Quantifying closeness of distributions of sums and maxima when tails are fat
Willekens, E.K.E.; Resnick, S.I.

Published: 01/01/1988

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal ?

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
Memorandum COSOR 88-11

Quantifying closeness of distributions
of sums and maxima when tails are fat
by
E. Willekens and S.I. Resnick

Eindhoven, April 1988
The Netherlands
QUANTIFYING CLOSENESS OF DISTRIBUTIONS OF SUMS AND MAXIMA WHEN TAILS ARE FAT

by

E. Willekens*
Eindhoven University of Technology

and

S.I. Resnick**
Cornell University

ABSTRACT

Let \( X_1, X_2, \ldots, X_n \) be \( n \) independent, identically distributed, non-negative random variables and put \( S_n = \sum_{i=1}^{n} X_i \) and \( M_n = \max_{1 \leq i \leq n} X_i \). Let \( \rho(X,Y) \) denote the uniform distance between the distributions of random variables \( X \) and \( Y \); i.e.,

\[
\rho(X,Y) = \sup_{x \in \mathbb{R}} |P(X \leq x) - P(Y \leq x)|.
\]

We consider \( \rho(S_n, M_n) \) when \( P(X_1 > x) \) is slowly varying and we provide bounds for the asymptotic behaviour of this quantity as \( n \to \infty \), thereby establishing a uniform rate of convergence result in Darling's law for distributions with slowly varying tails.

Keywords and phrases: slow variation, partial sums, partial maxima.

*Research supported by NSF Grant MCS 8501763 and by the Belgian National Fund for Scientific Research. Part of the research was carried out during a summer 1987 visit to the Department of Statistics, Colorado State University and grateful acknowledgement is made for their hospitality.

**Partially supported by NSF Grant MCS 8501763 at Colorado State University and at the end by the Mathematical Sciences Institute, Cornell University.
1. Introduction

Suppose that $X_1, X_2, \ldots$ is a sequence of non-negative, independent, identically distributed (i.i.d.) random variables with common distribution function (d.f.) $F$, and denote $F = 1 - F$. Put $S_n = \sum_{i=1}^{n} X_i$ and $M_n = \sqrt[n]{1} X_i$, $n = 1, 2, 3, \ldots$.

$F$ is said to be regularly varying at infinity with index $-\alpha$ ($\alpha > 0$) iff

\[
\lim_{x \to \infty} \frac{F(x)}{t^{-\alpha}} = t^{-\alpha}, \quad \text{for every } t > 0.
\]

If $\alpha = 0$ in (1.1), $F$ is called slowly varying. In the sequel, we will denote (1.1) as $F \in \mathcal{R}_{-\alpha}$.

If $F \in \mathcal{R}_{-\alpha}$ with $\alpha \neq 0$, it is well known that there exist linear normalizations such that $S_n$ and $M_n$ converge weakly to (different) non-degenerate limit laws. Moreover, the concept of regular variation is widely accepted to be the natural way of characterizing domains of attraction in these limit relations, see e.g. Doeblin [5], Feller [5], de Haan [4], Bingham et al [2], Resnick [11].

If $F$ is slowly varying ($\alpha = 0$), $EX^\rho = \infty$ for every $\rho > 0$ and Lévy [8] pointed out that for such distributions, every linear normalization of $S_n$ (or $M_n$) leads to a degenerate limit law. Hence one is forced to consider nonlinear normalizing functions and in this setup, Darling [3] showed that if $F \in \mathcal{R}_0$,

\[
nF(S_n) \Rightarrow E
\]

where $\Rightarrow$ denotes weak convergence and $E$ is an exponential random variable with parameter 1. Also

\[
nF(M_n) \Rightarrow E
\]
so that by uniform convergence,

\begin{equation}
\rho(S_n, M_n) := \sup_{x \geq 0} \left| P[\bar{n}F(S_n) \leq x] - P[\bar{n}F(M_n) \leq x] \right| \\
= \sup_{x \geq 0} \left| P(S_n \leq x) - P(M_n \leq x) \right| \to 0 \text{ as } n \to \infty.
\end{equation}

Another interpretation of this result is given in Resnick [10, section 5] where it is shown that

\[ a_n^{-1}(M_n, S_n) \Rightarrow (\xi, \xi) \]

where \( n \bar{F}(a_n) = 1 \), and \( \xi \) is such that \( P(\xi = 0) = e^{-1} = 1 - P(\xi = \infty) \).

Thus \( \bar{F} \in \mathcal{R}_0 \) implies that \( \rho(S_n, M_n) \to 0 \) as \( n \to \infty \) and in this paper we are interested in the rate of convergence to zero of \( \rho(S_n, M_n) \). In order to obtain a precise rate, it is natural to specify the manner in which \( \bar{F} \) is slowly varying. This is done in the next section where we discuss \( \Pi \)-varying tails. Section 3 contains the results on the rate of decay of \( \rho(S_n, M_n) \) under various conditions on \( \bar{F} \).

2. Preliminaries

From Karamata's Theorem ([2], [4], [6], [11]) it follows that \( \bar{F} \in \mathcal{R}_0 \) iff

\[ \frac{1}{x} \int_0^x u \bar{F}(u) = o(\bar{F}(x)) \ (x \to \infty). \]

We can specify the way in which \( \bar{F} \) is slowly varying by being more precise about the \( o \)-term in this relation. Therefore, suppose that
(2.1) \[ x^{-1} \int_{0}^{x} u dF(u) = V(1/(1 - F(x))), \]

where \( V \) is a non-negative measurable function such that \( xV(x) \to 0 \). More precise conditions on \( V \) will be given later.

In section 3 we show that (2.1) is a natural condition for obtaining a rate of convergence to zero of \( \rho(S_n, M_n) \). Here our first concern is to interpret the condition in (2.1) by translating it into an equivalent form containing only \( F \). In order to state the result, we introduce some necessary definitions and notations: A non-negative measurable function \( U \) is \( \Pi \)-varying (\( U \in \Pi \)) iff there exists a function \( b \in \mathcal{A}_0 \) such that

(2.2) \[ \lim_{x \to \infty} \frac{U(tx) - U(x)}{b(x)} = \log t. \]

(Cf. [2], [4], [11].) \( b \) is usually called an auxiliary function (a.f.) of \( U \) and it is shown in [4] that \( U \in \Pi \) iff \( x^{-1} \int_{0}^{x} s dU(s) \in \mathcal{A}_0 \) in which case we may take \( b(x) = x^{-1} \int_{0}^{x} s dU(s) \). If \( U \) is monotone, non-decreasing and right continuous, the inverse of \( U \) is defined as \( U^\leftarrow(x) = \inf\{y : U(y) \geq x\} \). It is well known that \( U \in \Pi \) with a.f. \( b \) iff \( U^\leftarrow \) is \( \Gamma \)-varying with a.f. \( f(x) = b(U^\leftarrow(x)) \); i.e.

(2.3) \[ \lim_{x \to \infty} \frac{U^\leftarrow(x + tf(x))}{U^\leftarrow(x)} = e^t \text{ for every } t \in \mathbb{R}. \]

(Cf. [2], [4], [11].) One can show (cf. [4]) that if \( f \) is the a.f. of a function in the class \( \Gamma \), then \( f \) is self-neglecting (\( f \in \mathcal{S}_N \)) (cf. [7]); i.e.

\[ \lim_{x \to \infty} \frac{f(x + uf(x))}{f(x)} = 1, \]
locally uniformly in $u \in \mathbb{R}$. Furthermore, if $f$ is any SN function we have

$$\exp\{\int_1^X (1/f(u))du\} \in \Gamma.$$  

The following relations between $\Pi$ and $\Gamma$ will be useful for later work.

**Lemma 2.1.** Suppose $U, H$ are non-decreasing on $(0, \infty)$.

A. (i) If $U \in \Gamma$ with a.f. $f(t) \in R_1 \cap SN$ then $\log U \in \Pi$ with a.f.

$$a(t) = t/f(t).$$

(ii) If $H \in \Pi$ with a.f. $H(t)L(t)/\log t$ where $t/L(e^t) \in SN$, then $H(e^t) \in \Gamma$

with a.f. $t/L(e^t)$.

B. (i) If $U \in \Gamma$ with a.f. $f \in R_1-\alpha$, $\alpha > 0$ then $\log U(x) \sim \alpha^{-1}x/f(x) \in R_\alpha$

(ii) If $H \in \Pi$ with a.f. $H(t)/\alpha \log t$ for some $\alpha > 0$ then $H(e^X) \in R_1/\alpha$.

C. (i) If $U(x) \to \infty$ and $U \in \Gamma$ with a.f. $f$ where $t^2/f(t) \in \Gamma$ with a.f. $h$, then

$$\log U \in \Gamma$$

with a.f. $h$.

(ii) If $H \in \Pi$ with a.f. $H(t)L(t)/\log t$ where $L(t) \to 0$ and $L(e^t) \in R_0$ then

$H(e^X) \in \Pi$ with a.f. $H(e^t)L(e^t)$.

**Proof.** (i) If $U \in \Gamma$, we have the Balkema--de Haan representation (cf. [11], for example)

$$U(x) = c(x)\exp\left\{\int_1^X (1/f_1(u))du\right\}$$

where $c(x) \to c > 0$ and $f_1 \sim f$, so that $f_1 \in R_1 \cap SN$. Hence

$$\log U(x) = \log c(x) + \int_1^X (1/f_1(u))du.$$  

(2.4)

Now $\int_1^X (1/f_1(u))du \in \Pi$ with a.f. $t/f_1(t) \to \infty$ because it is the integral of a

$-1-$varying function. Since
it follows that $\log U \in \Pi$.

(ii) Since we can always represent the a.f. of $H$ as $x^{-1} \int_0^x u dH(u) = H(x) - x^{-1} \int_0^x H(u) du$ we have for some function $b(x)$, $b(x) \to 1$, that

$$x^{-1} \int_0^x u dH(u) = b(x)H(x)L(x)/\log x$$

whence

$$\frac{H(x)}{\int_0^x H(u) du} = \left( x \left( 1 - \frac{b(x)L(x)}{\log x} \right) \right)^{-1}$$

and integrating from 1 to $x$ produces

$$\int_0^x H(u) du = c \exp \left\{ \int_1^x \left( 1 - \frac{b(s)L(s)}{\log s} \right)^{-1} ds \right\}.$$  

Since

$$\int_0^x H(u) du = xH(x) \left[ 1 - \frac{b(x)L(x)}{\log x} \right]$$

we get

$$H(x) = cx^{-1} \left[ 1 - \frac{b(x)L(x)}{\log x} \right]^{-1} \exp \left\{ \int_1^x \left( 1 - \frac{b(s)L(s)}{\log s} \right)^{-1} ds \right\}$$

$$= c \left[ 1 - \frac{b(x)L(x)}{\log x} \right]^{-1} \exp \left\{ \int_1^x \left( 1 - \frac{\log s}{b(s)L(s) - 1} \right)^{-1} ds \right\}.$$

and thus
(2.5) \[ H(e^x) = c(1 - x^{-1}b(e^x)L(e^x))^{-1}\exp\left\{ \int_0^x \left( \frac{y}{b(e^y)L(e^y)} - 1 \right)^{-1} \frac{dy}{y} \right\}. \]

Set \( f^*(x) = x(b(e^x)L(e^x))^{-1} \) and we get
\[ H(e^x) = c\left( (f^*(x) - 1)/f^*(x) \right)\exp\left\{ \int_0^x \frac{1}{(f^*(s) - 1)}ds \right\}. \]

Now observe that since the auxiliary function of \( H \) is \( H(x)L(x)/\log x \) we have
\[ H(x)/(H(x)L(x)(\log x)^{-1}) = \log x/L(x) \to -\infty \] (cf. [4], [11]) and thus \( f^*(x) \to -\infty \) whence \( (f^*(x) - 1)/f^*(x) \to 1 \) and \( f^*(x) - 1 \sim f^* \in SN \). Thus \( H(e^x) \in \Gamma \).

B. (i) From (2.4) and Karamata's Theorem
\[ \log U(x) \sim \alpha^{-1}x/f_1(x) \sim \alpha^{-1}x/f(x). \]

(ii) From (2.5) we have
\[ H(e^x) \sim c \exp\left\{ \int_0^x \frac{y}{(\alpha y/b(e^y)) - 1} \frac{dy}{y} \right\} \]
and since \( y/((\alpha y/b(e^y)) - 1) \to \alpha^{-1} \), the result follows from Karamata's representation of a regularly varying function ([2], [4], [11]).

C. (i) From (2.4) and the assumption \( U(x) \to \infty \) we have
\[ \log U(x) \sim \int_1^x (1/f_1(u))du \] where \( 1/f_1(u) = \gamma(u)/u^2 \) and \( \gamma \in \Gamma \) with a.f. h. Now \( \gamma \in \Gamma \) with a.f. h implies \( \gamma(u)/u^2 \in \Gamma \) with a.f. h and this in turn implies
\[ \int_1^x \gamma(u)/u^2du \in \Gamma \] with a.f. h (cf. [4], p. 45.).

(ii) From (2.5) it follows that
\[ H(e^x) \sim c \exp\left\{ \int_0^x b^*(s)L(e^s)/sds \right\} \]
where $b^*(s) \to 1$ and since $L(x) \to 0$ we get from the Karamata representation that $H(e^x) \in \mathcal{R}_0$. Because $H \in \Pi$ we may write ([1],[2])

$$H(x) = d(x) + \int_1^x a_1(s)/s \, ds$$

where $d = o(a_1)$ and $a_1(t) \sim H(t)L(t)/\log t$. Thus

$$H(e^x) = d(e^x) + \int_0^x a_1(e^y)dy$$

where

$$a_1(e^y) \sim H(e^y)L(e^y)/y \in \mathcal{R}_{-1}$$

and

$$\lim_{x \to \infty} d(e^x)/H(e^x)L(e^x) = \lim_{x \to \infty} \frac{d(x)a_1(x)}{a_1(x)H(x)L(x)} = \lim_{x \to \infty} \frac{d(x)}{a_1(x)\log x} = 0.$$  

Now $\int_0^x a_1(e^y)dy$, being the integral of a $-1$-varying function, is in $\Pi$ with a.f. $H(e^t)L(e^t)$ and the same is true of $H(e^x)$. \hfill \Box

We are now ready to formulate our theorem which interprets (2.1).

**Theorem 2.1.** Define $g = 1/(1-F)$ and consider the following relations:

(i) For some non-negative, measurable function $V$ satisfying $\lim_{x \to \infty} xvV(x) = 0$

(2.1) $x^{-1} \int_0^x udF(u) = V(g(x))$.

(ii) For some function $L(x) \geq 0$, $g \in \Pi$ with a.f. $g(x)L(x)/\log x$.

Equivalently we have for some $L \geq 0$ as $x \to \infty$

(2.6) $\frac{\bar{F}(tx)}{\bar{F}(t)} - 1 \sim (-\log x)(L(t)/\log t)$, $t \to \infty$. 
Then we have

A. \((i)\) holds and \(V \in \mathcal{R}_{-1} \) iff \((ii)\) holds and \(x/L(e^x) \in \text{SN}\).

B. \((i)\) holds and \(V \in \mathcal{R}_{-1-\alpha} \) \((\alpha > 0)\) iff \((ii)\) holds and \(\lim_{x \to \infty} L(x) = a^{-1}\).

C. \((i)\) holds and \(1/V \in \Gamma\) iff \((ii)\) holds, \(L(x) \to 0\), and \(L(e^x) \in \mathcal{R}_0\).

If one of the equivalences in A, B, or C holds, there is a function \(b(x) \to 1\) and \(\bar{F}\) is of the form \((c > 0)\)

\((\text{iii})\) \(\bar{F}(x) = c \left\{ 1 + \frac{b(x)L(x)}{\log x} \right\}^{-1} \exp \left\{ -\frac{x}{1} \left( \frac{b(u)L(u)}{\log u + b(u)L(u)} \right) du \right\} \)

and furthermore \(L\) and \(V\) determine each other asymptotically through the relation

\[ L(x) \sim g(x)V(g(x)) \log x. \]

**Proof.** Suppose \((2.1)\) holds for some function \(V(x)\) satisfying \(xV(x) \to 0\). Since from \((2.1)\)

\[ x(g^2(x) \int_0^x udF(u))^{-1} = \left( g^2(x)V(g(x)) \right)^{-1} \]

we get upon integrating with respect to \(dg(x)\) that for \(T \geq 1\)

\[ \int_1^T \frac{xdF(x)}{\int_0^x udF(u)} = \int_1^T \left( g^2(x)V(g(x)) \right)^{-1} dg(x) = \int \left( y^2V(y) \right)^{-1} dy \]

and since the left side is

\[ \log(\int_0^T \frac{1}{xdF(x)} / \int_0^1 xdF(x)) \]
we obtain for some $c > 0$. The representation
\[
\int_0^T x dF(x) = c \exp \left\{ \int_1^T (y^2 V(y))^{-1} dy \right\}.
\]
So using (2.1)
\[
(2.7) \quad x = \left( c/V(g(x)) \right) \exp \left\{ \int_1^x (y^2 V(y))^{-1} dy \right\}.
\]
Thus if we set
\[
(2.8) \quad H(x) = \left( c/V(x) \right) \exp \left\{ \int_1^x (y^2 V(y))^{-1} dy \right\}
\]
then
\[
x = H \circ g(x)
\]
and $g$ is the inverse of $H$.

To prove (A), suppose that both (2.1) holds and $V \in \mathcal{R}_{-1}$. Since $V \in \mathcal{R}_{-1}$ and $xV(x) \to 0$ it follows that $f(x) := x^2 V(x) \in SN$ since $f(x)/x = xV(x) \to 0$ and thus as $T \to \infty$
\[
\frac{f(t + xf(t))}{f(t)} = \frac{(t + xf(t))^2}{t^2} V(t + xf(t)) \to 1.
\]
Hence $H \in \Gamma$ with a.f. $f(x)$ whence $g \in \Pi$ with a.f. $f \circ g(x) = g^2(x)V(g(x))$. This proves (ii) and it remains to show
\[
x/L(e^X) \sim \frac{1}{g(e^X) V(g(e^X))} \in SN.
\]
However since $H \in \Gamma$ with a.f. $f \in SN \cap \mathcal{R}_1$ it follows from Lemma 2.1.A.(i) that $\log H \in \Pi$ with a.f. $a(t) = t/f(t) = 1/tV(t)$ and therefore $(\log H)^{-} \in \Gamma$ with a.f.
a((\log H)^r)(t) = 1/(\log H^r(t))V(\log H^r(t)) \in SN and the desired result follows since 
\(g(x) \sim H^r(x)\).

Suppose now that (ii) holds and \(x/L(e^x) \in SN\). We show (i) holds with 
\(V \in \mathcal{R}_{-1}\). We assume \(g \in \Pi\) with a.f. \(g(t)L(t)/\log t\) which implies \(F \in \Pi\) with a.f. 
\(\tilde{F}(t)L(t)/\log t\) whence

\[
\tilde{F}(t)L(t)/\log t \sim t^{-1} \int_0^t udF(u).
\]

From Lemma 2.1.A.(ii) we have \(g(e^x) \in \Gamma\) with a.f. \(x/L(e^x)\) whence by inversion 
\(\log g^r(y) \in \Pi\) with a.f. \(\log g^r(y)/L(g^r(y)) \in \mathcal{R}_0\) and thus we conclude

\[
V(t) := \frac{L(g^r(t))}{t \log g^r(t)} \in \mathcal{R}_{-1}.
\]

So we have

\[
V(g(t)) \sim \tilde{F}(t)L(t)/\log t \sim t^{-1} \int_0^t udF(u)
\]
as desired.

The derivation of (iii) is carried out as in Lemma 2.1.A.(ii).

B. Gwen (2.1) with \(V \in \mathcal{R}_{-1-a}\) we get from (2.6) that \(H(x) \in \Gamma\) with a.f. 
\(f(t) = t^2V(t)\) so \(H^r(x) \sim g(x) \in \Pi\) with a.f. \(g^2(t)V(g(t))\). From Lemma 1.B.(i) we have \(\log H(x) \sim \alpha^{-1}x/f(x) \in \mathcal{R}_\alpha\) so 
\(\log H(g(x)) \sim \log x \sim (\alpha g(x)V(g(x)))^{-1}\)
and so the a.f. of \(g\) is 
\(g^2(t)V(g(t)) \sim g(t)(\alpha \log t)^{-1}\)
as desired.
Conversely assume \( g \in \Pi \) with a.f. \( g(t)/\alpha \log t \). Then \( F \in \Pi \) with a.f. 
\[
\bar{F}(t)/\alpha \log t
\]
and so
\[
t^{-1} \int_0^t u dF(u) \sim \bar{F}(t)/\alpha \log t.
\]
From Lemma 1.B.ii we have \( g(x^t) \in \mathcal{R}_1/\alpha \) whence \( \log g^\tau(y) \in \mathcal{R}_\alpha \). So 
\[
V(t) := (at \log g^\tau(t))^{-1} \in \mathcal{R}_{-1-\alpha}
\]
and
\[
V(g(t)) \sim \bar{F}(t)/\alpha \log t \sim t^{-1} \int_0^t u dF(u)
\]
as desired.

C. Given (2.1) and \( 1/V \in \Gamma \) with a.f. \( h \) so that \( (1/V)^\tau \in \Pi \) with a.f. \( h \circ (1/V)^\tau \in \mathcal{R}_0 \). We use this to check that \( y^2 V(y) \in \text{SN} \). Note \( \lim_{t \to \infty} t^2 V(t)/h(t) = 0 \) since this limit equals 
\[
\lim_{y \to \infty} ((1/V)^\tau(y))^{2y^{-1}/h((1/V)^\tau)(y)}
\]
which is the limit of a function in \( \mathcal{R}_{-1} \). Therefore
\[
\lim_{t \to \infty} \frac{(t + xt^2 V(t))^{2V(t + xt^2 V(t))}}{t^2 V(t)} = \lim_{t \to \infty} (1 + xt^{-1} V(t))^{2V(t + xh(t)(t^2 V(t)/h(t))))/V(t)
\]
\[
= \exp\{-\lim_{t \to \infty} xt^2 V(t)/h(t)\} = 1
\]
which says that \( y^2 V(y) \in \text{SN} \). Furthermore \( t^2 V(t)/h(t) \to 0 \) implies \( V(t)/h(t) \to 0 \) and the above argument can be repeated to show \( V \in \text{SN} \). Thus \( H \) in (2.8) is in \( \Gamma \) with a.f. \( y^2 V(y) \) whence from Lemma 2.1.c.(i) \( \log H \in \Gamma \) with a.f. \( h \) and inverting we
conclude \( g \in \Pi \) (one desired conclusion) with a.f. \( g^2 V(g) \) and \( g(e^y) \in \Pi \) with a.f. \( h(g(e^y)) \in \mathcal{R}_0 \).

It remains to show that the a.f. of \( g \)

\[
g^2(x)V(g(x)) \sim g(x)L(x)/\log x
\]

where \( L(e^x) \in \mathcal{R}_0 \); i.e. we show

\[
xg(e^x)V(g(e^x)) \in \mathcal{R}_0.
\]

However \( 1/V \in \Gamma \) with a.f. \( h \) implies \( (x^2 V(x))^{-1} \in \Gamma \) with a.f. \( h \) so that ([4], p. 45)

\[
h(x) \sim x^2 V(x) \int_1^x 1/(y^2 V(y)) dy
\]

and from (2.8)

\[
h(x) \sim x^2 V(x) \log g^-(x)
\]

so that since \( h(g(e^x)) \in \mathcal{R}_0 \) we get

\[
h(g(e^x)) \sim g^2(e^x) V(g(e^x)) x \in \mathcal{R}_0
\]

and since \( g(e^x) \in \Pi \subset \mathcal{R}_0 \) we also get

\[
xg(e^x)V(g(e^x)) \in \mathcal{R}_0.
\]

Furthermore since \( h(t)/t \to 0 \) as a consequence of \( h \) being an auxiliary function, we have

\[
L(g(e^x)) \sim h(g(e^x))/g(e^x) \to 0
\]

whence \( L(x) \to 0 \).
Conversely, suppose \( g \in \Pi \) with a.f. \( g(x)L(x)/\log x \) where \( L(x) \to 0 \), \( L(e^x) \in \mathcal{R}_0 \). As in A and B we have

\[
\overline{F}(x)L(x)/\log x \sim x^{-1} \int_0^x udF(u)
\]

so it remains to check that

\[
V(x) := L(g^{-}(x))/(x \log g^{-}(x))
\]

satisfies \( 1/V \in \Gamma \). However from Lemma 2.1.C.(ii) \( g(e^x) \in \Pi \) with a.f. \( g(e^t)L(e^t) \) whence \( \log g^{-} \in \Gamma \) with a.f. \( tL(g^{-}(t)) =: h(t) \). This implies

\[
\log g^{-}(x)/(xL(g(x))) \in \Gamma \text{ with a.f. } h
\]

and further that

\[
x^2 \log g^{-}(x)/(xL(g(x))) = x \log g^{-}(x)/L(g^{-}(x)) = 1/V \in \Gamma
\]

with a.f. \( h \) as desired. \( \square \)

Theorem 2.1 informs us that condition (2.1) means \( F \) is \( \Pi \)-varying with a special form for the auxiliary function. In the next section we will show that (2.1) is a natural condition to obtain a rate of convergence for \( \rho(S_n, M_n) \).

3. Rates of convergence

Darling [3] showed that if \( \overline{F} \in \mathcal{R}_0 \),

\[
\mathbb{E}\frac{S_n}{M_n} - 1 \text{ as } n \to \infty
\]

Defining \( \epsilon_n^2 := \mathbb{E}\left[\frac{S_n}{M_n}\right] - 1 \), we thus have that \( \epsilon_n \to 0 \) as \( n \to \infty \). The first simple step expresses \( \rho(S_n, M_n) \) in terms of \( \epsilon_n \).
Lemma 3.1. Let \( \bar{F} \in \mathcal{A}_0 \). Then

\[
\rho(S_n, M_n) \leq \epsilon_n + \sup_{x \geq 0} (F^n(x) - F^n(x(1+\epsilon_n)^{-1})).
\]

Proof. We have for any \( x \geq 0 \),

\[
P(M_n > x) \leq P(S_n > x) = P(S_n > x, M_n^{-1} \cdot S_n > 1 + \epsilon_n)
+ P(S_n > x, M_n^{-1} \cdot S_n \leq 1 + \epsilon_n)
\leq P(M_n^{-1} \cdot S_n - 1 > \epsilon_n) + P(M_n(1 + \epsilon_n) > x).
\]

Since \( M_n^{-1} \cdot S_n - 1 \geq 0 \), we can apply Markov's inequality giving that

\[
P(M_n^{-1} \cdot S_n - 1 > \epsilon_n) \leq \frac{1}{\epsilon_n} E(M_n^{-1} \cdot S_n - 1) = \epsilon_n.
\]

Using this upper bound, we get that

\[
P(M_n > x) \leq P(S_n > x) \leq \epsilon_n + P(M_n > x(1+\epsilon_n)^{-1})
\]

whence

\[
0 \leq P(S_n > x) - P(M_n > x) \leq \epsilon_n + F^n(x) - F^n(x(1 + \epsilon_n)^{-1}).
\]

Taking suprema over \( x \) gives the result. \( \Box \)

It is clear from Lemma 3.1 that in order to bound \( \rho(S_n, M_n) \) we need to examine the two terms in the right hand side of (3.1). We first show that the conditions on \( F \) assumed in the previous section allow us to establish the precise asymptotic behaviour of \( \epsilon_n \) as \( n \to \infty \). This is done in the next lemma.
Lemma 3.2. Suppose (2.1) is satisfied.

(i) If \( V \in \mathcal{R}_{-1-\alpha}, \ 0 \leq \alpha, \) then \( \epsilon_n^2 \sim \Gamma(\alpha + 2) \cdot nV(n) \ (n \to \infty). \)

(ii) Set \( \Psi(x) = x^{-1}(-\log V)^{\gamma}(x^{-1}). \) If \(-\log V \in \mathcal{R}_\beta, \ \beta > 0\) then

\[
-\log \epsilon_n \sim \frac{1}{2} (1 + \beta^{-1})^{1/(1+\beta)} \psi^{-\gamma}(n) \ (n \to \infty) \quad \text{and} \quad \epsilon_n = \exp\{-W(n)\} \quad \text{where} \quad W \in \mathcal{R}_\beta/(1+\beta).
\]

Proof. We have from Darling [3] or from Maller and Resnick [9, Lemma 1.1] that

\[
\epsilon_n^2 = n(n-1) \int_0^\infty F^{n-2}(y) \left( y^{-1} \int_0^y u dF(u) \right) dF(y),
\]

and using (2.1) this becomes

\[
\epsilon_n^2 = n(n-1) \int_0^\infty F^{n-2}(y) \left( \frac{1}{F(y)} \right) dF(y).
\]

Define \( V_1 \) by \((0 < s < 1)\)

\[
V \left( \frac{1}{1-s} \right) = V_1 \left( -\log \frac{1}{s} \right)
\]

and set \( q(x) = -\log F(x), \ x \geq 0. \) Then

\[
\epsilon_{n+1}^2 = (n+1)n \int_0^\infty e^{-(n-1)q(y)} V_1 \left( \frac{1}{q(y)} \right) de^{-q(y)}
\]

\[
= (n+1)n \int_0^\infty e^{-ns} V_1 \left( \frac{1}{s} \right) ds
\]

and it seems irresistible to get the asymptotic behavior of \( \epsilon_n \) from well known Abel-Tauber theorems for Laplace transforms; see [2]. If \( V \in \mathcal{R}_{-1-\alpha}, \ \alpha \geq 0, \) it follows that \( V(x) \sim V_1(x) \ (x \to \infty), \) so that via standard methods [2],
\[ \epsilon_{n+1}^2 \sim nV(n) \cdot \Gamma(\alpha+2) \quad (n \to \infty). \]

This proves (i).

As for (ii), we use an Abel–Tauber theorem for Kohlbecker transforms [2, Theorem 4.12.11.9iii)] which immediately implies the result. \( \square \)

**Remarks.**
1. It would be worthwhile to establish a general Abel–Tauber theorem for Laplace transforms of functions in the class \( \Gamma \). Since this is not known, we concentrated in Lemma 3.2(ii) on the special case that \( -\log V \in \mathcal{R}_{\beta}, \beta > 0 \), which covers most cases.
2. We can get the converse assertions in Lemma 3.2(i) (or (ii)) by imposing a Tauberian condition on \( V \) (or \(-\log V\)), see Bingham et al. [2].

It is clear from Lemmas 3.1 and 3.2 that we can estimate \( \rho(S_n, M_n) \) if we bound the second term in the right hand side of (3.1).

**Lemma 3.3.** If (2.1) holds and either

\[ V \in \mathcal{R}_{-1-\alpha}, \quad \alpha \geq 0 \]

or

\[ 1/V \in \Gamma \quad \text{and} \quad -\log V \in \mathcal{R}_{\beta}, \quad \beta > 0 \]

then

\[ \sup_{x \geq 0} |F^n(x) - F^n(x(1+\epsilon_n)^{-1})| = o(\epsilon_n). \]

**Proof.** Clearly for every \( 0 \leq z \leq y \),

\[ F^n(y) - F^n(z) = \frac{y}{z} \int_z^n x^n(t) dF(t) \leq nF^{n-1}(y)(F(y) - F(z)). \]
From Theorem 2.1 we have $F \in \Pi$ with a.f. $V(g)$ and so given $\delta > 0$ there exists $x_0 = x_0(\delta)$ such that if $x \geq x_0$ we have

$$|F(x) - F(x(1+\epsilon_n)^{-1})| \leq (1+\delta)\log(1+\epsilon_n)V(g(x(1+\epsilon_n)^{-1}))$$

where we have used the fact that convergence in the definition of $\Pi$-variation is locally uniform. Combining this with (3.2) gives

$$F^n(x) - F^n(x(1+\epsilon_n)^{-1}) \leq nF^{n-1}(x)(F(x) - F(x(1+\epsilon_n)^{-1}))$$

$$\leq (1+\delta)nF^{n-1}(x)\log(1+\epsilon_n)V(g(x(1+\epsilon_n)^{-1})), \ x > x_0(\delta).$$

Therefore,

$$(3.3) \sup_{x \geq 0} |F^n(x) - F^n(x(1+\epsilon_n)^{-1})|$$

$$\leq nF^{n-1}(x_0) + (1+\delta)n \log(1+\epsilon_n) \cdot \sup_{x \geq x_0} F^{n-1}(x) V(g(x(1+\epsilon_n)^{-1})).$$

Since $x_0$ is a fixed number and $F(x_0) < 1$, it follows from Lemma 3.2 that

$$nF^{n-1}(x_0) = o(\epsilon_n) \ (n \to \infty).$$

We now consider the second term in the right hand side of (3.3). To prove that this is $o(\epsilon_n)$ obviously requires us to show that

$$\sup_{x \geq x_0} nF^{n-1}(y)V(g(x(1+\epsilon_n)^{-1})) \to 0 \ (n \to \infty).$$

Let $(x_n)_{n=1}^{\infty}$ be a sequence such that $x_n \to x_\infty$. 

If \( x_\infty < \infty \), clearly
\[
\text{nF}^{n-1}(x_n) V(g(x_n(1+\epsilon_n)^{-1})) \sim \text{nF}^{n-1}(x_\infty) V(g(x_\infty)) \to 0 \quad (n \to \infty).
\]
If \( x_\infty = \infty \), we use \( F = 1 - g^{-1} \) and

\[
\text{nF}^{n-1}(x_n) V(g(x_n(1+\epsilon_n)^{-1})) \sim \frac{n}{g(x_n)} e^{-n/g(x_n)} g(x_n) V(g(x_n)) \quad (n \to \infty).
\]

which tends to zero since \( xe^{-x} \) is bounded on \([0, \infty)\) and \( xV(x) \to 0 \) as \( x \to \infty \). This proves the lemma.

Combining Theorem 2.1 and Lemmas 3.1–3.3, we have proved the following theorem which gives a rate of convergence for \( \rho(S_n, M_n) \).

**Theorem 3.1.** Suppose that \( x^{-1} \int_0^X u dF(u) = V(1/(1 - F(x))) \) where \( xV(x) \to 0 \).

(i) If \( V \in R_{-1-\alpha} \), \( 0 < \alpha \), then

\[
\lim_{n \to \infty} \sup_{n} \rho(S_n, M_n)/(nV(n))^{1/2} \leq (\Gamma(\alpha + 2))^{1/2}
\]

(ii) Suppose \( 1/V \in \Gamma \) and \(-\log V \in R_{\beta}, \beta > 0\). Set \( \Psi(x) = x^{-1}(-\log V)^\gamma(x^{-1}) \) and \( W(x) = (1 + o(1)) \frac{1}{2} (1+\beta^{-1})^{1/(1+\beta)} / \Psi^\gamma(x) \) where \( o(1) \to 0 \) as \( x \to \infty \) so that \( W(x) \in R_{\beta/(1+\beta)} \). Then

\[
\lim_{n \to \infty} \sup_{n} \rho(S_n, M_n) \exp\{W(n)\} \leq 1.
\]

**Remarks.** 1. The \( o \)-term in Theorem 3.1(ii) stems from the fact that we only have an asymptotic expression for \(-\log \epsilon_n \) in Lemma 3.2(ii). If we want to specify this term we need more information on \( V \) which enables us to use an Abel-Tauber theorem with remainder for Kohlbecker transform in Lemma 3.2(ii).
2. We assumed in Theorem 2.1 that $V$ is regularly varying or that $1/V$ is $\Gamma$-varying. Clearly this can be generalized to $O(o)$-versions (see [2]), leading to $O(o)$-expressions for the behaviour of $\epsilon_n$ as $n \to \infty$. This then gives $O(o)$-type of results in Theorem 3.1.

We now give some examples.

1) Suppose $F(x) = (\log x)^{-\gamma}$, $x \geq e$, $\gamma > 0$. Then

$$F'(x) = \gamma (\log x)^{-\gamma-1}x^{-1} \in \mathcal{R}_{-1}$$

so that $F \in \Pi$ with a.f. $a(t) = (\log t)^{-\gamma-1}$. Since $g(x) = (\log x)^{\gamma}$ we have

$$V(x) = a(g^{-1}(x)) = \frac{\gamma}{x^{1+\gamma}} \in \mathcal{R}_{-1-\gamma^{-1}}$$

and therefore from Theorem 3.1

$$\limsup_{n \to \infty} \rho(S_n, M_n)n^{1/2} \leq (\gamma \Gamma(2+\gamma^{-1}))^{1/2}.$$ 

If $\gamma = 1$

$$\limsup_{n \to \infty} \sqrt{n} \rho(S_n, M_n) \leq \sqrt{2}.$$ 

2) If $F(x) = \exp\{-(\log x)^{\gamma}\}$, $x \geq 1$, $0 < \gamma < 1$, then

$$V(x) = \frac{\gamma-1}{x^{\gamma}} (\log x)^{\gamma}$$

so that

$$\limsup_{n \to \infty} \rho(S_n, M_n)(\log n)^{1-\gamma} \leq \gamma^{1/2}.$$ 

3) If $F(x) = (\log \log x)^{-\gamma}$, $x \geq e^e$, $\gamma > 0$, then

$$V(x) = (\gamma-1)x \gamma \leq e^{x^{1/\gamma}}$$
so that

\[
\limsup_{n \to \infty} p(S_n, M_n) \exp\left\{ \frac{1}{2} (1 + o(1))(1+\gamma)\gamma^{-\gamma/(1+\gamma)}n^{-1/(1+\gamma)} \right\} \leq 1.
\]

**Acknowledgement.** The authors take pleasure in thanking E. Omey and S. Rachev for helpful comments during the preparation of the paper.

**References**


