One-parameter co-semigroups on sequentially complete locally convex topological vector spaces

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One-parameter $c_0$-semigroups on sequentially complete locally convex topological vector spaces

by

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Introduction

In the past decades research has been done on one-parameter \(c_0\)-semigroups of continuous linear operators on complete, locally convex, vectorspaces, mainly with the intention to find alternatives for the celebrated Hille–Yosida theorem for one parameter \(c_0\)-semigroups on Banach spaces. But the simple example of the translation group on the Fréchet space \(C(\mathbb{R})\) of continuous functions from \(\mathbb{R}\) into \(\mathbb{C}\) illustrates that the resolvent set of the infinitesimal generator of a \(c_0\) group on a Fréchet space can be empty. So a characterization of the infinitesimal generator in terms of the resolvent is not likely and one replaces the notion of resolvent by the notion of asymptotic resolvent (cf. [Wa], [Ok]) and distributional resolvent (cf. [Kom]). Also we refer to [Bab] and [Dem].

The intentions of this paper are different. We analyse one-parameter \(c_0\)-semigroups \((\pi_t)_{t \geq 0}\) on locally convex vectorspaces \(V\) by exploring the relationship between \((\pi_t)_{t \geq 0}\) and the \(c_0\)-semigroup \((\sigma_t)_{t \geq 0}\) of translations on \(C(\mathbb{R}^+, V)\), the space of continuous functions from \(\mathbb{R}^+\) into \(V\). Techniques, used in the aforementioned papers, becomes more transparent and get a natural setting. Let us first introduce some notational conventions.

Throughout this paper by \(V\) a sequentially complete, locally convex, vector space is denoted. The locally convex topology of \(V\) is assumed to be brought about by a separating collection of seminorms, \(\{s_\nu \mid \nu \in \mathcal{D}\}\), where \(\mathcal{D}\) is a directed set such that for all \(\nu_1, \nu_2 \in \mathcal{D}\)

\[
\nu_1 \leq \nu_2 \Rightarrow \forall x \in V : s_{\nu_1}(x) \leq s_{\nu_2}(x). 
\]

Since each finite subset of \(\mathcal{D}\) has an upperbound a linear operator \(L\) from \(V\) into \(V\) is continuous if and only if

\[
(0.1) \quad \forall \nu \in \mathcal{D} \exists \rho \in \mathcal{D} \exists C > 0 \forall x \in V : s_\nu(Lx) \leq Cs_\rho(x). 
\]

Further, a collection \(\Lambda\) of linear operators on \(V\) is equicontinuous if and only if

\[
(0.2) \quad \forall \nu \in \mathcal{D} \exists \rho \in \mathcal{D} \exists C > 0 \forall \nu \in \mathcal{A} \forall x \in V : s_\nu(Lx) \leq Cs_\rho(x). 
\]

The concepts (0.1) and (0.2) are used frequently in this paper. For other elementary topics of the theory of locally convex vector spaces we refer to the monographs of Conway [Con], Schaefer [Sch] and Treves [Tre]. They are used without further reference.

Let \((\pi_t)_{t \geq 0}\) be a locally equicontinuous \(c_0\)-semigroup of continuous linear mappings on \(V\). Then the flow operator \(\mathcal{F}_x\) on \(V\), defined by

\[
(0.3) \quad (\mathcal{F}_x(t))(t) = \pi_t x, \quad t \in \mathbb{R}^+, \quad x \in V,
\]

is a continuous linear mapping from \(V\) into \(C(\mathbb{R}^+, V)\) where the latter space is endowed with the compact–open topology, i.e. the topology of uniform convergence on compacta. For \((\sigma_t)_{t \geq 0}\) denoting the translation semigroup on \(C(\mathbb{R}^+, V)\)

\[
(0.4) \quad (\sigma_t f)(\tau) = f(t + \tau), \quad t, \tau \in \mathbb{R}^+, \quad f \in C(\mathbb{R}^+, V),
\]

is a continuous linear mapping from \(C(\mathbb{R}^+, V)\) into \(C(\mathbb{R}^+, V)\).
there is the intertwining relation

\[(0.5) \quad \sigma_t \mathcal{F}_r = \mathcal{F}_r \pi_t, \quad t \in \mathbb{R}^+ .\]

The basic idea of the paper is first to study properties of the special semigroup \((\sigma_t)_{t \geq 0}\) on \(C(\mathbb{R}^+, V)\) which then carry over to arbitrary locally equicontinuous semigroups \((\sigma_t)_{t \geq 0}\) by means of the flow operator \(\mathcal{F}_r\) using (0.5). We sketch briefly the results of this paper.

First we introduce integration and differentiation in \(C(\mathbb{R}^+, V)\). For that we introduce \(\text{bv}_c(\mathbb{R}^+)\) as the collection of all left continuous functions from \(\mathbb{R}\) into \(C\), with bounded variation so that for some \(T > 0\) dependent of the choice of \(\mu\)

\[
\mu(t) = 0, \quad t \leq 0 \text{ and } \mu(t) = \mu(T), \quad t \geq T .
\]

Each \(\mu \in \text{bv}_c(\mathbb{R}^+)\) induces an integration operator \(I[\mu]\) on \(C(\mathbb{R}^+, V)\),

\[
I[\mu]f = \int_{\mathbb{R}^+} f(\tau) d\mu(\tau), \quad \text{(Riemann–Stieltjes)},
\]

and \(I[\mu]\) is continuous from \(C(\mathbb{R}^+, V)\) into \(V\). With this concept of integration, we define the concept of differentiation, introducing the spaces \(C^k(\mathbb{R}^+, V)\) of \(k\)-times differentiable functions from \(\mathbb{R}^+\) into \(V\) together with the differentiation operator \(\mathcal{D}\). We prove that \(\mathcal{D}\) is the infinitesimal generator of the semigroup \((\sigma_t)_{t \geq 0}\). One of the key results is the following

\[(0.6) \quad \text{Let } p \text{ be a polynomial. Then } p(\mathcal{D}) \text{ with domain } C^k(\mathbb{R}^+, V) \text{ where } k = \text{degree}(p) \text{ is closed as a densely defined operator in } C(\mathbb{R}^+, V). \text{(cf. theorem 2.5.)}\]

For the proof of this statement, we show as an auxiliary result that \(p(\mathcal{D})\) has a continuous right inverse and that its null space

\[
\{ f \in C^k(\mathbb{R}^+, V) \mid p(\mathcal{D}) f = 0 \}
\]

is closed.

The set \(\text{bv}_c(\mathbb{R}^+)\) is a convolution ring with identity and without zero divisors. We introduce the convolution operators on \(C(\mathbb{R}^+, V)\),

\[(0.7) \quad \sigma[\mu]f = \int_{\mathbb{R}^+} \sigma_t f d\mu(t), \quad f \in C(\mathbb{R}^+, V), \quad \mu \in \text{bv}_c(\mathbb{R}^+)\]

and with it the concept of approximate identity, i.e. a sequence \((\mu_k)_{k \in \mathbb{N}}\) in \(\text{bv}_c(\mathbb{R}^+) \cap C^\infty(\mathbb{R})\) satisfying

\[(0.8) \quad \sigma[\mu_k]f \to f \text{ as } k \to \infty \]

for all \(f \in C(\mathbb{R}^+, V)\). The collection \(\text{bv}_c(\mathbb{R}^+) \cap C^\infty(\mathbb{R})\) consists of mollifiers, so
By (0.8) and (0.9), $C^\infty(\mathbb{R}^+, V)$ is sequentially dense in $C(\mathbb{R}^+, V)$. The above results for the translation semigroup $(\sigma_t)_{t\geq 0}$ are applied to prove the following for a locally equicontinuous $c_0$-semigroup $(\pi_t)_{t\geq 0}$ on $V$.

(0.10) Let $(\delta_x)$ denote the infinitesimal generator of $(\pi_t)_{t\geq 0}$ with domain $\text{dom}(\delta_x)$. Then for all $k \in \mathbb{N}$, $x \in \text{dom}(\delta^k_x) \Rightarrow \mathcal{F}_x x \in C^k(\mathbb{R}^+, V)$. (Cf. Lemma 5.1.)

(0.11) Let $p$ be a polynomial. Then $p(\delta_x)$ with domain $\text{dom}(\delta^k_x)$, $k = \text{degree}(p)$, is closed as a densely defined operator in $V$. (Cf. Theorem 5.4.)

(0.12) For $\mu \in \text{bv}_c(\mathbb{R}^+)$ define $\pi[\mu]$ on $V$ by

$$\pi[\mu]x = \int_{\mathbb{R}^+} \pi_t x \, d\mu(t), \quad x \in V.$$ 

Then $\sigma[\mu]\mathcal{F}_x = \mathcal{F}_x \pi[\mu]$, and if $\mu \in C^\infty(\mathbb{R})$

$$\forall x \in V : \pi[\mu]x \in \text{dom}^\infty(\delta_x).$$

(0.13) There exists a sequence $(\mu_k)_{k \in \mathbb{N}}$ in $\text{bv}_c(\mathbb{R}^+) \cap C^\infty(\mathbb{R})$ such $\pi[\mu_k]x \to x$ as $k \to \infty$ for all $x \in V$. (Cf. Lemma 4.7.)

(0.14) Let $M \subset V$ be a closed subspace with $\pi_t(M) \subseteq M$ for all $t \in \mathbb{R}^+$. Then $\pi[\mu](M) \subseteq M$ for all $\mu \in \text{bv}_c(\mathbb{R}^+)$ and $M \cap \text{dom}^\infty(\delta_x)$ is dense in $M$. (Cf. Lemma 6.2 and 6.3.)

(0.15) Let $K$ be a densely defined closed linear operator in $V$ satisfying

$$\forall t \in \mathbb{R}^+ \pi_t(\text{dom}(K)) \subseteq \text{dom}(K), \quad \text{dom}^\infty(\delta_x) \subseteq \text{dom}(K)$$

and

$$\forall t \in \mathbb{R}^+ \forall x \in \text{dom}(K) K \pi_t x = \pi_t K x.$$ 

Then $\text{dom}^\infty(\delta_x)$ is a core for $K$, i.e.

$$\text{graph}(K) = \{ [x; Kx] \mid x \in \text{dom}^\infty(\delta_x) \}.$$ 

(Cf. Theorem 6.7.)

1 The space $C(\mathbb{R}^+, V)$, integration

By $C(\mathbb{R}^+, V)$ we denote the vector space of all continuous functions from $\mathbb{R}^+$ into $V$. Here $\mathbb{R}^+$ is the closed semi–infinite interval $[0, \infty)$. So a function $f : \mathbb{R}^+ \to V$ belongs to $C(\mathbb{R}^+, V)$ if and only if

$$\forall t \in \mathbb{R}^+ \forall \nu \in D \forall \epsilon > 0 \exists \delta > 0 \forall s \in \mathbb{R} : |t - s| < \delta \Rightarrow s_\nu(f(t) - f(s)) < \epsilon.$$
The triangle inequality ensures that \( t \mapsto s_\nu(f(t)) \) is continuous from \( \mathbb{R}^+ \) into \( \mathbb{R}^+ \) for each \( \nu \in \mathbb{D} \). Hence on \( C(\mathbb{R}^+, V) \) the following seminorms are defined: for \( \nu \in \mathbb{D} \) and \( K \subset \mathbb{R}^+ \) compact,

\[
s_{\nu,K}(f) := \max_{t \in K} s_\nu(f(t)) ;
\]

\( C(\mathbb{R}^+, V) \) is endowed with the related locally convex topology. In fact, this is the compact-open topology; so a net in \( C(\mathbb{R}^+, V) \) is convergent iff it converges uniformly on each compact subset \( K \) of \( \mathbb{R}^+ \).

Sequential completeness of \( V \) transfers to the space \( C(\mathbb{R}^+, V) \).

**Proposition 1.1.** The locally convex space \( C(\mathbb{R}^+, V) \) is sequentially complete.

Also, as in the classical situation with \( V = C \) we have

**Proposition 1.2.** For each \( K \subset \mathbb{R}^+ \) compact and each \( f \in C(\mathbb{R}^+, V) \), the restriction \( f|_K \) is uniformly continuous from \( K \) into \( V \), i.e.

\[
\forall \nu \in \mathbb{D} \quad \forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall t, s \in K : |t - s| < \delta \Rightarrow s_\nu(f(t) - f(s)) < \epsilon .
\]

Next we introduce Riemann–Stieltjes integration on \( C(\mathbb{R}^+, V) \). As in [DS] we define the space \( bV(\mathbb{R}^+) \) consisting of all left-continuous functions \( \mu \) from \( \mathbb{R} \) into \( C \) with \( \mu(t) = 0 \), \( t \leq 0 \) for which there is \( A > 0 \) such that for any \( m \in \mathbb{N} \) and ordered \( m \)-tuple, \( 0 = t_0 < t_1 < ... < t_m \),

\[
\sum_{j=1}^{m} |\mu(t_j) - \mu(t_{j-1})| \leq A .
\]

Then \( \text{var}(\mu) \), the variation of \( \mu \), is the infimum of all constants \( A \) which satisfy (*)'. By \( b\text{v}_c(\mathbb{R}^+) \) the subspace of \( bV(\mathbb{R}^+) \) is denoted, consisting of those \( \mu \in bV(\mathbb{R}^+) \) for which \( T > 0 \) exists such that \( \mu(t) = \mu(T) \) for \( t \geq T \).

Now take a fixed \( \mu \in b\text{v}_c(\mathbb{R}^+) \), with corresponding \( T > 0 \). Let \( \mathcal{P}[0,T] \) denote the set of all partitions of \([0,T]\) with the usual partial ordering and define for \( \alpha \in \mathcal{P}[0,T] \), \( \alpha = < m; t_0, ..., t_m > \) the linear operator \( I_{\mu,\alpha} \) on \( C(\mathbb{R}^+, V) \) by

\[
I_{\mu,\alpha} f = \sum_{j=1}^{m} (\mu(t_j) - \mu(t_{j-1})) f(t_{j-1}) .
\]

Then \( I_{\mu,\alpha} \) is continuous from \( C(\mathbb{R}^+, V) \) into \( V \). In particular with

\[
\alpha_n = < 2^n; t_{n,0}, ..., t_{n,2^n} > , \quad t_{n,j} = \frac{j}{2^n} T ,
\]

we have \( \alpha_n \prec \alpha_{n+1} \) and for each \( f \in C(\mathbb{R}^+, V) \) due to the uniform continuity of \( f \) on \([0,T]\), the sequence \((I_{\mu,\alpha_n} f)\) is Cauchy in \( V \). Let \( I_{\mu} f \) denote its limit. Again the uniform continuity of \( f \) on \([0,T]\) guarantees that the net \((I_{\mu,\alpha} f)_{\alpha \in \mathcal{P}[0,T]}\) converges to \( I_{\mu} f \). Instead of \( I_{\mu} f \) we use also the more suggestive integral notation
From the above construction it follows that for each \( \nu \in \mathcal{M} \), \( \mu \in \text{bv}_c(\mathbb{R}^+) \) and \( f \in C(\mathbb{R}^+, V) \)

\[
\int \nu f d\mu \quad \text{or} \quad \int f(\tau) d\mu(\tau).
\]

with \( K = [0, T], T \) sufficiently large. So \( I_\mu \) is a continuous linear operator from \( C(\mathbb{R}^+, V) \) into \( V \). Besides, for each continuous linear functional \( \mathcal{L} \) on \( V \) and \( f \in C(\mathbb{R}^+, V) \)

\[
(\mathcal{L} \circ I_\mu)(f) = \int_{\mathbb{R}^+} \mathcal{L}(f(t)) d\mu(t)
\]
as an ordinary Riemann–Stieltjes integral.

By taking \( \mu_{a,b} \in \text{bv}_c(\mathbb{R}^+) \) with

\[
\mu_{a,b}(t) = \begin{cases} 
0 , & t < a , \\
 t - a , & a \leq t < b , \\
b - a , & t \geq b , 
\end{cases}
\]

we define for \( f \in C(\mathbb{R}^+, V) \)

\[
\int_{a}^{b} f(\tau) d\tau := \int_{\mathbb{R}^+} f d\mu_{a,b}.
\]

We shall use frequently the following simple observation

**Proposition 1.3.** For each complex valued \( \varphi \in C(\mathbb{R}) \) and \( f \in C(\mathbb{R}^+, V) \), \( t \mapsto \varphi(t) f(t) \) from \( \mathbb{R}^+ \) into \( V \) is continuous.

We proceed by introducing the "primitivation" operator \( J \) on \( C(\mathbb{R}^+, V) \). For \( f \in C(\mathbb{R}^+, V) \) we let \( Jf : \mathbb{R}^+ \to V \) be defined by

\[
(Jf)(t) = \int_{0}^{t} f(\tau) d\tau , \quad t \in \mathbb{R}^+ .
\]

Then \( Jf \in C(\mathbb{R}^+, V) \), because for \( t, s \in [0, a], a > 0, \)

\[
s_\nu(Jf(t) - Jf(s)) \leq |t - s| s_\nu(0,a)(f) ,
\]

which shows also that \( J \) is continuous, from \( C(\mathbb{R}^+, V) \) into \( C(\mathbb{R}^+, V) \).

Next we show that \( J \) is injective. Let \( Jf = 0 \). Then for all continuous linear functionals \( \mathcal{L} \) on \( V \) and all \( t \in \mathbb{R}^+ \)
\[ 0 = \mathcal{L}(J f(t)) = \int_0^t \mathcal{L}(f(\tau))d\tau. \]

So from ordinary calculus we get \( \mathcal{L}(f(t)) = 0 \) for all \( t \in \mathbb{R}^+ \) and \( \mathcal{L} \in V^* \). We conclude that \( f = 0 \).

It follows that for all \( k \in \mathbb{N} \) the operator \( J^k \) is injective, and because for all \( \mathcal{L} \in V^* \)

\[ \mathcal{L}(J^k f(t)) = \int_0^t \int_0^t \cdots \int_0^t \mathcal{L}(f(t_k))dx_k\ldots dx_1 = \int_0^t \frac{(t - \tau)^{k-1}}{(k-1)!} \mathcal{L}(f(\tau))d\tau \]

we see that

\[ (J^k f)(t) = \int_0^t \frac{(t - \tau)^{k-1}}{(k-1)!} f(\tau)d\tau. \]

For \( \lambda \in C \) we define \( \mathcal{E}_\lambda \) on \( C(\mathbb{R}^+, V) \) by

\[ (\mathcal{E}_\lambda f)(t) = e^{-\lambda t} f(t), \quad t \in \mathbb{R}^+, \]

and \( J(\lambda) \) by

\[ J(\lambda) = \mathcal{E}_{-\lambda} J \mathcal{E}_\lambda. \]

Then \( \mathcal{E}_\lambda \) and \( J(\lambda) \) are continuous from \( C(\mathbb{R}^+, V) \) into \( C(\mathbb{R}^+, V) \) with \( J(\lambda)^k = \mathcal{E}_\lambda J^k \mathcal{E}_{-\lambda} \) so that

\[ (J(\lambda)^k f)(t) = \int_0^t \frac{(t - \tau)^{k-1}}{(k-1)!} e^{\lambda(t-\tau)} f(\tau)d\tau. \]

2 The spaces \( C^k(\mathbb{R}^+, V) \), differentiation

In this section we define the concept of differentiation in \( C(\mathbb{R}^+, V) \).

By \( C^k(\mathbb{R}^+, V) \) we denote the subspace of \( C(\mathbb{R}^+, V) \) consisting of all \( f \in C(\mathbb{R}^+, V) \) for which there exist a \( V \)-valued polynomial \( q \) of degree \( \leq k - 1 \)

\[ q(t) = x_0 + tx_1 + \cdots + t^{k-1}x_{k-1} \]

with \( x_0, \ldots, x_{k-1} \in V \), and a \( g \in C(\mathbb{R}^+, V) \) such that

\[ f = q + J^k g. \]

If \( f \) can be represented this way, then this representation is unique. Indeed,
\[ q + J^k g = 0 \]

implies that for all \( t \in \mathbb{R}^+ \) and \( \mathcal{L} \in V^* \)
\[
\mathcal{L}(q(t)) + \mathcal{L}(J^k g(t)) = 0.
\]

With the integral representation of the second term in mind, we see that \( k \) times differentiation yields \( \mathcal{L}(g(t)) = 0, \ t \in \mathbb{R}^+, \mathcal{L} \in V^* \). So \( g = 0 \) and therewith \( q = 0 \).

**Definition 2.1.** For \( \varphi \in C(\mathbb{R}^+, C) \) and \( x \in V \) by \( \varphi \otimes x \) we mean the function in \( C(\mathbb{R}^+, V) \) defined by \( (\varphi \otimes x)(t) = \varphi(t)x \).

**Definition 2.2.** The differentiation operator \( \mathcal{D} \) in \( C(\mathbb{R}^+, V) \) with domain \( \text{dom}(\mathcal{D}) = C^1(\mathbb{R}^+, V) \) is defined by
\[
\mathcal{D}f = g : \Leftrightarrow f = \psi_0 \otimes f(0) + Jg.
\]

Here \( \psi_0(t) = 1, \ t \in \mathbb{R}^+ \).

Inductively \( \mathcal{D}^k \) is defined by
\[
\text{dom}(\mathcal{D}^k) = \{ f \in \text{dom}(\mathcal{D}^{k-1}) | \mathcal{D}^{k-1}f \in \text{dom}(\mathcal{D}) \}
\]
with
\[
\mathcal{D}^k f = \mathcal{D}(\mathcal{D}^{k-1} f).
\]

From the definition of \( \mathcal{D} \) and \( J \) we see that \( J \) maps \( C(\mathbb{R}, V) \) into \( \text{dom}(\mathcal{D}) \) with \( \mathcal{D}J = I \) (the identity). Applying this and elementary calculus it follows inductively that for all \( k \in \mathbb{N} \)
\[
\text{dom}(\mathcal{D}^k) = C^k(\mathbb{R}^+, V)
\]
and
\[
\mathcal{D}^k f = g : \Leftrightarrow f = \sum_{j=0}^{k-1} \psi_j \otimes (\mathcal{D}^j f)(0) + J^k g
\]
where \( \psi_j(t) = \frac{t^j}{j!} \) (cf. Riemann remainder formula).

On each of the spaces \( C^k(\mathbb{R}^+, V) \) we impose the locally convex topology brought about by the seminorms
\[
s_{\nu,K}^k(f) = \sum_{j=0}^{k} s_{\nu,K}(\mathcal{D}^j f).
\]

Thus for \( 0 \leq \ell \leq k \), the operator \( \mathcal{D}^\ell \) from \( C^k(\mathbb{R}^+, V) \) into \( C^{k-\ell}(\mathbb{R}^+, V) \) is continuous. Also, \( J^\ell \) from \( C^k(\mathbb{R}^+, V) \) into \( C^{k+\ell}(\mathbb{R}^+, V) \) is continuous.

The definition of \( \mathcal{D} \) and \( J(\lambda) \) and the observation that \( \mathcal{D} - \lambda I = \mathcal{E}_- \mathcal{D} \mathcal{E}_\lambda \) yield the following algebraic relations
\[(\mathcal{D} - \lambda)^{\ell} J(\lambda)^{k} = \begin{cases} J(\lambda)^{k-\ell} & \text{for } k \geq \ell \\ (\mathcal{D} - \lambda)^{\ell-k} & \text{for } \ell > k \end{cases} \]

Now almost by definition of \(\mathcal{D}^{k}\), for all \(f \in C^{k}(\mathbb{R}^{+}, V)\)

\[J^{k} \mathcal{D}^{k} f = f - \sum_{j=0}^{k-1} \psi_{j} \otimes (\mathcal{D}^{j} f)(0)\]

and so for all \(\lambda \in \mathcal{C}\)

\[J(\lambda)^{k} (\mathcal{D} - \lambda)^{k} f = \mathcal{E}_{-\lambda} J^{k} \mathcal{D}^{k} \mathcal{E}_{\lambda} f = f - \sum_{j=0}^{k-1} \psi_{j,\lambda} \otimes ((\mathcal{D} - \lambda)^{j} f)(0)\]

where \(\psi_{j,\lambda}\) denotes the Bohl function

\[\psi_{j,\lambda} = \frac{t^{j}}{j!} e^{\lambda t} .\]

From this we see that for each \(m \in \mathbb{N}\), \(\lambda \in \mathcal{C}\) and \(f \in C^{m}(\mathbb{R}^{+}, V)\),

\[(\mathcal{D} - \lambda)^{m} f = 0 \iff f \in \text{span}\{\psi_{j,\lambda} \otimes x \mid x \in V, j = 0, \ldots, m-1\} .\]

We shall extend this result replacing \((\mathcal{D} - \lambda)^{m}\) by \(p(\mathcal{D})\) for any polynomial \(p\).

So let \(p : \mathbb{R} \rightarrow \mathcal{C}\) be a polynomial with zeroes \(\lambda_{j}, j = 1, \ldots, r\) and respective multiplicities \(m_{j}\).

Define the polynomial \(p_{kJ}\) by

\[p_{kJ}(t) = \frac{p(t)}{(t - \lambda_{j})^{k}} , t \in \mathbb{R}, k = 1, \ldots, m_{j}, j = 1, \ldots, r .\]

Then there are \(a_{kJ} \in \mathcal{C}\) such that

\[\forall t \in \mathbb{R} : \sum_{j=1}^{r} \sum_{k=1}^{m_{j}} a_{kJ} p_{kJ}(t) = 1 .\]

Define \(K\) on \(C(\mathbb{R}^{+}, V)\) by

\[K = \sum_{j=1}^{r} \sum_{k=1}^{m_{j}} a_{kJ} J(\lambda_{j})^{k} .\]

Then \(K\) maps \(C^{d}(\mathbb{R}^{+}, V)\) into \(C^{d}(\mathbb{R}^{+}, V)\) with \(d\) the degree of \(p\), and for \(f \in C^{d}(\mathbb{R}^{+}, V)\)

\[p(\mathcal{D}) K f = \sum_{j=1}^{r} \sum_{k=1}^{m_{j}} a_{kJ} p_{kJ}(\mathcal{D}) f = f .\]
Next we compute $Kp(D)$. First observe that for $f \in C^d(\mathbb{R}, V)$,

$$J(\lambda_j^k p(D)f) = J(\lambda_j^k[(D - \lambda_j)^k p_{kj}(D)f] = p_{kj}(D)f - \sum_{i=0}^{k-1} \psi_i, \lambda_j \otimes (p_{k-i,j}(D)f)(0)$$

and so

$$Kp(D)f = f - \sum_{j=1}^{r} \sum_{k=1}^{m_j} a_{kj} \sum_{i=0}^{k-1} \psi_i, \lambda_j \otimes (p_{k-i,j}(D)f)(0) = f - \sum_{j=1}^{r} \sum_{i=0}^{m_j-1} \psi_i, \lambda_j \otimes (r_{ij}(D)f)(0)$$

where $r_{ij}$ is the polynomial of degree $\leq d - 1$,

$$r_{ij} = \sum_{k=1}^{m_j-i} a_{k+i,j}p_{kj} .$$

Since $p(D)f = 0$ if and only if $Kp(D)f = 0$, we come to the following conclusion.

**Theorem 2.3.** Let $p$ be a polynomial with degree $d$, say. Then

$$\ker(p(D)) := \{ f \in C^d(\mathbb{R}^+, V) | p(D)f = 0 \} = \text{span}\{ \psi_i, \lambda_j \otimes x | x \in V, i = 0, ..., m_j - 1, j = 1, ..., r \}$$

where $\lambda_1, ..., \lambda_r$ are the zeros of $p$ with multiplicities $m_j$.

We intend to prove that for each polynomial the differential operator $p(D)$ with domain $C^d(\mathbb{R}^+, V)$, $d = \text{degree}(p)$, is closed as a linear mapping in $C(\mathbb{R}^+, V)$. The closedness of $p(D)$ is used to prove closedness of $p(\delta_\tau)$ for any infinitesimal generator $\delta_\tau$ of a locally equicontinuous semigroup $(\pi_t)_{t \geq 0}$ Therefore the following auxiliary result, which implies that $\ker(p(D))$ is closed.

**Lemma 2.4.** Let $\varphi_1, ..., \varphi_n \in C(\mathbb{R}^+, C)$. Then the subspace $M$,

$$M = \text{span}\{ \varphi_j \otimes x | x \in V, j = 1, ..., n \}$$

is closed in $C(\mathbb{R}^+, V)$.

**Proof.** Without loss of generality assume that $\{ \varphi_1, ..., \varphi_n \}$ is independent in $C(\mathbb{R}^+, C)$.

**Claim.** There exist $t_1, ..., t_n \in \mathbb{R}^+$ such that the matrix $(\varphi_j(t_i))_{i,j=1}^{n}$ is invertible.

**Proof.** For $n = 1$ the claim is obviously true. Now suppose the claim is valid for $n = m - 1$, and let $\{ \varphi_1, ..., \varphi_m \}$ be independent in $C(\mathbb{R}^+, C)$. Then there are $t_1, ..., t_{m-1} \in \mathbb{R}^+$ such that the matrix $(\varphi_j(t_i))_{i,j=1}^{m-1}$ is invertible. If there were no $t_m \in \mathbb{R}^+$ such that the matrix $(\varphi_j(t_i))_{i,j=1}^{m}$ is invertible, then for all $t \in \mathbb{R}^+$
\[ \varphi_j(t) = \sum_{i=1}^{m-1} \beta_i(t)\varphi_j(t_i), \quad j = 1, \ldots, m. \]

Hence

\[ \text{span}\{\varphi_j \mid j = 1, \ldots, m-1\} = \text{span}\{\beta_j \mid j = 1, \ldots, m-1\} \]

and

\[ \varphi_m = \sum_{i=0}^{m-1} \varphi_j(t_i)\beta_i, \]

a contradiction. \[\square\]

Now define the continuous linear mappings \( \Lambda_i, i = 1, \ldots, n, \) from \( C(\mathbb{R}^+, V) \) into \( V \) by

\[ \Lambda_i f = \sum_{j=1}^{n} a_{ij} f(t_j) \]

where \((a_{ij})\) denotes the inverse matrix of \((\varphi_j(t_i))_{i,j=1}^{n}. \) Let \((f_\alpha)\) be a net in \( M \) convergent to some \( f \in C(\mathbb{R}^+, V). \) Then

\[ f_\alpha = \sum_{i=1}^{n} \varphi_i \otimes \Lambda_i f_\alpha \]

and \( \lim_{\alpha} \Lambda_i f_\alpha = \Lambda_i f. \) Consequently

\[ f = \lim_{\alpha} \sum_{i=1}^{n} \varphi_i \otimes \Lambda_i f_\alpha = \sum_{i=1}^{n} \varphi_i \otimes \Lambda_i f \in M. \]

\[\square\]

We come to one of the main results of this paper.

**Theorem 2.5.** Let \( p : C \to C \) be a polynomial. Then \( p(D) \) with domain \( C^d(\mathbb{R}^+, V), \) \( d = \text{degree}(p), \) is a closed linear mapping in \( C(\mathbb{R}^+, V), \) i.e. the graph of \( p(D) \) is closed in the product vector space \( C(\mathbb{R}^+, V) \times C(\mathbb{R}^+, V), \) with respect to the product topology.

**Proof.** We may assume that \( p \) is monic, \( p(t) = \prod_{j=1}^{r}(t - \lambda_j)^{m_j}. \) So for \( f \in C^d(\mathbb{R}^+, V), \)

\[ p(D)f = \prod_{j=1}^{r}(D - \lambda_j)^{m_j}f. \]

Define
\[ \mathcal{R} = \prod_{j=1}^{r} J(\lambda_j)^{m_j} \]

The \( \mathcal{R} \) maps \( C(\mathbb{R}^+, V) \) into \( C(\mathbb{R}^+, V) \) continuously with for all \( f \in C(\mathbb{R}^+, V) \), \( \mathcal{R}f \in C^d(\mathbb{R}^+, V) \) and \( p(D) \mathcal{R}f = f \).

Let \( (f_\alpha) \) be a net in \( C^d(\mathbb{R}^+, V) \) for which there are \( f \) and \( g \) in \( C(\mathbb{R}^+, V) \) such that

\[ f_\alpha \rightarrow f \text{ and } p(D)f_\alpha \rightarrow g \text{ in } C(\mathbb{R}^+, V). \]

Then \( f_\alpha - \mathcal{R}p(D)f_\alpha \in \ker(p(D)) \) and so, since \( \ker(p(D)) \) is closed by Lemma 2.4, \( f - \mathcal{R}g \in \ker(p(D)) \). It follows that

\[ f = (f - \mathcal{R}g) + \mathcal{R}g \in \ker(p(D)) + C^d(\mathbb{R}^+, V) = C^d(\mathbb{R}^+, V) \]

and

\[ p(D)f = p(D)(f - \mathcal{R}g) + p(D)\mathcal{R}g = g. \]

\[ \square \]

**Corollary 2.6.** For each \( k \in \mathbb{N} \) the linear operator \( D^k \) with domain \( C^k(\mathbb{R}^+, V) \) is closed in \( C(\mathbb{R}^+, V) \).

**Corollary 2.7.** The locally convex topology of \( C^k(\mathbb{R}^+, V) \) brought about by the seminorms \( s_{\nu,K}^k \),

\[ s_{\nu,K}^k(f) = \sum_{j=0}^{k} s_{\nu,K}(D^j f) \]

equals the locally convex topology of \( C^k(\mathbb{R}^+, V) \) brought about by the seminorms \( \tilde{s}_{\nu,K}^k \)

\[ \tilde{s}_{\nu,K}^k(f) = s_{\nu,K}(f) + s_{\nu,K}(D^k f) \]

Moreover, \( C^k(\mathbb{R}^+, V) \) is sequentially complete with this topology.

**Proof.** Clearly, \( \tilde{s}_{\nu,K}^k(f) \leq s_{\nu,K}^k(f) \). Let \( (f_\alpha) \) be a net in \( C^k(\mathbb{R}^+, V) \) such that \( f_\alpha \rightarrow f \) and \( D^k f_\alpha \rightarrow D^k f \) in \( C(\mathbb{R}^+, V) \). Then \( f_\alpha - J^kD^k f_\alpha \rightarrow f - J^kD^k f \) and

\[ f_\alpha - J^kD^k f_\alpha = \sum_{j=0}^{k-1} \psi_j \otimes (D^j f_\alpha)(0). \]

It follows from Lemma 2.4 that

\[ (D^j f_\alpha)(0) \rightarrow (D^j f)(0) \]
and so

\((D^\ell(f_\alpha - J^kD^k f_\alpha))\)

is a convergent net in \(C(\mathbb{R}^+, V)\) for \(\ell = 0, \ldots, k - 1\). Since

\[ D^\ell f_\alpha = D^\ell (f_\alpha - J^kD^k f_\alpha) + J^{k-\ell}D^k f_\alpha , \]

the net \((D^\ell f_\alpha)\) is convergent in \(C(\mathbb{R}; V)\) with limit \(D^\ell f\). Let \((f_j)\) be a sequence in \(C^k(\mathbb{R}^+, V)\) such that both \((f_j)\) and \((D^k f_j)\) are Cauchy sequences in \(C(\mathbb{R}^+, V)\). Then there are \(f\) and \(g\) in \(C(\mathbb{R}^+, V)\) such that

\[ f_j \rightarrow f \quad \text{and} \quad D^k f_j \rightarrow g \quad \text{in} \quad C(\mathbb{R}^+, V) . \]

Since \(D^k\) is closed we get \(f \in C^k(\mathbb{R}^+, V)\) with \(D^k f = g\). \(\square\)

3 The translation semigroup, convolution

There is a third natural action to be defined on \(C(\mathbb{R}^+, V)\), namely translation. For \(t \geq 0\) we define \(\sigma_t\) on \(C(\mathbb{R}^+, V)\) by

\[(\sigma_t f)(s) = f(t + s) , \quad s \geq 0 .\]

Then \(\sigma_t\) is continuous on \(C(\mathbb{R}^+, V)\) for each \(t \geq 0\) with

\[ \sigma_t \sigma_s = \sigma_{t+s} , \quad \sigma_0 = I . \]

So \((\sigma_t)_{t \geq 0}\) is a one-parameter semigroup. Further, for \(t_0 \in \mathbb{R}^+\), \(\nu \in \mathcal{D}\), \(K \subset \mathbb{R}^+\) compact and for \(f \in C(\mathbb{R}^+, V)\)

\[ \lim_{t \rightarrow t_0} s_\nu K(\sigma_t f - \sigma_{t_0} f) = \lim_{t \rightarrow t_0} \max_{s \in K} s_\nu (f(s + t) - f(s + t_0)) = 0 . \]

due to the uniform continuity of \(f\) on compacta in \(\mathbb{R}^+\). Hence \((\sigma_t)_{t \geq 0}\) is a strongly continuous semigroup on \(C(\mathbb{R}^+, V)\).

Theorem 3.1. The differentiation operator \(\mathcal{D}\) with domain \(C^1(\mathbb{R}^+, V)\) is the infinitesimal generator of the semigroup \((\sigma_t)_{t \geq 0}\).

Proof. Let \(\delta_\sigma\) denote the infinitesimal generator of \((\sigma_t)_{t \geq 0}\).

- Let \(f \in C^1(\mathbb{R}^+, V)\) with \(\mathcal{D} f = g\). Then

\[ f(s) = f(0) + \int_0^s g(\tau) d\tau . \]
and for \( t > 0 \) and \( s \geq 0 \)

\[
\frac{(\sigma_t f)(s) - f(s)}{t} - g(s) = \frac{1}{t} \int_s^{s+t} (g(\tau) - g(s))d\tau .
\]

So for \( \nu \in \mathcal{D} \),

\[
s_\nu \left( \frac{(\sigma_t f)(s) - f(s)}{t} - g(s) \right) \leq \max_{\tau \in [s,s+t]} s_\nu (g(\tau) - g(s))
\]

and for \( K \subset \mathbb{R}^+ \) compact

\[
s_{\nu,K} \left( \frac{\sigma_t f - f}{t} - g \right) \leq \max_{s \in K} \max_{\tau \in [s,s+t]} s_\nu (g(\tau) - g(s)) .
\]

Since \( g \) is uniformly continuous on \( K = [0,t] \) we see that

\[
\lim_{t \to 0} s_{\nu,K} \left( \frac{\sigma_t f - f}{t} - g \right) = 0 .
\]

Therefore \( f \in \text{dom}(\delta_\sigma) \) with \( \delta_\sigma f = g = Df \).

Let \( f \in \text{dom}(\delta_\sigma) \) with \( \delta_\sigma f = g \), i.e.

\[
g = \lim_{t \to 0} \frac{\sigma_t f - f}{t} \quad \text{in } C(\mathbb{R}^+, V) .
\]

Now for all \( s \in \mathbb{R}^+ \)

\[
f(s) = \lim_{t \to 0} \frac{1}{t} \int_s^{s+t} f(\tau)d\tau
\]

in \( V \) and so

\[
f(s) - f(0) = \lim_{t \to 0} \frac{1}{t} \left[ \int_s^{s+t} f(\tau)d\tau - \int_0^t f(\tau)d\tau \right]
\]

\[
= \lim_{t \to 0} \frac{1}{t} \int_0^s (\sigma_t f - f)(\tau)d\tau = \int_0^s g(\tau)d\tau .
\]

We see that \( f \in C^1(\mathbb{R}^+, V) \) with \( Df = g \).

For all \( f \in C(\mathbb{R}^+, V) \) the function \( t \mapsto \sigma_t f \) belongs to \( C(\mathbb{R}^+, C(\mathbb{R}^+, V)) \), whence we have its Riemann–Stieltjes integral for each \( \mu \in \text{bv}_c(\mathbb{R}) \), and define

\[
(3.2) \quad \sigma[\mu] f := \int_{\mathbb{R}^+} \sigma_t f \, d\mu(\tau) .
\]
So \( \sigma[\mu] \) is linear operator from \( C({\mathbb R}^+, V) \) into \( C({\mathbb R}^+, V) \). Since

\[
s_{\nu,K}(\sigma[\mu]f) \leq \text{var}(\mu)s_{\nu,K}(f)
\]

where \( \tilde{K} = K + [0, T] \) with \( T \) so large that \( \mu(t) = \mu(T) \) for \( t > T \), the operator \( \sigma[\mu] \) is continuous.

By definition \( \mu \mapsto \sigma[\mu] \) is a linear map. Further, it can be checked that

\[
\sigma[\mu_1]\sigma[\mu_2] = \sigma[\mu_1 * \mu_2]
\]

with

\[
(\mu_1 * \mu_2)(s) = \int_0^s \mu_1(s - \sigma)d\mu_2(\sigma).
\]

Define \( H_t \in \text{bv}_c({\mathbb R}^+) \), \( t \geq 0 \), by

\[
H_t(s) = \begin{cases} 
0 & 0 \leq s \leq t \\
1 & s > t.
\end{cases}
\]

Then \( \sigma[H_t] = \sigma_t \).

**Lemma 3.3.** The linear span, \( \text{span}\{\sigma_t \ | \ t \in {\mathbb R}^+\} \), is strongly sequentially dense in \( \{\sigma[\mu] \mid \mu \in \text{bv}_c({\mathbb R}^+)\} \), i.e. for each \( \mu \in \text{bv}_c({\mathbb R}^+) \) there exists a sequence \( (\mu_k)_{k \in {\mathbb N}} \) in \( \text{span}\{H_t \ | \ t \in {\mathbb R}^+\} \) such that for all \( f \in C({\mathbb R}^+, V) \)

\[
\lim_{k \to \infty} \sigma[\mu_k]f = \sigma[\mu]f.
\]

**Proof.** Let \( \mu \in \text{bv}_c({\mathbb R}^+) \) and \( T > 0 \) such that \( \mu(t) = \mu(T), \ t > T \). Define

\[
t_{ki} = \frac{i}{2^k} T, \quad i = 0, 1, ..., 2^k, \ k \in {\mathbb N},
\]

and

\[
\mu_k = \sum_{i=1}^{2^k} (\mu(t_{ki}) - \mu(t_{ki-1}))H_{t_{ki} - t_{ki-1}}.
\]

Then

\[
(\sigma[\mu] - \sigma[\mu_k])f(t) = \sum_{i=1}^{2^k} \int_{t_{ki-1}}^{t_{ki}} (f(t + \tau) - f(t + t_{ki-1})d\mu(\tau).
\]

So for \( \nu \in {\mathcal D} \) and \( K \subset {\mathbb R} \) compact
\[ s_{v,K}(\sigma[\mu]f - \sigma[\mu_k]f) \leq \max_{i \in \{1, \ldots, 2^n\}} \max_{t_i \in K} \max_{r \in [t_{i-1}, t_i]} s_v(f(t + \tau) - f(t + t_{i-1})) \]

The right hand side tends to zero as \( k \to \infty \), because \( f \) is uniformly continuous on compacta in \( \mathbb{R}^+ \).

We write \( \text{bv}_c^\infty(\mathbb{R}^+) \) instead of \( \text{bv}_c(\mathbb{R}^+) \cap C^\infty(\mathbb{R}) \). For \( \mu \in \text{bv}_c^\infty(\mathbb{R}^+) \), its derivatives \( \mu^{(k)} \) are \( C^\infty \)-functions on \( \mathbb{R}^+ \) with compact support, and \( \mu^{(k)}(0) = 0 \). So for \( f \in C(\mathbb{R}^+, V) \)

\[
(\sigma[\mu]f)(t) = \int_{\mathbb{R}^+} \mu'(\tau)f(t + \tau)d\tau = \int_{\mathbb{R}} \mu'(\tau - t)f(\tau)d\tau .
\]

We see that \( \sigma[\mu]f \in \bigcap_{k \in \mathbb{N}} C^k(\mathbb{R}^+, V) =: C^\infty(\mathbb{R}^+, V) \) with

\[
D^k\sigma[\mu]f = (-1)^k\sigma[\mu^{(k)}]f .
\]

A sequence \( (\mu_n)_{n \in \mathbb{N}} \) in \( \text{bv}_c^\infty(\mathbb{R}^+) \) is said to be an approximative identity whenever for all \( f \in C(\mathbb{R}^+, V) \)

\[
\sigma[\mu_n]f \to f \text{ as } n \to \infty .
\]

Let \( \varphi \) be a \( C^\infty \)-function on \( \mathbb{R} \), \( \varphi(t) \geq 0 \), with compact support contained in \( \mathbb{R}^+ \) and \( \int_{\mathbb{R}^+} \varphi(\tau)d\tau = 1 \). Define

\[
(3.4) \quad \mu_n(t) = n \int_0^t \varphi(n\tau)d\tau , \quad t \geq 0 , \quad n \in \mathbb{N} .
\]

Then \( \mu_n \in \text{bv}_c^\infty(\mathbb{R}^+) \) and for \( t \geq 0 \)

\[
(\sigma[\mu_n]f - f)(t) = \int_{\mathbb{R}^+} [n\varphi(n\tau)f(t + \tau) - \varphi(\tau)f(t)]d\tau = \int_{\mathbb{R}^+} \varphi(\tau)(f(t + \tau/n) - f(t))d\tau .
\]

So we find

\[
s_{v,K}(\sigma[\mu_n]f - f) \leq \max_{a \in [0,\infty]} s_{v,K}(\sigma_a f - f)
\]

and the right-hand side tends to zero. From this we see the existence in \( \text{bv}_c^\infty(\mathbb{R}) \) of an approximative identity. And it follows that \( C^\infty(\mathbb{R}^+, V) \) is sequentially dense in \( C(\mathbb{R}, V) \). In fact there is the more general result.

**Lemma 3.5.** Let \( M \) be a closed subspace of \( C(\mathbb{R}^+, V) \) with \( \sigma_t(M) \subseteq M \) for all \( t \in \mathbb{R}^+ \). Then \( M \cap C^\infty(\mathbb{R}^+, V) \) is sequentially dense in \( M \).

**Proof.** For all \( \mu \in \text{span}\{H_t \mid t \in \mathbb{R}^+\} \), \( \sigma[\mu](M) \subseteq M \) and so by Lemma 3.3 and the fact that \( M \) is closed for all \( \mu \in \text{bv}_c(\mathbb{R}^+) \), \( \sigma[\mu](M) \subseteq M \). Now let \( (\mu_n) \) be an approximate identity in \( \text{bv}_c^\infty(\mathbb{R}^+) \). Then for \( f \in M, \sigma[\mu_n]f \in M \cap C^\infty(\mathbb{R}^+, V) \), and \( \sigma[\mu_n]f \to f \). \( \square \)
4 One parameter $c_0$-semigroups, the flow operator

Let $(\pi_t)_{t \geq 0}$ be a one-parameter semigroup of continuous linear mappings on $V$; so for all $t_1 \geq 0$ and $t_2 \geq 0$

$$\pi_{t_1} \pi_{t_2} = \pi_{t_1 + t_2}, \quad \pi_0 = I.$$

To each $x \in V$ we associate its flow $\mathcal{F}_x x : \mathbb{R}^+ \rightarrow V$ by defining $(\mathcal{F}_x x)(t) = \pi_t x, \ t \geq 0$.

The semigroup $(\pi_t)_{t \geq 0}$ is said to be strongly continuous at $t = 0$, or a $c_0$-semigroup, if for all $x \in V$

$$\lim_{t \to 0} \pi_t x = x.$$

Since for each $t \geq 0$ the operator $\pi_t$ is continuous on $V$ it follows that the flow $\mathcal{F}_x x$ is a right continuous function from $\mathbb{R}^+$ into $V$. We want to have continuity of $\mathcal{F}_x x$ for each $x \in V$, so that $\mathcal{F}_x$ is a linear operator from $V$ into $C(\mathbb{R}^+, V)$. Therefore the following definition.

**Definition 4.1.** A one-parameter semigroup $(\pi_t)_{t \geq 0}$ is said to be locally equicontinuous if for all $K \subset \mathbb{R}^+$ compact the collection

$$\{\pi_t \mid t \in K\}$$

is equicontinuous, i.e.

$$\forall \nu \in \mathcal{D} \exists \rho \in \mathcal{D} \exists C > 0 \forall t \in K \forall x \in V : s_{\nu}(\pi_t x) \leq C s_{\rho}(x).$$

**Lemma 4.2.** Let $(\pi_t)_{t \geq 0}$ be a locally equicontinuous $c_0$-semigroup. Then $\mathcal{F}_x x \in C(\mathbb{R}^+, V)$ for all $x \in V$ and $\mathcal{F}_x : V \rightarrow C(\mathbb{R}^+, V)$ is continuous.

**Proof.** Let $t_0 > 0$. Then $\mathcal{F}_x x$ is right continuous at $t = t_0$. Now for $0 \leq t < t_0$ and $\nu \in \mathcal{ID}$

$$s_{\nu}(\pi_t x - \pi_{t_0} x) = s_{\nu}(\pi_t (x - \pi_{t_0 - t} x)).$$

Take $K_0 = [0, t_0]$. Then there is $\bar{\nu} \in \mathcal{ID}$ and $C > 0$ such that

$$s_{\nu}(\pi_t x - \pi_{t_0} x) \leq C s_{\bar{\nu}}(x - \pi_{t_0 - t} x).$$

We see that $\mathcal{F}_x x$ is left continuous at $t = t_0$.

Reformulation in terms of $\mathcal{F}_x$ of the local equicontinuity property of $(\pi_t)_{t \geq 0}$ yields

$$\forall \nu \in \mathcal{D} \forall K \subset \mathbb{R}^+, \text{compact} \exists C > 0 \exists \bar{\nu} \in \mathcal{D} \forall x \in V s_{\bar{\nu}, K}(\mathcal{F}_x x) \leq C s_{\rho}(x)$$
expressing the continuity of $F_x$.

We approach this aspect of a $c_0$-semigroup from a different angle. Therefore we recall the following theorem, cf. [Tre], p. 347, Theorem 33.1, for barreled locally convex spaces.

Let $E$ be a barreled locally convex space and $F$ a locally convex space. Then a subset $H$ of continuous linear operators from $E$ into $F$ is equicontinuous if and only if $H$ is bounded for the topology of pointwise (= strong) convergence.

Now suppose $(\pi_t)_{t \geq 0}$ is a $c_0$-semigroup on $V$ such that $F_x x \in C(\mathbb{R}^+, V)$ for all $x \in V$, and suppose that $V$ is barreled. Let $K \subset \mathbb{R}^+$ be compact. Then for all $\nu \in D$ and $x \in V$

$$\sup_{t \in K} s_{\nu}(\pi_t x) < \infty$$

because $t \mapsto s_{\nu}(\pi_t x)$ is continuous on $\mathbb{R}^+$. So the set $\{\pi_t \mid t \in K\}$ is bounded for the topology of pointwise convergence, whence equicontinuous. Thus we derived

**Theorem 4.3.** (cf. [Komj. proposition 1.1]). Suppose that $V$ is barreled. Let $(\pi_t)_{t \geq 0}$ be a semigroup on $V$ such that for each $x \in V$ its flow $F_x x \in C(\mathbb{R}^+, V)$. Then $F_x$ is continuous from $V$ into $C(\mathbb{R}^+, V)$ or equivalently $(\pi_t)_{t \geq 0}$ is a locally equicontinuous semigroup.

In the remaining part of this paper we consider a fixed $c_0$-semigroup $(\pi_t)_{t \geq 0}$ such that $F_x$ is a continuous linear operator from $V$ into $C(\mathbb{R}^+, V)$.

**Definition 4.4.** By $\Delta_0$ the continuous linear mapping from $C(\mathbb{R}^+, V)$ into $V$ is denoted that satisfies $\Delta_0 f = f(0)$. So $\Delta_0 F_x$ is the identity on $V$.

**Definition 4.5.** For each $\mu \in bvc(\mathbb{R}^+)$ the linear operator $\pi[\mu]$ on $V$ is defined by

$$\pi[\mu] = \Delta_0 \sigma[\mu] F_x .$$

It follows from this definition that for all $\mu \in bvc(\mathbb{R}^+)$, $x \in V$

$$\pi[\mu] x = \int_{\mathbb{R}^+} (\sigma_+ F_x x)(0) d\mu(\tau) = \int_{\mathbb{R}^+} \pi_+ x d\mu(\tau) .$$

The reader can check that $\pi_t \pi[\mu] = \pi[\mu] \pi_t$ and so

$$\pi[\mu_1] \pi[\mu_2] = \Delta_0 \sigma[\mu_1] F_x \pi[\mu_2] = \Delta_0 \sigma[\mu_1] \sigma[\mu_2] F_x = \pi[\mu_1 \ast \mu_2] .$$

We observe further that for all $t \in \mathbb{R}^+$, $\pi[H_t] = \pi_t$.

In section 3 we presented some results for the convolution operators $\sigma[\mu]$ on $C(\mathbb{R}^+, V)$. They have the following consequences for the operator $\pi[\mu]$ on $V$. 

Lemma 4.6. The linear span of the set \( \{ \pi_t \mid t \in \mathbb{R}^+ \} \) is strongly sequentially dense in \( \{ \pi[\mu] \mid \mu \in \text{bv}_c(\mathbb{R}^+) \} \).

**Proof.** Let \( \mu \in \text{bv}_c(\mathbb{R}^+) \). According to Lemma 3 there is a sequence \( (\mu_k) \) in \( \text{span}\{ H_t \mid t \geq 0 \} \) such that \( \sigma[\mu_k]f \to \sigma[\mu]f \) as \( k \to \infty \) for all \( f \in C(\mathbb{R}^+, V) \). Then for all \( x \in V \)

\[
\pi[\mu_k]x = \Delta_0 \sigma[\mu_k]F_x x \to \Delta_0 \sigma[\mu]F_x x = \pi[\mu]x \,.
\]

\( \Box \)

Lemma 4.7. Let \( (\mu_n) \) be an approximate identity in \( \text{bv}_c^\infty(\mathbb{R}^+) \). Then for all \( x \in V \),

\[ \pi[\mu_n]x \to x. \]

5 One parameter \( c_0 \)-semigroups, the infinitesimal generator

We recall our assumption that \( (\pi_t)_{t \geq 0} \) is a locally equicontinuous \( c_0 \)-semigroup on \( V \).

By \( \text{dom}(\delta_\pi) \) the subspace of \( V \) is denoted consisting of all \( x \in V \) for which the limit

\[ \delta_\pi x := \lim_{t \to 0} \frac{1}{t}(\pi_tx - x) \]

exists in \( V \). The linear operator \( \delta_\pi : \text{dom}(\delta_\pi) \to V \), thus defined is called the infinitesimal generator of the semigroup \( (\pi_t)_{t \geq 0} \). Inductively, \( \text{dom}(\delta_\pi^k) \) is defined,

\[ \text{dom}(\delta_\pi^k) = \{ x \in \text{dom}(\delta_\pi^{k-1}) \mid \delta_\pi^{k-1}x \in \text{dom}(\delta_\pi) \} \]

and

\[ \delta_\pi^k x = \delta_\pi(\delta_\pi^{k-1} x). \]

Besides,

\[ \text{dom}^\infty(\delta_\pi) := \bigcap_{k=1}^\infty \text{dom}(\delta_\pi^k). \]

Lemma 5.1. Let \( k \in \mathbb{N} \) and \( x \in V \). Then \( x \in \text{dom}(\delta_\pi^k) \) if and only if \( F_\pi x \in C^k(\mathbb{R}^+, V) \) (= \( \text{dom}(D^k) \)). If so, then \( F_\pi D_\pi^k = D^k F_\pi \).

**Proof.** Having proved the assertion for \( k = 1 \), the case \( k > 1 \) can be dealt with using a straightforward induction argument.

Now suppose \( F_\pi x \in C^1(\mathbb{R}^+, V) \). Then by Theorem 3.1

\[ \lim_{t \to 0} \frac{1}{t}(\sigma_t F_\pi x - F_\pi x) = D F_\pi x \]
and consequently

\[ \lim_{t \to 0} \frac{1}{t} (\Delta_0 \sigma_t F_x z - \Delta_0 F_x z) = \Delta_0 D F_x z . \]

So \( z \in \text{dom}(\delta_\sigma) \) and \( \delta_\sigma z = \Delta_0 D F_x z \). Suppose \( x \in \text{dom}(\delta_\sigma) \). Then the continuity of \( F_x \) yields

\[ \lim_{t \to 0} \frac{1}{t} (\sigma_t F_x z - F_x z) = F_x \delta_\sigma z . \]

So \( F_x z \in \text{dom}(D) = C^1(\mathbb{R}^+, V) \) by Theorem 3.1 with \( D F_x z = F_x \delta_\sigma z \).

Corollary 5.2. Let \( x \in V \). Then \( x \in \text{dom}^\infty(\delta_\sigma) \) if and only if \( F_x x \in C^\infty(\mathbb{R}^+, V) \).

Lemma 5.3. \( \text{dom}^\infty(\delta_\sigma) \) is dense in \( V \).

Proof. Take an approximate identity \((J_n)_n\) in \( bV \in \text{dom}^\infty(\mathbb{R}^+) \). Then for all \( x \in V \) for all \( n \in \mathbb{N} \)

\[ \sigma[J_n] F_x x \in C^\infty(\mathbb{R}^+, V) \]

and so \( \pi[J_n] x \in \text{dom}^\infty(\delta_\sigma) \). Now observe that \( \pi[J_n] x \to x \) as \( n \to \infty \).

In literature one can find results concerning the closedness of \( \delta^k \) as a linear operator in \( V \) only for Banach spaces \( V \). The proofs are based on certain fractional norm inequalities. The result we present now, seems new even for Banach spaces; certainly its proof is surprisingly simple after the preparations of Sections 1-3.

Theorem 5.4. Let \( p \) be a polynomial, \( p(z) = a_0 + a_1 z + \ldots + a_d z^d, a_d \neq 0 \). Then the linear operator \( p(\delta_\sigma) \),

\[ p(\delta_\sigma) = a_0 I + a_1 \delta_\sigma + \ldots + a_d \delta_\sigma^d \]

with domain \( \text{dom}(\delta^d_\sigma) \) is closed as a densely defined linear operator in \( V \).

Proof. By definition \( x \in \text{dom}(\delta^d_\sigma) \) implies that \( x \in \text{dom}(\delta^k_\sigma) \) for all \( k = 1, \ldots, d \), and so \( p(\delta_\sigma) \) is well-defined.

Let \((x_\alpha)_\alpha \in I\) be a net in \( \text{dom}(\delta^d_\sigma) \) for which there are \( x, y \in V \) such that

\[ x_\alpha \to x \text{ and } p(\delta_\sigma) x_\alpha \to y \text{ in } V . \]

Then continuity of \( F_x \) ensures that

\[ F_x x_\alpha \to F_x x \text{ and } F_x p(\delta_\sigma) x_\alpha \to F_x y \text{ in } C(\mathbb{R}^+, V) . \]

Since \( p(D) \) with domain \( C^d(\mathbb{R}^+, V) \) is closed and since \( F_x p(\delta_\sigma) x_\alpha = p(D) F_x x_\alpha \), we obtain \( F_x x \in C^d(\mathbb{R}^+, V) \) and \( p(D) F_x x = F_x y \). Consequently, \( x \in \text{dom}(\delta^d_\sigma) \) by Lemma 5.1 and \( y = \Delta_0 p(D) F_x x = p(\delta_\sigma) x \).

Theorem 5.5. For each \( k \in \mathbb{N} \) the operator \( \delta^k_\sigma \) is closed. The vector space \( \text{dom}(\delta^k_\sigma) \) endowed with the graph topology, i.e. the locally convex topology brought about by the seminorms
\[ \delta_t^k(x) = s_\nu(x) + s_\nu(\delta_t^k x) \]

is sequentially complete. Also, as a consequence, \( \text{dom}^\infty(\delta_x) = \bigcap_{k \in \mathbb{N}} \text{dom}(\delta_t^k) \) with the intersection topology is sequentially complete. Both \( \text{dom}(\delta_t^k) \) and \( \text{dom}^\infty(\delta_x) \) are invariant under the action of the semigroup \((\pi_t)_{t \geq 0}\) with

\[ \delta_t^k \pi_t x = \pi_t \delta_t^k x, \quad x \in \text{dom}(\delta_t^k). \]

The operator \( F_* \) maps \( \text{dom}(\delta_t^k) \) into \( C^k(\mathbb{R}^+, V) \) continuously.

**Proof.** That \( \delta_t^k \) is closed follows from Theorem 5.4. Besides for all \( x \in \text{dom}(\delta_t^k) \), \( \pi_t \delta_t^k x = \Delta_0 \sigma_t D^k F_* x = \Delta_0 D^k \sigma_t F_* x = \delta_t^k \pi_t x \). Remains to check that \( \text{dom}(\delta_t^k) \) is sequentially complete. So let \((x_j)\) be a Cauchy sequence in \( \text{dom}(\delta_t^k) \). Then \((x_j)\) and \((\delta_t^k x_j)\) are Cauchy sequences in \( V \). So there are \( x \) and \( y \) in \( V \) such that \( x_j \to x \) and \( \delta_t^k x_j \to y \) as \( j \to \infty \). We conclude that \( x \in \text{dom}(\delta_t^k) \) with \( \delta_t^k x = y \). \( \Box \)

### 6 One-parameter semigroups, invariance

**Definition 6.1.** A subspace \( M \) of \( V \) is said to be \((\pi_t)\)-invariant if for all \( t \geq 0 \), \( \pi_t(M) \subseteq M \).

**Lemma 6.2.** Let \( M \) be a closed \((\pi_t)\)-invariant subspace of \( V \). Then \( \pi[\mu](M) \subseteq M \) for all \( \mu \in \text{bv}_c(\mathbb{R}^+) \).

**Proof.** For all \( \mu \in \text{span}\{H_t \mid t \geq 0\} \) we have \( \pi[\mu](M) \subseteq M \). So the assertion follows from Lemma 4.6 and the closedness of \( M \). \( \Box \)

**Lemma 6.3.** Let \( M \) be a closed \((\pi_t)\)-invariant subspace of \( V \). Then \( M \cap \text{dom}^\infty(\delta_x) \) is dense in \( M \).

**Proof.** Let \((\mu_n)\) in \( \text{bv}_c(\mathbb{R}^+) \) be an approximate identity. Then for each \( x \in M \), \( \pi[\mu_n]x \in M \cap \text{dom}^\infty(\delta_x) \) and \( \pi[\mu_n]x \to x \) as \( n \to \infty \). \( \Box \)

As we saw in the previous section, \( \pi_t(\text{dom}^\infty(\delta_x)) \subseteq \text{dom}^\infty(\delta_x) \) with \( \pi_t \delta_t^k x = \delta_t^k \pi_t x \) for all \( x \in \text{dom}^\infty(\delta_x) \) and \( t \geq 0 \). Let \( \tilde{\pi}_t \) denote the restriction of \( \pi_t \) to \( \text{dom}^\infty(\delta_x) \). Then \( (\tilde{\pi}_t)_{t \geq 0} \) is a locally equicontinuous \( C_0 \)-semigroup on the sequentially complete locally convex space \( \text{dom}^\infty(\delta_x) \) with, of course, \( \delta_x = \delta_x|_{\text{dom}^\infty(\delta_x)} \). Also the corresponding flow operator \( F_* \) from \( \text{dom}^\infty(\delta_x) \) into \( C(\mathbb{R}^+, \text{dom}^\infty(\delta_x)) \) is the restriction of \( F_* \) to \( \text{dom}^\infty(\delta_x) \). It follows that for all \( \mu \in \text{bv}_c(\mathbb{R}^+) \)

\[ \tilde{\pi}[\mu] = \Delta_0 \sigma[\mu] F_* = \pi[\mu]|_{\text{dom}^\infty(\delta_x)}. \]

If \( M_0 \) is a closed subspace of \( \text{dom}^\infty(\delta_x) \) with \( \tilde{\pi}_t(M_0) \subseteq M_0 \) for all \( t \geq 0 \), then by Lemma 6.2

\[ \pi[\mu](M_0) = \tilde{\pi}[\mu](M_0) \subseteq M_0. \]
We arrive at the following extension of Lemma 6.3.

**Theorem 6.4.** Let $M_0$ be a closed subspace of the locally convex space $\text{dom}^\infty(\delta_\pi)$ satisfying $\pi_t(M_0) \subseteq M_0$ for all $t \geq 0$. Let $M$ denote the closure of $M_0$ in $V$. Then $M_0 = M \cap \text{dom}^\infty(\delta_\pi)$.

**Proof.** We observe that $M$ is $(\pi_t)$-invariant and that $M_0 \subseteq M \cap \text{dom}^\infty(\delta_\pi)$. Let $x \in M \cap \text{dom}^\infty(\delta_\pi)$. Then there is a net $(x_\alpha)$ in $M_0$ with $x_\alpha \to x$ in $V$. For each $\mu \in \text{bv}_c^\infty(\mathbb{R}^+)$, $(\pi[\mu]x_\alpha)$ converges to $\pi[\mu]x$ in $\text{dom}^\infty(\delta_\pi)$, because $\delta_\pi^k \pi[\mu] = (-1)^k \pi[\mu^{(k)}]$. Further $\pi[\mu]x_\alpha = \pi[\mu]x_\alpha \in M_0$ for all $\alpha \in I$, so that $\pi[\mu]x \in M_0$. Letting $(\mu_n)$ be an approximate identity in $\text{bv}_c^\infty(\mathbb{R}^+)$ we have $\pi[\mu_n]x \to x$ in $\text{dom}^\infty(\delta_\pi)$ so that $x \in M_0$. \hfill \Box

**Corollary 6.5.** Let $M_0 \subseteq \text{dom}^\infty(\delta_\pi)$ be $(\pi_t)$-invariant with closure $M$ in $V$. Then $M \cap \text{dom}^\infty(\delta_\pi)$ is the closure of $M_0$ in $\text{dom}^\infty(\delta_\pi)$.

Next we present some results on closed operators in $V$ which commute with each $\pi_t$, $t \geq 0$.

**Definition 6.6.** A linear operator $K$ in $V$ with domain $\text{dom}(K)$ is said to be $(\pi_t)$-invariant if $\pi_t(\text{dom}(K)) \subseteq \text{dom}(K)$ and $K \pi_t x = \pi_t K x$ for all $x \in \text{dom}(K)$ and $t \geq 0$.

On the vector space $V \oplus V$ consisting of all pairs $[x; y]$ with $x \in V$ and $y \in V$ we impose the direct sum topology brought about by the seminorms

$$s_{\nu}[x; y] = s_{\nu}(x) + s_{\nu}(y).$$

Then $V \oplus V$ is sequentially complete. Also we introduce the one parameter semigroup $(\pi_t + \pi_t)_{t \geq 0}$ on $V \oplus V$,

$$\pi_t + \pi_t [x; y] = [\pi_t x; \pi_t y].$$

It can be checked readily that $(\pi_t + \pi_t)_{t \geq 0}$ is a locally equicontinuous $c_0$- semigroup on $V \oplus V$ with infinitesimal generator $\delta_{\pi \oplus \pi}$ satisfying

$$\text{dom}(\delta_{\pi \oplus \pi}) = \text{dom}(\delta_\pi^k) \oplus \text{dom}(\delta_\pi^k)$$

$$\delta_{\pi \oplus \pi}^k [x; y] = [\delta_\pi^k x; \delta_\pi^k y].$$

Similarly,

$$(\pi + \pi)[\mu][x; y] = [\pi[\mu]x; \pi[\mu]y].$$

Now the above definition expresses that a linear operator $K$ in $V$ is $(\pi_t)$-invariant if and only if its graph,

$$\text{graph}(K) = \{[x; K x] \mid x \in \text{dom}(K)\}$$

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is \((\pi_t \oplus \pi_t)\)-invariant.
This observation leads us to the following theorem.

**Theorem 6.7.** Let \(\mathcal{K}\) be a closed \((\pi_t)\)-invariant linear operator in \(V\).

i) For all \(\mu \in \text{bv}_c(\mathbb{R}^+)\) and \(x \in \text{dom}(\mathcal{K})\)

\[
\pi[\mu]x \in \text{dom}(\mathcal{K}) \quad \text{and} \quad \mathcal{K}\pi[\mu]x = \pi[\mu]\mathcal{K}x .
\]

Assume \(\text{dom}^\infty(\delta_\ast) \subset \text{dom}(\mathcal{K})\).

ii) \(\mathcal{K}(\text{dom}^\infty(\delta_\ast)) \subset \text{dom}^\infty(\delta_\ast)\) and \(\mathcal{K}|_{\text{dom}^\infty(\delta_\ast)}\) is closed as a linear operator from \(\text{dom}^\infty(\delta_\ast)\) into \(\text{dom}^\infty(\delta_\ast)\).

iii) \(\text{graph}(\mathcal{K})\) is the closure in \(V \oplus V\) of

\[
\text{graph}(\mathcal{K}|_{\text{dom}^\infty(\delta_\ast)}) = \{[x; \mathcal{K}x] \mid x \in \text{dom}^\infty(\delta_\ast)\} .
\]

**Proof.**

i) As observed \(\text{graph}(\mathcal{K})\) is a closed \((\pi_t \oplus \pi_t)\)-invariant subspace of \(V \oplus V\). So the result follows from Lemma 6.2.

ii) Let \((\mu_n)\) be an approximate identity in \(\text{bv}_c^\infty(\mathbb{R}^+)\). Let \(x \in \text{dom}^\infty(\delta_\ast)\) and \(k \in \mathbb{N}\). Then we have

\[
\delta_\ast^k\pi[\mu_n]Kx = (-1)^k\mathcal{K}\pi(\mu_n)[k]x = \mathcal{K}\pi(\mu_n)[k]\delta_\ast^kx = \pi[\mu_n]\mathcal{K}\delta_\ast^kx .
\]

Since \(\delta_\ast^k\) is closed we obtain \(Kx \in \text{dom}(\delta_\ast^k)\) and \(\delta_\ast^kKx = K\delta_\ast^kx\). Since \(k \in \mathbb{N}\) was arbitrary, \(Kx \in \text{dom}^\infty(\delta_\ast)\). Further \(\mathcal{K}|_{\text{dom}^\infty(\delta_\ast)}\) is closed because

\[
\text{graph}(\mathcal{K}|_{\text{dom}^\infty(\delta_\ast)}) = \text{graph}(\mathcal{K}) \cap \text{dom}^\infty(\delta_\ast \oplus \delta_\ast) .
\]

iii) This assertion is a consequence of Lemma 6.3. \(\square\)

### 7 One-parameter c₀-groups, in summary

In this section we describe some aspects of \(c_0\)-groups \((\gamma_t)_{t \in \mathbb{R}}\) on sequentially complete locally convex topological vector spaces \(V\). The collection \(\{\gamma_t \mid t \in \mathbb{R}\}\) consists of continuous linear operators from \(V\) into \(V\) satisfying

\[
\gamma_{t_1 + t_2} = \gamma_{t_1} \gamma_{t_2} , \quad \gamma_0 = I \quad \text{and} \quad \lim_{t \to 0} \gamma_t x = x, \quad \text{for} \ t_1, t_2 \in \mathbb{R} \quad \text{and} \ x \in V .
\]

The corresponding flow operator \(\mathcal{F}_\gamma\) on \(V\) defined by
\((F_t x)(t) = \gamma_t x, \quad t \in \mathbb{R}\)

maps \(V\) into \(C(\mathbb{R}, V)\), the space of continuous functions from \(\mathbb{R}\) into \(V\). We endow \(C(\mathbb{R}, V)\) with the compact-open topology brought about by the seminorms

\[ s_{\nu,K}(f) = \max_{t \in K} s_{\nu}(f(t)) \]

where \(\nu \in ID, K \subset \mathbb{R}\) compact and \(\{s_{\nu} | \nu \in ID\}\) is the collection seminorms describing the topology of \(V\). The space \(C(\mathbb{R}, V)\) is sequentially complete. Introducing the translations \((\sigma_t)_{t \in \mathbb{R}}\) on \(C(\mathbb{R}, V)\) by

\[ (\sigma_t f)(s) = f(s + t), \quad s \in \mathbb{R}, f \in C(\mathbb{R}, V) \]

we have the intertwining relation

\[ F_t \gamma_t = \sigma_t F_\gamma, \quad t \in \mathbb{R}. \]

Instead of \(bv_c(\mathbb{R}^+)\), introduced in connection with semigroups, we deal now with \(bv_c(\mathbb{R})\), the space of all left continuous functions \(\mu\) on \(\mathbb{R}\) of bounded variation for which there exists \(T > 0\) such that

\[ \mu(t) = 0 \text{ for } t \leq -T \]

and

\[ \mu(t) = \mu(T) \text{ for } t \geq T. \]

Each \(\mu \in bv_c(\mathbb{R})\) yields an integration operator \(I_\mu\) from \(C(\mathbb{R}, V)\) into \(V\),

\[ I_\mu f = \int \limits_{\mathbb{R}} f(\tau) d\mu(\tau) \]

where the integral is of Riemann–Stieltjes type. Thus we can introduce the convolution operators

\[ \sigma[\mu] f = \int \limits_{\mathbb{R}} \sigma_\tau f d\mu(\tau). \]

The vector space \(bv_c(\mathbb{R})\) is a convolution ring with identity and without zero divisors, with convolution defined by

\[ (\mu_1 * \mu_2)(t) = \int \limits_{\mathbb{R}} \mu_1(t - \tau) d\mu_2(\tau). \]

In this section we present results only. Proofs are completely similar to the proof of the corresponding results for the translation semigroup on \(C(\mathbb{R}^+, V)\). In this respect we also refer to [Eij].

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• The one-parameter group \((\sigma_t)_{t \in \mathbb{R}}\) is a \(c_0\)-group on \(C(\mathbb{R}, V)\).

• For \(f \in C(\mathbb{R}, V)\) introduce \(Jf\) by

\[
(Jf)(t) = \int_0^t f(\tau) d\tau, \quad t \in \mathbb{R}.
\]

Then \(J\) on \(C(\mathbb{R}, V)\) is continuous and injective. By \(C^k(\mathbb{R}, V)\) the subspace of \(C(\mathbb{R}, V)\) is denoted consisting of all \(f\) for which there are a \(V\)-valued polynomial \(p\) of degree \(k\) and \(g \in C(\mathbb{R}, V)\) such that

\[
f = p + J^k g.
\]

This \(g\) is called the \(k\)-th derivative of \(f\) and so denoted by \(D^k f\). The differentiation operator \(D\) is thus well defined and equals the infinitesimal generator of the group \((\sigma_t)_{t \in \mathbb{R}}\).

- For each complex polynomial \(p\), the operator \(p(D)\) with natural domain \(C^d(\mathbb{R}, V)\), where \(d = \text{degree}(p)\), is closed as a linear operator in \(C(\mathbb{R}, V)\).

- Let \(\mu \in bV_c(\mathbb{R}) := bV_c(\mathbb{R}) \cap C^\infty(\mathbb{R})\). Then \(\mu' \in C_c^\infty(\mathbb{R})\) and \(\sigma[\mu]\) maps \(C(\mathbb{R}, V)\) into \(C^\infty(\mathbb{R}, V)\). Using an approximate identity it follows that \(C^\infty(\mathbb{R}, V)\) is dense in \(C(\mathbb{R}, V)\).

- The linear span of the set \(\{\sigma_t \mid t \in \mathbb{R}\}\) is strongly dense in \(\{\sigma[\mu] \mid \mu \in bV_c(\mathbb{R})\}\).

Of course, the results on one-parameter \(c_0\)-semigroups have their analogues for one-parameter groups. The proofs of the now mentioned results have the same style of argumentation as the proofs given in Section 4, 5 and 6. See also [Eij].

Let \((\gamma_t)_{t \in \mathbb{R}}\) be a one parameter \(c_0\)-group of continuous linear mappings on \(V\) with corresponding flow operator \(F_\gamma\) from \(V\) into \(C(\mathbb{R}, V)\).

- If \(V\) is barreled, then \((\gamma_t)_{t \in \mathbb{R}}\) is locally equicontinuous, or, equivalently, then \(F_\gamma\) is continuous. (Cf. Theorem 4.3.)

Assume in the rest of the statements that \(F_\gamma\) is continuous.

- Define the operator \(\delta_\gamma\) in \(V\) to be the infinitesimal generator of \((\gamma_t)_{t \in \mathbb{R}}\).

\[
x \in \text{dom}(\delta_\gamma) : \Leftrightarrow \delta_\gamma x := \lim_{t \to 0} \frac{\gamma_t x - x}{t}
\]

exists.

- Then \(x \in \text{dom}(\delta^k_\gamma)\) if and only if \(F_\gamma x \in C^k(\mathbb{R}, V)\), \(k \in \mathbb{N}\). Let \(\text{dom}^\infty(\delta_\gamma) = \bigcap_{k \in \mathbb{N}} \text{dom}(\delta^k_\gamma)\). Then \(\text{dom}^\infty(\delta_\gamma)\) is dense in \(V\).

- For each \(\mu \in bV_c(\mathbb{R})\) the operator \(\gamma[\mu]\)
\[ \gamma[\mu]x = \int_{\mathbb{R}} \gamma_\tau x \, d\mu(\tau) \]

is well-defined and continuous on \( V \). The linear span of the set \( \{ \gamma_t \mid t \in \mathbb{R} \} \) is dense in \( \{ \gamma[\mu] \mid \mu \in \text{bv}_c(\mathbb{R}) \} \). Let \( (\mu_n) \) be an approximate identity. Then \( \gamma[\mu_n] \to I \) strongly on \( V \).

- Let \( M \) be a closed subspace of \( V \) with \( \gamma_t(M) = M \) for all \( t \in \mathbb{R} \). Then \( \gamma[\mu](M) \subseteq M \) and \( M \cap \text{dom}^\infty(\delta_\gamma) \) is dense in \( M \). Let \( M_0 \) be a closed subspace of \( \text{dom}^\infty(\delta_\gamma) \) (with respect to its natural topology) and let \( M \) denote the closure of \( M_0 \) in \( V \). Then \( M_0 = M \cap \text{dom}^\infty(\delta_\gamma) \).

- Let \( K \) be a closed \((\gamma_t)\)-invariant operator in \( V \) with domain \( \text{dom}(K) \). Then for all \( \mu \in \text{bv}_c(\mathbb{R}) \) and \( x \in \text{dom}(K) \),

\[ \gamma[\mu]x \in \text{dom}(K) \text{ and } K\gamma[\mu]x = \gamma[\mu]Kx . \]

If \( \text{dom}^\infty(\delta_\gamma) \subseteq \text{dom}(K) \), then \( K(\text{dom}^\infty(\delta_\gamma)) \subseteq \text{dom}^\infty(\delta_\gamma) \) and

\[ \text{graph}(K) = \{ (x;Kx) \mid x \in \text{dom}^\infty(\delta_\gamma) \} . \]

**References**


