1$ and that, by part ii), $\lim_{c \to 0} S_c = 2$ for $\rho = 1$.

**Lemma 4.3** For each $c \in (0,1)$, and each $\rho \geq 1 + \varepsilon$, $R(\rho, \delta, \varepsilon)$ is strictly decreasing in $\varepsilon$. Further, for $\alpha > \rho$, $R(\rho_m(c, \varepsilon), \delta, \varepsilon)$ is, for each $c \in (0,1)$, strictly decreasing in $\varepsilon$.

**Proof.** The proof is straightforward. □

In order to apply the above subset selection technique in practice, tables of $R(\rho, \delta, \varepsilon)$, $R_m(c, \varepsilon)$, $\max\{R(\rho, \delta, \varepsilon)|\rho \geq 1 + \varepsilon\}$, $S_c$ and of $a(R_m, c, \varepsilon)$ and $b(R_m, c, \varepsilon)$ as functions of $\rho, c, \varepsilon$ and $R_m$ are needed. Such tables have been computed for the case where $\rho = 2$ and $\varepsilon = 0$ and can be found in Section 5. How these tables can be used in the case where $\varepsilon > 0$ is also explained in Section 5.

## 5 Tables

In this section, tables of the values of $R(\rho, \delta, 0)$, $R_m(c, 0)$, $\max\{R(\rho, \delta, 0)|\rho \geq 1\}$, $S_c$ and of $a(R_m, c, 0)$ and $b(R_m, c, 0)$ are given for several values of $\rho, c$ and $R_m$ for the cases where $\rho = 2$ and $\alpha = 3, 5, 10$, respectively.

For $\rho = 2$ and $\rho > 1$ we get

$$R(\rho, \delta, 0) = C(\alpha)(\rho - 1)^2 \int_0^{\infty} \frac{t^{\alpha - 1}}{(1 + t)^{2\alpha}} dt.$$

Using the transformation $x = t/(t+1)$ and putting $\lambda = \rho/(\rho + c)$, we get for $\alpha = 3$

$$R(\rho, \delta, 0) = (\rho - 1)^2(1 - 10\lambda^3 + 15\lambda^4 - 6\lambda^5),$$

for $\alpha = 5$

$$R(\rho, \delta, 0) = (\rho - 1)^2(1 - 126\lambda^5 + 420\lambda^6 - 540\lambda^7 + 315\lambda^8 - 70\lambda^9),$$

and for $\alpha = 10$

$$R(\rho, \delta, 0) = (\rho - 1)^2(1 - 92378\lambda^{10} + 755820\lambda^{11} - 2771340\lambda^{12} +$$

$$+ 5969040\lambda^{13} - 8314020\lambda^{14} + 7759752\lambda^{15} +$$

$$- 4849845\lambda^{16} + 1956240\lambda^{17} - 461890\lambda^{18} + 48620\lambda^{19}).$$

These values of $R(\rho, \delta, 0)$ are given in Table 1. Values of $R(\rho, \delta, \varepsilon)$ can be obtained from Table 1 by noting that for all $\rho \geq 1, c \in (0,1]$ and $\varepsilon \geq 0$

$$R(\rho, \delta, \varepsilon) = P \left( T > \frac{\rho}{c} \right) \left( \frac{\rho}{1 + \varepsilon} - 1 \right)^\rho I(\rho > 1 + \varepsilon)$$
\[ P \left( T > \frac{p^*}{c^*} \right) \left( p^* - 1 \right)^p I\left( p^* > 1 \right) \]

\[ = R(p^*, \delta_c, 0), \]

where, as before, \( p^* = \rho/(1 + \varepsilon) \) and \( c^* = c/(1 + \varepsilon) \).

Also, the next relation may be useful for practical applications:

\[ R(\rho, \delta_c, \varepsilon) = P \left( T > \frac{p}{c} \right) (\rho - 1)^p I(\rho > 1) \left( \frac{p^* - 1}{\rho^* - 1} \right)^p I(\rho^* > 1) \]

\[ = R(\rho, \delta_c, 0) \left( \frac{p^* - 1}{\rho^* - 1} \right)^p I(\rho^* > 1). \]

Table 1. \( R(\rho, \delta_c, 0) \) for some values of \( \rho \) and \( c \)

\[ \alpha = 3 \]

| \( \rho \)  | \( c = 1 \) | \( .9 \) | \( .8 \) | \( .7 \) | \( .6 \) | \( .5 \) | \( .4 \) | \( .3 \) | \( .2 \) |
|------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1.5        | .079        | .069        | .058        | .047        | .036        | .026        | .017        | .009        | .003        |
| 2          | .210        | .177        | .145        | .114        | .084        | .058        | .035        | .018        | .007        |
| 2.5        | .326        | .269        | .215        | .165        | .119        | .080        | .048        | .023        | .008        |
| 3          | .414        | .337        | .265        | .200        | .142        | .093        | .054        | .026        | .009        |
| 3.5        | .478        | .384        | .299        | .222        | .155        | .100        | .057        | .027        | .009        |
| 4          | .521        | .415        | .319        | .235        | .163        | .104        | .059        | .027        | .009        |
| 4.5        | .550        | .435        | .332        | .242        | .166        | .105        | .059        | .027        | .009        |
| 5          | .568        | .446        | .338        | .244        | .167        | .104        | .058        | .027        | .009        |
| 5.5        | .582        | .452        | .339        | .242        | .163        | .101        | .055        | .025        | .008        |
| 6          | .578        | .445        | .331        | .235        | .157        | .096        | .052        | .023        | .007        |
| 7          | .565        | .433        | .320        | .225        | .149        | .091        | .049        | .022        | .007        |
| 8          | .548        | .418        | .307        | .215        | .142        | .086        | .046        | .020        | .006        |
| 9          | .529        | .401        | .294        | .205        | .135        | .081        | .043        | .019        | .006        |
| 10         | .509        | .385        | .281        | .195        | .128        | .077        | .041        | .018        | .006        |
| 11         | .489        | .369        | .268        | .186        | .122        | .073        | .039        | .017        | .005        |
| 12         | .470        | .354        | .257        | .178        | .116        | .069        | .037        | .016        | .005        |
| 13         | .452        | .340        | .246        | .170        | .110        | .066        | .035        | .015        | .005        |
| 14         | .435        | .326        | .235        | .162        | .105        | .063        | .033        | .014        | .004        |

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Table 1. $R(\rho, \delta, 0)$ for some values of $\rho$ and $c$
(continued)

\[ \alpha = 5 \]

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\[ \alpha = 10 \]

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In Table 2, the maximum value of \( R(\rho, \delta_c, 0) \), as a function of \( \rho \), as well as the maximizing value, \( \rho_m(c, 0) \), of \( \rho \) are given for some values of \( c \) and \( \alpha \). Values of \( \rho_m(c, \varepsilon) \) can be obtained from Table 2 by using the fact that, for all \( c \in (0, 1] \) and \( \varepsilon \geq 0 \),

\[
\rho_m(c, \varepsilon) = (1 + \varepsilon)\rho_m\left(\frac{c}{1 + \varepsilon}, 0\right).
\]

This can be seen as follows. First note that (see (5.1))

\[
R(\rho(1 + \varepsilon), \delta_c, \varepsilon) = R(\rho, \delta_{c^*}, 0),
\]

where, again, \( c^* = c/(1 + \varepsilon) \). Now, \( R(\rho, \delta_{c^*}, 0) \) is maximized, in \( \rho \), by \( \rho_m(c^*, 0) \). So, \( R(\rho(1 + \varepsilon), \delta_c, \varepsilon) \) is maximized, in \( \rho \), by a \( \rho \) satisfying \( \rho(1 + \varepsilon) = \rho_m(c, \varepsilon) \), from which the result follows immediately.

Values of \( \max\{R(\rho, \delta_c, \varepsilon)|\rho \geq 1 + \varepsilon\} \) can be obtained from Table 2 by noting that

\[
R(\rho_m(c, \varepsilon), \delta_c, \varepsilon) = R(\rho_m(c^*, 0)(1 + \varepsilon), \delta_c, \varepsilon) = R(\rho_m(c^*, 0), \delta_{c^*}, 0).
\]

**Table 2** \( \rho_m(c, 0) \) and \( M = \max_{\rho \geq 1} R(\rho, \delta_c, 0) \) for some values of \( c \) and \( \alpha \)

<table>
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In Table 3 the expected subset size, \( \mathcal{E}_c S_c \), is given for \( \alpha = 3, 5 \) and 10 and for some values of \( \rho \) and \( c \), where

\[
\mathcal{E}_c S_c = 1 + C(\alpha) \int_0^{\rho/c} \frac{t^{\alpha-1}}{(1 + t)^{2\alpha}} dt = 1 + C(\alpha) \int_{\rho/c}^{\rho/(\rho + c)} x^{\alpha-1} (1 - x)^{\alpha-1} dx.
\]

These integrals can be computed in a way analogous to the one used for the computation of \( R \).
Table 4 contains, for values of \( \alpha > p = 2 \), some values of \( a(R_o, c, 0) \) and \( b(R_o, c, 0) \) defined by

\[
R(a(R_o, c, 0), c, 0) = R(b(R_o, c, 0), c, 0) = R_o.
\]

Values of \( a(R_o, c, \varepsilon) \) and of \( b(R_o, c, \varepsilon) \) can be obtained from Table 4 by noting that

\[
a(R_o, c, \varepsilon) = (1 + \varepsilon)a(R_o, c^*, 0) \quad \text{and} \quad b(R_o, c, \varepsilon) = (1 + \varepsilon)b(R_o, c^*, 0).
\]

This follows from the fact that, by (5.1),

\[
R(a(R_o, c, \varepsilon), \delta, 0) = \left( \frac{a(R_o, c, \varepsilon)}{1 + \varepsilon} \right) \delta, 0)
\]

and that, by the definition of \( a(R_o, c, \varepsilon) \),

\[
R(a(R_o, c^*, 0), \delta, 0) = R_o.
\]

The proof for \( b(R_o, c, \varepsilon) \) is analogous.

### Table 3. \( \mathcal{E}_\delta S_c \) for some values of \( \rho \) and \( c \)

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Table 3. $E_\theta S_c$ for some values of $\rho$ and $c$
(continued)

$\alpha = 5$

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Table 4. \( a(c) = a(R_0, c, 0) \) and \( b(c) = b(R_0, c, 0) \) for some values of \( c \) and \( R_0 \)

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References


Paul van der Laan
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands

Constance van Eeden
Moerland 19
Broek in Waterland
The Netherlands