Between controllable and uncontrollable

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Abstract

The distance between a system \((A,B)\) and the set of uncontrollable systems is the minimum of the smallest singular value of \([\lambda I - A,B]\) with respect to \(\lambda\).
INTRODUCTION

In [1], Paige shows that the traditional methods, providing "yes/no" answers, for testing the controllability of a system are not satisfactory in the sense that they may lead to wrong conclusions. Therefore a controllability measure, based on a continuous metric rather than a discrete metric seems to be useful. Such a measure can be constructed using the distance between a system and the set of uncontrollable systems. This is shown in [2] where also an algorithm for the computation of this distance is given.

In this paper it is shown that this distance can be computed by minimizing a function of one complex variable. In order to present the concepts and the computational procedures without getting involved in a number of non illustrative details which arises inevitably if all systems have to be real, we will deal with complex systems. The strictly real case can be developed along the lines of [2].

RESULTS

The set systems, having n states and m inputs, is \( \text{SYS}_{n,m} \),

\[
\text{SYS}_{n,m} = \left\{ (A,B) \mid A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times m} \right\}.
\]

In order to be able to measure the distance between two systems (A,B) and (F,G) in \( \text{SYS}_{n,m} \), we use the metric \( d \), defined by

\[
d <(A,B),(F,G)> = \| [A-F, B-G] \|_2
\]

where \( \| \|_2 \) is the spectral norm (see [3]). (The Frobenius norm leads to the same results.)

Because we want to compute the distance between a system (A,B) and the set
of uncontrollable systems we define \( \text{UNCO}_{n,m} \) (the set of uncontrollable system in \( \text{SYS}_{n,m} \)),

\[
\text{UNCO}_{n,m} = \left\{ (F,G) \mid (F,G) \in \text{SYS}_{n,m}, \exists \lambda \in \mathbb{C} : \text{rank}[\lambda I - F, G] < n \right\}.
\]

We will also need \( \text{UNCO}_{n,m,\lambda} \), the subset of \( \text{UNCO}_{n,m} \) having \( \lambda \in \mathbb{C} \) as an uncontrollable mode,

\[
\text{UNCO}_{n,m,\lambda} = \left\{ (F,G) \mid (F,G) \in \text{SYS}_{n,m}, \text{rank}[\lambda I - F, G] < n \right\}.
\]

Observe that

\[
\text{UNCO}_{n,m} = \bigcup_{\lambda \in \mathbb{C}} \text{UNCO}_{n,m,\lambda} \tag{1}
\]

The distance between a system \((A,B)\) and \(\text{UNCO}_{n,m}\) may now be defined to be (this number may be interpreted as a controllability measure)

\[
d < (A,B), \text{UNCO}_{n,m} > = \inf_{(F,G) \in \text{UNCO}_{n,m}} d < (A,B), (F,G) >.
\]

We now have

\[
d < (A,B), \text{UNCO}_{n,m} > = \inf_{\lambda \in \mathbb{C}} d < (A,B), \text{UNCO}_{n,m,\lambda} >. \text{ See (1)}. \]

For each \( \lambda \in \mathbb{C} \) we obtain

\[
d < (A,B), \text{UNCO}_{n,m,\lambda} > = \inf_{(F,G) \in \text{UNCO}_{n,m,\lambda}} d < (A,B), (F,G) >
\]

and

\[
\inf_{(F,G) \in \text{UNCO}_{n,m,\lambda}} \| [A - F, B - G] \|_2 = \sigma_{\min} [\lambda I - A, B] \tag{2}
\]

where \( \sigma_{\min} [\lambda I - A, B] \) is the (possibly multiple) smallest singular value of \( [\lambda I - A, B] \). See [3] for the equality in (2) as a property of the smallest singular value.
Because $\sigma_{\min} [\lambda I - A, B]$ is a continuous function of $\lambda$ and because $\sigma_{\min} [\lambda I - A, B] \to \infty$ for $|\lambda| \to \infty$, we may conclude with the following theorem.

**THEOREM.** The distance between $(A, B) \in \text{SYS}^n_m$ and the set of uncontrollable systems is

$$d < (A, B), \text{UNCO}^n_m > = \min_{n,m} \sigma_{\min} [\lambda I - A, B].$$

**Proof.** Obvious from the foregoing.

**Remarks.**
- Generally, the system $(F, G) \in \text{UNCO}^n_m$, such that $d < (A, B), \text{UNCO}^n_m > = d < (A, B), (F, G) >$, is complex.
- The distance between a system $(A, B)$ and the set of non-stabilizable systems in $\text{SYS}^n_m$ can be computed by minimizing over $\mathbb{C}_+ = \{c \in \mathbb{C}, \text{Re}(c) \geq 0\}$, instead of over $\mathbb{C}$, in the theorem. Generalizations similar to this one are easily made.

A number of examples can be found in [2].

**REFERENCES**

