Generalized Fischer-Fock spaces

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GENERALIZED
FISCHER-FOCK SPACES
by
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Summary
This paper is on functional Hilbert spaces of entire analytic functions which extend the class of Fischer-Fock spaces. They are related with Bargmann's description of Schwarz' test space of rapidly decreasing $C^\infty$-functions and its dual the space of tempered distributions.
Preliminaries

Let \( \mathcal{P} \) denote the collection of all entire analytic functions \( f \) for which all derivatives \( f^{(n)}(0) \), \( n = 0, 1, 2, \ldots \), in \( z = 0 \), are strictly positive. For each \( f \in \mathcal{P} \) the function \( K_f \) on \( \mathbb{C} \times \mathbb{C} \) defined by

\[
K_f(z,w) = f(zw) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (zw)^n, \quad z,w \in \mathbb{C}
\]

is of positive type. To \( K_f \) there is associated precisely one functional Hilbert space \( H[f] \), cf. [Ar]. The Hilbert space \( H[f] \) consists of all entire analytic functions \( \phi \) with the property that

\[
\| \phi \|_f^2 := \sum_{n=0}^{\infty} \frac{|\phi^{(n)}(0)|^2}{n! f^{(n)}(0)} < \infty.
\]

The functions \( \phi \in H[f] \) satisfy the estimation

\[
| \phi(z) |^2 \leq f(|z|^2) \| \phi \|_f^2, \quad z \in \mathbb{C}.
\]

The normalized monomials \( \left( \frac{f^{(n)}(0)}{n!} \right)^{\frac{1}{n}} z^n \) establish an orthonormal basis in \( H[f] \).

In \( \mathcal{P} \) we introduce an order relation by

\[
f_1 \leq f_2 : \iff \exists \lambda > 0 : \lambda f_2 - f_1 \in \mathcal{P}.
\]

As one can readily check, \( f_1 \leq f_2 \) implies that \( H[f_1] \) can be continuously injected into \( H[f_2] \). Further, the class \( \mathcal{P} \) is closed with respect to addition, \( f_1 + f_2 \), and joint multiplication, \( f_1 f_2 \). In this connection we mention the following interesting result of Burbea, cf. [Bu]:

Let \( \phi_j \in H[f_j] \), \( j = 1, 2 \). Then \( \phi_1 \phi_2 \in H[f_1 f_2] \) and

\[
\| \phi_1 \phi_2 \|_{f_1 f_2} \leq \| \phi_1 \|_{f_1} \| \phi_2 \|_{f_2}.
\]

In this paper we concentrate on the confluent hypergeometric functions \( f_{a,b,c} \in \mathcal{P} \), \( a,b,c > 0 \), defined by

\[
f_{a,b,c}(z) = \sum_{n=0}^{\infty} \frac{(a)_n (cz)^n}{(b)_n n!}, \quad z \in \mathbb{C}.
\]

(We use Pochhammer's symbol \( (r)_n = \frac{\Gamma(r+n)}{\Gamma(r)} \), \( r \in \mathbb{R} \).)

The functions \( f_{a,b,c} \) satisfy the order relation

\[
f_{a,b,c} \leq f_{\tilde{a},\tilde{b},\tilde{c}}
\]

in case
* $c < \bar{c}$ and $a, b$, $\bar{a}$ and $\bar{b}$ arbitrary,
* $c = \bar{c}$ and $a - a \geq \bar{b} - b$.

The space $H[f_{1,1,1}]$ is the classical Fischer-Fock space or Bargmann space, cf. [NeSh] and [Ba1]. The functional Hilbert space $H[f_{1,b,c}]$ are introduced in [Bu], where they are called generalized $(b,c)$-Fischer spaces. So $H[f_{a,b,c}]$ may be called the generalized $(a,b,c)$-Fischer space.

1. Generalized $(a,b,c)$-Fischer spaces

For $\kappa, \mu \in \mathbb{R}$, let $W_{\kappa,\mu}$ denote the Whittaker function of the second kind which for $|\mu - \kappa| > \frac{1}{2}$ satisfies

$$W_{\kappa,\mu}(t) = \frac{1}{\Gamma\left(\frac{1}{2} + \mu - \kappa\right)} t^{\mu + \frac{3}{2}} \exp(-\frac{1}{2} t) \int_0^\infty e^{-s} s^{\mu - \kappa - \frac{1}{2}} (1 + s)^{\mu + \frac{1}{2}} ds,$$

cf. [MOS], p.313. So for each $\kappa, \mu \in \mathbb{R}$ with $|\mu - \kappa| > \frac{1}{2}$ the function $W_{\kappa,\mu}$ is positive on $(0,\infty)$.

Consider the following integral relations, cf. [MOS], p. 316,

$$(1.1) \quad \int_0^\infty t^{n+v-1} \exp(-\frac{1}{2} ct) W_{\kappa,\mu}(ct) \, dt = \frac{\Gamma\left(\frac{1}{2} + \mu + v + n\right) \Gamma\left(\frac{1}{2} - \mu + v + n\right)}{\Gamma(1 - \kappa + na + n)} \left[ \frac{1}{c} \right]^{n+v}.$$

We set

$$(1.2) \quad G_{a,b,c}(t) = \frac{c^v \Gamma(b)}{\pi \Gamma(a)} t^{v-1} \exp(-\frac{1}{2} ct) W_{\kappa,\mu}(ct)$$

with

$$\kappa = \frac{b-2a+2}{2}, \quad \mu = \frac{b-1}{2}, \quad v = \frac{b}{2}.$$

Then from (1.1) we deduce

$$(1.3) \quad \int_0^\infty t^n G_{a,b,c}(t) \, dt = \frac{1}{\pi} \frac{(b)_n}{(a)_n} \frac{n!}{c^n}, \quad n \in \mathbb{N}_0.$$

Next we introduce the space $F_{a,b,c}$ of all entire analytic functions $\phi$ for which the integral

$$\int_{\mathbb{R}^2} |\phi(x+iy)|^2 G_{a,b,c}(x^2+y^2) \, dx \, dy$$

is finite. With the natural inner product $(\cdot, \cdot)_{a,b,c},$
\[(\phi, \psi)_{a,b,c} = \int_{\mathbb{R}^2} \phi(x + iy) \overline{\psi(x + iy)} G_{a,b,c}(x^2 + y^2) \, dx \, dy,\]

\(F_{a,b,c}\) is a Hilbert space.

\(1.4\) Theorem.
The Hilbert space \(F_{a,b,c}\) equals the functional Hilbert space \(H[fa,b,c]\), i.e.

\[
\forall \omega \in \mathcal{C}: \phi(\omega) = \int_{\mathbb{R}^2} \phi(z) \frac{1}{F_1(a,b,c \overline{\omega} z)} G_{a,b,c} (|z|^2) \, dx \, dy, \quad z = x + iy.
\]

\[
(\phi, \psi)_{a,b,c} = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{n!} \frac{\psi^{(n)}(0)}{(b)_n} \frac{1}{(a)_n c^n}.
\]

Proof.
From relation (1.3) we deduce that the normalized monomials \(u_n(a,b,c; z) = \left\{ \frac{\phi_{(a)} n^a}{(b)_n n!} \right\}^{1/2} z^n, z \in \mathcal{C}\), establish an orthonormal set in \(F_{a,b,c}\). Already we know that the \(u_n(a,b,c)\) establish an orthonormal basis in \(H[fa,b,c]\).

Now for \(\phi \in F_{a,b,c}\) the series

\[
\sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{n!} z^n
\]

converges to \(\phi\) uniformly on each disc \(D_r = \{ z \in \mathcal{C} \mid |z| \leq r \} \). So we have

\[
\begin{align*}
\int_{D_r} \left[ \sum_{n,m=0}^{\infty} \frac{\phi^{(n)}(0)}{n!} \frac{\phi^{(m)}(0)}{m!} (x + iy)^n (x - iy)^m \right] G_{a,b,c}(x^2 + y^2) \, dx \, dy = \\
= \sum_{n,m=0}^{\infty} \frac{\phi^{(n)}(0)}{n!} \frac{\phi^{(m)}(0)}{m!} \int_{D_r} (x + iy)^n (x - iy)^m G_{a,b,c}(x^2 + y^2) \, dx \, dy = \\
= \sum_{n=0}^{\infty} \left( \frac{1}{n!} \phi^{(n)}(0) \right)^2 \left[ \pi \int_0^r G_{a,b,c}(t) \, dt \right].
\end{align*}
\]

Letting \(r \to \infty\) we obtain the identity

\[
\|\phi\|^2_{a,b,c} = \sum_{n=0}^{\infty} \frac{1}{n!} \left| \frac{\phi^{(n)}(0)}{(a)_n c^n} \right|^2 \frac{(b)_n}{(a)_n c^n}.
\]

Thus the result follows.
(1.5) Special cases.

* \( a = b = 1, \ c > 0 \)

\[
G_{1,1,c} (x^2 + y^2) = \exp \left[ -c(x^2 + y^2) \right].
\]

The space \( F_{1,1,c} \) equals the Bargmann-Fock space with reproducing kernel

\[
f_{1,1,c} (z \overline{w}) = \exp[-c \, z \, \overline{w}]
\]

* \( a = 1, \ b, c > 0 \)

\[
G_{1,b,c} (x^2 + y^2) = c^{b/2} \Gamma(b) \, t^{b/2-1} \exp[-\frac{1}{2} \, c \, W_{\frac{b}{2}, \frac{b-1}{2}} (ct)]
\]

\[
= c^{b/2} \Gamma(b) \, (x^2 + y^2)^{b-1} \exp[-c(x^2 + y^2)].
\]

The space \( F_{1,b,c} \) equals the generalized \((b,c)\)-Fischer space with reproducing kernel

\[
f_{1,b,c}(z \overline{w}) = {}_1 F_1 (1,b;c \, z \, \overline{w})
\]

* \( b = 1, \ a, c > 0 \)

\[
G_{a,1,c} (x^2 + y^2) = \frac{c^{\frac{a}{2}}}{\Gamma(a)} \, (x^2 + y^2)^{-\frac{a}{2}} \exp[-\frac{1}{2} \, c \, (x^2 + y^2)] \, W_{3-2a,0}(c(x^2 + y^2))
\]

\[
= \frac{c^{\frac{a}{2}}}{\Gamma(a)} \, \exp[-c(x^2 + y^2)] \, L_{1-a} (c(x^2 + y^2))
\]

where

\[
L_{1-a}(t) = \sum_{n=0}^{\infty} \frac{(a-1)_n}{n!} \frac{t^n}{n!}, \quad t \in \mathbb{R}.
\]

(1.6) Corollary.

Let \( a, b, c > 0 \). The functions \( \phi \in F_{a,b,c} \) satisfy the following growth estimate

\[
|\phi(z)| = O(|z|^{a-b} \exp(\frac{1}{2} \, c \, |z|^2)), \quad |z| > 1.
\]

Proof.

By (0.3) for each \( z \in \mathbb{C} \) we have

\[
|\phi(z)| \leq ||\phi||_{a,b,c} \left( {}_1 F_1 (a,b;c \, |z|^2) \right)^{1/2}.
\]

So the result follows from the asymptotics of the confluent hypergeometric functions for large values of the argument.

\[
\]

(1.7) Corollary.

Let \( a, b, c > 0 \). Then \( \phi \in F_{a,b,c} \) iff \( \phi \) is entire analytic and
\[ \sum_{n=0}^{\infty} \frac{|\phi^{(n)}(0)|^2}{n!} \frac{(n+1)^{b-a}}{c^n} < \infty. \]

**Proof.**

Since the limit \( \lim_{n \to \infty} (n+1)^{b-a} \frac{(b)_n}{(a)_n} \) exists, the assertion is a consequence of the previous theorem.

\[ \]  

(1.8) **Corollary.**

Let \( c > 0 \) and let \( a, b, \bar{a}, \bar{b} > 0 \) with \( a - b = \bar{a} - \bar{b} \). Then \( F_{a,b,c} = F_{\bar{a},\bar{b},c} \) as function spaces with equivalent inner products.

On \( F_{a,b,c} \times F_{\bar{b},\bar{a},c-1} \) we introduce the sesquilinear form

\[ \langle \phi, \psi \rangle_{a,b,c} = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0) \overline{\psi^{(n)}(0)}}{n!} \]

which is well-defined because

\[ \sum_{n=0}^{\infty} \left| \frac{\phi^{(n)}(0) \overline{\psi^{(n)}(0)}}{n!} \right| \leq \| \phi \|_{a,b,c} \| \psi \|_{b,a,c}. \]

Since for each \( r > 0 \)

\[ \frac{1}{\pi} \int_{|x+iy| \leq r} \psi(x+iy) \phi(x+iy) \exp[-(x^2+y^2)] \, dx \, dy = \]

\[ = \sum_{n=0}^{\infty} \frac{\psi^{(n)}(0) \overline{\phi^{(n)}(0)}}{n!} \cdot \frac{1}{n!} \int_{0}^{r} t^n e^{-t} \, dt \]

it follows letting \( r \to \infty \) that

\[ \langle \psi, \phi \rangle = \frac{1}{\pi} \int_{\mathbb{R}^2} \psi(x+iy) \overline{\phi(x+iy)} \exp(-x^2-y^2) \, dx \, dy. \]

\[ \]  

2. **Projective and inductive limits**

Let \( A \) denote the vector space of all entire analytic functions endowed with the Frechet topology generated by the norms

\[ \| \phi \|_k = \sup_{n \in \mathbb{N}} \left| \phi^{(n)}(0) \right| \frac{k^n}{n!}. \]

Dual to \( A \) is the vector space \( E \) of all entire analytic functions of exponential type. So \( \psi \in E \) if there are \( K > 0 \) and \( c > 0 \) such that
The space $E$ is a countable inductive limit of Banach spaces. To be more specific,

$$E = \bigcup_{k=1}^{\infty} E_k$$

where $E_k$ is the subspace of $E$ consisting of all $\psi \in E$ with the property that

$$\sup_{n \in \mathbb{N}} |\psi^{(n)}(0)| k^{-n} < \infty.$$  

The spaces $E$ and $A$ are each other's strong duals where the duality is established by the sesquilinear form

$$<\psi, \phi> = \sum_{n=0}^{\infty} \frac{\psi^{(n)}(0) \overline{\phi^{(n)}(0)}}{n!}, \quad \psi \in E, \phi \in A.$$  

As in (1.10) we have

$$<\psi, \phi> = \frac{1}{\pi} \int_{\mathbb{R}^2} \psi(x+iy) \overline{\phi(x+iy)} \exp(-x^2-y^2) \, dx \, dy.$$  

The spaces $E$, $F_{1,1,1}$ and $A$ constitute a Gelfand triple,

$$(2.1) \quad E \hookrightarrow F_{1,1,1} \hookrightarrow A$$

where $<\psi, \phi> = <\psi, \phi>_{1,1,1}$ for all $\psi \in E$ and $\phi \in F_{1,1,1}$, cf. [AnVa]. The space $E$ is about the smallest space that contains the coherent states $e_w$, $e_w(z) = \exp(w \cdot z)$. So for all $\phi \in A$

$$\phi(w) = <e_w, \phi>, \quad w \in \mathbb{C}.$$  

The following lemma indicates that the $F_{a,b,c}$ give rise to continuous scales of Hilbert spaces.

$$(2.2) \text{Lemma.}$$

The continuous and dense inclusion $F_{a,b,c} \hookrightarrow F_{\tilde{a},\tilde{b},\tilde{c}}$ holds true in the following cases

* $c < \tilde{c}$ and $a, b, \tilde{a}$ and $\tilde{b}$ arbitrary,
* $c = \tilde{c}$ and $\tilde{a} - a \geq \tilde{b} - b$.

$\text{Proof.}$

 Cf. assertion (0.6) of the preliminaries.  

Clearly all functional Hilbert spaces $F_{a,b,c}$ are contained in $A$ and contain $E$ as a dense subspace. The triple (2.1) extends in the following obvious way

$$E \hookrightarrow F_{a,b,c} \hookrightarrow F_{\tilde{a},\tilde{b},\tilde{c}} \hookrightarrow F_{1,1,1} \hookrightarrow F_{\tilde{b},\tilde{a},\tilde{c}} \hookrightarrow F_{b,a,\frac{1}{\tilde{c}}} \hookrightarrow A.$$  

Here each space on the left hand side is in duality with a space on the right hand side, where the
duality is established by the form

\[
\frac{1}{\pi} \int_{\mathbb{R}^2} \psi(x + iy) \overline{\phi(x + iy)} \exp(-x^2 - y^2) \, dx \, dy.
\]

The monomials \( u_n(z) = \frac{z^n}{\sqrt{n!}} \) form an orthonormal basis in \( F_{1,1,1} \), consisting of eigenfunctions of the differential operator \( R = z \frac{d}{dz} + 1 \) with eigenvalues \( n + 1, n \in \mathbb{N}_0 \). In the next lemma we describe the relation between the spaces \( F_{a,b,c} \) and the self-adjoint operator \( R \).

(2.3) Lemma.

* Let \( 0 < c < 1 \) and let \( a, b > 0 \)

\[
F_{a,b,c} = R^{1/2} (a-b) \exp[\frac{1}{2} (\log c) R] (F_{1,1,1}).
\]

* Let \( b \geq a > 0 \)

\[
F_{a,b,1} = R^{1/2} (a-b) (F_{1,1,1}).
\]

* Let \( a > b > 0 \)

\( F_{a,b,1} \) is the completion of \( F_{1,1,1} \) with respect to the norm \( \phi \mapsto \| R^{1/2} (b-a) \exp[\frac{1}{2} (\log c) R] \phi \|_{1,1,1} \), \( \phi \in F_{a,b,1} \).

* Let \( c > 1 \) and let \( a, b > 0 \)

\( F_{a,b,c} \) is the completion of \( F_{1,1,1} \) with respect to the norm \( \phi \mapsto \| R^{1/2} (b-a) \exp[-\frac{1}{2} (\log c) R] \phi \|_{1,1,1} \), \( \phi \in F_{a,b,c} \).

Proof.

For each \( \phi \in F_{1,1,1} \) we have

\[
\phi = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{\sqrt{n!}} u_n.
\]

So the assertions are consequences of Corollary (1.7) and the spectral theorem for self-adjoint operators.

We consider the following chains of Hilbert spaces

\[
\{ F_{a,1,1} \mid a > 0 \}, \quad \{ F_{1,b,1} \mid b > 0 \}, \quad \{ F_{1,1,c} \mid c > 0 \}.
\]

They yield the following inductive / projective limits, which fit in the general set up of the paper [EGK].
The projective limit $\bigcap_{b > 0} F_{1,b,1}$ which is in strong duality with the inductive limit $\bigcup_{a > 0} F_{a,1,1}$.

The inductive limit $\bigcup_{0 < c < 1} F_{1,1,c}$ which is in strong duality with the projective limit $\bigcup_{c > 1} F_{1,1,c}$.

(2.4) Lemma.

The projective limit $\bigcap_{b > 0} F_{1,b,1}$ equals the space of all $C^\infty$-vectors of the operator $R$, i.e.

$$D^\infty(R) := \bigcap_{n=1}^{\infty} D(R^n) = \bigcap_{b > 0} F_{1,b,1}.$$ 

The inductive limit $\bigcup_{0 < c < 1} F_{1,1,c}$ equals the space of all analytic vectors of the operator $R$, i.e.

$$D^\infty(R) := \bigcup_{t > 0} e^{-tr}(F_{1,1,1}) = \bigcup_{0 < c < 1} F_{1,1,c}.$$ 

The operator $R$ is unitarily equivalent to the positive self-adjoint operator $H$ in $L_2(\mathbb{R})$ defined by $H = \frac{1}{2} ( -\frac{d^2}{dx^2} + x^2 + 1 )$. Indeed, let $\psi_n$, $n = 0, 1, 2, \cdots$, denote the $n$-th Hermite function defined by the formula

$$\psi_n(x) = \frac{(-1)^n}{(\sqrt{\pi} n! 2^n)^{1/2}} e^{\frac{1}{2}x^2} \left( \frac{d}{dx} \right)^n (e^{-x^2}).$$

The functions $\psi_n$ establish an orthonormal basis in $L_2(\mathbb{R})$. They are eigenfunctions of the self-adjoint operator $H$ with

$$H \psi_n = (n + 1) \psi_n.$$ 

Now the linear operator $A$ on $L_2(\mathbb{R})$ defined by

$$(Af)(z) = \int_{\mathbb{R}} A(z,x) f(x) \, dx$$

where

$$A(z,x) = \pi^{-1/4} \exp\left[-\frac{1}{2} (x^2 + x^2) + \sqrt{2} zx \right],$$

maps $L_2(\mathbb{R})$ unitarily onto $F_{1,1,1}$. In particular,
\[ A \psi_n = u_n, \ n = 0, 1, 2, \ldots \]

and

\[ A H A^* = R. \]

The Hermite functions are also eigenfunctions of the Fourier transformation \( F \) on \( L_2(\mathbb{R}) \), viz. \( IF \psi_n = (i)^n \psi_n \). Thus in a natural way Fourier invariant test- and distribution spaces arise from series expansions with respect to the Hermite functions. We mention Schwarz' test space \( S \) of \( C^\infty \)-functions of rapid decrease and the Gelfand-Shilov spaces \( S_{\alpha}^a, \alpha \geq \frac{1}{2} \). Namely, the following characterizations are valid.

(2.5) Lemma.

* The space \( S \) consists of precisely all square integrable functions \( \phi \) for which \( (\phi, \psi_n)_{L_2} = O(n^{-k}) \) for all \( k \in \mathbb{N} \).

* For each \( \alpha \geq \frac{1}{2} \), the space \( S_{\alpha}^a \) consists of precisely all square integrable functions \( \phi \) for which \( (\phi, \psi_n) = O(\exp(-n^{2\alpha} t)) \) for some \( t > 0 \). In particular,

\[ S_{1/2}^a = D_{H_0}(H) = \bigcup_{t > 0} e^{-dH}(L_2(\mathbb{R})). \]

Proof.

Cf. [Si] and [Go]. \( \Box \)

Consequently, we have the following results.

(2.6) Corollary.

* For each \( b > 0 \) the image of \( H^{-b}(L_2(\mathbb{R})) \) under \( A \) equals \( F_{1,b+1,1} \). In particular,

\[ A(S) = \bigcap_{b > 0} F_{1,b+1,1}. \]

* For each \( t > 0 \) the image of \( e^{-dH}(L_2(\mathbb{R})) \) under \( A \) equals \( F_{1,1, e^{-t}} \). In particular

\[ A(S_{1/2}^t) = \bigcup_{0 < c < 1} F_{1,1,c}. \]

* For each \( a > 0 \), let \( H^a(L_2(\mathbb{R})) \) denote the completion of \( L_2(\mathbb{R}) \) with respect to the norm \( f \mapsto \| H^{-a} f \|_{L_2(\mathbb{R})} \). Then \( A \) extends to \( H^a(L_2(\mathbb{R})) \) with \( A(H^a(L_2(\mathbb{R}))) = F_{a+1,1,1} \). In particular,

\[ A(S') = \bigcup_{a > 0} F_{a,1,1}. \]

Remark. These results are in correspondence with the results stated in [Ba].
References


