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Conditions for one-step convergence of the local defect correction method for elliptic problems

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We analyse the convergence behavior of the local defect correction (LDC) method for elliptic problems. It is shown that the iteration is driven by the difference between the local coarse grid solution and the coarse grid restriction of the fine grid solution. We list the conditions on the differential operator and the discretizations used under which LDC will converge in a single iteration. The theoretical results are verified in numerical experiments.

1 Introduction

Partial differential equations with solutions that have highly localized properties appear in many application areas, e.g. combustion, shock hydrodynamics, and transport in porous media. Such problems require a fine grid only in the region(s) of high activity, whereas elsewhere a coarser grid suffices. We consider a discretization method for elliptic boundary value problems. In this technique, the local defect correction (LDC) method, the discretization on the composite grid is based on a combination of standard discretizations on uniform grids with different spacings that cover different parts of the domain. The coarse grid must cover the entire domain, and its spacing is chosen in agreement with the relatively smooth behavior of the solution outside the high activity areas. Apart from this global coarse grid, one or more local fine grids are used that are also uniform, each of which covers only a (small) part of the domain and contains a high activity region. The grid spacings of the local grids are chosen in agreement with the behavior of the continuous solution in that part of the domain.

The LDC method is an iterative process: the basic global discretization is improved by the local discretizations defined in subdomains. The update of the coarse grid solution is achieved by adding a defect correction term to the right hand side of the coarse grid problem. At each iteration step, the process yields a discrete approximation of the continuous solution on the composite grid. The discrete problem that is actually being

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solved is an implicit result of the iterative process. Therefore, the LDC method is both
a discretization method and an iterative solution method.

In this paper we study the convergence behavior of the LDC method. We formulate
the iteration in terms of the approximate values at the coarse grid point located at
the interface between coarse and fine grids only. It will be shown that the iteration
is driven by the difference between the local coarse grid solution and the projection
of the local fine grid solution on the coarse grid. In particular, the iteration has reached
its fixed point if the coarse grid solution and the fine grid solutions coincide. We study
some special cases where this occurs after only one correction step. The cases consid-
ered are of interest for the analysis of time-dependent LDC [13]. The results in this
paper are in agreement with previous studies on the convergence behavior of the LDC
method [2].

LDC is closely related to methods like the multi-level adaptive techniques (MLAT) [4]
and the fast adaptive composite grid (FAC) method [9,11]. MLAT uses a multigrid-like
hierarchy of (local) grids and uses coarser grids as correction grids for accelerating con-
vergence. The finer grids are used as correction grids to improve accuracy, because the
fine grid solution is used to approximate the coarse grid discretization error [8]. For
the FAC method in a variational setting, convergence results have been given by [9].
The variational theory is extended to the finite volume element method in [10]. Xu [17]
previews an abstract framework and general convergence theory for a wide range of it-
terative methods, among which domain decomposition, multigrid and multilevel meth-
ods. He groups the algorithms in parallel and successive subspace correction methods
and shows in particular that FAC is equivalent to classic multigrid with smoothing in
the area of refinement only.

An analysis of LDC in combination with finite difference discretizations is presented
in [5, 6]. The method is combined with finite volume discretizations in [3] and with fi-
nite elements in [16]. In [7, 14] it is studied with different grid types. The method
is successfully applied to a combustion problem in [1]. A generalization to time-de-
dependent problem is presented in [12].

This paper is organized as follows: in Section 2, we outline the LDC method for
two grids with an a priori chosen area of refinement. In Section 3 we study under
which conditions ‘superconvergence’ (the LDC iteration will reach its fixed point in
one iteration) will occur. In two numerical experiments, Section 4, we choose the dis-
cretization methods such that they satisfy the conditions found and we observe indeed
convergence after a single iteration.

2 The LDC method for elliptic problems

Consider the elliptic boundary value problem

\[
\begin{aligned}
\begin{cases}
Lu = f, & \text{in } \Omega, \\
u = g, & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]  

(2.1)

In (2.1), \(L\) is a linear elliptic differential operator, and \(f\) and \(g\) are the source term and
Dirichlet boundary condition, respectively. To discretize (2.1), we first choose a global
coarse grid (grid size \(H\)), which we denote by \(\Omega^H\). An initial approximation \(u_0^H\) on \(\Omega^H\)
can be found by solving the system

\[
L^H u_0^H = f^H,
\]

(2.2)

which is a discretization of boundary value problem (2.1). In (2.2), the right hand
side \(f^H\) incorporates the source term \(f\) as well as the Dirichlet boundary condition \(g\).
Figure 1: A global coarse and a local fine grid. The darker area is the area of high activity $\Omega_1$. The interface $\Gamma$ is dashed. Circles, not located on the domain boundary, are nodes of the coarse grid. Diamonds, not located on the domain boundary or the interface, are nodes of the local fine grid.

We assume $L^H$ to be invertible. We will not be specific about the elliptic operator $L$ and the discrete operator $L^H$ in this section; it may help to think of them as $L = \Delta$, and $L^H$ the standard five-point stencil approximating $\Delta$. Here, $\Delta$ denotes the two-dimensional Laplacian operator.

Now, assume that the continuous solution $u$ of (2.1) has a high activity region in some (small) part of the domain. This high activity of $u$ may be captured by discretizing (2.1) on a composite grid. To this end, we choose $\Omega_1 \subset \Omega$ such that the high activity region of $u$ is contained in $\Omega_1$. In $\Omega_1$, we choose a local fine grid (grid size $h$), which we denote by $\Omega_1^h$. The fine grid is chosen such that $\Omega^H \cap \Omega_1 \subset \Omega_1^h$, i.e., grid points of the global coarse grid that lie in the area of refinement belong to the local fine grid too.

In order to formulate a discrete problem on $\Omega_1^h$, we have to define artificial boundary conditions on $\Gamma$, the interface between $\Omega_1$ and $\Omega \setminus \Omega_1$, see Figure 1. We prescribe artificial Dirichlet boundary conditions by applying an interpolation operator $P^{h,H}$. The operator $P^{h,H}$ maps function values at grid points of the coarse grid that lie on the interface, denoted by $\Gamma^H$, to function values at grid points of the fine grid that lie on the interface, denoted by $\Gamma^h$. If we denote the vector space of grid functions on $\Gamma^H$ by $G(\Gamma^H)$, and likewise introduce $G(\Gamma^h)$, we have $P^{h,H} : G(\Gamma^H) \rightarrow G(\Gamma^h)$. In practice, we take $P^{h,H}$ to be the linear interpolation operator on the interface.

In this way, we find the following approximation $u^{h}_{i} = 0$, on $\Omega_1^h$:

$$L^h u^h_{i} = f^h_{i} - B^h_{i} P^{h,H}(u^H_{i}).$$ (2.3)

In (2.3), the matrix $L^h_{i}$ is a discrete approximation of the differential operator $L$ on the subdomain $\Omega_1$, and the first term on the right hand side $f^h_{i}$ incorporates the source term $f$ as well as the Dirichlet boundary condition $g$ on $\partial \Omega_1 \setminus \Gamma$ given in (2.1). The dependence on the coarse grid approximation via the artificial Dirichlet boundary condition is made explicit by the second term on the right hand side. We assume $L^h_{i}$ to be invertible.

When boundary value problem (2.1) has been discretized and solved on a coarse grid, and when an area of the coarse grid has been refined and a local solution has been calculated on the finer grid, we can use the local fine grid solution to update
Figure 2: A global coarse grid and its partitioning. The darker area is the area of high activity $\Omega_l$. The interface $\Gamma$ is dashed. We have $\Omega_l^H = \{1\}$, $\Gamma^H = \{2, 4, 5\}$, and $\Omega_c^H = \{3, 6, 7, 8, 9\}$. Grid points on the boundary are labeled $\partial$.

the coarse grid approximation. This update can be achieved by projecting the more accurate fine grid solution onto the local coarse grid, and by calculating the residual of the projected solution; this residual is an estimate for the local discretization error of the coarse grid discretization. The estimate is used to formulate a modified discrete problem on the coarse base grid. This is considered in detail in what follows.

The grid points of the coarse grid will be partitioned as follows

$$\Omega^H = \Omega_l^H \cup \Gamma^H \cup \Omega_c^H,$$  \hspace{1cm} (2.4)

where $\Omega_l^H := \Omega^H \cap \Omega_l$, $\Gamma^H := \Omega^H \cap \Gamma$ and $\Omega_c^H := \Omega^H \setminus (\Omega_l^H \cup \Gamma^H)$. See Figure 2. Using this partitioning of $\Omega^H$, we set

$$u^H := \begin{pmatrix} u_l^H \\ u_\Gamma^H \\ u_c^H \end{pmatrix}.\hspace{1cm} (2.5)$$

We will apply this same partitioning for other grid functions in $G(\Omega^H)$. Assuming that the stencil at grid point $(x, y)$ involves (at most) function values at $(x + iH, y + jH)$ with $i, j \in \{-1, 0, 1\}$, we can partition the discrete operator $L^H$ as

$$L^H = \begin{pmatrix} L_l^H & B_{l,r}^H & 0 \\ B_{l,r}^H & L_{l,c}^H & B_{l,c}^H \\ 0 & B_{c,r}^H & L_{c}^H \end{pmatrix}.\hspace{1cm} (2.6)$$

Using this decomposition, the coarse grid initialization (2.2) may also be written as

$$\begin{pmatrix} L_l^H & B_{l,r}^H & 0 \\ B_{l,r}^H & L_{l,c}^H & B_{l,c}^H \\ 0 & B_{c,r}^H & L_{c}^H \end{pmatrix} \begin{pmatrix} u_{l,0}^H \\ u_{\Gamma,0}^H \end{pmatrix} = \begin{pmatrix} f_l^H \\ f_{\Gamma}^H \end{pmatrix}.\hspace{1cm} (2.7)$$

After one coarse grid solve and one local fine grid solve, we have found the coarse grid approximations $u_{l,0}^H$, $u_{\Gamma,0}^H$, and $u_{c,0}^H$, and the fine grid approximation $u_{l,0}^h$. We have one approximation at each grid point $(x, y)$ of $\Gamma^H$ and $\Omega_c^H$, namely $u_{l,0}^H(x, y)$ and $u_{c,0}^H(x, y)$, respectively, and we have two approximations at each grid point $(x, y)$ of $\Omega_l^H$, ...
namely both \( u_{I,0}^H(x, y) \) and \( u_{I,0}^L(x, y) \). Of these last two approximations, \( u_{I,0}^L(x, y) \) is considered more accurate, as it is calculated on a finer grid. We try to improve the coarse grid approximation by using the approximation \( u_{I,0}^L(x, y) \) calculated on the local fine grid to estimate the local discretization error of the coarse grid discretization. For the description of this step, we introduce the operator \( R_{H,h} : G(\Omega_h^I) \rightarrow G(\Omega_h^I) \) as the restriction from \( \Omega_h^I \) onto \( \Omega_h^I \), viz. \( (R_{H,h}u_h^I)(x,y) = u_h^I(x,y) \) for all \( u_h^I \in G(\Omega_h^I) \), \((x,y) \in \Omega_h^I \).

If we would substitute the projection on \( \Omega_h^I \) of the exact solution \( u \) of boundary value problem (2.1) into the coarse grid discretization (2.2), we would find the local discretization error or defect \( d^H \), given by

\[
L^H (u_{\Omega_h^I}) = f^H + d^H.
\]

In particular, we would find the following local defect on \( \Omega_h^I \):

\[
L^H (u_{\Omega_h^I}) + B_{I,r}^H (u_{\Gamma_{\Omega_h^I}}) = f^H + d^H.
\]

If we would know the values of the defect \( d^H \), we could use them to find a better approximation on the coarse grid. This could be achieved by putting the defect vector on the right hand side of (2.2). However, as we do not know the exact solution of the boundary value problem, we can calculate neither \( d^H \) nor \( d^H \). What we can do though, is to use the approximation \( u_{I,0}^L \) calculated on the local fine grid to estimate \( d^H \).

Using (2.9), we find

\[
d^H = L^I (u_{\Omega_h^I}) + B_{I,r}^L (u_{\Gamma_{\Omega_h^I}}) - f^I \approx L^H (R_{H,h}u_{I,0}^L(x,y)) + B_{I,r}^H u_{I,0}^L(x,y) - f^H =: d_{I,0}^H.
\]

Using (2.9), we find an estimate of the local discretization error of the coarse grid discretization at all points of \( \Omega_h^I \). Therefore, we can update the coarse grid approximation by placing the estimate (2.9) at the right hand side of the coarse grid equation (2.2) or (2.7). This leads to the coarse grid correction step to find \( u_{I,i}^H \), \( i = 1, \ldots \), on the coarse grid

\[
L^H u_{I,i}^H = \begin{pmatrix} f^H + d_{I,i-1}^H \\ f^H \\ f^H \\ f^H \end{pmatrix} = \begin{pmatrix} L^H (R_{H,h}u_{I,i-1}^L(x,y)) + B_{I,r}^H u_{I,i-1}^L(x,y) \\ f^H \\ f^H \end{pmatrix}.
\]

Summarizing, we have found the following iterative method.

**Algorithm 2.1**

**Two-grid LDC algorithm with a priori chosen area of refinement**

**Initialization**

- Solve the basic coarse grid problem (2.2).
- Solve the local fine grid problem (2.3) with \( i = 0 \).

**Iteration, \( i = 1, 2, \ldots \)**

- Solve the updated coarse grid problem (2.10).
- Solve the local fine grid problem (2.3).

**3 The convergence properties of LDC**

The following lemma shows that once the coarse and the fine grid approximations coincide in the common points between coarse and fine grid, the LDC algorithm converges and a fixed point of the iteration has been reached.
Lemma 3.1 If $u^H_{i,k} = R^{H,h} v^h_{i,k}$ for a certain index $k$, then Algorithm 2.1 converges and it has reached a fixed point, i.e.,

$$u^H_i = u^H_k, \quad u^h_{i,i} = u^h_{i,k},$$

for all $i = k, k+1, \ldots$

Proof. Assume that $u^H_{i,k} = R^{H,h} v^h_{i,k}$ for a certain $k$. Then from (2.9) we have that

$$d^H_{i,k} = L^H_{i} R^{H,h} u^h_{i,k} + B^H_{i,r} u^h_{i,k} - f^H_{i} = L^H_{i} u^H_{i,k} + B^H_{i,r} u^H_{i,k} - f^H_{i} = d^H_{i,k-1},$$

where the last identity comes from the first row of (2.10). As a consequence

$$L^H_{i} u^H_{k+1} = \begin{pmatrix} f^H_i + d^H_{i,k} \\ f^H_i \\ f^H_c \\ f^H_c \end{pmatrix} = \begin{pmatrix} f^H_i + d^H_{i,k-1} \\ f^H_i \\ f^H_c \\ f^H_c \end{pmatrix} = L^H_{i} u^H_k.$$  \hspace{1cm} (3.3)

Because we have assumed $L^H_{i}$ to be invertible, we have $u^H_{k+1} = u^H_k$ for all grid points in the global coarse grid. As $f^H \subset \Omega^H$, we have $u^H_{r,k+1} = u^H_{r,k}$. By induction, we find $u^H_i = u^H_k$ and $u^h_{i,i} = u^h_{i,k}$ for all $i = k, k+1, \ldots$  \hspace{1cm} \Box

For some special cases, the LDC iteration will reach its fixed point in one step. We now study under which conditions this ‘superconvergence’ will occur. To that end we define the differences between the local coarse grid solution and the projection of the local fine grid solution $v^H_{i,i} \in G(\Omega^H)$, viz.

$$v^H_{i,i} := u^H_{i,i} - R^{H,h} u^h_{i,i-1},$$

as well as the differences between successive approximations, viz.

$$v^H_{i,i} := u^H_{i,i} - u^H_{i,i-1}, \quad v^h_{i,i} := u^h_{i,i} - u^h_{i,i-1}.$$  \hspace{1cm} (3.4)

Using these definitions, we can rewrite the first equation in (2.10), namely

$$L^H_{i} u^H_{i,i} + B^H_{i,r} u^h_{i,i} = L^H_{i} R^{H,h} u^h_{i,i-1} + B^H_{i,r} u^H_{i,i-1},$$

as

$$L^H_{i} v^H_{i,i} + B^H_{i,r} v^h_{i,i} = 0.$$  \hspace{1cm} (3.5)

If we write (2.3) for $i$ and for $i-1$ and then subtract one from the other, we obtain

$$L^H_{i} v^H_{i,i} + B^H_{i,r} v^h_{i,i} = 0.$$  \hspace{1cm} (3.6)

Expressions (3.7) and (3.8) can be interpreted as a coarse and a fine grid discretization of the local continuous homogeneous problem

$$\begin{cases}
L \omega = 0, & \text{in } \Omega, \\
\omega = \bar{\omega}, & \text{on } \Gamma, \\
\omega = 0, & \text{on } \partial \Omega \setminus \Gamma.
\end{cases}$$  \hspace{1cm} (3.9)

In (3.9) the boundary condition $\bar{\omega}$ is such that

$$\bar{\omega}|_{\Gamma^H} = p^{h,H} v^H_{r,i},$$

and, as a logical consequence,

$$\tilde{\omega}|_{\Gamma^H} = v^H_{r,i}.$$  \hspace{1cm} (3.10)
If the two discretizations (3.7) and (3.8) are such that the coarse and fine grid solutions coincide at the common points, i.e.

\[ v^H_{l;i} = R^{H,h} v^h_{l;i}, \]  

then the following identity holds

\[ u^H_{l;i} - R^{H,h} u^h_{l;i} = R^{H,h} \left( u^h_{l;i} - u^h_{l;i-1} \right), \]  

or

\[ u^H_{l;i} = R^{H,h} u^h_{l;i}. \]  

This means, see Lemma 3.1, that the LDC algorithm has converged and the iteration stops.

There is one particular situation in which property (3.12) is already verified for \( i = 1 \): this is when the numerical scheme we adopt to discretize the original boundary value problem (2.1) on \( \Omega^H \) and \( \Omega^h \) is such that, if applied to (3.9), it always produces the exact analytical solution of the homogeneous problem. By always, we mean for any boundary condition \( \omega \) and any choice of the grid size. In one dimension this is for example the case when \( L = d^2/dx^2 \) and the numerical discretization is performed by centered differences or by any other method that integrates linear functions exactly. Another one-dimensional example is a convection-diffusion equation discretized by the exponential scheme described for instance in [15, page 86]. In all these examples property (3.12) is already verified for \( i = 1 \) and the LDC algorithm converges in exactly one iteration. We summarize this result in the following theorem.

**Theorem 3.2** If the discretization schemes used for the coarse and fine grids solve the local homogeneous boundary value problem (3.9) exactly, the LDC iteration will converge in one step.

**Remark 3.3** Note that it is not surprising that the convergence behavior of LDC is ruled by the local homogeneous problem (3.9). In [3, Lemma 4], in fact, the properties of the LDC converged solution are proved on the basis of certain assumptions on the solution of the local coarse grid homogeneous system.

The results found are consistent with those presented in [2], where the LDC iteration is studied through the iteration matrix \( M \). The matrix \( M \) can be written as a matrix product that contains the subproduct [2, Theorem 2]

\[ D := \left[ B^H_{l,f} - L^H_{l,f} R^{H,h} \left( L^h_{l} \right)^{-1} B^h_{l,f} P^{h,H} \right] = L^H_{l} \left[ (L^H_{l})^{-1} B^H_{l,f} - R^{H,h} (L^h_{l})^{-1} B^h_{l,f} P^{h,H} \right]. \]  

(3.15)

The term \( (L^H_{l})^{-1} B^H_{l,f} \), see the first row of (2.7), is a local coarse grid solution of the homogeneous system (3.9), while \( R^{H,h} (L^h_{l})^{-1} B^h_{l,f} P^{h,H} \), see (2.8), represents the coarse grid restriction of a fine grid solution of (3.9). Clearly if these two terms coincide, \( D \) and hence the iteration matrix \( M \) is zero and the LDC algorithm converges in one iteration.

4 **Numerical experiments**

In this section we verify the theoretical results of Section 3 by means of two numerical experiments. In the first one the LDC method is applied to the following one-
Figure 3: Numerical results of the LDC algorithm applied to boundary value problem (4.1). The solid curve in each figure in the continuous solution $u$.

<table>
<thead>
<tr>
<th>iteration</th>
<th>$\epsilon_l^H$</th>
<th>$\delta_l^H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 0$</td>
<td>$5.1 \cdot 10^{-1}$</td>
<td>$3.2 \cdot 10^{-1}$</td>
</tr>
<tr>
<td>$i = 1$</td>
<td>$4.9 \cdot 10^{-3}$</td>
<td>$4.4 \cdot 10^{-16}$</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>$4.9 \cdot 10^{-3}$</td>
<td>$4.4 \cdot 10^{-16}$</td>
</tr>
</tbody>
</table>

Table 1: Numerical results of the LDC algorithm applied to boundary value problem (4.1).
dimensional diffusion problem
\[
\begin{aligned}
-\frac{d^2 u}{dx^2} &= f, \quad \text{in } \Omega = (0, 1), \\
\ u(0) &= u_0, \quad u(1) = u_1.
\end{aligned}
\] (4.1)

In (4.1) the source term \( f \) and the boundary conditions \( u_0 \) and \( u_1 \) are chosen in such a way that
\[
u(x) = \frac{1}{2} \left( \tanh \left( 50 (x - 1/8) \right) + 1 \right).
\] (4.2)

The exact analytical solution of boundary value problem (4.1) has a high activity area around \( x = 1/8 \) and is smooth in the rest of the domain. For this reason we solve the problem with LDC. We define \( H = 1/20, \ N = 1/H \) and we let the global coarse grid be
\[
\Omega^{H} := \{ jH \mid j = 1, 2, \ldots, N - 1 \}.
\] (4.3)

The area of refinement is chosen as \( \Omega_1 = (0, \gamma), \) with \( \gamma = 0.3, \) and the local fine grid is taken as
\[
\Omega^{h}_1 := \{ jh \mid j = 1, 2, \ldots, n - 1 \},
\] (4.4)

with \( h = 1/100 = H/5, \ n = \gamma/h. \) Both on the coarse and on the fine grid the second derivative is approximated by the standard three-point centered differences scheme.

In our one-dimensional setting the interpolation operator \( p^{h,H} \) reduces to the identity function. The numerical results of this numerical experiment are illustrated in Figure 3, while in Table 1 we report the two parameters \( e_i^{H} \) and \( \delta_i^{H} \) defined as follows:
\[
e_i^{H} := \max \left( \left| u_i^H - u_{i,H} \right| \right), \quad \delta_i^{H} := \max \left( \left| u_{i-1}^H - R^{H,h} u_i^h \right| \right).
\] (4.5)

The parameter \( e_i^{H} \) is the maximum of the absolute error of the coarse grid approximation \( u_i^H, \) while \( \delta_i^{H} \) is the maximum absolute difference between the coarse and the fine grid approximation in the common points between \( \Omega_1^H \) and \( \Omega_l^h. \) From Table 1 we can see that the initial coarse grid error is reduced by two orders of magnitude after just one LDC iteration. Furthermore, we can notice that after the first iteration the LDC algorithm has reached a fixed point: this is indicated by the fact that \( \delta_i^{H} \) is of the same order as the machine precision. Indeed, as foreseen theoretically, if we perform one more LDC iteration the coarse grid approximation does not change anymore and its error is not reduced any further. The LDC algorithm converges in exactly one iteration because the centered differences scheme is able to integrate the homogeneous problem
\[
\begin{aligned}
-\frac{d^2 u}{dx^2} &= 0, \quad \text{in } \Omega_1 = (0, \gamma), \\
\ u(0) &= 0, \quad u(\gamma) = C.
\end{aligned}
\] (4.6)

In the second numerical experiment we consider the one-dimensional convection-diffusion problem
\[
\begin{aligned}
\nu \frac{du}{dx} - \lambda \frac{d^2 u}{dx^2} &= f, \quad \text{in } \Omega = (0, 1), \\
\ u(0) &= u_0, \quad u(1) = u_1.
\end{aligned}
\] (4.7)

In (4.7) we take \( \nu = 10 \) and \( \lambda = 1. \) The source term \( f \) and the boundary conditions are chosen in such a way that the exact analytical solution of the problem is the same as in the previous numerical experiment. Also the global coarse grid \( \Omega^{H} \), the area of refinement \( \Omega_1 \) and the local fine grid \( \Omega_l^h \) are the same as before. However, this
Figure 4: Numerical results of the LDC algorithm applied to boundary value problem (4.7). The solid curve in each figure in the continuous solution $u$.

Table 2: Numerical results of the LDC algorithm applied to boundary value problem (4.7).

<table>
<thead>
<tr>
<th>iteration</th>
<th>$\epsilon_{l}^{Hi}$</th>
<th>$\delta_{l}^{Hi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 0$</td>
<td>$4.6 \cdot 10^{-1}$</td>
<td>$4.5 \cdot 10^{-1}$</td>
</tr>
<tr>
<td>$i = 1$</td>
<td>$5.4 \cdot 10^{-3}$</td>
<td>$2.9 \cdot 10^{-16}$</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>$5.4 \cdot 10^{-3}$</td>
<td>$2.2 \cdot 10^{-16}$</td>
</tr>
</tbody>
</table>
time the differential operator is discretized by the exponential scheme, described for example in [15, page 86]. Using the exponential scheme, the first coarse approximation is computed solving the system

\[
\begin{align*}
    a_{-1} u_0^H(x_{j-1}) + a_0 u_0^H(x_j) + a_1 u_0^H(x_{j+1}) &= f_j, & j = 1, 2, \ldots, N - 1, \\
    u_0^H(x_0) &= u_0, & u_0^H(x_N) = u_1,
\end{align*}
\] (4.8)

where \( x_j := jH \) and

\[
    a_{-1} := \frac{\nu}{H} \frac{\exp(vH/\lambda)}{1 - \exp(vH/\lambda)}, \quad a_1 := \frac{1}{H} \frac{1}{1 - \exp(vH/\lambda)}, \quad a_0 := -(a_{-1} + a_1). \] (4.9)

Analogous expressions hold for the discretization on the local fine grid. When applied to the integration of the homogeneous problem

\[
\begin{align*}
    \nu \frac{d}{dx} - \lambda \frac{d^2 u}{dx^2} &= 0, & \text{in } \Omega_1 = (0, \gamma), \\
    u(0) &= 0, & u(\gamma) = C,
\end{align*}
\] (4.10)

the exponential scheme produces the exact solution for any value of \( C \) and any grid size. Therefore also in this second numerical experiment we expect the LDC algorithm to converge in one iteration. The results in Figure 4 and Table 2 confirm this: \( \delta^1 \) is of the order of the machine accuracy which means that indeed the LDC algorithm has reached its fixed point in one iteration. Also note that, like before, in just one iteration the maximum of the initial coarse grid error is reduced by two orders of magnitude.

References


