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ON THE GENERICITY OF STABILIZABILITY FOR TIME-DELAY SYSTEMS*

LUC C. G. J. M. HABETS†

Abstract. Conditions for the stabilizability of time-delay systems with incommensurable point delays by dynamic state feedback are known in the literature. In this paper it is shown that these conditions are satisfied generically.

Although an algebraic approach is used to describe the class of all time-delay systems with point delays, the concept of genericity is formulated in a topological framework. In the metric space consisting of all parametrizations of time-delay systems, the subset of all stabilizable systems is an open and dense subset.

The proof is given for the commensurable delay case first. It is shown that the incommensurable delay case is not significantly more difficult and that the same arguments prove also that systems with incommensurable time-delays are generically stabilizable.

Key words. time-delay systems with point delays, stabilizability, genericity

AMS subject classifications. 93B25, 93D15, 15A54

1. Introduction. Time-delay systems with point delays can be seen as rather straightforward generalizations of ordinary linear time-invariant systems. In the delay case, $\dot{x}(t)$, the derivative of the evolution variable x at time t , and $y(t)$, the output y at time t , do not depend only on the evolution variable x and the input u at time t but also on the evolution variable and input from specific time instants in the past. Let $\sigma_1, \dots, \sigma_k$ denote k delay operators with incommensurable time-delays $\tau_1, \dots, \tau_k$, acting on the trajectories of the evolution variable and the input:

$$(1) \quad \sigma_i x(t) = x(t - \tau_i), \quad \sigma_i u(t) = u(t - \tau_i), \quad (i = 1, \dots, k).$$

Then a system with k *incommensurable* time-delays $\tau_1, \dots, \tau_k$ can be written as

$$(2) \quad \begin{cases} \dot{x}(t) = A(\sigma_1, \dots, \sigma_k)x(t) + B(\sigma_1, \dots, \sigma_k)u(t), \\ y(t) = C(\sigma_1, \dots, \sigma_k)x(t) + D(\sigma_1, \dots, \sigma_k)u(t), \end{cases}$$

where $A(\sigma_1, \dots, \sigma_k)$, $B(\sigma_1, \dots, \sigma_k)$, $C(\sigma_1, \dots, \sigma_k)$, and $D(\sigma_1, \dots, \sigma_k)$ are polynomial matrices in the delay operators $\sigma_1, \dots, \sigma_k$ of appropriate dimensions. Note that the state of this system at time t is not the evolution variable $x(t)$ but the time-trajectory $\{x(\xi) \mid \xi \in [t - T, t]\}$ of this evolution variable. Here T denotes the length of the largest time-delay occurring in (2).

After substitution of indeterminates $s_1, \dots, s_k$ for the delay operators $\sigma_1, \dots, \sigma_k$ in (2), the system $\Sigma = (A(s_1, \dots, s_k), B(s_1, \dots, s_k), C(s_1, \dots, s_k), D(s_1, \dots, s_k))$ over the polynomial ring $\mathbb{R}[s_1, \dots, s_k]$ is obtained. Together with the delays $\tau_1, \dots, \tau_k$, this quadruple of matrices is a complete description of the delay system (2); since the time-delays $\tau_1, \dots, \tau_k$ are incommensurable, there is a 1-1 correspondence between time-delay systems of the form (2) and systems $\Sigma = (A(s_1, \dots, s_k), B(s_1, \dots, s_k), C(s_1, \dots, s_k), D(s_1, \dots, s_k))$ over the ring $\mathbb{R}[s_1, \dots, s_k]$.

To study the concept of internal stability for time-delay systems, consider the differential-difference equation for the evolution variable x , and assume that no input is applied. Then the system is called *internally stable* if, independent of the given initial conditions, the evolution

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†Institut für Dynamische Systeme, Department of Mathematics, University of Bremen, P.O. Box 330 440, D-28334 Bremen, Germany (luc@mathematik.uni-Bremen.de).

variable $x(t)$ tends to zero for $t \rightarrow \infty$. According to [7, Cor. 4.1, p. 182], this notion of stability is equivalent to the following condition on the matrix $A(s_1, \dots, s_k)$:

$$\forall \lambda \in \overline{\mathbb{C}^+} : \det(\lambda I - A(e^{-\tau_1 \lambda}, \dots, e^{-\tau_k \lambda})) \neq 0.$$

Here $\tau_1, \dots, \tau_k$ are the time-delays of the delay operators $\sigma_1, \dots, \sigma_k$ corresponding to the indeterminates $s_1, \dots, s_k$, and $\mathbb{C}^+$ denotes the open complex right half plane.

If a system is not internally stable, this property may be achieved by a proper choice of a static or dynamic feedback compensator. Completely analogous to the case of systems without delays, this so-called stabilizability problem can be split into two dual parts: the problem of stabilization by (static or dynamic) *state* feedback and the detectability problem. In the rest of this paper we confine ourselves to the problem of stabilizability by state feedback and therefore assume that $C = I$ and $D = 0$.

In the literature, the problem of stabilizability of time-delay systems has been solved in at least two different ways. Surprisingly, both the infinite-dimensional systems approach and the systems over rings approach yield the same conditions for the solvability of this problem. However, there are also important differences between these results. In the infinite-dimensional systems approach, a static state feedback (possibly containing distributed time-delays) suffices to achieve internal stability, whereas in the algebraic approach a dynamic feedback compensator (containing only point delays) is required for this.

THEOREM 1.1 (see [12], [3], [14], [4]). *Consider a time-delay system Σ :*

$$(3) \quad \dot{x}(t) = A(\sigma_1, \dots, \sigma_k)x(t) + B(\sigma_1, \dots, \sigma_k)u(t),$$

where σ_i ($i = 1, \dots, k$) denotes the delay operator with time-delay τ_i and where $A(\sigma_1, \dots, \sigma_k)$ and $B(\sigma_1, \dots, \sigma_k)$ are matrices of polynomials in the delay operators $\sigma_1, \dots, \sigma_k$, of size $n \times n$ and $n \times m$, respectively. Substitute indeterminates $s_1, \dots, s_k$ for $\sigma_1, \dots, \sigma_k$ and regard $\Sigma = (A(s_1, \dots, s_k), B(s_1, \dots, s_k))$ as a linear system over the polynomial ring $\mathcal{R} = \mathbb{R}[s_1, \dots, s_k]$. Then the following three conditions are equivalent:

- (i) Σ is internally stabilizable by a dynamic state feedback compensator only containing point delays,
- (ii) Σ is internally stabilizable by a static state feedback, possibly containing distributed time-delays,

$$(4) \quad \text{(iii)} \quad \forall z \in \overline{\mathbb{C}^+} : \text{rank}(zI - A(e^{-\tau_1 z}, \dots, e^{-\tau_k z}) \mid B(e^{-\tau_1 z}, \dots, e^{-\tau_k z})) = n.$$

Rank condition (iii), which can be seen as a generalization of the well-known Hautus test (see [8]) to the case of time-delay systems with point delays, is the starting point of this paper. We shall prove that this condition is generically satisfied on the parameter-space describing all time-delay systems with point delays of the form (3). This means that condition (iii) is very weak; it is satisfied for most time-delay systems.

The condition of reachability for systems over polynomial rings (see, e.g., [9], [16]) can be stated as a rank condition in almost the same way as the stabilizability condition (for a short proof see, for example, [5]). In [11], Lee and Olbrot prove that this condition is generically satisfied if and only if the number of inputs to the system is larger than the number of indeterminates of the polynomial ring (i.e., the number of incommensurable time-delays). Their approach is completely algebraic; they compare the number of polynomial equations that have to be satisfied with the number of unknowns and apply some results from algebraic geometry to prove their result (except on some hypersurfaces in the parameter-space of all time-delay systems, the reachability condition is always satisfied).

At first sight, this approach also looks very promising for solving the genericity problem of stabilizability for time-delay systems. In this case, however, each indeterminate s_i in the polynomial ring corresponds to a delay operator σ_i of length τ_i , and in the Laplace domain

σ_i and τ_i are interrelated via an exponential function. In this way some extra (exponential) equations are obtained that are probably enough to remove the condition on the number of inputs. Unfortunately, this method fails because we are now dealing with both polynomial and exponential equations, which do not fit into the algebraic-geometric framework any more.

In this paper we choose a completely different approach; we describe the concept of genericity in a topological way. In this setting, a certain property is called *generic* if it holds on an open and dense subset of the parameter-space describing all time-delay systems. However, before we can speak of open or dense subsets, we first have to introduce a topology on this space. This topology formalizes our intuitive ideas on the following question: when are the parametrizations of two time-delay systems said to be close to each other? In §2 this topological framework is treated in more detail.

Then all tools are available to prove that the set of stabilizable time-delay systems is open, which is described in our §3. The proof of denseness is more involved. In §4 we start with some preliminary results on matrices over the ring of analytic functions. These are used in §5 to show that the set of stabilizable time-delay systems is indeed a dense subset of the parameter-space describing all time-delay systems.

Remark 1.2. In the rest of this paper it is always tacitly assumed that we are dealing with time-delay systems with commensurable delays. This implies that there is only one delay operator σ required to describe the system equations (2). In general, this situation is much simpler than the incommensurable delay case. Fortunately this distinction does not make any difference for the approach we take to the problem. All results are easily generalized to the incommensurable delay case because the assumption of the presence of only one time-delay operator is never used explicitly. This assumption is only made to simplify notation to highlight the really important ideas more clearly. In §6 we return to this subject briefly and explain why the methods developed in this paper are also applicable in the incommensurable delay case.

2. A topological framework for time-delay systems. This section is devoted to the introduction of a topology on the parameter-space describing all time-delay systems with commensurable time-delays. This topology reflects our intuitive notion of the concept of genericity. Also the space of all 2-dimensional polynomials is equipped with a suitable norm. These polynomials, and especially characteristic polynomials, play a vital role in the characterization of stability. Some of the topological aspects of this relationship are discussed in more detail.

Consider a triple $\Sigma = (A(s), B(s), \tau)$, with $A(s) \in \mathbb{R}[s]^{n \times n}$, $B(s) \in \mathbb{R}[s]^{n \times m}$, and $\tau \in \mathbb{R}^+$. After substitution of the delay operator σ with time-delay τ for the indeterminate s , such a triple is a complete description of the time-delay system:

$$(5) \quad \begin{cases} \dot{x}(t) = A(\sigma)x(t) + B(\sigma)u(t), \\ \sigma x(t) = x(t - \tau), & \sigma u(t) = u(t - \tau). \end{cases}$$

On the other hand, the triple $\Sigma = (A(s), B(s), \tau)$ can be seen as a point in the parameter-space

$$(6) \quad \mathcal{V} = \{(A(s), B(s), \tau) \mid A(s) \in \mathbb{R}[s]^{n \times n}, B(s) \in \mathbb{R}[s]^{n \times m}, \tau \in \mathbb{R}^+\}.$$

Clearly, to each element of $\mathcal{V}$ there corresponds a time-delay system as defined in (5). By imposing a metric on each of the three components of $\mathcal{V}$, the parameter-space $\mathcal{V}$ is turned into a metric space, and thereby its topology is fixed. We start with the introduction of a norm on polynomial matrices in $\mathbb{R}[s]^{p \times q}$.

Let $P(s)$ be a $p \times q$ polynomial matrix over $\mathbb{R}[s]$. Then there exists an $\ell \in \mathbb{N} \cup \{0\}$ and real matrices $P_0, P_1, \dots, P_\ell$, with $P_\ell \neq 0$, such that

$$P(s) = \sum_{i=0}^{\ell} P_i s^i.$$

This ℓ is called the *degree* of the polynomial matrix $P(s)$ and is denoted by $\ell = \deg(P(s))$. Defining $P_i := 0$ for $i > \ell$, we can map the polynomial matrix $P(s)$ to the sequence $(P_i)_{i=0}^{\infty}$ of real matrices. In this way we obtain an explicit description of $P(s)$ in terms of its parameters. In fact, there is a 1-1 correspondence between polynomial matrices and the space $\ell_0(\mathbb{R}^{p \times q})$ consisting of all real matrix sequences with only a finite number of nonzero elements (i.e., matrices with at least one nonzero entry), via the bijection

$$\psi : \ell_0(\mathbb{R}^{p \times q}) \rightarrow \mathbb{R}[s]^{p \times q} : \psi((P_i)_{i=0}^{\infty}) = \sum_{i=0}^{\infty} P_i s^i.$$

The space $\ell_0(\mathbb{R}^{p \times q})$ is easily turned into a normed space by defining the norm of $(P_i)_{i=0}^{\infty}$ by

$$\|(P_i)_{i=0}^{\infty}\| = \sum_{i=0}^{\infty} \|P_i\|,$$

where $\|P_i\|$ is the operator induced norm of the real matrix P_i . It is evident that the same norm can also be used for polynomial matrices.

DEFINITION 2.1. Let $P(s)$ be a $p \times q$ matrix over $\mathbb{R}[s]$. Let $(P_i)_{i=0}^{\infty} \in \ell_0(\mathbb{R}^{p \times q})$ be such that

$$(7) \quad P(s) = \sum_{i=0}^{\infty} P_i s^i.$$

Then the norm of $P(s)$ is defined as

$$(8) \quad \|P(s)\|_{pm} := \sum_{i=0}^{\infty} \|P_i\|,$$

where $\|P_i\|$ is the operator induced matrix norm of P_i for all $i \in \mathbb{N} \cup \{0\}$.

The norm $\|\cdot\|_{pm}$ for polynomial matrices has a very important property. In the Introduction we have seen that for the investigation of the stability properties of a time-delay system, the exponential function $e^{-\tau z}$ has to be substituted for the indeterminate s in a polynomial matrix $A(s)$. Since for all $z \in \overline{\mathbb{C}^+}$, the norm $|e^{-\tau z}|$ is bounded above by 1, the norm $\|P(s)\|_{pm}$ of the polynomial matrix $P(s)$ is a uniform upper bound for the norm of $P(e^{-\tau z})$ in the closed right half plane.

LEMMA 2.2. Let $P(s) \in \mathbb{R}[s]^{p \times q}$. Then for all $\tau > 0$ and for all $z \in \overline{\mathbb{C}^+}$, we have

$$\|P(e^{-\tau z})\| \leq \|P(s)\|_{pm}.$$

With condition (iii) of Theorem 1.1 in mind, we see that Lemma 2.2 has a very interesting consequence for square polynomial matrices.

COROLLARY 2.3. Let $A(s) \in \mathbb{R}[s]^{n \times n}$. Then for all $\tau > 0$ and $z \in \overline{\mathbb{C}^+}$, and for all $w \in \mathbb{C}$ satisfying $|w| > \|A(s)\|_{pm}$, we have

$$\text{rank}(wI - A(e^{-\tau z})) = n.$$

Using Definition 2.1, the parameter-space $\mathcal{V}$ may be equipped with a suitable metric.

DEFINITION 2.4. Let $\Sigma_1 = (A_1(s), B_1(s), \tau_1)$ and $\Sigma_2 = (A_2(s), B_2(s), \tau_2)$ be two elements of the parameter-space $\mathcal{V}$. Then the distance between Σ_1 and Σ_2 is defined as

$$(9) \quad d_{\mathcal{V}}(\Sigma_1, \Sigma_2) := \|A_1(s) - A_2(s)\|_{pm} + \|B_1(s) - B_2(s)\|_{pm} + |\tau_1 - \tau_2|.$$

With this distance function $d_{\mathcal{V}}(\cdot, \cdot)$, the parameter-space $\mathcal{V}$ becomes a metric space.

Once the topology on the parameter-space $\mathcal{V}$ has been fixed, the concept of genericity is easily defined. For each triple $\Sigma = (A(s), B(s), \tau)$ in $\mathcal{V}$, it is possible to check the stabilizability of the corresponding time-delay system using Theorem 1.1. Let

$$S := \{(A(s), B(s), \tau) \in \mathcal{V} \mid \forall z \in \overline{\mathbb{C}^+} : \text{rank}(zI - A(e^{-\tau z})|B(e^{-\tau z})) = n\}$$

be the set of all stabilizable delay systems. Then the property of stabilizability is called *generic* if the set S is an open and dense subset of the parameter-space $\mathcal{V}$. In the topology on $\mathcal{V}$ generated by the metric $d_{\mathcal{V}}(\cdot, \cdot)$, this implies that the set S covers almost the whole space $\mathcal{V}$:

(i) S is open. A stabilizable time-delay system remains stabilizable after a small perturbation of the parameters describing the system (i.e., the property of stabilizability is a robust property).

(ii) S is a dense subset of $\mathcal{V}$. Every element $\Sigma \in \mathcal{V}$ can be approximated arbitrarily close by a sequence of stabilizable systems (i.e., a sequence in S).

We see that the topology generated by the metric of Definition 2.4 leads to a formal definition of genericity that looks very natural and that is completely in accordance with our intuitive notion of this concept.

In almost the same way as for polynomial matrices, it is possible to regard the polynomial ring $\mathbb{R}[s, z]$ as a linear space and to define a norm on this space.

DEFINITION 2.5. Let $p(s, z) \in \mathbb{R}[s, z]$, and write $p(s, z)$ as

$$(10) \quad p(s, z) = \sum_{i=0}^{\ell} \sum_{j=0}^k p_{ij} s^j z^i.$$

Then the norm of $p(s, z)$ is defined as

$$(11) \quad \|p(s, z)\|_p := \sum_{i=0}^{\ell} \sum_{j=0}^k |p_{ij}|.$$

With this norm, $\mathbb{R}[s, z]$ becomes a normed ring.

Analogously to the polynomial matrix case, there exists a 1-1 correspondence between polynomials $p(s, z)$ in two variables and exponential polynomials of the form $p(e^{-\tau z}, z)$. Characteristic polynomials of this form determine the stabilizability of a time-delay system. From this point of view, the norm of Definition 2.5 has several interesting properties. For example, the norm $\|p(s, z)\|_p$ is a good measure for the magnitude of $|p(e^{-\tau z}, z)|$ in a bounded part of the closed right half plane.

LEMMA 2.6. Let $p(s, z) \in \mathbb{R}[s, z]$, and assume that the degree of p in z is n , i.e.,

$$p(s, z) = \sum_{i=0}^n \sum_{j=0}^k p_{ij} s^j z^i,$$

and there exists a $j \in \{0, \dots, k\}$ such that $p_{nj} \neq 0$. Let $M > 1$ and $\varepsilon > 0$. If

$$(12) \quad \|p(s, z)\|_p < \varepsilon \cdot \frac{M-1}{M^{n+1}-1},$$

then

$$(13) \quad \forall \tau > 0 \quad \forall z \in \overline{\mathbb{C}^+} \text{ s.t. } |z| \leq M : |p(e^{-\tau z}, z)| < \varepsilon.$$

Finally there is a clear relationship between polynomial matrices in one indeterminate on the one hand and 2-dimensional polynomials on the other. In this relationship the characteristic polynomial plays the leading role. In the rest of this paper we need only the following result.

PROPOSITION 2.7. *Let $A(s) \in \mathbb{R}[s]^{n \times n}$. Then*

$$\begin{aligned} \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall B(s) \in \mathbb{R}[s]^{n \times n} : \\ \|A(s) - B(s)\|_{pm} < \delta \implies \|\det(zI - A(s)) - \det(zI - B(s))\|_p < \varepsilon. \end{aligned}$$

According to Proposition 2.7, the map χ ,

$$(14) \quad \chi : \mathbb{R}[s]^{n \times n} \longrightarrow \mathbb{R}[s, z] : \chi(A(s)) = \det(zI - A(s)),$$

is continuous with respect to the norms on $\mathbb{R}[s]^{n \times n}$ and $\mathbb{R}[s, z]$ as defined in (8) and (11), respectively. The validity of this result follows from the fact that the determinant of a matrix is a sum of products of its entries. Since both addition and multiplication are continuous operations, a proof of Proposition 2.7 follows straightforwardly.

Remark 2.8. With the norms and metrics defined in this section, none of the spaces $\mathcal{V}$, $\mathbb{R}[s]^{p \times q}$, or $\mathbb{R}[s, z]$ becomes a complete metric space. All these spaces basically consist of sequences (of scalars or matrices) with a finite number of nonzero elements. However, we imposed a sort of ℓ_1 -norm on these spaces that does not distinguish between sequences with a finite and an infinite number of nonzero elements. Therefore it is easy to construct a Cauchy sequence that does not converge.

Fortunately, this somewhat unsatisfactory situation is not troublesome because completeness is never used in the proofs of our genericity result. Moreover, this problem may be solved by introducing so-called *inductive limit topologies* (see, e.g., [1, Chap. IV, §5]). For the study of genericity of more general concepts of stabilizability, this topology is indispensable (see [6, §3.3]), but in our case, inductive limit topologies would make things unnecessarily complicated.

3. On the robustness of the property of stabilizability. In this section the first part of our genericity result is proved. Based on the topological framework introduced in the previous section, it is shown that the subset S of $\mathcal{V}$, consisting of all parametrizations of stabilizable time-delay systems, is an open subset of $\mathcal{V}$. In practice this means that stabilizability of a time-delay system is a robust property: it is preserved after small perturbations of the parameters. In this section an upper bound is derived for the distance between a nominal stabilizable system and all the perturbed systems that are allowed. If the distance between a perturbed system and the nominal system is smaller than this upper bound, the perturbed system is still stabilizable. Since this upper bound is always larger than zero, this immediately implies that S is open.

From Theorem 1.1 we know that the stabilizability condition for time-delay systems is a full rank condition on a matrix in the variable z , which has to be satisfied for all $z \in \overline{\mathbb{C}^+}$. Now the proof of the main theorem of this section is based on the fact that in $\mathbb{C}^{n \times (n+m)}$ the set of all matrices of full row rank is open, i.e., a full row rank matrix in $\mathbb{C}^{n \times (n+m)}$ remains of full row rank after small perturbations of its entries.

THEOREM 3.1. *Let $\Sigma_0 = (A_0(s), B_0(s), \tau_0)$ be a point in $\mathcal{V}$, and assume that the time-delay system (5) corresponding to Σ_0 is stabilizable, i.e.,*

$$\forall z \in \overline{\mathbb{C}^+} : \text{rank}(zI - A_0(e^{-\tau_0 z})|B_0(e^{-\tau_0 z})) = n.$$

Then there exists a $\rho > 0$ such that all systems Σ in the ball around Σ_0 with radius ρ ,

$$\mathcal{B}(\Sigma_0, \rho) := \{\Sigma \in \mathcal{V} \mid d_{\mathcal{V}}(\Sigma, \Sigma_0) < \rho\},$$

are stabilizable.

Proof. First of all there exists an $\ell \in \mathbb{N}$ such that $A_0(s)$ and $B_0(s)$ can be written as

$$A_0(s) = \sum_{i=0}^{\ell} A_i s^i, \quad B_0(s) = \sum_{i=0}^{\ell} B_i s^i.$$

Next define G as

$$(15) \quad G := \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0 \text{ and } |z| \leq \|A_0(s)\|_{pm} + 1\}.$$

Since the delay system corresponding to $\Sigma_0 = (A_0(s), B_0(s), \tau_0)$ is stabilizable, it follows from [3] or [14] that the matrix $(zI - A_0(e^{-\tau_0 z})|B_0(e^{-\tau_0 z}))$ has a right-inverse $T(z)$ that is analytic on $\overline{\mathbb{C}^+}$. Now G is a compact subset of $\overline{\mathbb{C}^+}$, so $T(z)$ is bounded on G , and thus

$$(16) \quad K := \max\{\|T(z)\| \mid z \in G\}$$

is well defined.

Choose

$$(17) \quad \rho := \min \left(1, \frac{1}{4K}, \frac{1}{\|A_0(s)\|_{pm} + 1} \cdot \frac{1}{4K\ell} \cdot \min \left(\frac{1}{\|A_0(s)\|_{pm}}, \frac{1}{\|B_0(s)\|_{pm}} \right) \right),$$

then clearly $\rho > 0$. We show that all systems in $\mathcal{B}(\Sigma_0, \rho)$ are stabilizable.

Let $\Sigma = (A(s), B(s), \tau) \in \mathcal{V}$ be such that $d_{\mathcal{V}}(\Sigma, \Sigma_0) < \rho$. The proof that Σ is stabilizable, i.e., that

$$\forall z \in \overline{\mathbb{C}^+} : \operatorname{rank}(zI - A(e^{-\tau z}) | B(e^{-\tau z})) = n,$$

is divided into two parts: the case $|z| > \|A_0(s)\|_{pm} + 1$ and the case $|z| \leq \|A_0(s)\|_{pm} + 1$.

Let $z \in \overline{\mathbb{C}^+}$, and assume that $|z| > \|A_0(s)\|_{pm} + 1$. Because $d_{\mathcal{V}}(\Sigma, \Sigma_0) < \rho$, we have

$$\|A(s)\|_{pm} \leq \|A_0(s)\|_{pm} + \|A(s) - A_0(s)\|_{pm} < \|A_0(s)\|_{pm} + \rho.$$

Using (17) it follows that $|z| > \|A_0(s)\|_{pm} + 1 \geq \|A_0(s)\|_{pm} + \rho > \|A(s)\|_{pm}$ and, according to Corollary 2.3 (with $w = z$), this implies that

$$\operatorname{rank}(zI - A(e^{-\tau z})) = n.$$

But then certainly $\operatorname{rank}(zI - A(e^{-\tau z}) | B(e^{-\tau z})) = n$.

The second part of the proof is more complicated. Let $z \in \overline{\mathbb{C}^+}$, $|z| \leq \|A_0(s)\|_{pm} + 1$. We start by proving that

$$(18) \quad \|(zI - A(e^{-\tau z}) | B(e^{-\tau z})) - (zI - A_0(e^{-\tau_0 z}) | B_0(e^{-\tau_0 z}))\| < \frac{1}{K}.$$

First note that

$$(19) \quad \begin{aligned} & \|(zI - A(e^{-\tau z}) | B(e^{-\tau z})) - (zI - A_0(e^{-\tau_0 z}) | B_0(e^{-\tau_0 z}))\| \\ & \leq \|A_0(e^{-\tau_0 z}) - A(e^{-\tau z})\| + \|B(e^{-\tau z}) - B_0(e^{-\tau_0 z})\|. \end{aligned}$$

Now clearly

$$(20) \quad \|A_0(e^{-\tau_0 z}) - A(e^{-\tau z})\| \leq \|A_0(e^{-\tau_0 z}) - A_0(e^{-\tau z})\| + \|A_0(e^{-\tau z}) - A(e^{-\tau z})\|.$$

With Lemma 2.2 it is easy to see that the second term in (20) is bounded from above. Since $\|A(s) - A_0(s)\|_{pm} \leq d_V(\Sigma, \Sigma_0) < \rho$ and $\rho \leq \frac{1}{4K}$, we obtain

$$(21) \quad \|A_0(e^{-\tau z}) - A(e^{-\tau z})\| \leq \|A_0(s) - A(s)\|_{pm} < \rho \leq \frac{1}{4K}.$$

To estimate the other term, we apply the mean value theorem:

$$(22) \quad \begin{aligned} \|A_0(e^{-\tau_0 z}) - A_0(e^{-\tau z})\| &= \left\| \sum_{i=0}^{\ell} A_i \cdot (e^{-i\tau_0 z} - e^{-i\tau z}) \right\| \leq \sum_{i=0}^{\ell} \|A_i\| \cdot i|z| \int_{\tau_0}^{\tau} |e^{-i\xi z}| d\xi \\ &\leq \|A_0(s)\|_{pm} \cdot \ell \cdot (\|A_0(s)\|_{pm} + 1) \cdot \rho < \frac{1}{4K}, \end{aligned}$$

where in the last inequality (17) was used.

Completely analogously we can prove that

$$(23) \quad \|B(e^{-\tau z}) - B_0(e^{-\tau_0 z})\| \leq \|B(e^{-\tau z}) - B_0(e^{-\tau z})\| + \|B_0(e^{-\tau z}) - B_0(e^{-\tau_0 z})\| < \frac{1}{2K}.$$

Combining the previous inequality with (19)–(22), we get (18).

Now recall that $(zI - A_0(e^{-\tau_0 z}) \mid B_0(e^{-\tau_0 z}))$ is right-invertible, with right-inverse $T(z)$. Moreover $\|T(z)\| \leq K$. So

$$(24) \quad \|(zI - A(e^{-\tau z}) \mid B(e^{-\tau z})) - (zI - A_0(e^{-\tau_0 z}) \mid B_0(e^{-\tau_0 z}))\| < \frac{1}{K} \leq \frac{1}{\|T(z)\|}.$$

Finally, according to [10, p. 399], (24) implies that the matrix $(zI - A(e^{-\tau z}) \mid B(e^{-\tau z}))$ is right-invertible. This completes the proof. $\square$

Remark 3.2. Although Theorem 3.1 seems to have much in common with the result of Pandolfi in [13, §5], there are several differences. First of all, Pandolfi's result is obtained within the more general framework of distributed parameter systems, of which the class of time-delay systems considered in this paper is only a small subclass. Moreover, in the setting of Pandolfi, perturbations of systems are perturbations of the linear operators describing the system, and robustness of stabilizability is studied in this context. In our approach, perturbations are described within the metric space $\mathcal{V}$ of all parametrizations. Although Pandolfi's result holds in a much more general setting, our result is more specialized to capture the concept of genericity for the class of time-delay systems with point delays.

Remark 3.3. For robustness of stabilizability it is crucial that the rank condition for stabilizability, $\text{rank}(zI - A(e^{-\tau z}) \mid B(e^{-\tau z})) = n$, has to be satisfied only on the half plane $\mathbb{C}^+$. In [13, §4] it is shown that modal controllability, i.e., the property that the same rank condition is satisfied on the whole complex plane, is not robust. In that situation, the division of the proof into two parts (the cases $|z| > \|A_0(s)\|_{pm} + 1$ and $|z| \leq \|A_0(s)\|_{pm} + 1$) is of no use because the norm of $A(e^{-\tau z})$ may become arbitrarily large when $\text{Re } z \rightarrow -\infty$.

4. Some results on matrices over the ring of analytic functions. In the second part of the proof of our genericity result, matrices of analytic functions play an important role. The relationship between the rank of these matrices and their determinants is of special interest. This section can be seen as an intermezzo in which this relationship between rank and determinant is studied using projection matrices.

The first lemma describes how projections can be helpful for the computation of the determinant of a matrix.

LEMMA 4.1. *Let A_1 and A_2 be two arbitrary square matrices, and let E be a projection. Define $\rho(E) := \text{rank}(E)$. Let α be an indeterminate. Then*

$$(25) \quad \det(\alpha E A_1 + (I - E)A_2) = \alpha^{\rho(E)} \cdot \det(E A_1 + (I - E)A_2).$$

Proof. Choose a basis $\{x_1, \dots, x_n\}$ such that $\text{range}(E) = \langle x_1, \dots, x_{\rho(E)} \rangle$ and $\text{range}(I - E) = \langle x_{\rho(E)+1}, \dots, x_n \rangle$. Let $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ denote the matrix of $E A_1 + (I - E)A_2$ with respect to this new basis, where B_1 consists of the first $\rho(E)$ rows of B and B_2 consists of the last $n - \rho(E)$ rows. Then $\begin{pmatrix} \alpha B_1 \\ B_2 \end{pmatrix}$ is the matrix of $\alpha E A_1 + (I - E)A_2$ with respect to this basis. Hence

$$\begin{aligned} \det(\alpha E A_1 + (I - E)A_2) &= \det \begin{pmatrix} \alpha B_1 \\ B_2 \end{pmatrix} = \alpha^{\rho(E)} \cdot \det \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \\ &= \alpha^{\rho(E)} \cdot \det(E A_1 + (I - E)A_2). \quad \square \end{aligned}$$

Let $Q(z)$ be an $n \times n$ matrix over the ring of analytic functions on $\mathbb{C}$, i.e., all entries of $Q(z)$ are analytic functions in z . Define

$$(26) \quad p(z) := \det(Q(z)).$$

It is clear that in a point $\lambda \in \mathbb{C}$, the matrix $Q(\lambda)$ is of full rank if and only if $p(\lambda) \neq 0$. Also, when $p(\lambda) = 0$, it is possible to obtain more precise information on the rank of $Q(\lambda)$ from the determinant function $p(z)$, by using a suitable projection E .

PROPOSITION 4.2. *Let $Q(z)$ be an $n \times n$ matrix of analytic functions and $p(z) = \det(Q(z))$. Assume that for a certain $\lambda \in \mathbb{C}$, $p(\lambda) = 0$. Define the matrix of analytic functions $Q_1(z)$ as*

$$Q_1(z) := \frac{Q(z) - Q(\lambda)}{z - \lambda} = \sum_{j=1}^{\infty} \frac{1}{j!} Q^{(j)}(\lambda) (z - \lambda)^{j-1}.$$

Let E be a projection such that $E Q(\lambda) = 0$. Then

$$(27) \quad p(z) = (z - \lambda)^{\rho(E)} \cdot \det(E Q_1(z) + (I - E)Q(z)).$$

Moreover, if $\rho(E) = k$, then

$$(28) \quad \begin{cases} p^{(j)}(\lambda) = 0 & \text{for } j = 1, \dots, k-1, \\ p^{(k)}(\lambda) = k! \cdot \det(E Q'(\lambda) + (I - E)Q(\lambda)). \end{cases}$$

Proof. $Q(z)$ can be written as $Q(z) = Q(\lambda) + (z - \lambda)Q_1(z)$. Therefore

$$\begin{aligned} p(z) &= \det(E Q(z) + (I - E)Q(z)) = \det((z - \lambda)E Q_1(z) + (I - E)Q(z)) \\ &= (z - \lambda)^{\rho(E)} \cdot \det(E Q_1(z) + (I - E)Q(z)), \end{aligned}$$

where in the last step Lemma 4.1 is used. The result on the derivatives of $p(z)$ in λ when $\rho(E) = k$ is an easy consequence of (27) and the definition of $Q_1(z)$. $\square$

COROLLARY 4.3. *Let $Q(z)$ be an $n \times n$ matrix of analytic functions and $p(z) = \det(Q(z))$. Then*

$$(29) \quad \forall \lambda \in \mathbb{C} : \left[\begin{array}{c} p(\lambda) = 0 \\ p'(\lambda) \neq 0 \end{array} \right] \implies \text{rank}(Q(\lambda)) = n - 1.$$

Proof. Let $\lambda \in \mathbb{C}$ be such that $p(\lambda) = 0$ and $p'(\lambda) \neq 0$. Choose a projection E with $\text{range}(Q(\lambda)) = \ker(E)$. Since $Q(\lambda)$ is singular, $\rho(E) = \text{rank}(E) \geq 1$. According to Proposition 4.2 we have

$$p(z) = (z - \lambda)^{\rho(E)} \cdot \det(EQ_1(z) + (I - E)Q(z)).$$

Suppose that $\rho(E) > 1$. Then $p'(\lambda) = 0$. This contradicts our assumption, and therefore, $\rho(E) = 1$. This implies that $\dim(\text{range}(Q(\lambda))) = n - 1$. $\square$

Remark 4.4. From Proposition 4.2 it is clear that ($p(\lambda) = 0$ and $p'(\lambda) \neq 0$) is a sufficient condition for $Q(\lambda)$ to have rank $n - 1$, but it is not a necessary one. It is also possible that $\text{rank}(Q(\lambda)) = n - 1$ while $p'(\lambda) = 0$. In that case the matrix $EQ'(\lambda) + (I - E)Q(\lambda)$ is singular.

In §5 we are especially interested in matrices $Q(z)$ of analytic functions for which the determinant $p(z)$ has only simple zeros. According to Corollary 4.3 this implies that

$$\text{if } p(\lambda) = 0, \text{ then } \text{rank}(Q(\lambda)) = n - 1.$$

Let $Q(z)$ be given, and assume that $\lambda \in \mathbb{C}$ is such that $p(\lambda) = \det(Q(\lambda)) = 0$ and also $p'(\lambda) = 0$. Then it is possible to perturb $Q(z)$ in such a way that λ becomes a simple zero of $p(z)$. However, to prove this result, we first need a lemma that describes how a constant matrix can be perturbed to increase its rank.

LEMMA 4.5. *Let A be an $n \times n$ matrix over $\mathbb{C}$, and assume that $\text{rank}(A) = \ell$. For each $j \in \{1, \dots, n - \ell\}$, there exists a matrix $B \in \mathbb{R}^{n \times n}$ satisfying the following properties:*

$$(i) \quad \|B\| = 1 \text{ and } \text{rank}(B) = j,$$

$$(ii) \quad \forall \alpha, \beta \neq 0 : \text{range}(\alpha A + \beta B) = \text{range}(A) \oplus \text{range}(B).$$

Proof. Let $e_1, \dots, e_n$ denote the standard basis in $\mathbb{C}^n$. Then there exists a permutation $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that

$$(30) \quad \text{range}(A) = \langle Ae_{\pi(1)}, \dots, Ae_{\pi(\ell)} \rangle.$$

Choose vectors $e_{i_1}, \dots, e_{i_{n-\ell}}$ from the standard basis satisfying

$$(31) \quad \langle e_{i_1}, \dots, e_{i_{n-\ell}} \rangle \oplus \text{range}(A) = \mathbb{C}^n.$$

Let $j \in \{1, \dots, n - \ell\}$, and define B as

$$\begin{cases} Be_{\pi(k)} = 0 & \text{for } k = 1, \dots, \ell, \ell + j + 1, \dots, n, \\ Be_{\pi(k)} = e_{i_{k-\ell}} & \text{for } k = \ell + 1, \dots, \ell + j. \end{cases}$$

With this choice of B , it is obvious that (i) is satisfied.

From the construction of B it is immediately clear that $\text{range}(A) \cap \text{range}(B) = \{0\}$. Moreover, the inclusion $\text{range}(\alpha A + \beta B) \subset \text{range}(A) + \text{range}(B)$ is trivial. So, to prove (ii), we only have to show the correctness of the other inclusion.

Let $x_1 \in \text{range}(A)$. Then there exists a $y_1 \in \langle e_{\pi(1)}, \dots, e_{\pi(\ell)} \rangle$ such that $x_1 = Ay_1$. But clearly $By_1 = 0$. Hence

$$(\alpha A + \beta B) \left(\frac{1}{\alpha} y_1 \right) = Ay_1 + \frac{\beta}{\alpha} By_1 = x_1,$$

and $x_1 \in \text{range}(\alpha A + \beta B)$.

Let $x_2 \in \text{range}(B)$. Then there exists a $y_2 \in \langle e_{\pi(\ell+1)}, \dots, e_{\pi(\ell+j)} \rangle$ such that $By_2 = x_2$. Since $Ay_2 \in \text{range}(A)$, there exists a $y_3 \in \langle e_{\pi(1)}, \dots, e_{\pi(\ell)} \rangle$ such that $Ay_2 = Ay_3$. Now

$$(\alpha A + \beta B) \cdot \frac{1}{\beta}(y_2 - y_3) = \frac{\alpha}{\beta}(Ay_2 - Ay_3) + By_2 - By_3 = By_2 = x_2,$$

and $x_2 \in \text{range}(\alpha A + \beta B)$. This completes the proof of (ii). $\square$

At this stage all ingredients to prove the main result of this section are available. This result describes how a matrix of analytic functions may be perturbed to reduce the multiplicity of one of the zeros of its determinant to 1.

PROPOSITION 4.6. *Let $Q(z)$ be an $n \times n$ matrix of analytic functions, and define $p(z) = \det(Q(z))$. Assume that $\lambda \in \mathbb{C}$ satisfies $p(\lambda) = 0$. Let $g(z)$ be an analytic function such that $g'(\lambda) \neq 0$.*

Then for each $\varepsilon > 0$ there exists an $n \times n$ polynomial matrix $\Delta(s)$ over $\mathbb{R}[s]$ that satisfies the following properties (where $\tilde{Q}(z) := Q(z) + \Delta(g(z))$ and $\tilde{p}(z) := \det(\tilde{Q}(z))$):

- (i) $\|\Delta(s)\|_{pm} < \varepsilon$,
- (ii) $\deg(\Delta(s)) \leq 1$ if $g(\lambda)$ is real, and $\deg(\Delta(s)) \leq 2$ if $g(\lambda)$ is complex,
- (iii) $\tilde{p}(\lambda) = 0$ and $\tilde{p}'(\lambda) \neq 0$.

Proof. If $p'(\lambda) \neq 0$, the proof is trivial: take $\Delta(s) = 0$.

Assume $p'(\lambda) = 0$. Let $\varepsilon > 0$. If $\text{rank}(Q(\lambda)) = n - 1$, define $B_1 := 0$. Otherwise, choose a matrix B_1 according to Lemma 4.5, with $\|B_1\| = 1$ and $\text{rank}(B_1) = n - 1 - \text{rank}(Q(\lambda))$ in such a way that

$$\forall \alpha \neq 0 : \text{range}(Q(\lambda) + \alpha B_1) = \text{range}(Q(\lambda)) \oplus \text{range}(B_1).$$

This implies that for all $\alpha \neq 0$, $\text{rank}(Q(\lambda) + \alpha B_1) = n - 1$.

Fix $\alpha := \frac{1}{3}\varepsilon$ and apply Lemma 4.5 again, but now to the matrix $Q(\lambda) + \alpha B_1$. In this way we find a matrix B_2 (possibly depending on α), satisfying $\|B_2\| = 1$, $\text{rank}(B_2) = 1$, and

$$\forall \beta \neq 0 : \text{range}(Q(\lambda) + \alpha B_1 + \beta B_2) = \text{range}(Q(\lambda)) \oplus \text{range}(B_1) \oplus \text{range}(B_2).$$

So for every $\beta \neq 0$, the matrix $(Q(\lambda) + \alpha B_1 + \beta B_2)$ has rank n .

Let E denote the projection on $\text{range}(B_2)$ along $\text{range}(Q(\lambda) + \alpha B_1)$, so that $E(Q(\lambda) + \alpha B_1) = 0$ and $EB_2 = B_2$. Then $\rho(E) = \text{rank}(E) = 1$. Define $Q_\alpha(z) := Q(z) + \alpha B_1$ and $p_\alpha(z) := \det(Q_\alpha(z))$. So $p_\alpha(\lambda) = \det(Q(\lambda) + \alpha B_1) = 0$ and using formula (28) from Proposition 4.2 we obtain

$$p'_\alpha(\lambda) = \det(EQ'_\alpha(\lambda) + (I - E)Q_\alpha(\lambda)) = \det(EQ'(\lambda) + (I - E)Q_\alpha(\lambda)).$$

First note that $\ker(EQ'(\lambda) + (I - E)Q_\alpha(\lambda)) \subset \ker(Q_\alpha(\lambda))$. Moreover, we know that $\dim(\ker(Q_\alpha(\lambda))) = 1$. Therefore the problem can be divided into two different cases.

Case 1. $\ker(EQ'(\lambda) + (I - E)Q_\alpha(\lambda)) = \{0\}$. Then $p'_\alpha(\lambda) = \det(EQ'(\lambda) + (I - E)Q_\alpha(\lambda)) \neq 0$, and $\Delta(s) := \alpha B_1$ satisfies (ii), (iii), and also (i) because $\|\Delta(s)\|_{pm} = \|\alpha B_1\| \leq \frac{1}{3}\varepsilon$.

Case 2. $\ker(EQ'(\lambda) + (I - E)Q_\alpha(\lambda)) = \ker(Q_\alpha(\lambda))$. If $g(\lambda)$ is real, define for all $\beta \in \mathbb{R} \setminus \{0\}$

$$\Delta_\beta(s) := \alpha B_1 + \beta(s - g(\lambda))B_2;$$

if $g(\lambda)$ is complex, define for all $\beta \in \mathbb{R} \setminus \{0\}$

$$\Delta_\beta(s) := \alpha B_1 + \beta(s - g(\lambda))(s - \overline{g(\lambda)})B_2.$$

Then in each case $\Delta_\beta(s) \in \mathbb{R}[s]^{n \times n}$, and moreover (ii) is satisfied.

Let $\beta \in \mathbb{R} \setminus \{0\}$ and define $\tilde{Q}(z) := Q(z) + \Delta_\beta(g(z))$. Then $\tilde{Q}(\lambda) = Q(\lambda) + \alpha B_1$, and in both the real and the complex cases there exists a $\gamma \neq 0$ such that $\tilde{Q}'(\lambda) = Q'(\lambda) + \gamma B_2$. Since $\tilde{Q}(\lambda) = Q(\lambda) + \alpha B_1$ is singular, we still have that $\tilde{p}(\lambda) = 0$, and according to Proposition 4.2,

$$\tilde{p}'(\lambda) = \det(E\tilde{Q}'(\lambda) + (I - E)\tilde{Q}(\lambda)) = \det(E(Q'(\lambda) + \gamma B_2) + (I - E)Q_\alpha(\lambda)).$$

Assume that $x \in \ker(E(Q'(\lambda) + \gamma B_2) + (I - E)Q_\alpha(\lambda))$. Then $x \in \ker(Q_\alpha(\lambda))$. So by assumption, $x \in \ker(EQ'(\lambda) + (I - E)Q_\alpha(\lambda))$. Moreover we have that $EB_2 = B_2$, and thus we obtain

$$\gamma B_2 x = (E(Q'(\lambda) + \gamma B_2) + (I - E)Q_\alpha(\lambda))x - (EQ'(\lambda) + (I - E)Q_\alpha(\lambda))x = 0.$$

So $(Q_\alpha(\lambda) + \gamma B_2)x = 0$. By construction $Q_\alpha(\lambda) + \gamma B_2 = Q(\lambda) + \alpha B_1 + \gamma B_2$ has full rank, and thus $x = 0$. This implies that $\text{rank}(E(Q'(\lambda) + \gamma B_2) + (I - E)Q_\alpha(\lambda)) = n$. Therefore $\tilde{p}'(\lambda) \neq 0$, and $\tilde{Q}(z)$ satisfies condition (iii) for all $\beta \neq 0$.

To satisfy (i), we choose

$$\beta := \frac{1}{4} \varepsilon \cdot \min\left(\frac{1}{|g(\lambda)|}, 1\right),$$

when $g(\lambda)$ is real, and

$$\beta := \frac{1}{8} \varepsilon \cdot \min\left(\frac{1}{|g(\lambda) + \overline{g(\lambda)}|}, \frac{1}{g(\lambda) \cdot \overline{g(\lambda)}}, 1\right),$$

when $g(\lambda)$ is complex. Then it is easily verified that $\|\Delta_\beta(s)\|_{pm} < \varepsilon$. This completes the proof. $\square$

Remark 4.7. When the matrix $Q(z)$ of analytic functions has the property that $\overline{Q(\bar{z})} = Q(z)$, its determinant $p(z)$ also has that property. This implies that λ is a zero of $p(z)$ of multiplicity k if and only if $\bar{\lambda}$ is a zero of $p(z)$ of the same multiplicity. Note also that if $\overline{g(\bar{z})} = g(z)$ and $g(\lambda)$ is complex, the reduction process described in the proof of Proposition 4.6 reduces the multiplicity of both the zeros λ and $\bar{\lambda}$ to 1 in only one step. Although in general a perturbation matrix $\Delta(s)$ of degree 2 is needed to fix this problem, this matrix handles both zeros λ and $\bar{\lambda}$ at the same time.

Remark 4.8. Corollary 4.3 and Proposition 4.6 are formulated in a very general context of matrices over analytic functions, but in the next section they are only used for a very specific case. It is clear that for the time-delay system corresponding to the point $\Sigma = (A(s), B(s), \tau) \in \mathcal{V}$, the matrix $(zI - A(e^{-\tau z}))$ is very important for its stabilizability properties. Therefore the results of this section are applied to the case $Q(z) = (zI - A(e^{-\tau z}))$ and $g(z) = e^{-\tau z}$. Then clearly $g'(\lambda) = -\tau e^{-\tau \lambda} \neq 0$ for all $\lambda \in \mathbb{C}$. In this perspective, Proposition 4.6 describes how the matrix $A(s)$ has to be perturbed within the normed ring $\mathbb{R}[s]^{n \times n}$ in such a way that $(zI - (A(e^{-\tau z}) + \Delta(e^{-\tau z})))$ satisfies the condition of Corollary 4.3.

5. Approximation by stabilizable time-delay systems. In this section, the second and final part of our genericity result is proven. We show that the subset S of $\mathcal{V}$, consisting of all parametrizations of stabilizable time-delay systems, is a dense subset of $\mathcal{V}$. This means that in any arbitrary small neighborhood of a point $\Sigma \in \mathcal{V}$, corresponding to a nonstabilizable time-delay system, there is a point $\tilde{\Sigma} \in S \subset \mathcal{V}$ that describes a stabilizable time-delay system. In this section such an approximation by stabilizable time-delay systems is constructed explicitly.

The main idea of the proof is as follows. Let a point $\Sigma = (A(s), B(s), \tau) \in \mathcal{V}$ be given such that the corresponding time-delay system is not stabilizable. First of all it may be shown (use, e.g., Corollary 2.3) that for all matrices $\tilde{A}(s) \in \mathbb{R}[s]^{n \times n}$, the analytic function $\tilde{p}(z) = \det(zI - \tilde{A}(e^{-\tau z}))$ has only a finite number of zeros in $\mathbb{C}^+$. Using Rouché's theorem,

Corollary 4.3, and Proposition 4.6, it is possible to prove that for every $\varepsilon > 0$, there exists a matrix $A_\varepsilon(s) \in \mathbb{R}[s]^{n \times n}$ such that $\|A(s) - A_\varepsilon(s)\|_{pm} < \frac{1}{2}\varepsilon$ and

$$(32) \quad \forall z \in \overline{\mathbb{C}^+} : [\text{rank}(zI - A_\varepsilon(e^{-\tau z})) < n \implies \text{rank}(zI - A_\varepsilon(e^{-\tau z})) = n - 1].$$

So, in all points $z \in \overline{\mathbb{C}^+}$ where the matrix $(zI - A_\varepsilon(e^{-\tau z}))$ loses rank, it loses only rank 1. This loss of rank has to be compensated by the matrix $B(s)$. Therefore this matrix has to be perturbed in such a way that the perturbed version $B_\varepsilon(s)$ satisfies the inequality $\|B_\varepsilon(s) - B(s)\|_{pm} < \frac{1}{2}\varepsilon$ and is such that

$$(33) \quad \forall z \in \overline{\mathbb{C}^+} : [\text{rank}(zI - A_\varepsilon(e^{-\tau z})) < n \implies \text{rank}(zI - A_\varepsilon(e^{-\tau z}) \mid B_\varepsilon(e^{-\tau z})) = n].$$

Since the analytic function $p_\varepsilon(z) = \det(zI - A_\varepsilon(e^{-\tau z}))$ has only a finite number of zeros in the closed right half plane, it is possible to satisfy this condition. In this way we find a stabilizable time-delay system $\Sigma_\varepsilon = (A_\varepsilon(s), B_\varepsilon(s), \tau)$ such that $d_V(\Sigma, \Sigma_\varepsilon) < \varepsilon$, and the proof is complete.

The rest of this section is devoted to a detailed elaboration of the scheme of the proof given above. The first lemma (which can be seen as a direct consequence of Corollary 2.3) describes the location of the zeros of the analytic function $p(z) = \det(zI - A(e^{-\tau z}))$ corresponding to the square polynomial matrix $A(s)$.

LEMMA 5.1 (see [7, p. 18]). *Let $A(s) \in \mathbb{R}[s]^{n \times n}$ and $\tau > 0$ be given. Then the analytic function $p(z) = \det(zI - A(e^{-\tau z}))$ has only a finite number of zeros in the closed right half plane $\overline{\mathbb{C}^+}$. Moreover, all the zeros of $p(z)$ in $\overline{\mathbb{C}^+}$ are located within the semi-disc*

$$(34) \quad D := \{z \in \mathbb{C} \mid \text{Re } z \geq 0 \text{ and } |z| \leq \|A(s)\|_{pm}\}.$$

In Lemma 5.1, the role of the right half plane $\overline{\mathbb{C}^+}$ is not crucial. By shifting to the left and to the right it is possible to show that $p(z) = \det(zI - A(e^{-\tau z}))$ has a finite number of zeros in any arbitrary right half plane.

In the proof of the main results of this section, we often assume that the function $p(z) = \det(zI - A(e^{-\tau z}))$ has no zeros on the boundary of $\mathbb{C}^+$, i.e., $p(z)$ has no zeros on the imaginary axis. Fortunately, this is not really a restriction. By an arbitrarily small perturbation of the matrix $A(s)$ corresponding to $p(z) = \det(zI - A(e^{-\tau z}))$, it is possible to shift the zeros of $p(z)$ in the horizontal direction. In this way we can prove the following result.

PROPOSITION 5.2. *Let $A(s) \in \mathbb{R}[s]^{n \times n}$ and $\tau > 0$ be given. Let $\varepsilon > 0$. Then there exists a polynomial matrix $A_1(s) \in \mathbb{R}[s]^{n \times n}$ satisfying the following properties:*

- (i) $\|A(s) - A_1(s)\|_{pm} < \varepsilon$,
- (ii) $\deg(A_1(s)) = \deg(A(s))$,
- (iii) *the characteristic function $p_1(z) := \det(zI - A_1(e^{-\tau z}))$ has no zeros on the imaginary axis.*

The next theorem is a restatement of a well-known result from complex analysis. It plays a crucial role in the rest of this section because it describes how small perturbations of an analytic function influence the location of its zeros.

THEOREM 5.3 (Rouché's theorem (see, e.g., [15, Thm. 10.43])). *Let f and g be two functions that are analytic inside and on a Jordan curve $\mathcal{J}$. Suppose that f and g have no zeros on $\mathcal{J}$. Denote by N_f and N_g the total number of zeros of f and g inside $\mathcal{J}$, also counting multiplicities. Then*

$$(35) \quad [\forall z \in \mathcal{J} : |f(z) - g(z)| < |f(z)|] \implies N_g = N_f.$$

Let $\mathcal{J}$ be a Jordan curve, and let f and g be two functions satisfying the conditions of Theorem 5.3. Define $\delta := \min\{|f(z)| \mid z \in \mathcal{J}\}$. Then the condition $|f(z) - g(z)| < \delta$ implies

that f and g have the same number of zeros inside $\mathcal{J}$. This observation is exploited in the next lemma.

LEMMA 5.4. *Let $A(s) \in \mathbb{R}[s]^{n \times n}$ and $\tau > 0$ be given. Let $\mathcal{J}$ be a Jordan curve in $\overline{\mathbb{C}^+}$ such that $p(z) = \det(zI - A(e^{-\tau z}))$ has no zeros on $\mathcal{J}$. Then there exists an $\bar{\varepsilon} > 0$ such that for all polynomial matrices $\tilde{A}(s) \in \mathbb{R}[s]^{n \times n}$ satisfying $\|A(s) - \tilde{A}(s)\|_{pm} < \bar{\varepsilon}$, the characteristic function $\tilde{p}(z) := \det(zI - \tilde{A}(e^{-\tau z}))$ corresponding to $\tilde{A}(s)$ has the same number of zeros within $\mathcal{J}$ as $p(z)$ (counting multiplicities) and no zeros on $\mathcal{J}$.*

Proof. Define $p_c(s, z) := \det(zI - A(s))$. Then $p_c(s, z) \in \mathbb{R}[s, z]$, and the degree of $p_c(s, z)$ in z is n . Define

$$(36) \quad \delta := \min\{|p(z)| \mid z \in \mathcal{J}\},$$

and $M := 1 + \max\{|z| \mid z \in \mathcal{J}\}$. Now apply Proposition 2.7. Choose an $\bar{\varepsilon} > 0$ in such a way that for all matrices $\tilde{A}(s) \in \mathbb{R}[s]^{n \times n}$ satisfying $\|A(s) - \tilde{A}(s)\|_{pm} < \bar{\varepsilon}$, the following inequality holds:

$$(37) \quad \|p_c(s, z) - \tilde{p}_c(s, z)\|_p < \delta \frac{M-1}{M^{n+1}-1}.$$

Here $\tilde{p}_c(s, z)$ denotes the characteristic polynomial $\det(zI - \tilde{A}(s))$ of $\tilde{A}(s)$, which is also of degree n in z . We show that for $\bar{\varepsilon}$ the claim of Lemma 5.4 holds.

Let $\tilde{A}(s) \in \mathbb{R}[s]^{n \times n}$ be such that $\|A(s) - \tilde{A}(s)\|_{pm} < \bar{\varepsilon}$. First apply Lemma 2.6 to $r(s, z) := p_c(s, z) - \tilde{p}_c(s, z)$ and use inequality (37). In this way we obtain

$$(38) \quad \forall z \in \overline{\mathbb{C}^+}, |z| \leq M : |p(z) - \tilde{p}(z)| < \delta.$$

So in particular $|p(z) - \tilde{p}(z)| < \delta$ for all $z \in \mathcal{J}$. Apparently $\tilde{p}(z)$ has no zeros on $\mathcal{J}$. (Otherwise there would be a $\lambda \in \mathcal{J}$ such that $|p(\lambda)| < \delta$, which contradicts definition (36).) Finally, since both $p(z)$ and $\tilde{p}(z)$ are analytic functions without zeros on $\mathcal{J}$, Rouché's theorem and formulae (36) and (38) imply that $p(z)$ and $\tilde{p}(z)$ have the same number of zeros inside the Jordan curve $\mathcal{J}$ (counting multiplicities). $\square$

Lemma 5.4 indicates that small perturbations of the matrix $A(s)$ affect the zeros of $p(z)$ only slightly: they cannot cross the Jordan curve $\mathcal{J}$. The idea is now to perturb $A(s)$ in such a way that the multiple zeros of $p(z)$ inside $\mathcal{J}$ become simple without changing the total number of zeros inside $\mathcal{J}$. In this approach, Rouché's theorem (in the disguised form of Lemma 5.4) again plays an important role.

PROPOSITION 5.5. *Let $A(s) \in \mathbb{R}[s]^{n \times n}$ and $\tau > 0$ be given. Let $\mathcal{J}$ be a Jordan curve in $\overline{\mathbb{C}^+}$, and assume that $p(z) = \det(zI - A(e^{-\tau z}))$ has no zeros on $\mathcal{J}$. Choose $\bar{\varepsilon} > 0$ such that Lemma 5.4 is satisfied. Let N_p denote the total number of zeros of $p(z)$ within $\mathcal{J}$, counting multiplicities. Then for all $\varepsilon \in (0, \bar{\varepsilon})$ there exists a matrix $\tilde{A}(s) \in \mathbb{R}[s]^{n \times n}$ such that*

- (i) $\|A(s) - \tilde{A}(s)\|_{pm} < \varepsilon$,
- (ii) $\deg(\tilde{A}(s)) \leq \max(\deg(A(s)), 2)$,
- (iii) *the analytic function $\tilde{p}(z) := \det(zI - \tilde{A}(e^{-\tau z}))$ has N_p zeros within $\mathcal{J}$ and all these zeros are simple.*

Proof. Let $\varepsilon \in (0, \bar{\varepsilon})$. Then it follows from Lemma 5.4 that for all $\tilde{A}(s) \in \mathbb{R}[s]^{n \times n}$ satisfying $\|A(s) - \tilde{A}(s)\|_{pm} < \varepsilon < \bar{\varepsilon}$, the number of zeros of $\tilde{p}(z) = \det(zI - \tilde{A}(e^{-\tau z}))$ inside $\mathcal{J}$ is equal to N_p . Let L_p denote the number of simple zeros of $p(z)$ within $\mathcal{J}$. The proposition is proved with the following induction argument:

$$\forall i \in \{0, 1, \dots, N_p - L_p\} \exists A_i(s) \in \mathbb{R}[s]^{n \times n} \text{ such that}$$

$$(1) \|A(s) - A_i(s)\|_{pm} \leq \frac{2^i - 1}{2^i} \cdot \varepsilon,$$

 FIG. 5.1. Location of the zeros inside a Jordan curve $\mathcal{J}$.

(2) $\deg(A_i(s)) \leq \max(\deg(A(s)), 2)$,

(3) the analytic function $p_i(z) = \det(zI - A_i(e^{-\tau z}))$ has at least $L_p + i$ simple zeros within $\mathcal{J}$, i.e., $L_{p_i} \geq L_p + i$, where L_{p_i} denotes the number of simple zeros of $p_i(z)$ enclosed by $\mathcal{J}$.

When $i = 0$, this is trivial. Choose $A_0(s) = A(s)$.

Induction step. Suppose that for certain $i \in \{0, 1, \dots, N_p - L_p - 1\}$ we have found a matrix $A_i(s)$ satisfying (1)–(3). If $L_{p_i} \geq L_p + i + 1$, choose $A_{i+1}(s) = A_i(s)$, and we are ready.

Next assume that $L_{p_i} = L_p + i$. Since $i < N_p - L_p$, we know that at least one of the N_p zeros of $p_i(z)$ inside $\mathcal{J}$ is a multiple zero. Let $\lambda_j, j \in \{1, \dots, \ell\}$, denote all distinct zeros of $p_i(z)$ in $\mathcal{J}$. Then there exists a $\rho > 0$ such that the circles $\mathcal{C}_j$ defined by

$$(39) \quad \mathcal{C}_j = \{z \in \mathbb{C} \mid |z - \lambda_j| = \rho\}$$

neither intersect one another nor the Jordan curve $\mathcal{J}$ (see Figure 5.1). Apply Lemma 5.4 to each of these circles $\mathcal{C}_j$. Then for all $j = 1, \dots, \ell$, we find an $\bar{\varepsilon}_j > 0$ such that for all $\hat{A}(s) \in \mathbb{R}[s]^{n \times n}$, the inequality $\|A(s) - \hat{A}(s)\|_{pm} < \bar{\varepsilon}_j$ implies that $p_i(z)$ and $\hat{p}(z) = \det(zI - \hat{A}(e^{-\tau z}))$ have the same number of zeros within $\mathcal{C}_j$ and no zeros on $\mathcal{C}_j$. Define $\hat{\varepsilon} := \min\{\bar{\varepsilon}_j \mid j = 1, \dots, \ell\}$.

Assume, without loss of generality, that λ_1 is a multiple zero of $p_i(z)$. Apply Proposition 4.6 to $Q(z) = (zI - A_i(e^{-\tau z}))$ with $g(z) = e^{-\tau z}$ and $\lambda = \lambda_1$. Clearly $g'(\lambda_1) = -\tau e^{-\tau \lambda_1} \neq 0$, so there exists a polynomial matrix $\Delta(s) \in \mathbb{R}[s]^{n \times n}$, with $\deg(\Delta(s)) \leq 2$, in norm bounded by

$$\|\Delta(s)\|_{pm} < \min \left(\hat{\varepsilon}, \frac{1}{2^{i+1}} \cdot \varepsilon \right)$$

and such that $\tilde{p}(z) = \det(Q(z) + \Delta(e^{-\tau z}))$ has only a simple zero in $z = \lambda_1$. We show that $A_{i+1}(s) := A_i(s) + \Delta(s)$ meets the requirements (1)–(3), with i replaced by $i + 1$.

(1) and (2) are very straightforward:

$$\begin{aligned} \|A(s) - A_{i+1}(s)\|_{pm} &\leq \|A(s) - A_i(s)\|_{pm} + \|A_i(s) - A_{i+1}(s)\|_{pm} \\ &\leq \frac{2^i - 1}{2^i} \cdot \varepsilon + \frac{1}{2^{i+1}} \cdot \varepsilon = \frac{2^{i+1} - 1}{2^{i+1}} \cdot \varepsilon, \end{aligned}$$

and $\deg(A_{i+1}(s)) \leq \max(\deg(A_i(s)), 2) \leq \max(\deg(A(s)), 2)$.

(3) Since $\|A_{i+1}(s) - A_i(s)\|_{pm} < \hat{\varepsilon}$, we can apply Lemma 5.4 to each of the circles $\mathcal{C}_j$ defined in (39) separately. In this way we determine that for all $j \in \{1, \dots, \ell\}$, the number of zeros of $p_{i+1}(z)$ within $\mathcal{C}_j$ is equal to the number of zeros of $p_i(z)$ within $\mathcal{C}_j$ (counting multiplicities). This implies that the L_{p_i} circles containing a simple zero of $p_i(z)$ also contain exactly one (simple) zero of $p_{i+1}(z)$. Moreover, the multiple zero λ_1 has become simple by construction, and thus

$$L_{p_{i+1}} \geq L_{p_i} + 1 = L_p + i + 1.$$

This completes the proof of the induction argument. The correctness of Proposition 5.5 follows immediately by taking $\hat{A}(s) = A_{N_p - L_p}(s)$. $\square$

Proposition 5.5 shows that the matrix perturbations introduced in Proposition 4.6 can be used successively to reduce the multiplicity of zeros to 1. Rouché's theorem guarantees not only that the total number of zeros within the Jordan curve $\mathcal{J}$ remains constant but also that simple zeros remain simple. Combining Propositions 5.2 and 5.5, we can finish the first part of the proof as indicated in the introduction of this section by an appropriate choice of the Jordan curve $\mathcal{J}$.

THEOREM 5.6. *Let $A(s) \in \mathbb{R}[s]^{n \times n}$ and $\tau > 0$ be given. Then for all $\varepsilon > 0$ there exists a matrix $A_\varepsilon(s) \in \mathbb{R}[s]^{n \times n}$ such that*

- (i) $\|A(s) - A_\varepsilon(s)\|_{pm} < \varepsilon$,
- (ii) $\deg(A_\varepsilon(s)) \leq \max(\deg(A(s)), 2)$,
- (iii) $\forall \lambda \in \mathbb{C}^+ : \text{rank}(\lambda I - A_\varepsilon(e^{-\tau\lambda})) \geq n - 1$.

Proof. Let $\varepsilon > 0$. Choose according to Proposition 5.2 a matrix $A_1(s) \in \mathbb{R}[s]^{n \times n}$, of the same degree as $A(s)$, satisfying $\|A(s) - A_1(s)\|_{pm} < \frac{1}{2}\varepsilon$ and such that $p_1(z) = \det(zI - A_1(e^{-\tau z}))$ has no zeros on the imaginary axis.

Define $R := \|A_1(s)\|_{pm} + 1$ and the Jordan curve $\mathcal{J}$, as depicted in Figure 5.2, by

$$(40) \quad \mathcal{J} := \{z \in \mathbb{C} \mid (\text{Re } z = 0 \text{ and } |z| < R) \text{ or } (\text{Re } z \geq 0 \text{ and } |z| = R)\}.$$

So, according to Lemma 5.1, all zeros of $p_1(z) = \det(zI - A_1(e^{-\tau z}))$ in $\overline{\mathbb{C}^+}$ are located inside the Jordan curve $\mathcal{J}$. Let N_{p_1} denote this number of zeros of $p_1(z)$ within $\mathcal{J}$ (counting multiplicities). We choose $\bar{\varepsilon} > 0$ such that Lemma 5.4 is satisfied and apply Proposition 5.5 with $\tilde{\varepsilon} := \frac{1}{2} \cdot \min(1, \varepsilon, \bar{\varepsilon})$. Then we find a matrix $A_\varepsilon(s) \in \mathbb{R}[s]^{n \times n}$ such that

- (1) $\|A_1(s) - A_\varepsilon(s)\|_{pm} \leq \frac{1}{2} \cdot \min(1, \varepsilon, \bar{\varepsilon}) \leq \frac{1}{2}\varepsilon$,
- (2) $\deg(A_\varepsilon(s)) \leq \max(\deg(A_1(s)), 2)$,
- (3) $p_\varepsilon(z) = \det(zI - A_\varepsilon(e^{-\tau z}))$ has N_{p_1} zeros within $\mathcal{J}$ that are all simple.

Clearly, the matrix $A_\varepsilon(s)$ satisfies both (i) and (ii), so we have to prove only (iii). Since $\|A_\varepsilon(s)\|_{pm} < \|A_1(s)\|_{pm} + \frac{1}{2}$, Lemma 5.1 implies that $p_\varepsilon(z)$ has no zeros in $\overline{\mathbb{C}^+}$ outside $\mathcal{J}$. Moreover, since $\|A_1(s) - A_\varepsilon(s)\|_{pm} < \bar{\varepsilon}$, we know from Lemma 5.4 that $p_\varepsilon(z)$ has no zeros on $\mathcal{J}$. Therefore all zeros of $p_\varepsilon(z)$ in $\overline{\mathbb{C}^+}$ are located within $\mathcal{J}$. According to (3), all these zeros are simple and thus we have

$$(41) \quad \forall z \in \overline{\mathbb{C}^+} : [p_\varepsilon(z) = 0 \implies p'_\varepsilon(z) \neq 0].$$

 FIG. 5.2. The Jordan curve $\mathcal{J}$.

Now, let $\lambda \in \overline{\mathbb{C}^+}$, and assume that $\text{rank}(\lambda I - A_\varepsilon(e^{-\tau\lambda})) < n$. Then $p_\varepsilon(\lambda) = \det(\lambda I - A_\varepsilon(e^{-\tau\lambda})) = 0$, and thus according to (41), $p'_\varepsilon(\lambda) \neq 0$. Applying Corollary 4.3 to the matrix $(zI - A_\varepsilon(e^{-\tau z}))$, we obtain

$$\text{rank}(\lambda I - A_\varepsilon(e^{-\tau\lambda})) = n - 1.$$

This completes the proof. $\square$

In the second part of this section we are concerned with perturbations of the matrix $B(s)$. Suppose that a point $\Sigma = (A(s), B(s), \tau) \in \mathcal{V}$ is given. First perturb $A(s)$ in such a way that for $A_\varepsilon(s)$ conditions (i)–(iii) of Theorem 5.6 are satisfied. From Lemma 5.1 it follows that the analytic function $p_\varepsilon(z) = \det(zI - A_\varepsilon(e^{-\tau z}))$ has only a finite number of zeros in $\overline{\mathbb{C}^+}$, say $\lambda_1, \dots, \lambda_k$. We know that for each $i \in \{1, \dots, k\}$, $\text{rank}(\lambda_i I - A_\varepsilon(e^{-\tau\lambda_i})) = n - 1$, and therefore the left-kernel of the matrix $(\lambda_i I - A_\varepsilon(e^{-\tau\lambda_i}))$, i.e., the linear subspace of $\mathbb{C}^n$ consisting of all row vectors x^T such that $x^T \cdot (\lambda_i I - A_\varepsilon(e^{-\tau\lambda_i})) = 0$, is 1-dimensional. So for each $i \in \{1, \dots, k\}$, this left-kernel is spanned by one row vector $v_i^T \in \mathbb{C}^n$. Now $(\lambda_i I - A_\varepsilon(e^{-\tau\lambda_i}) \mid B(e^{-\tau\lambda_i}))$ has rank n if and only if

$$(42) \quad v_i^T \cdot B(e^{-\tau\lambda_i}) \neq 0.$$

So, to achieve stabilizability, we have to perturb $B(s)$ in such a way that for the perturbed version $B_\varepsilon(s)$ the following holds:

$$(43) \quad \forall i \in \{1, \dots, k\} : v_i^T \cdot B_\varepsilon(e^{-\tau\lambda_i}) \neq 0.$$

To find such a perturbation of $B(s)$, we first look for a vector b that is not perpendicular to a given finite set of vectors.

LEMMA 5.7. *Let the column vectors $v_1, \dots, v_k \in \mathbb{C}^n$ be given, and assume that they are all nonzero. Then there exists a vector $b \in \mathbb{R}^n$ such that*

$$\forall i \in \{1, \dots, k\} : v_i^T \cdot b \neq 0.$$

Proof. First define for all $i = 1, \dots, k$ the linear spaces

$$V_i := \{x \in \mathbb{R}^n \mid v_i^T \cdot x = 0\}.$$

Since all vectors v_i are nonzero, the sets V_i are linear subspaces of $\mathbb{R}^n$, with dimension smaller than or equal to $n - 1$. This implies that each V_i is a nowhere dense subset of $\mathbb{R}^n$. Application of Baire's category theorem (see, for example, [15, Thm. 5.6 and Rem. 5.7]) yields

$$\mathbb{R}^n \neq \bigcup_{i=1}^k V_i. \quad \square$$

Intuitively, the result of Lemma 5.7 is clear. The vectors $v_1, \dots, v_k$ correspond to linear subspaces $V_1, \dots, V_k$ in $\mathbb{R}^n$ of dimension smaller than or equal to $n - 1$. Now we simply have to pick a vector $b \in \mathbb{R}^n$ that is not an element of one of these subspaces $V_1, \dots, V_k$. Since we only consider a finite number of subspaces, this is a rather easy task.

Lemma 5.7 makes it possible to find a perturbation of the matrix $B(s)$ that is suitable for our purpose. This result is stated in the next lemma.

LEMMA 5.8. *Let the vectors $v_1, \dots, v_k \in \mathbb{C}^n$ and $b_1, \dots, b_k \in \mathbb{C}^n$ be given. Assume that for all $i \in \{1, \dots, k\}$: $\|v_i\| = 1$. Then for all $\varepsilon > 0$ there exists a vector $\beta \in \mathbb{R}^n$ such that*

- (i) $\|\beta\| < \varepsilon$,
- (ii) $\forall i \in \{1, \dots, k\} : v_i^T \cdot (b_i + \beta) \neq 0$.

Proof. Let $\varepsilon > 0$. Choose, according to Lemma 5.7, a vector $\gamma \in \mathbb{R}^n$ such that $v_i^T \cdot \gamma \neq 0$ for all $i \in \{1, \dots, k\}$. If for all $i \in \{1, \dots, k\}$ we have $v_i^T \cdot b_i = 0$, then $\beta = \frac{1}{2}\varepsilon \cdot \frac{\gamma}{\|\gamma\|}$ satisfies the claim. Otherwise, choose a $\rho \in (0, \min\{|v_i^T \cdot b_i| \mid v_i^T \cdot b_i \neq 0, i = 1, \dots, k\})$, and define

$$\beta := \frac{1}{2} \cdot \min(\varepsilon, \rho) \cdot \frac{1}{\|\gamma\|} \cdot \gamma.$$

Then (i) is clear: $\|\beta\| \leq \frac{1}{2} \cdot \varepsilon \cdot 1 < \varepsilon$. To prove (ii), let $i \in \{1, \dots, k\}$. If $v_i^T \cdot b_i = 0$, then

$$v_i^T \cdot (b_i + \beta) = v_i^T \cdot \beta = \frac{1}{2} \cdot (v_i^T \gamma) \cdot \frac{1}{\|\gamma\|} \cdot \min(\varepsilon, \rho) \neq 0.$$

On the other hand, if $v_i^T \cdot b_i \neq 0$, then

$$|v_i^T \cdot (b_i + \beta)| = |v_i^T b_i + v_i^T \beta| \geq |v_i^T b_i| - |v_i^T \beta| \geq \rho - \|v_i\| \cdot \|\beta\| \geq \rho - \frac{1}{2}\rho > 0.$$

So, in either case, $v_i^T \cdot (b_i + \beta) \neq 0$. $\square$

At this point, the proof outlined in the introduction of this section is almost complete. We have only to state and prove the main result.

THEOREM 5.9. *Let $\Sigma = (A(s), B(s), \tau) \in \mathcal{V}$ be given. For all $\varepsilon > 0$ there exists a point $\tilde{\Sigma} = (\tilde{A}(s), \tilde{B}(s), \tilde{\tau}) \in \mathcal{V}$ such that*

- (i) $d_{\mathcal{V}}(\Sigma, \tilde{\Sigma}) < \varepsilon$,
- (ii) $\deg(\tilde{A}(s)) \leq \max(\deg(A(s)), 2)$ and $\deg(\tilde{B}(s)) = \deg(B(s))$,
- (iii) *the time-delay system corresponding to $\tilde{\Sigma}$ is stabilizable, i.e.,*

$$\forall z \in \overline{\mathbb{C}^+} : \text{rank}(zI - \tilde{A}(e^{-\tilde{\tau}z}) \mid \tilde{B}(e^{-\tilde{\tau}z})) = n.$$

Proof. Let $\varepsilon > 0$. First apply Theorem 5.6 to $A(s)$, and choose a matrix $\tilde{A}(s) \in \mathbb{R}[s]^{n \times n}$ such that

- (1) $\|A(s) - \tilde{A}(s)\|_{pm} < \frac{1}{2}\varepsilon$,
- (2) $\deg(\tilde{A}(s)) \leq \max(\deg(A(s)), 2)$,
- (3) $\forall z \in \overline{\mathbb{C}^+} : \text{rank}(zI - \tilde{A}(e^{-\tau z})) \geq n - 1$.

According to Lemma 5.1, the function $\tilde{p}(z) = \det(zI - \tilde{A}(e^{-\tau z}))$ has only a finite number of zeros in $\overline{\mathbb{C}^+}$, say $\lambda_1, \dots, \lambda_k$. Only in these points, $(zI - \tilde{A}(e^{-\tau z}))$ loses rank, but still $\text{rank}(zI - \tilde{A}(e^{-\tau z})) = n - 1$. So the left-kernel of $(zI - \tilde{A}(e^{-\tau z}))$ is one-dimensional for all $z \in \{\lambda_1, \dots, \lambda_k\}$. Choose vectors $v_1, \dots, v_k$ of norm 1 in $\mathbb{C}^n$, spanning these left-kernels:

$$\forall i \in \{1, \dots, k\} : \text{span}(v_i) = \{x \in \mathbb{C}^n \mid x^\top \cdot (\lambda_i I - \tilde{A}(e^{-\tau \lambda_i})) = 0\}.$$

Denote for all $i \in \{1, \dots, k\}$ the first column of $B(e^{-\tau \lambda_i})$ by b_i . According to Lemma 5.8, there exists a $\beta \in \mathbb{R}^n$ such that $\|\beta\| < \frac{1}{2}\varepsilon$ and $v_i^\top \cdot (b_i + \beta) \neq 0$ for all $i = 1, \dots, k$.

Define $\tilde{B}(s)$ as the sum of $B(s)$ and the $n \times m$ matrix $(\beta \mid 0)$ consisting of the column β , completed with zeros:

$$\tilde{B}(s) := B(s) + (\beta \mid 0).$$

Then (i) and (ii) obviously hold, and we need to show only that $\tilde{\Sigma} = (\tilde{A}(s), \tilde{B}(s), \tau)$ satisfies (iii).

For this, let $z \in \overline{\mathbb{C}^+}$. If $z \notin \{\lambda_1, \dots, \lambda_k\}$, then $\text{rank}(zI - \tilde{A}(e^{-\tau z})) = n$, so certainly $\text{rank}(zI - \tilde{A}(e^{-\tau z}) \mid \tilde{B}(e^{-\tau z})) = n$.

Otherwise, suppose that $z = \lambda_i$ for certain $i \in \{1, \dots, k\}$. Let $x \in \mathbb{C}^n$ be such that

$$(44) \quad x^\top \cdot (\lambda_i I - \tilde{A}(e^{-\tau \lambda_i}) \mid \tilde{B}(e^{-\tau \lambda_i})) = 0.$$

Hence, $x^\top$ is an element of the left-kernel of $(\lambda_i I - \tilde{A}(e^{-\tau \lambda_i}))$, and there exists an $\alpha \in \mathbb{C}$ such that $x = \alpha \cdot v_i$. Now the first column of $\tilde{B}(e^{-\tau \lambda_i})$ is $b_i + \beta$, and

$$0 = x^\top \cdot (b_i + \beta) = \alpha v_i^\top \cdot (b_i + \beta) = \alpha \cdot [v_i^\top \cdot (b_i + \beta)].$$

We conclude that $\alpha = 0$. This completes the proof. $\square$

From Theorem 5.9 it follows directly that the subset S of $\mathcal{V}$, consisting of all parametrizations of stabilizable time-delay systems, is a dense subset of $\mathcal{V}$. Note that the conditions on the degrees of $\tilde{A}(s)$ and $\tilde{B}(s)$ are essential. According to Theorem 5.9, it is possible to construct a sequence of time-delay systems $(\Sigma_i)_{i=1}^\infty = (A_i(s), B_i(s), \tau_i)_{i=1}^\infty$ converging to $\Sigma = (A(s), B(s), \tau)$ (in the sense of §2) with the property that

$$\forall i \in \mathbb{N} : \deg(A_i(s)) \leq \max(\deg(A(s)), 2) \text{ and } \deg(B_i(s)) = \deg(B(s)).$$

This means that to achieve stabilizability, we have to perturb only a finite number of all parameters describing the original system Σ . Construction of a sequence of stabilizable systems converging to Σ , but with an increasing degree in s , is of no use for our genericity result because this requires systems with time-delays of constantly increasing length. Since we can always obtain a stabilizable system using perturbations of an a priori given degree, the result of Theorem 5.9 also holds within the framework of so-called inductive limit topologies, mentioned at the end of §2.

At this stage, our conjecture on the genericity of stabilizability for time-delay systems is reduced to a simple corollary from Theorems 3.1 and 5.9.

THEOREM 5.10. *Time-delay systems of the form (5) are generically stabilizable in the following sense: the subset S of the parameter-space $\mathcal{V}$, consisting of all parametrizations $\Sigma = (A(s), B(s), \tau)$ of time-delay systems satisfying*

$$\forall z \in \overline{\mathbb{C}^+} : \text{rank}(zI - A(e^{-\tau z}) \mid B(e^{-\tau z})) = n,$$

is an open and dense subset of the metric space $\mathcal{V}$.

6. Generalization to the case of incommensurable time-delays. In §§2–5, a derivation of our genericity result is given for systems with commensurable time-delays. This restriction was made only for notational convenience; the incommensurable delay case is not significantly more difficult. In this section we point out that with exactly the same arguments as before, the genericity result can also be proved for the more general class of systems with incommensurable time-delays.

In the algebraic terminology, systems with k incommensurable time-delays, given by $\tau_1, \dots, \tau_k$, are modeled as systems over the ring $\mathbb{R}[s_1, \dots, s_k]$, where the indeterminate s_i corresponds to the delay operator σ_i with time-delay τ_i . To apply a topological approach to our genericity problem, first a parameter-space $\mathcal{W}$ (the incommensurable version of $\mathcal{V}$) has to be introduced. Denoting $\mathbb{R}[s_1, \dots, s_k]$ by $\mathcal{R}$, $\mathcal{W}$ is defined as

$$\mathcal{W} := \{\Sigma = (A, B, (\tau_1, \dots, \tau_k)) \mid A \in \mathcal{R}^{n \times n}, B \in \mathcal{R}^{n \times m}, \tau_i \in \mathbb{R}^+ (i = 1, \dots, k)\}.$$

In the same way as in the commensurable delay case, a matrix over $\mathbb{R}[s_1, \dots, s_k]^{p \times q}$ can be seen as a k -dimensional sequence of $p \times q$ matrices over $\mathbb{R}$, with only a finite number of nonzero elements. So, application of an ℓ_1 -norm is possible, and in this way Definition 2.1 may be generalized. In the same way, polynomials in more than two indeterminates can be treated.

With these generalized definitions of the norms, the results of §2 remain valid. Most of these results rely on the fact that for all $z \in \overline{\mathbb{C}^+}$: $|e^{-\tau z}| \leq 1$. Since all time-delays τ_i are strictly larger than zero, we still have

$$(45) \quad \forall i \in \{1, \dots, k\} \quad \forall \tau_i > 0 \quad \forall z \in \overline{\mathbb{C}^+} : |e^{-\tau_i z}| \leq 1,$$

and the same proofs may be applied. The only difficulty left is the result on the continuity of the map χ from a polynomial matrix to its characteristic polynomial. Here exponentials do not play a role, but for this result the number of indeterminates is not significant at all, and therefore it also holds in the incommensurable delay case.

The results of §3 are easily generalized, as far as perturbations of the matrices $A(s_1, \dots, s_k)$ and $B(s_1, \dots, s_k)$ are concerned. Perturbations of the lengths of the time-delays are more complicated. However, because of (45), all perturbations of time-delays can be treated successively. In each step i ($i = 1, \dots, k$), the exponentials $e^{-\tau_1 z}, \dots, e^{-\tau_{i-1} z}$ and $e^{-\tau_{i+1} z}, \dots, e^{-\tau_k z}$, corresponding to all the other time-delays except τ_i , are bounded above by 1 in absolute value because we assume that $z \in \overline{\mathbb{C}^+}$. Therefore exactly the same techniques as in formula (22) may be applied successively for each τ_i separately to arrive at the desired result.

Section 4 is already put in a general context, so here nothing has to be done. Note however that in Proposition 4.6 only one time-delay is required to achieve an appropriate perturbation of the matrix $Q(z)$.

In the first part of §5 we are now dealing with analytic functions of the form

$$p(z) = \det(zI - A(e^{-\tau_1 z}, \dots, e^{-\tau_k z})).$$

The assumption on the absence of zeros on the imaginary axis can be removed in almost the same way as stated in Proposition 5.2. Trivially, Rouché's theorem is still valid, and it is also easily seen that all zeros of $p(z) = \det(zI - A(e^{-\tau_1 z}, \dots, e^{-\tau_k z}))$ in $\overline{\mathbb{C}^+}$ are contained in a compact subset of $\overline{\mathbb{C}^+}$. Therefore, Lemma 5.4 still holds and the same process of successively reducing the order of the zeros to 1 can be used. Again, Rouché's theorem guarantees that the total number of zeros in $\overline{\mathbb{C}^+}$ remains constant and that simple zeros remain simple. Moreover, the results of §4 imply that the condition on the degree of $A(s_1, \dots, s_k)$ is satisfied. Perturbations of the matrix $B(s)$ can be obtained in exactly the same way as described for the

commensurable delay case. Therefore Theorem 5.9 may also be generalized to delay systems with incommensurable time-delays.

Summarizing, we conclude that our genericity result for the stabilizability of time-delay systems with commensurable time-delays also holds in the incommensurable delay case. This final conclusion is stated in the last theorem.

THEOREM 6.1. *Time-delay systems with incommensurable time-delays of the form*

$$\dot{x}(t) = A(\sigma_1, \dots, \sigma_k)x(t) + B(\sigma_1, \dots, \sigma_k)u(t),$$

where σ_i ($i = 1, \dots, k$) denotes the delay operator corresponding to a time-delay τ_i , are generically stabilizable in the following sense: the subset of the parameter-space

$$\mathcal{W} = \{(A(s_1, \dots, s_k), B(s_1, \dots, s_k), (\tau_1, \dots, \tau_k)) \mid A(s_1, \dots, s_k) \in \mathbb{R}[s_1, \dots, s_k]^{n \times n}, \\ B(s_1, \dots, s_k) \in \mathbb{R}[s_1, \dots, s_k]^{n \times m} \text{ and } \forall i \in \{1, \dots, k\} : \tau_i > 0\},$$

consisting of all parametrizations $\Sigma = (A(s_1, \dots, s_k), B(s_1, \dots, s_k), (\tau_1, \dots, \tau_k))$ of time-delay systems satisfying

$$\forall z \in \overline{\mathbb{C}^+} : \text{rank}(zI - A(e^{-\tau_1 z}, \dots, e^{-\tau_k z}) \mid B(e^{-\tau_1 z}, \dots, e^{-\tau_k z})) = n,$$

is an open and dense subset of the metric space $\mathcal{W}$.

7. Conclusions. In this paper it was shown that time-delay systems with commensurable or incommensurable time-delays are generically stabilizable. First, an algebraic approach was used to model time-delay systems with point delays. For this class of systems, a topological framework was introduced to formalize the concept of genericity. In this setting it was shown that the set of stabilizable time-delay systems is an open and dense subset of the parameter-space describing all time-delay systems. This means that stabilizability is a robust property; it is preserved after small perturbations of the parameters. Moreover, a nonstabilizable time-delay system can be approximated arbitrarily close by a sequence of stabilizable time-delay systems. Therefore the property of stabilizability is very weak; it is generic in the sense described above.

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