A Model for Positive Corona Inception From Charged Ellipsoidal Thundercloud Hydrometeors

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Abstract Lightning is observed to incept in thundercloud electric fields below the threshold value $E_\text{th}$ for discharge initiation. To explain this, the local enhancement of the electric field by hydrometeors is considered. The conditions for the onset of positive corona discharges are studied in air for ellipsoidal geometries. A hydrometeor is simulated as an individual charged conductor in zero ambient field; there is only a field generated by the charge on the hydrometeor surface. By doing so, the feasibility of corona inception from ellipsoidal hydrometeors can be formulated based on the self-sustaining condition of electron avalanches. For hydrometeor dimensions of a few millimeters and typical thundercloud pressure, values above 2.4 $E_\text{th}$ are found for the onset electric field at the tip of the ellipsoid. From simulations the required ambient electric field for corona onset from an uncharged hydrometeor can then be derived. This results in values between 0.07 $E_\text{th}$ and 0.8 $E_\text{th}$ for semi-axes aspect ratios between 0.01 and 1. The charge required on the hydrometeor surface for corona onset for typical pressures is minimum for semi-axes aspect ratios increasing from 0.12 to 0.22 for decreasing hydrometeor volume from 42 to 4.2 mm$^3$. For representative simulated hydrometeors, onset charges are between 495 and 2,430 pC. For a hydrometeor volume of 4.2 mm$^3$ results are comparable to measured precipitation charges for aspect ratios between 0.16 and 0.32. From the results it is concluded that corona onset from ellipsoidal hydrometeors of a realistic volume can be achieved in thundercloud conditions for certain aspect ratios.

1. Introduction

One of the greatest unanswered questions in lightning physics is how lightning is initiated in a thunderstorm (Dwyer & Uman, 2014; Mazur, 2016; Petersen et al., 2008). Though not taken at the exact location of lightning initiation, from in situ measurements it appears that lightning initiates in thundercloud electric fields which are considerably lower than the breakdown electric field required for the inception of electric discharges (Marshall et al., 1995; Stolzenburg & Marshall, 2009). One of the most popular theories explaining how this is possible is the hydrometeor theory (Crabb & Latham, 1974; Dawson & Duff, 1970; Mazur, 2016; Petersen et al., 2008).

Other options are in situ sampling inadequacies, and a lowered threshold field for electron runaway, or for positive streamer formation (Williams, 2006). The hydrometeor theory states that hydrometeors—ice and water particles in thunderclouds—locally enhance the electric field, such that the breakdown field is exceeded, and lightning inception is enabled. The role of hydrometeors in lightning inception has been investigated in laboratory experiments (Petersen et al., 2015; Coquillat et al., 1995; R. F. Griffiths & Latham, 1974) with a main focus on corona onset, which is the initial stage of the formation of a lightning leader.

A corona discharge is only possible when a critical voltage, the onset voltage $V_\text{on}$, on the electrode is reached. Equivalently, the electric field $E$ in the ionization region should exceed the breakdown threshold field $E_\text{b}$ and thus the field on the surface of the electrode should exceed the onset field $E_\text{on}$. Historically, the onset field in air is computed using Peek’s law. Formulated from empirical observations, this law gives the onset field as a function of radius in cylindrical coordinates (Lowke & D’Alessandro, 2003; Monroín et al., 2018; Peek, 1915). The onset field has, for example, been used to predict corona on high voltage transmission lines (J. Xu et al., 2018; M. Xu et al., 2012; Yamazaki & Olsen, 2004). The breakdown field scales linearly with pressure, and at a typical thundercloud altitude pressure of 0.4 atm this field has a value of about 10 kV/cm (Raizer, 1991).

The main mechanism of electrical breakdown is the electron avalanche. For electric fields above $E_\text{b}$, electrons can multiply by means of impact ionization of air molecules, thereby forming avalanches. In order to have a self-sustaining discharge, a constant source of seed electrons is required, which can be supplied by...
The first group of electrons that collides with the gas molecules and leads to photoionization is known as the primary electron avalanche, and the subsequently formed second group of electrons that can give further photoionization is known as the secondary electron avalanche (see Figure 1). In air, the gas molecules that are dominant in emitting photons after collisions with free electrons are nitrogen molecules, and the gas molecules that are predominantly photoionized by these photons are oxygen molecules. It should be noted that avalanche formation is a stochastic process, such that electron multiplication can also take place in fields (slightly) below the breakdown field. These contributions are briefly investigated but otherwise neglected in this work.

The onset of a corona discharge is typically defined by the discharge becoming self-sustaining. The self-sustaining criterion that is often applied is the amount of photons produced by the secondary avalanche being at least equal to those produced by the primary avalanche (Liu, Dwyer, & Rassoul, 2012; Naidis, 2005). This condition is also adapted by the current work, which closely follows the structure of the work by Liu, Dwyer, and Rassoul (2012).

Depending on electrode polarity, corona discharges can be positive or negative. A popular hypothesis for lightning initiation is that the development of a positive streamer system, developed from a seed positive streamer from the corona on a hydrometeor, precedes and leads to negative breakdown (C. Phelps, 1974; C. T. Phelps & Griffiths, 1976; Loeb, 1966; L. P. Babich et al., 2016; Petersen et al., 2008). Therefore, positive corona discharges are of great interest when investigating the initial stage of lightning initiation.

Laboratory experiments have revealed that the onset of a corona discharge strongly depends on the size and shape of the hydrometeor. In their study on corona initiation from small ice crystals, Petersen et al. (2015) reported that the onset field $E_0$ decreases with hydrometeor length and that ice crystals with sharper tips promote glow coronae while inhibiting positive streamer formation. Moreover, they noted that the onset field increases linearly with the relative gas density $\delta = N/N_0$ (where $N$ and $N_0$ are the actual and standard gas densities), meaning $E_0 \sim pT$, with $p$ the pressure and $T$ the temperature. The decrease of the onset field with size is also found in many point-to-

![Figure 1](https://agupubs.onlinelibrary.wiley.com/doi/10.1029/2021JD035505)
plane and rod-to-plane experiments using metal electrodes (D’Alessandro & Berger, 1999; Kip, 1938; Nasser & Heiszler, 1974; Schumann, 1923; Waters & Stark, 1975), which are observed to give corona onset voltages very similar to ice electrodes (Bandel, 1951).

In simulations, similar conclusions were reached. Dubinova et al. (2015) investigated discharge inception conditions for (relatively large with lengths of a few cm) dielectric ellipsoidal hydrometeors and concluded that an increase in hydrometeor length yields stronger field enhancement, as does a decrease in hydrometeor tip radius. Hence, a longer, sharper hydrometeor generally requires a lower background electric field for the initiation of a discharge. Likewise, in simulations of streamer initiation from charged water drops L. P. Babich et al. (2016) found a lower threshold ambient field for larger drop sizes. Dubinova et al. (2015) also observed an optimal semi-axes aspect ratio for inception; though longer hydrometeors produce a higher electric field, the probability of discharge initiation decreases when they become too sharp, because the field enhancement becomes too localized at the tip. As this ratio fixes the ellipsoidal hydrometeor’s shape, an optimal shape can be determined. Simulations (Riousset et al., 2020) also show the experimentally observed linear pressure dependence of discharge initiation. This is expected, as it follows from the pressure dependence of corona breakdown field.

In addition to size, shape and air density, the onset of a corona discharge has been found to depend on the orientation, surface features and initial charge of the hydrometeor. R. F. Griffiths and Latham (1974) concluded from experimental studies on ice particles that onset fields in thundercloud regions are probably in the range of 400–500 kV/cm, which was later corrected by R. Griffiths (1975) to 350–450 kV/cm when taking into account the effect of charge on ice particles. Furthermore, R. F. Griffiths and Latham (1974) suggested that continuous corona discharges could be generated from thundercloud ice crystals at temperatures above −18°C only. Of course, the gas density increases with decreasing temperature, explaining the subsequent increase of the onset field. Moreover, the surface conductivity decreases with decreasing temperature such that corona onset becomes less likely (Petersen et al., 2006; R. F. Griffiths & Latham, 1974), and generally smaller ice crystals are formed at lower temperatures (Petersen et al., 2006). In 2006, Petersen et al. (2006) demonstrated that corona discharges can initiate in temperatures down to −38°C, showing that corona and streamer discharges can initiate from hydrometeors at thundercloud altitudes relevant for lightning initiation. Moreover, from numerical simulations L. Babich et al. (2017) observed that the required charge on hydrometeors at these representative temperatures and altitudes agrees with measured thundercloud precipitation charges, which are generally between 10 and 200 pC and for a small fraction of hydrometeors between 200 and 400 pC (Marshall & Winn, 1982). In a follow-up study, L. P. Babich and Bochkov (2018) showed that for ice needles with lengths in the order of 1 cm, streamer formation can take place. However, additional field enhancement is required, for example, by relativistic runaway electron avalanches.

To conclude, these studies reveal that the onset of a corona discharge from a hydrometeor depends on its size, shape and surface charge, and on environmental conditions such as pressure, temperature and the ambient electric field. Experimental results and in situ measurements indicate the essential role of hydrometeors in lightning initiation. These findings are supported by simulations of lightning inception from ice and water particles.

The comparison of experimental work on corona onset from ice point electrodes to measurements on metal point electrodes has shown the corona onset voltage to be very comparable (Bandel, 1951). This has also been applied to numerical studies. As ice has a high dielectric constant (~93) at low frequencies and a low dielectric constant of 3 at high frequencies (Evans, 1965) with the crossover around 10^4 Hz (Mavrovic et al., 2018), simulating hydrometeors as metals is appropriate when the electric field varies slowly over time (Gao et al., 2020; Sadighi et al., 2015). This is the case for the field the first electrons experience during streamer inception, meaning this metal approximation is applicable to investigation of corona inception. Using a conductor model, Gao et al. (2020) simulated a hydrometeor as an isolated conducting sphere in an ambient field and found corona discharges occur in ambient fields below the breakdown field. Hydrometeors were also modeled as perfect conductors by Liu, Kosar, et al. (2012), who studied streamer formation from the tip of a conducting object simulated as an ionization column in nonzero ambient electric field. They found that streamers can form well below the breakdown threshold field. Sadighi et al. (2015) also simulate a hydrometeor as an ionization column, and conclude initiation of stable streamers from thundercloud hydrometeors in a 0.3 E_b electric field is possible, in case of enhanced ambient ionization levels in front of the streamer, which can for example, be created by corona discharges. To simulate
the onset of a positive corona discharge from a metal electrode in air, Naidis (2005) introduced a model giving a corona inception criterion taking into account the ambient pressure and the size and shape of the electrode. This model was applied to spherical and cylindrical electrodes, and later revisited by Liu, Dwyer, and Rassoul (2012) for the spherical case.

The main goal of this paper is to extend the model introduced by Naidis (2005) to include another representative shape, the prolate spheroid, as ice and water particles in a thundercloud can have a wide variety of shapes depending on thundercloud conditions. Their sizes range from a few micrometers to several centimeters (MacGorman et al., 1998). The size distribution of hydrometeors is little investigated within thunderclouds due to difficulties of in situ measurements (Mazur, 2016), but it is expected that the extreme cases of several centimeters are rare, and that a millimeter range is more representative (Gardiner et al., 1985; Weinheimer et al., 1991). The shape of the hydrometeor in the direction perpendicular to the thundercloud electric field has a negligible contribution to the field enhancement. More precisely, the enhancement at the tips is mainly determined by the length of the hydrometeor and the radius of curvature of the tip of the hydrometeor (Dubinova et al., 2015; Köhn & Ebert, 2015). Taking this into account, it can prove fruitful to investigate ellipsoidal hydrometeors. More specifically, assuming cylindrically symmetric thundercloud conditions, a prolate ellipsoid of revolution, or prolate spheroid, is considered.

Thus, the purpose of this study is to simulate positive corona discharges originating from a positively charged spheroidal hydrometeor tip. In doing so, the feasibility of lightning initiation from a spheroidal hydrometeor is studied. The simulation of this configuration is done using the model for the onset of positive corona discharges introduced by Naidis (2005) and further elaborated by Liu, Dwyer, and Rassoul (2012). The investigated hydrometeor is isolated and without ambient electric field. Thus, there is only an electric field generated by the charge on the hydrometeor, which differs from realistic lightning occurrences, where there is also an external field present due to the large-scale charge distribution. However, the effects of the field induced by a charged particle can already reveal a lot about the role of particle shape and size in discharge inception. Hence, for the charged hydrometeor the dependence of corona onset on its semi-axes aspect ratio and volume is reported for various ambient pressures by varying its major and minor axes.

2. Model Description

As elaborated, a corona discharge is the result of electron multiplication via direct impact ionization within avalanches. The resulting avalanches are seeded by electrons supplied through photoionization. Taking loss by attachment processes into account, $\alpha = \eta$ defines electrical breakdown, where $\alpha$ is the number of ionizing collisions per unit length and $\eta$ the number of electron attachments per unit length, both depending on the electric field. Formulating the net ionization coefficient $\alpha_{\text{eff}}(E) = \alpha(E) - \eta(E)$, the breakdown field $E_b$ is defined by $\alpha_{\text{eff}}(E_b) = 0$.

The number of photons produced by a primary avalanche is denoted by $N_1$, and those produced by a secondary avalanche by $N_2$. $N_2$ depends on $N_1$ through $N_2 = \gamma N_1$, where $\gamma$ is the mean number of photons from the secondary avalanche produced by one of the photons from the primary avalanche (see Figure 1). Naidis (2005) formulates the criterion for corona inception as the secondary avalanche producing at least as many photons as the primary avalanche, so $N_2 = N_1$, or equivalently $\gamma = 1$. Then, the discharge is self-sustaining; it can proceed without external ionization sources. This criterion does not take into account the stochastic nature of discharge inception. The region around the hydrometeor where the breakdown field is exceeded is sufficiently small such that individual electron avalanches, which have an intrinsically random nature, should be considered. Here, the randomness is neglected, and therefore also the contributions from outside this region, as only the total amount of electrons in the avalanche is investigated. The inclusion of stochastic effects would soften the criterion, as then electrons can “tunnel” to higher energies (Rutjes, 2018).

For point and wire electrodes, most of the electrons and photons are produced near the surface of the electrodes (Naidis, 2005). Following Naidis (2005) and Liu, Dwyer, and Rassoul (2012), it is therefore a reasonable assumption that all primary photons that lead to photoionization are produced at the electrode surface. To satisfy the self-sustaining condition, on average each primary photon must give rise to at least one secondary photon at the same location. This assumption overestimates the effect of photoionization, as the effect of photons on
electron production is now maximized. Therefore, this paper studies only the minimum conditions for the onset of a corona discharge within this assumption. Besides inducing photoionization and thereby triggering secondary electron avalanches, a photon can also fall back to the electrode surface or leave the ionization region and consequently not contribute to the secondary avalanche. Different factors, such as the photon absorption probability, affect this balance and thus play a role in satisfying the $\gamma = 1$ criterion for positive corona onset.

To formulate the $\gamma = 1$ criterion, a spherical coordinate system $(r, \theta, \phi)$ is introduced with its origin at the surface of the electrode. This is illustrated in Figure 2 for the ellipsoidal electrode, with major axis $a$ and minor axis $b$, considered in this paper. In the region near the electrode tip the electric field $E$ reaches its maximum. Consequently, the number of electrons in the primary avalanche and the probability of photon emission are also maximum near the tip. For simplicity, it is assumed that the primary photon is emitted at the origin of the spherical coordinate system. Taking into account all possible directions in which this photon can move, the photon absorption region can be defined as the part of the ionization region ($E \geq E_k$) where $\theta \leq \pi/2$. In other words, the photon absorption region is the region that can be reached by the photon and where the field is sufficiently high such that an electron avalanche can be created. This region, highlighted in deep yellow in Figure 2, is thus the region of interest for the initiation of a corona discharge.

The corona inception criterion $\gamma = 1$, derived by Naidis (2005) using the above self-sustaining criterion, is then formulated as

$$\gamma \approx \xi \beta (\rho_0) \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin \theta d\theta \int_0^{r_{\text{max}}(E)} r^2 P(r) \left[ \exp \left( \int_{r_0(\theta, \phi)}^r a_{\text{eff}}(\rho, E) d\rho \right) - 1 \right] dr = 1.$$ \hspace{1cm} (1)

The coordinates $\rho, r, \theta$ and $\phi$ are defined in Figure 2. Because of the cylindrical symmetry of the prolate spheroid, there is no $\phi$-dependence. Besides the spherical coordinate system $(r, \theta, \phi)$ with the origin at the tip of the ellipsoid, the coordinate $\rho$, which is given by the direction of the electric field and starts from the $z$-axis, is introduced as well, as is the radial coordinate $\rho'$ from the center of the ellipsoid.

The term $\xi$ is the ionization probability of an oxygen molecule at photon absorption. The distance $\rho_{0\beta}(r, \theta, \phi)$ is the distance between the point of photoionization (equivalent to the position of photon absorption) and the symmetry axis of the ellipsoid along the direction of the electric field in the point of photoionization. It is thus the length of the line along the $\rho$ coordinate that ends at the point of photon absorption (see Figure 2). Similarly, the distance from the symmetry axis of the ellipsoid to its surface along the surface electric field direction is given by $\rho_0$ (for a sphere this would be its radius). The position where the electric field has decreased to the breakdown field $E_k$ is given by $r_{\text{max}}$ in the spherical coordinate system $(r, \theta, \phi)$. Naidis (2005) uses the expression for the photon absorption probability $P(r)$ in air where photoionization of oxygen molecules takes place at absorption of radiation of wavelengths 98–102.5 nm, emitted by nitrogen molecules (Zhelezniak et al., 1982)

$$P(r) = \frac{\exp(-\kappa_1 r \delta) - \exp(-\kappa_2 r \delta)}{4 \pi r^3 \log (\kappa_2/\kappa_1)},$$ \hspace{1cm} (2)

where $\kappa_1 = 5.6$ cm$^{-1}$ and $\kappa_2 = 320$ cm$^{-1}$. The term $\xi \beta (\rho_0)$ can be found from

$$\xi \beta = \left( 0.03 + \frac{3.78}{E} \right) \frac{\delta_1}{\delta + \delta_1},$$ \hspace{1cm} (3)

where $\delta_1 = 0.04$ and $E$ is the electric field in kV/cm (Zhelezniak et al., 1982). Here $\beta$ is the coefficient of production of ionizing photons scaled to the net ionization coefficient $\alpha_{\text{eff}}$. Because of its weak dependence on the electric field and the high fields at the electrode surface, $\beta$ is approximated by its value $\beta(\rho_0)$ at the surface. To apply the corona inception criterion $\gamma = 1$ to a prolate spheroid, analytical expressions should be derived for
the distances $r_{\text{mat}}$, $\rho_{0r}$ and $\rho_{\text{ab}}$ and the electric field $E$, on which the ionization probability and the net ionization coefficient depend.

To determine $\rho_{\text{ab}}$, the direction of the electric field is needed. This direction is given by the bisector of the two straight lines from the focal points of the prolate spheroid to the observation point (Curtright et al., 2020). Using various trigonometric relations, which are given in Supporting Information S1, it can be derived that

$$
\rho_{\text{ab}} = \sqrt{2\rho_{1}^2 \left( \frac{4V_a^2 + b^2 (\rho_{12}^2 + \rho_{12}^2 \cos(\theta)) + \rho_{12}^2 \cos(\theta) \rho_{12}^2 + \rho_{12}^2 \cos(2\theta)}{\rho_{1}^2 + \rho_{12}^2} \right)}, \tag{4}
$$

with $\rho_1$ and $\rho_2$ the straight lines from the two focal points of the ellipsoid to the observation point (see also supporting information S1) given by

$$
\rho_{1,2} = \sqrt{r^2 + \left( a \mp \sqrt{a^2 - b^2} \right)^2 \cos(\theta) + 2 \left( a \mp \sqrt{a^2 - b^2} \right) \rho_{12}} \tag{5}
$$

Using the derived expression for $\rho_{\text{ab}}$, the distance $\rho_0$ can be formulated. This is done by formulating the equation of the ellipsoid with the origin at its tip, using the coordinate system $(r, \theta, \phi)$. By solving the ellipsoid equation

$$
\sum_{k=1}^{3} \frac{x_k^2}{a_k^2} + \Theta (\vec{r}) = 1 \quad \text{rewritten in the considered coordinates) for } r \text{ and substituting } r \text{ in the expression for } \rho_{\text{ab}}, \rho_{\text{ab}} \text{ is constrained to the surface of the ellipse and thus } \rho_0 \text{ is obtained. Because the surface of an ellipsoid is quadratic, two expressions for } r \text{ are obtained and therefore two expressions for } \rho_0 \text{. These expressions are valid separately for } \theta \leq \pi/2 \text{ and } \theta > \pi/2 \text{ and are given in Appendix A.}
$$

The electric field of a conducting ellipsoid has been derived analytically by Köhn and Ebert (2015) for the prolate spheroid case and by Curtright et al. (2020) for arbitrary dimensions. The derivation of the electric field strength $E$ yields

$$
E(x, y, z) = \frac{Q}{4\pi\epsilon_0} \left( \prod_{k=1}^{3} \frac{1}{\sqrt{a_k^2 + \Theta (\vec{r})}} \right) \sqrt{\sum_{k=1}^{3} \frac{x_k^2}{a_k^2 + \Theta (\vec{r})^2}} \tag{6}
$$

where $Q$ is the total charge on the ellipsoid surface, $\epsilon_0$ is the vacuum permittivity, $\Theta (\vec{r})$ the equipotential surfaces and $a_1 = a_2 = b$, $a_3 = a_4 = b$ and $a_5 = a_6 = a$ are the semiaxes of the considered spheroid of Figure 2. Moreover, the $\Theta$-equipotentials follow from

$$
\sum_{k=1}^{3} \frac{x_k^2}{a_k^2 + \Theta (\vec{r})^2} = 1, \text{ for } \Theta (\vec{r}) > 0. \tag{7}
$$

The above electric field expression can be rewritten in the considered coordinates $(\rho', r, \theta, \phi)$ as defined in Figure 2. Here $r$ can be converted to $\rho'$ using the trigonometric relation $\rho' = \sqrt{a^2 + r^2 + 2a \cos(\theta)}$. The field is also reformulated to contain the electric field at the ellipsoid tip $(z = a \text{ in Equation } 6) E_0 = \frac{Q}{4\pi\epsilon_0 \gamma r}$. To obtain the final expression for the electric field, the $\rho'$ coordinate is converted to the $\rho$ coordinate along the electric field direction as required for the $\gamma = 1$ criterion. This is done using the derived $\rho_{\text{ab}}$. In order to have analytically solvable equations in this derivation, $r$ is not converted to $\rho'$ in the conversion from $\rho'$ to $\rho$. This means some ambiguity remains in the expression of the electric field $E = E(\rho, r, \theta)$ and $\phi$). As eventually the equation $\gamma = 1$ is solved numerically, this ambiguity is not a problem as long as the resulting $\alpha_{\text{eff}}$ (which is calculated using the electric field) behaves correctly. The used formulation is

$$
E(\rho, r, \theta) = \frac{2b^2 E_0 \rho'}{\sqrt{a^2 + b^2 + q + \rho'^2 \left( -a^2 + b^2 + q + \rho'^2 \right)} + \sqrt{b^2 + 2a \cos(\theta) r^2 (\cos(2\theta)) + q \rho'^2}} \tag{8}
$$

with the shorthand $q = \sqrt{\frac{2a^2 (a^2 - b^2) (a^2 + 2a \cos(\theta) r^2 + 2b \cos(2\theta))}{a^2 + 2a \cos(\theta) r^2 + 2b \cos(2\theta)}} + \left( a^2 - b^2 \right)^2 + \rho'^4$ and with
with \( p = 2a^2 + 2b^2 + 2b^2 r^2 \). The derivations of these expressions are presented in Supporting Information S1.

Finally, the distance \( r_{\text{max}} \) from the tip of the prolate spheroid to the position where \( E = E_k \) can be determined. Because there is no explicit solution for \( r_{\text{max}} \) in the considered geometry, this is done by approximating the surface \( E = E_k \) as forming an ellipsoid surface near the tip, as is validated in simulations in Supporting Information S1. Then, finding \( r_{\text{max}} \) specifically for \( \theta = 0 \) and \( \theta = \pi/2 \) is sufficient to obtain \( r_{\text{max}} \) for arbitrary \( \theta \). These expressions are found by reformulating the electric field in terms of \( r \) and \( \theta \) only and solving \( E(r, \theta = 0) = E_k \text{ and } E(r, \theta = \pi/2) = E_k \), the latter leading to a case known as “Causus irreducibilis” (Wantzel, 1843). This yields an analytical expression for \( r_{\text{max}} \), which can be validated using the aforementioned simulations:

\[
 r_{\text{max}}(\theta = \pi/2) = \frac{r_{\text{max}}(\theta = 0)2(2a + r_{\text{max}}(\theta = 0))\sin^2(\theta) + r_{\text{max}}(\theta = \pi/2)^2(2a + r_{\text{max}}(\theta = 0))\cos^2(\theta) - ar_{\text{max}}(\theta = \pi/2)\cos(\theta)}{r_{\text{max}}(\theta = 0)(2a + r_{\text{max}}(\theta = 0))\sin^2(\theta) + r_{\text{max}}(\theta = \pi/2)^2\cos^2(\theta)} 
\]

with \( r_{\text{max}}(\theta = 0) = \sqrt{a^2 + \frac{b^2 (E_0 - E_k)}{E_k}} - a \) and \( r_{\text{max}}(\theta = \pi/2) = \frac{1}{2\sqrt{3}} \sqrt{\frac{E_k}{G}} \) derived in Supporting Information S1, where also the expressions for \( F \) and \( G \) are presented.

Using the now known required expressions, the surface electric field \( E_0 \) at the tip of the ellipsoidal hydrometeor required for the onset of a positive corona discharge can be computed by numerically solving \( \gamma = 1 \) (Equation 1) (Liu, Dwyer, & Rassoul, 2012) at the known electric field distribution \( E(\rho, r, \theta) \) for different values of the relative gas density \( \delta \) and major and minor axes \( a \) and \( b \).

Then, the found onset field \( E_0 \) can be used to evaluate the ionization integral \( K \) given by

\[
 K = \int_{\rho_0}^{\rho_c} a_{\alpha\beta}(\rho, E) d\rho, \quad (11) 
\]

where \( \rho_c = \rho_0(r_{\text{max}}, \theta, \phi) \) gives the position of the breakdown field \( E_\alpha \). Exponentiation of \( K \) yields the number of electrons produced by an avalanche from the edge of the ionization region to the surface of the hydrometeor. Equation 11 is thus a criterion for the onset of a positive corona discharge with \( K \) a threshold value that needs to be reached to enable initiation. It is important to note that the above integration is taken along the field line from the surface of the electrode to the edge of the ionization region, because the avalanche follows the direction of the electric field. In this work the ionization integral is considered along \( \theta = 0 \), where \( \rho_0 = b^2 \phi a \) and \( \rho_c = \sqrt{\frac{b^2 (E_0 - E_k)}{E_k} + \frac{b^2 (E_0 - E_k)}{E_k}} \).

Per the convention used by Naidis (2005), the model is set up to output the onset field \( E_{\alpha\beta} \) from which the onset voltage \( V_0 \), onset charge \( Q \), and ionization integral \( K \) can be derived. This order, supported by the mentioned historical relevance of the onset field, is thus kept in the following results section.

As stated, the used model gives the minimum condition for the onset of a corona discharge. Besides assuming all photons are emitted at the surface, it neglects the presence of space charge created in the discharge. Furthermore, the onset criterion is only imposed on the secondary avalanche; further avalanches are assumed to take place when this criterion is satisfied. While these factors generally increase the threshold for corona inception, including its stochastic nature would lower this threshold. The validity of the model depends on the relevant dimensions. For the model to be reliable, the largest photon absorption length \( (r_{\text{max}}) \) should be smaller than the length of the ellipsoid. Otherwise, the equilibrium between the ionization coefficients with the local electric field cannot be guaranteed.

3. Results and Discussions

3.1. The Effects of Varying Aspect Ratio and Volume of Spheroidal Hydrometeors on the Corona Inception Criterion

To calculate the required effective ionization coefficient \( \alpha_{\alpha\beta} \) in Equation 1, we need to use the air plasma-chemical reactions which are listed in Table 1. All electron impact ionization, excitation, elastic and attachment reactions...
(except three-body attachment) that are included in the list were obtained from the Itikawa database (Itikawa database, www.lxcat.net, retrieved on 15 Sep 2020., n.d.; Itikawa, 2005, 2008). The three-body attachment with O\textsubscript{2} as the third body was obtained from the Phelps database (Phelps database, www.lxcat.net, retrieved on 15 Sep 2020., n.d.) and scaled to the different δ. Next, the reactions were used as input for BOLSIG+ (Hagelaar & Pitchford, 2005; BOLSIG+ solver ver. Windows 12/2019, n.d.) to calculate the ionization and attachment coefficients. The results are depicted in Figure 3 and the effective ionization coefficient is defined as the subtraction of the attachment coefficient from the ionization coefficient. A breakdown field of 133.8 Td, corresponding to 36.0 kV/cm at atmospheric pressure, is found. Curves extracted from the work by Liu, Dwyer, and Rassoul (2012) are also depicted for comparison. They employ a breakdown field of 32.0 kV/cm, which is smaller as three-body attachment is not included.

Using the corona inception criterion, Equation 1 is solved numerically in MATLAB for varying hydrometeor volume

\[ V = \frac{4}{3} \pi a^3 \]

, with

C = \frac{ab^2}{a^3} \]

the volume parameter, and varying aspect ratios \( b/a \). The aspect ratio \( b/a \) is considered instead of, for example, the major axis \( a \), such that the effects of varying volume and shape can be investigated separately.

The studied hydrometeor geometries have volume parameters of \( C = 1, 5, 10, 50, \) and 100 mm\textsuperscript{3} and aspect ratios from \( b/a = 0.01 \) to 1, where \( b/a = 1 \) represents a sphere (\( a = b \)). The three largest sizes were selected for future comparison with experimental work performed alongside this study (Mirpour, 2021), and are used to visualize dependencies on hydrometeor shape and size. In addition, because their sizes better represent thundercloud hydrometeors (Gardiner et al., 1985; Weinheimer et al., 1991), results for \( C = 1 \) mm\textsuperscript{3} and \( C = 5 \) mm\textsuperscript{3} are computed for a smaller selection of aspect ratios between \( b/a = 0.01 \) and 1 and are presented only in text. In the spherical limit these volume parameters give hydrometeor diameters of 2.0 and 3.4 mm respectively. First, positive corona inception is investigated at

![Figure 3. Reduced attachment (\( \eta/N \)) and ionization (\( \alpha/N \)) coefficients as a function of reduced electric field in an N\textsubscript{2}:O\textsubscript{2} = 80:20 mixture at \( \delta = 1 \). The breakdown field is determined where \( \alpha - \eta = 0 \), which in our study is 133.8 Td. The coefficients as a function of reduced electric field employed by Liu, Dwyer, and Rassoul (2012) are also depicted.](image-url)

**Table 1**

<table>
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<th>Reaction Type</th>
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<th>( e^- + N_2 \rightarrow 2e^- + N_2^+ )</th>
<th>( e^- + N_2 \rightarrow 2e^- + N^+ + N )</th>
<th>( e^- + N_2 \rightarrow 3e^- + N^2+ + N )</th>
<th>( e^- + O\rightarrow 2e^- + O_2^+ )</th>
<th>( e^- + O\rightarrow 2e^- + O^+ + O )</th>
<th>( e^- + O\rightarrow 3e^- + O^2+ + O )</th>
<th>( e^- + O_2 \rightarrow O^- + O )</th>
<th>( e^- + O_2 \rightarrow O^- + O_2^* )</th>
<th>( e^- + N_2 \rightarrow e^- + N_2^+ )</th>
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atmospheric pressure ($\delta = 1$). The onset field $E_0$ at the tip of the ellipsoid, found directly from solving Equation 1, is presented in Figure 4a. It can be seen that $E_0$ decreases with volume for a fixed aspect ratio. For the smallest hydrometeor depicted, $C = 10 \text{ mm}^3$, the onset field at $b/a = 0.045$ is 366 kV/cm, while for the largest hydrometeor, $C = 100 \text{ mm}^3$, this is 248 kV/cm. For $C = 1 \text{ mm}^3$ and $C = 5 \text{ mm}^3$ these values are 127 kV/cm and 108 kV/cm, respectively. The decrease of the onset field with increasing volume is expected, because a larger hydrometeor, simulated as an electrode, provides a larger distance across which electrons can contribute to avalanches. For the same electric field at the electrode tip, a larger electrode has a larger ionization region.

From Figure 4a it can also be concluded that for a fixed volume, a sharper ellipsoid has a larger onset field $E_0$ at its tip. Because a sharper ellipsoid has less surface near the photon absorption region, a larger $E_0$ is needed to meet the inception criterion. In the spherical limit, $b/a = 1$, the onset field for $C = 10 \text{ mm}^3$ is about 17% larger than that for $C = 100 \text{ mm}^3$, and for the much sharper tip at $b/a \approx 0.015$ this difference has increased to about 70%. The onset field thus increases much stronger with sharpness for a smaller hydrometeor, which is expected as a smaller object has more surface area compared to its volume. The onset field of the ellipsoid is well-approximated by that of a sphere of radius $\rho_0 = b^2/a$ (radius of curvature of the ellipsoid tip), following the model by Naidis (2005). An example comparison between these onset fields is given in Supporting Information S1 for the $C = 100 \text{ mm}^3$ ellipsoid.

It should be noted, however, that Figure 4a does not tell us the whole story. This onset field gives only the electric field value at the tip of the ellipsoid. Moreover, as charges on a conductor tend to move away from each other as much as possible on its surface, the electric field is enhanced more strongly near a sharper tip. Hence, even though a sharper ellipsoid has a larger $E_0$, this does not necessarily mean corona inception from sharper hydrometeors in thunderstorms is less likely. On the contrary, Petersen et al. (2015) observed sharper hydrometeors promote glow coronae. R. F. Griffiths and Latham (1974) suggested in their paper on coronae from ice hydrometeors that the onset ambient field decreases with increasing combined length of the liquid filament, which was confirmed by Crabb and Latham (1974), who also found that the elongated filament resulting from raindrop collision promotes corona onset. This seems to contradict the decrease of $E_0$ with elongation in Figure 4a, but taking into account the mentioned effect of only considering the tip this discrepancy is explained. To draw clearer conclusions, other quantities such as potential, surface charge and the ionization integral should be considered as well.

From the onset field $E_0$ the onset voltage along the major axis (from the tip to infinity) can be calculated by the integration of the electric field. This onset voltage $V_0$ is shown in Figure 4b. The inception voltage increases.
with hydrometeor volume. This is also found by Liu, Dwyer, and Rassoul (2012) for a spherical electrode. Additionally, Figure 4b shows that the onset voltage is lower for a sharper ellipsoid. In the spherical limit, the largest hydrometeor \( C = 100 \text{ mm}^3 \) requires 30 kV for corona onset, while the smallest hydrometeor \( C = 1 \text{ mm}^3 \), not depicted) requires 9 kV. For a very sharp tip the volume dependence is less noticeable, and the onset voltage is about 4 kV for all considered hydrometeor sizes.

Besides the onset voltage \( V_0 \), the onset charge \( Q \) can also be derived from the onset field \( E_0 \) through \( \frac{Q}{A A A A 0} = \frac{4 \pi \pi \pi \pi}{2 \epsilon_{0}} \). The onset charge, which is the total charge on the electrode surface, is depicted in Figure 5. A size-dependent optimum aspect ratio \( b/a \) is observed at which the onset charge is lowest. While a sharper ellipsoid has a higher onset field and thus requires more charge at the tip to reach this \( E_0 \), a larger fraction of the total charge is collected at its tip because of the optimization of charge separation. In simulations of corona inception from hydrometeors modeled as dielectrics in an external electric field, Dubinova et al. (2015) also found a size-dependent aspect ratio for which the onset background field is minimum. From Figure 5 the range of onset charge for hydrometeors with volumes between 42 and 420 \( \text{mm}^3 \) is found to be 2,367 pC to 15,467 pC at atmospheric pressure. For the smallest hydrometeor volume considered, 4.2 \( \text{mm}^3 \), the onset charge ranges from a minimum of 746 pC at \( b/a = 0.081 \) to a maximum of 1,543 pC. Moreover, for a volume of 21 \( \text{mm}^3 \), corresponding to \( C = 5 \text{ mm}^3 \), the values range from 1,673 pC to 2,792 pC.

Finally, the ionization integral \( K \) along the major axis can be calculated from the onset field through Equation 11. The result is presented in Figure 6, and the computed values for \( C = 1 \text{ mm}^3 \) and \( C = 5 \text{ mm}^3 \) agree with the observed trend. Here, the approximation using a sphere of radius \( b^2/a \) does not yield comparable results. At a fixed volume, the ionization integral decreases with \( b/a \), meaning that less electrons are required in an avalanche from the edge of the photon absorption region to the electrode surface. To interpret these results the dependence of the photon absorption area and length on the electrode dimensions are studied in COMSOL for some data points, of which the results are given in Table 2. From this data it can be concluded that for a fixed aspect ratio, a smaller electrode has a smaller photon absorption area and length, as does a sharper electrode for a fixed volume. However, for a

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**Figure 5.** The onset charge for positive coronae at the tip of the ellipsoidal hydrometeor for \( C = 10, 50, \) and 100 \( \text{mm}^3 \) for varying aspect ratio \( b/a \) at atmospheric pressure.

**Figure 6.** The ionization integral along the major axis (\( \theta = 0 \)) for positive corona onset at the tip of the ellipsoidal hydrometeor for \( C = 10, 50, \) and 100 \( \text{mm}^3 \) for varying aspect ratio \( b/a \) at atmospheric pressure.
very sharp electrode the photon absorption area and length are approximately equal, as can be seen for $bla = 0.014$ in Table 2.

Because an ellipsoid with smaller $bla$ has a smaller photon absorption region, photons are absorbed closer to the electrode compared to its size, such that stronger avalanches are required to satisfy the inception criterion. A similar argument was made by Naidis (2005) to explain the ionization integral dependence on radius for a spherical and cylindrical electrode. Comparing the data points for different volumes, two regions can be discerned in Figure 6, separated by a cross-over point around $bla = 0.55$. At large $bla$, where $K$ drops below 14, the largest ellipsoid has the largest value for the ionization integral, again because photons are absorbed closer to the electrode with respect to its size. When $K$ increases above 14 for decreasing $bla$ it is observed that the smallest ellipsoid has the largest $K$ value. An explanation for this could be that when $bla$ becomes small enough, the photon absorption region becomes so small that its absolute size instead of its relative size determines the value of the ionization integral. Stronger avalanches are then required for a smaller electrode. For very small $bla$ the data points for different volumes appear to converge again. A likely explanation is that when the ellipsoid becomes very sharp, a photon is absorbed sufficiently close to the tip, such that the total volume of the electrode has no effect; only the sharpness of the tip determines the value of the ionization integral. This is supported by the photon absorption area being approximately equal for the different volumes at $bla = 0.014$ in Table 2.

### 3.2. Variation of the Corona Inception Criterion With Pressure

Next, the dependence of corona onset from an ellipsoidal hydrometeor on the ambient pressure is investigated by varying the relative gas density $\delta$. More specifically, the values $\delta = 10$, 1, and 0.1, analogous to the works by Naidis (2005) and Liu, Dwyer, and Rassoul (2012), and $\delta = 0.5$, representative for thundercloud altitudes, are considered. The volume parameter is fixed at $C = 10 \text{ mm}^3$. The aspect ratio $bla$ is varied again from $bla = 0.01$ to 1. The results for the onset field at the hydrometeor tip are shown in Figure 7a. For $\delta = 0.5$ the onset field decreases with aspect ratio from $1,247 \text{ kV/cm}$ to $44 \text{ kV/cm}$, whereas for $\delta = 0.1$ the decrease is from $977 \text{ kV/cm}$ to $12 \text{ kV/cm}$. As expected, a higher pressure leads to a higher onset field $E_{\text{p}}$. As explained by Liu, Dwyer, and Rassoul (2012), at a higher pressure more of the excited nitrogen molecules responsible for emitting the ionizing photons are quenched, leading to a lower photon production, such that a higher field is required. To briefly examine how the results are affected by the aforementioned photoionization outside of the ionization region, the computations are redone at thundercloud pressure with an integration upper limit of $10r_{\text{max}}$ instead of $r_{\text{max}}$. It follows that the difference in outcome is generally well below 1%, only rising above 5% for $\delta = 0.5$ for the smallest depicted volume parameter $C = 10 \text{ mm}^3$, and only for very blunt tips, nearing $bla \approx 1$. Neglecting this stochastic effect thus seems justified.

Similarly, the onset charge increases with pressure, as depicted in Figure 7b. For $C = 10 \text{ mm}^3$ the onset charge is between 547 and 2,400 pC for $\delta = 0.1$ and between 1,580 pC and 3,065 pC for $\delta = 0.5$. For $\delta = 0.5$ the onset charge is minimal at an aspect ratio of 0.12. A pressure above atmospheric pressure, at $\delta = 10$, is not representative for thunderstorms, but is included for completeness. Not depicted, but as stated more typical for thunderclouds, are the onset charge values for $C = 1 \text{ mm}^3$ and $C = 5 \text{ mm}^3$. Onset charges range from 495 to 1,371 pC for hydrometeors of $C = 1 \text{ mm}^3$, and from 1,116 pC to 2,430 pC for $C = 5 \text{ mm}^3$. The minimum onset charge values of 495 and 1,116 pC are achieved at aspect ratios of 0.22 and 0.14, respectively.

Again, the ionization integral $K$ can be calculated from the onset field and is plotted in Figure 7c. The pressure dependence can be explained as before; due to increased quenching of excited nitrogen molecules at higher pressures the photon production is lowered. Therefore, stronger avalanches are required to satisfy the inception criterion.
3.3. Dependence of the Derived Ambient Electric Field on the Aspect Ratio for Thundercloud Pressure

An estimation of the ambient field $E_{bg}$ required for corona onset can be derived from the onset field $E_0$ at the hydrometeor tip. This is done by simulating the hydrometeor as a conductor without surface charge in an ambient electric field in COMSOL, and increasing this field until the determined $E_0$ is obtained at the tip. The relative gas density of $\delta = 0.5$ and volume parameter of $C = 10 \text{ mm}^3$ are chosen. The results are presented in Figure 8. It is seen that the required background field $E_{bg}$ is below the breakdown field $E_k$, between 0.07 $E_k$ and 0.8 $E_k$, and is lowest for the sharpest hydrometeor tips.

While these two configurations are not directly comparable, as the ambient field would influence the field distribution and therefore the outcome of the numerical simulations, the result provides insight on corona initiation. Were the ambient field present in the first place, less charge would be required on the hydrometeor to satisfy the self-sustaining condition. Thus, the calculated onset field without inclusion of an ambient field is higher. Despite this, values of the ambient field below the breakdown field are found.
4. Conclusions and Outlook

The corona inception criterion set up by Naidis (2005) is applied through numerical simulations to spheroidal electrodes of various dimensions at different pressures. By doing so, the theoretical onset of a positive corona from an ellipsoidal hydrometeor is studied. It is found that the onset electric field at the hydrometeor tip decreases with hydrometeor volume and tip bluntness and increases with pressure. For a hydrometeor of 42 mm³ volume \( C = ab^2 = 10 \text{ mm}^3 \) and thundercloud pressure \( (\delta = 0.5) \), the onset field at the tip varies approximately from \( 2.4E_k \) (limiting case sphere) to \( 70E_k \) (sharp est case considered), where \( E_k \) is the breakdown field. To accentuate, the high field of \( 70E_k \) falls off rapidly when moving away from the very sharp tip. Because of this rapid decay, the very high field is required to ensure sufficient area for photon absorption. However, the value remains unrealistically high and is included only for completeness. The aspect ratio of 0.01 at which this value is found is also the extreme case where the hydrometeor takes on a needle-like shape. Furthermore, the onset field at the tip is not deemed representative for the likeliness of corona onset as it does not provide information on the entire surface. These values were also obtained without the inclusion of an ambient electric field. The onset voltage is investigated as well. As we can observe, sharper hydrometeors need a lower voltage to initiate a discharge.

Another way to better predict the feasibility of corona onset in thundercloud electric fields is by the derivation of the required ambient electric field \( E_{bg} \) to have an enhanced electric field of \( E_k \) at the tip of the hydrometeor \( (C = 1,000 \text{ mm}^3) \). The values are calculated at \( \delta = 0.5 \), where \( E_k = 18.0 \text{ kV/cm} \), with the breakdown field of 36.0 kV/cm at ground pressure.

Whereas the onset field only provides information on the hydrometeor tip and the onset voltage only on the major axis, the onset charge is the total charge on the hydrometeor surface. This onset charge reveals, depending on hydrometeor volume, an optimal semi-axes aspect ratio of the ellipsoidal hydrometeor for which the least amount of charge is required for positive corona onset. As this optimum was not found for the onset field or onset voltage, this suggests that considering only the major axis may not be sufficient when investigating corona onset conditions. Interestingly, in their study on lightning inception from hydrometeors, simulated as dielectrics in an ambient electric field, Dubinova et al. (2015) obtain a length-dependent optimum aspect ratio of the hydrometeor that requires the lowest ambient field for discharge inception. In addition, the obtained results can be compared to measured precipitation charges. Generally the hydrometeor charge is measured below 400 pC (Marshall & Winn, 1982). These charges were measured for estimated hydrometeor diameters between 1 and 3 mm, close to the 2.0 and 3.4 mm diameters of the simulated \( C = 1 \text{ mm}^3 \) and \( C = 5 \text{ mm}^3 \) hydrometeors in the spherical limit. For these simulated volumes and a relative gas density of \( \delta = 0.5 \) the onset charge is found to be between 495 and 2,430 pC. The minimum onset charge of 495 pC for \( C = 1 \text{ mm}^3 \), which is higher than measured in situ, is found at an aspect ratio of 0.22. In their simulations on spherical hydrometeors using the same corona inception criterion as this paper, Liu, Dwyer, and Rassoul (2012) have shown that the onset charge varies over several orders of magnitude in the estimated size range of hydrometers. For spherical hydrometeors of 2 mm diameter, the simulated onset charge was about three times larger than for a 1 mm diameter. Checking a few aspect ratios for \( C = 0.125 \text{ mm}^3 \) (corresponding to a 1 mm diameter for a sphere) indeed yields charges below 170 pC for a large aspect ratio range. For the simulated \( C = 1 \text{ mm}^3 \) hydrometeors, the difference with the highest measured precipitation charges is less than 100 pC between aspect ratios of 0.16 and 0.32. Thus, for a selection of shapes of representative hydrometeor volumes the simulated and measured charges are comparable. The considered configuration is an isolated hydrometeor with zero ambient field. Interaction between hydrometeors (Jánský & Pasko, 2020 and references therein) and a non-zero ambient field could lower the amount of charge required for corona inception, which explains why the found onset charge is somewhat higher than expected from in situ measurements.

![Figure 8](image_url)
Besides the onset charge, the ionization integral \( K \) also displays different behavior in different \( b/a \) regions. For hydrometeors with very blunt tips, close to a spherical shape, a larger hydrometeor has a larger \( K \) value for onset, as photons are absorbed closer to the hydrometeor with respect to its size. However, for hydrometeors with sufficiently sharp tips, the absolute size of the photon absorption region seems to be more important than its relative size, such that a smaller hydrometeor has a larger value of the ionization integral. For any ellipsoidal shape, the value of the ionization integral is larger at higher pressures.

To investigate the validity of the results, the approximations and assumptions of the model should be evaluated. First, the distance \( r_{\text{max}} \) from the tip to the edge of the photon absorption region should be smaller than the hydrometeor length. Using the expression derived in Supporting Information S1, it is found that for all data points the maximum ratio of this distance to length is \( r_{\text{max}}/L = 0.2 \), meaning this condition for the model to hold is satisfied. Furthermore, the presence of space charges is ignored in the model, leading to an overestimation of the electric field magnitude. When the ionization integral \( K \), or equivalently number of electrons in the avalanche, is large enough, the perturbation of the electric field by the space charge becomes comparable to the magnitude of the electric field itself, such that space charge cannot be neglected. This is accompanied by the transformation of the avalanche into a streamer. In literature, it is often taken that \( K \) should be below 14–22 (Naidis, 2005; Raizer, 1991) for the perturbation of the electric field by space charge to be neglected. In the results, the value of \( K \) is below this threshold for sufficiently blunt hydrometeors. Near \( b/a = 0.55 \) in Figure 6, which is also the cross-over point of the three curves, this value rises above 14. Hence, for sharper ellipsoids possibly more physics should be added to the model to obtain more accurate results.

In the model of the current work, it is assumed that there are sufficient free electrons present for the primary electron avalanche. To be able to draw conclusions on whether corona onset is possible in thunderclouds, it should be considered how these free electrons are supplied, and if this supply is large enough. The source of free electrons for lightning initiation is a widely researched subject, see for example, (Dubinova, 2016; Rutjes et al., 2019).

From the above considerations, it can be concluded that lightning initiation from a spheroidal hydrometeor is feasible. While the onset field at the tip of the charged hydrometeor without ambient field was not found to be below the breakdown field in the considered configuration, the derived onset ambient electric field for the uncharged hydrometeor is lower than this threshold. Further enhancement could be provided by the interaction between hydrometeors. For representative dimensions and pressures, the amount of charge required for corona onset provided by the model is comparable to measured hydrometeor charges. From our results, it appears that only considering the major axis is not sufficient to reach conclusions on this matter. To further investigate the corona onset from hydrometeors using this model, more physics could be included. Most importantly, the thundercloud ambient electric field could be added to the model. Furthermore, the method can be applied to a hydrometeor cluster. The role of humidity, which was studied by Liu, Dwyer, and Rassoul (2012) for spherical hydrometeors, and the low-temperature environment can also be investigated. Finally, the model could be adjusted to account for space charge effects.

**Appendix A: Derivation of the Distance \( \rho_0 \)**

To find the distance \( \rho_0 \) from the major axis to the surface of the ellipsoid along the surface electric field direction, the equation defining the ellipsoid (with the origin at the tip of the ellipsoid)

\[
\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{(z + a)^2}{a^2} = 1
\]

is reformulated in spherical coordinates, which yields

\[
\frac{r^2 \sin^2(\theta)}{b^2} + \frac{(a + r \cos(\theta))^2}{a^2} = 1.
\]

Solving Equation A2 for \( r \) gives two solutions, valid separately for \( \theta \leq \pi/2 \) and \( \theta > \pi/2 \), namely
as the range $\theta \leq \pi/2$ is the ionization region, which only encompasses the tip, $r = 0$, of the ellipsoidal surface (see also Figure 2). Substituting these solutions into the expression for $\rho_{ab}$ (Equation 4 and Supporting Information S1), thus constraining $\rho_{ab}$ to the surface of the ellipsoid, gives

$$\rho_0 = \frac{b^2}{a} \left\{ \begin{array}{ll}
\frac{4 \left( 2a \left( \sqrt{a^2 - b^2} - a \right) + b^2 + p_1 \right)}{\left( \sqrt{a^2 - b^2} - a \right) + b^2 + p_1} & \theta \leq \pi/2, \\
\frac{2a \left( \sqrt{a^2 - b^2} - a \right) + b^2 + p_1}{8} & \theta > \pi/2,
\end{array} \right.$$  

with

$$p_1 = 8 \rho_1 p_2 \left( r = \frac{2ab^2 \cos(\theta)}{(a^2 - b^2) \cos^2(\theta) - a^2} \right)$$

and

$$p_2 = -\rho_1 \left( r = \frac{2ab^2 \cos(\theta)}{(a^2 - b^2) \cos^2(\theta) - a^2} \right) - 2a \left( \sqrt{a^2 - b^2} - a \right) - b^2$$

**Data Availability Statement**

The data generated for this study, in order to model corona onset through hydrometeors, is made available at the 4TU. ResearchData data repository with open access via https://doi.org/10.4121/16820980.v1. This data is also displayed in the figures presented in this study. The MATLAB scripts used to generate the data for this paper and the full derivations of the indicated expressions are included in Supporting Information S1.

**References**


Xu, M., Tan, Z., & Li, K. (2012). Modified peck formula for calculating positive dc corona inception electric field under variable humidity. IEEE Transactions on Dielectrics and Electrical Insulation, 19(4), 1377–1382. https://doi.org/10.1109/TDEI.2012.6260014
