A new approach to turning point theory

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Abstract

Consider the linear ODE \( \varepsilon \frac{d z}{dt} = A(t, \varepsilon) z + f(t, \varepsilon) \). Often turning points are related to eigenvalues of \( A \), changing sign on some (small) interval, as a function of \( t \). This is a somewhat deceptive model as was noted by others (notably Wasow) as well. For vector valued ODEs the directions of solutions may make the situation more complex. In particular, realizing that the (reasonable) requirement of well-conditioning (uniformly in \( \varepsilon \)) implies a dichotomy of the solution space of the model it is possible to get more insight in what a turning point is and what it cannot be. The results are directly generalizable to non-linear problems.

Some results from ongoing research based on the Riccati method will also be discussed.

1 Preliminaries

In this paper we consider \( n \)-dimensional linear singular perturbation problems of the form

\[
\varepsilon \frac{dz}{dt} = A(t, \varepsilon) z + f(t, \varepsilon), \quad t \in [-1, 1], \tag{1a}
\]

where \( \varepsilon \) is a small parameter, \( 0 < \varepsilon < \varepsilon_0 \) (\( \varepsilon_0 \) fixed). Moreover we assume that the boundary conditions (BCs)

\[
B_{-1}^c x(-1) + B_1^c x(1) = b_\varepsilon \tag{1b}
\]

are such that the boundary value problem (BVP) (1a,b) is uniformly well-conditioned. By this we mean that for each \( 0 < \varepsilon < \varepsilon_0 \) the BVP has a unique solution and that small perturbations in the data have just a minor influence on the solution.

The assumption of well-conditioning implies for instance that the family of solution spaces \( S^\varepsilon \) of the homogeneous part of (1a) is uniformly dichotomic (cf. [2]), which is defined by

Definition 1.1

The family of solution spaces \( S^\varepsilon \) of the homogeneous ODE

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\[
\begin{align*}
\epsilon \frac{dx}{dt} &= A(t, \epsilon) x, \quad (t, \epsilon) \in [-1, 1] \times (0, \epsilon_0), \\
\end{align*}
\] (2)

is uniformly dichotomic if there exist constants \(k (1 \leq k < n)\) and \(K\) (of moderate size) such that, for each \(\epsilon \in (0, \epsilon_0)\), there exists a fundamental solution \(Y\) of (2) satisfying

\[
\begin{align}
\|Y(t, \epsilon) [I_k 0]^{-1}(s, \epsilon)\| &\leq K, \quad -1 \leq t \leq s \leq 1, \\
\|Y(t, \epsilon) [0 0]^{-1}(s, \epsilon)\| &\leq K, \quad 1 \geq t \geq s \geq -1.
\end{align}
\] (3a) (3b)

Such a uniform dichotomy implies that the solution space \(S^e\) of (2) can be split in two subspaces. Write \(Y(t, \epsilon) = \begin{bmatrix} Y_1(t, \epsilon) & Y_2(t, \epsilon) \end{bmatrix}\) and define the solution subspaces \(S_1^e\) and \(S_2^e\) by

\[
\begin{align}
S_1^e(t) &:= \left\{ Y_1(t, \epsilon) c_1 \mid c_1 \in \mathbb{R}^k \right\}, \quad t \in [-1, 1], \\
S_2^e(t) &:= \left\{ Y_2(t, \epsilon) c_2 \mid c_2 \in \mathbb{R}^{n-k} \right\}, \quad t \in [-1, 1].
\end{align}
\] (4a) (4b)

Then (3a) implies that a solution in \(S_1^e\) is nowhere fast decreasing, while (3b) implies that a solution in \(S_2^e\) is nowhere fast increasing. Therefore \(S_1^e\) is called the dominant solution subspace and \(S_2^e\) the dominated solution subspace.

Moreover we shall make the following not really restrictive

**Assumption 1.2**

\[
\min_{0 \neq \phi_1, \phi_2} \left\{ \frac{\|\phi_1(-1, \epsilon)\|}{\|\phi_1(1, \epsilon)\|} \right\} \rightarrow \infty \quad \text{as} \quad \epsilon \rightarrow 0.
\]

This extra assumption implies that any solution in \(S_1^e\) has to be somewhere fast increasing and therefore \(S_1^e\) cannot contain solutions that are smooth in norm everywhere.

In the sequel we shall use the following

**Definition 1.3**

A mode \(z_1 \in S_1^e\), respectively \(z_2 \in S_2^e\), is called a normalized basis mode if \(\|z_1(1, \epsilon)\| = 1\), respectively \(\|z_2(-1, \epsilon)\| = 1\).

We finish this section with the remark that all the obtained results can simply be generalised to non-linear problems. In that case we have to consider the corresponding Jacobian.
2 The definition of a turning point

Very often the notion of a turning point is related to a change in sign of an eigenvalue of the matrix $A$, considered as a function of $t$. Though this might be the case indeed, this is not contradicting the statement that the underlying well-conditioned problem has a (uniform) dichotomy. In order to illustrate what may happen study the following simple problem first.

Consider the ODE

$$\epsilon \frac{d^2 u}{dt^2} + 2t \frac{du}{dt} = 0, \quad t \in [-1, 1],$$

with the BCs given by

$$u(-1) = b_1 \quad \text{and} \quad u(1) = b_2.$$  \hspace{1cm} (5a)

We transform this to a first order system by writing

$$z_1 = \sqrt{\pi \epsilon} \frac{du}{dt} \quad \text{and} \quad z_2 = u.$$  \hspace{1cm} (6)

So we obtain

$$\epsilon \frac{d}{dt} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -2t & 0 \\ \sqrt{\pi} / \epsilon & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad t \in [-1, 1],$$

with obvious BCs. Define

$$E(t, \epsilon) = e^{-t^2 / \epsilon} \quad \text{and} \quad I(t, \epsilon) = \frac{1}{\sqrt{\pi \epsilon}} \int_{-\infty}^{t} E(\tau, \epsilon) d\tau.$$  \hspace{1cm} (7)

The two basis solutions $y_1$ and $y_2$, given by

$$y_1(t, \epsilon) = \begin{pmatrix} E(t, \epsilon) \\ I(t, \epsilon) \end{pmatrix} \quad \text{and} \quad y_2(t, \epsilon) = \begin{pmatrix} -E(t, \epsilon) \\ 1 - I(t, \epsilon) \end{pmatrix}$$  \hspace{1cm} (8)

do not make small angles as $\epsilon \to 0$, for any $t$. Moreover, $y_1(t, \epsilon)$ is fast increasing for $t \leq 0$ and smooth for $t > \sqrt{\epsilon}$, whereas $y_2(t, \epsilon)$ is smooth for $t < -\sqrt{\epsilon}$ and fast decreasing for $t > \sqrt{\epsilon}$. Hence, we have a uniform dichotomy. Note, however, that in this case not only the non-trivial eigenvalue of the system matrix in (7) changes sign at $t = 0$, but that also the corresponding eigensystem rotates very fast (and finally degenerates) around $t = 0$. In order to obtain a well-conditioned problem this rotational activity is essential. Hence, the often correct idea that there exist basis solutions which directions follow the eigenvectors (as a function of $t$) does not make sense here.

Now consider the general linear singularly perturbed homogeneous ODE

$$\epsilon \frac{dz}{dt} = A(t, \epsilon) z, \quad t \in [-1, 1].$$  \hspace{1cm} (9)

Any solution $z$ of (9) can be decomposed as

$$z = w q,$$  \hspace{1cm} (10)
where \( w = \| x \| \) is the size of \( x \) and \( q \) indicates the direction of \( x \) on the unit sphere. Then we have

\[
\frac{dq}{dt\epsilon} = A(t, \epsilon) q - q \lambda(t, \epsilon), \quad t \in [-1, 1],
\]

(11a)

\[
\frac{dw}{dt\epsilon} = \lambda(t, \epsilon) w, \quad t \in [-1, 1],
\]

(11b)

where

\[
\lambda(t, \epsilon) = q^T(t, \epsilon) A(t, \epsilon) q(t, \epsilon).
\]

(11c)

We now like to call a point \( t = \xi \) a turning point if either \( \epsilon^{-1}\lambda(t, \epsilon) \) or \( q(t, \epsilon) \) changes an order of magnitude at \( \xi \) for some normalized basismode \( x(t, \epsilon) \) (see [3] and also [6]).

As an application consider the previous example, e.g. the solution \( y_1 \). We may characterize the direction by the tangent or the cotangent of the solution in the \((1, 2)\)-plane with respect to the abscissa. For \( h > 0 \) fixed we obtain

\[
I(-h, \epsilon)/E(-h, \epsilon) \to 0, \quad \epsilon \to 0
\]

\[
E(h, \epsilon)/I(h, \epsilon) \to 0, \quad \epsilon \to 0.
\]

Hence, the direction switches faster than \( O(h) \) around \( t = 0 \). We also have that

\[
\lambda(t, \epsilon) = \frac{\sqrt{\epsilon/\pi} IE - 2tE^2}{I^2 + E^2}.
\]

It is easy to see that \( \epsilon^{-1}\lambda(-h, \epsilon) \) is very large and \( \epsilon^{-1}\lambda(h, \epsilon) \to 0 \), for fixed \( h \) and \( \epsilon \to 0 \). Hence, \( \epsilon^{-1}\lambda \) switches faster than \( O(h) \) around \( t = 0 \). So we have a turning point both with respect to direction and with respect to size.

### 3 Turning points and the Riccati method

A particularly interesting method for singularly perturbed problems is the Riccati method (cf. [3]). For simplicity we restrict ourselves to the second order case here. As is well known the solution of the (quadratic) Riccati ODE is then effectively the tangent of the direction of a dominant mode. In case of a turning point this direction, and consequently also its tangent, changes drastically. The question arises whether an IVP integrator, used to solve the Riccati equation, will be able to follow this (potentially whimsical) behaviour described in section 2. (It goes without saying that this integrator should be implicit).

Consider more specifically the second order scalar problem

\[
\frac{d^2u}{dt^2} - a(t) \frac{du}{dt} = 0, \quad t \in [-1, 1],
\]

(12)

with \( a(t) < 0 \) (for all \( t \in [-1, 1] \)), subject to the BCs

\[
u(-1) + \epsilon \frac{du}{dt}(-1) = 0 \quad \text{and} \quad u(1) \text{ given}.
\]

(13)
We form a first order system by writing $x_1 := u$ and $x_2 := e \frac{du}{dt}$, so

$$
\varepsilon \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & a(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad t \in [-1, 1],
$$

(14)

subject to $x_1(-1) + x_2(-1) = 0$ and $x_1(1)$ given.

Using the Riccati transformation $T(t, e) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we can obtain a decoupled upper triangular system for a new variable, provided (cf. [5])

$$
\frac{dr}{dt} = (a(t) - r) r, \quad t \in [-1, 1], \quad \text{subject to } r(-1, e) = -1.
$$

(15)

Hence, as $\varepsilon \downarrow 0$ we see that $r(t, e) \equiv 0$ is a stable (forced) solution of (15) and there exists an unstable (forced) solution $r(t, e) \approx a(t)$ (i.e. stable from $t = 1$ backward). Consequently, if $a(-1) > r(-1, e) = -1$ then $r$ will rapidly become very large negative and if $a(-1) < -1$ then $r$ will rapidly approach 0 (as $\varepsilon \downarrow 0$).

Let us e.g. consider the case $a(-1) < -1$ and consider the Euler Backward method to find an approximate solution $\{r_i\}$, where $r_i \approx r(t_i)$ and $\{t_i\}$ a certain mesh. For a given approximation $r_i$, we obtain

$$
r_{i+1} = -\frac{1}{2} \left( \frac{\varepsilon}{h_i} - a(t_{i+1}) \right) + \varepsilon r_i \frac{1}{h_i} + \frac{1}{4} \left( \frac{\varepsilon}{h_i} - a(t_{i+1}) \right)^2, \quad (h_i = t_{i+1} - t_i).
$$

(16)

Most likely the routine will choose such a sign in (16) that $|r_{i+1} - r_i|$ is smallest. If $h_i$ is comparatively large it may easily happen that $a(t_{i+1}) - 2r_i > \frac{\varepsilon}{h_i}$ and so the wrong choice (a minus sign) in (16) will occur. Thus, if $r_i$ is close to $a(t_{i+1})$ then $r_{i+1}$ will be close to $a(t_{i+1})$ too, contradicting the analytical prediction that $r(t, e) \to 0$ rapidly.

The phenomenon described above is related to the superstability property of the (otherwise very useful) BDF methods (cf. [1]): because of the strong damping for $h_i \gg \varepsilon$, both in stable and unstable situations, the routine is not able to find out that it may be on an analytically meaningless solution curve.

There are a number of reasons why the numerical solution of a Riccati equation may go astray. Since the latter equation is second order it makes sense to assume that there exist two solutions of this Riccati equation, $\eta(t, e)$ and $\xi(t, e)$, which characterise the direction of a forward stable and a backward stable mode, respectively. A first reason then may be that $r(-1, e)$ is too close to $\xi(-1, e)$ as $\varepsilon \to 0$. We remark, however, that this relates to a very ill-conditioned problem.

Another possibility is that the discrete solution $\{r_i\}$ (obtained via a BDF method) switches from a "solution curve" close to $\{\eta(t_i, e)\}$ to a "curve" close to $\{\xi(t_i, e)\}$ (see Figure 3.1).

This may happen if the local error, caused by the integration method (being proportional to some power of $h_i$) is of the order of the distance $|\eta(t_i, e) - \xi(t_i, e)|$. A potentially high activity of $\eta$ and $\xi$ on $(t_i, t_{i+1})$ may then not be noticed. This kind of (directional) turning points require a more sophisticated way of integrating the Riccati equation (see [3]) or a more general transformation, about which will be reported elsewhere (see also [4]).
References


