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Partial Element Equivalent Circuit (PEEC) models for on-chip passives and interconnects

Menno E. Verbeek

Abstract

The accurate simulation of the electromagnetic behaviour of interconnects and on-chip inductors is becoming more and more important for chip designs. For this purpose, we investigate the use of the electric field integral equation, which can be used to construct a full-wave electromagnetic model. To facilitate the combined analysis of the chip's circuit and the electromagnetic behaviour, the electromagnetic model can be formulated as an equivalent circuit. We have conducted some simple experiments to see the method at work and to get an idea of the frequencies for which retardation effects are important.

1 Introduction

With increasing clock speeds, the electromagnetic (EM) behaviour of interconnects on chips is becoming more and more important for the functionality of the chip, and the simple transmission line models are often insufficient. With the development of on-chip inductive elements, full-wave EM simulations can not be avoided any more. The challenge is not only to compute the EM behaviour, but also to integrate the results of such an EM simulation with the simulation of the chip’s circuit elements. This integration is easiest if the EM simulation results in the form of an equivalent circuit for the interconnect and/or on-chip inductive elements.

There are several EM simulation techniques. In [1], a very nice overview of the different methods is given. A popular EM simulation method is the Finite-Difference Time-Domain (FDTD) method [2] using a Yee lattice [3]. In this method the Maxwell equations are solved approximately in the time-domain using a finite-difference scheme for the electric and magnetic fields.

The disadvantage of these time-domain methods is that the integration of the results with frequency-domain analysis of the chips circuit is non-trivial. To facilitate this integration, the terms in spatial discretisation of the Maxwell equations for the FDTD method can be represented by circuit elements. This leads to a Finite Element Equivalent Circuit (FEEC) for the EM structure, which can be trivially integrated with the chip’s circuit for analysis by a circuit simulator.

A finite-element or finite-difference discretisation for the EM simulation can lead to very large numbers of degrees of freedom, which will considerably slow down the analysis of the chips circuit. To reduce the size of the EM model, we can use an integral equation formulation of the Maxwell equations. In this context, one usually uses the Electric Field Integral Equation (EFIE), which will be derived in section 2. The EFIE is often discretised using the so-called Method of Moments (MoM) [4], which is a Galerkin discretisation method. The spatial discretisation of the EFIE can also be interpreted as an equivalent circuit, as we will show in section 4. In contrast to the FEEC, this is called the Partial Element Equivalent Circuit (PEEC) and was first introduced in [5]. Later it was extended to include dielectric regions [6]. The integral equation approach usually has much less degrees of freedom, leading to relatively small but dense interaction matrices in comparison to the large and sparse interaction matrices that arise in the differential equation approach. Still, the PEEC model is often used in combination with model order reduction methods to further reduce the number of degrees of freedom.

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In this paper, we will review the use of the EFIE to formulate a 3-dimensional full wave model for on-chip passive and interconnect patterns, and the formulation of the corresponding PEEC equivalent circuit.

2 The Electric Field Integral Equation (EFIE)

This section will start with a short description of the derivation of the EFIE. For more background information on electrodynamics see [7]. More theory on integral equations can be found in [8].

The behaviour of macroscopic charges, currents and electromagnetic fields is completely described by the macroscopic Maxwell equations

\[ \nabla \times \mathbf{H} - \frac{\partial}{\partial t} \mathbf{D} = \mathbf{J} \quad \nabla \cdot \mathbf{D} = \rho \]  
\[ \nabla \times \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = 0 \quad \nabla \cdot \mathbf{B} = 0 \]  

These should be augmented with equations for the microscopic behaviour of the medium. If the medium is isotropic (i.e. no directional dependency), this can be summarised with

\[ \mathbf{D} = \varepsilon \mathbf{E} \quad \text{and} \quad \mu \mathbf{H} = \mathbf{B} \]  

in which \( \varepsilon \) is the dielectric constant of the medium while \( \mu \) is the magnetic permeability.

It is convenient to express the electric and magnetic fields in terms of the vector potential \( \mathbf{A} \) and scalar potential \( \phi \). They are defined by

\[ \mathbf{B} = \nabla \times \mathbf{A} \]  
\[ \mathbf{E} = \frac{\partial}{\partial t} \mathbf{A} - \nabla \phi \]  

This definition leaves some freedom of choice for the potentials. Here we use this freedom to satisfy the Lorentz gauge

\[ \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial}{\partial t} \phi = 0 \]  

where \( c = (\varepsilon \mu)^{-1/2} \). The Lorentz gauge condition (2.5) still leaves some freedom of choice for \( \mathbf{A} \) and \( \phi \) but this will not bother us.

In the Lorentz gauge, the Maxwell equations can be written as a set of wave equations

\[ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \phi = -\frac{1}{\varepsilon} \rho \]  
\[ \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} = -\mu \mathbf{J} \]  

This shows that the potentials, and thus the fields, are propagated with the speed \( c \), which is therefore the speed if light in the medium. Note that the wave equations (2.6) by itself are not coupled. They are coupled by the Lorentz gauge (2.5), or equivalently by the continuity equation

\[ \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} \rho = 0 \]  

which expresses the conservation of charge.

Using the Green function for the wave equations (2.6), we can write the potentials as an integral over the sources \( \rho \) and \( \mathbf{J} \). In the case where \( \varepsilon \) and \( \mu \) can be assumed constant, the Green function is given by

\[ G(\mathbf{x}, t, \mathbf{x}', t') = \frac{\delta(t - t' - |\mathbf{x} - \mathbf{x}'|/c)}{4\pi |\mathbf{x} - \mathbf{x}'|} \]
The Dirac delta function in the numerator is the result of the finite light speed and results in a time delay. Using this Green function, the potentials can be written as

\[ A(x, t) = \frac{\mu}{c} \int G(x, t, x', t') J(x', t') d^3x' dt' \]  \hspace{1cm} (2.9)

\[ \phi(x, t) = \frac{1}{\epsilon} \int G(x, t, x', t') \rho(x', t') d^3x' dt' . \]  \hspace{1cm} (2.10)

The electric field can be reconstructed using the definition (2.4),

\[ E(x, t) = -\mu \frac{\partial}{\partial t} \int G(x, t, x', t) J(x', t') d^3x' dt' - \frac{1}{\epsilon} \nabla \int G(x, t, x', t') \rho(x', t') d^3x' dt' + E^E(x, t) , \]  \hspace{1cm} (2.11)

where we restricted the spatial part of the integral to \( \Gamma \), the region of space where the source terms \( \rho \) and \( J \) are non-zero. In our application, \( \Gamma \) consists of the conducting regions. The first term on the right-hand side of equation (2.11) is the inductive part while the second term is the capacitive part of \( E \). The electric field due to external sources \( E^E \) must be added to get the total electric field \( E \).

Inside a conductor, the electric field and the current density are directly related by Ohm’s law

\[ J(x, t) = \sigma(x) E(x, t) , \]  \hspace{1cm} (2.12)

in which \( \sigma \) is the conductance. Combining this with the electric field given by equation (2.11), we get the Electric Field Integral Equation (EFIE)

\[ \mu \frac{\partial}{\partial t} \int G(x, t, x', t) J(x', t') d^3x' dt' + \frac{1}{\epsilon} \nabla \int G(x, t, x', t') \rho(x', t') d^3x' dt' + \frac{J(x, t)}{\sigma(x)} = E^E(x, t) \quad \forall x \in \Gamma . \]  \hspace{1cm} (2.13)

The combination of the EFIE (2.13) and the continuity equation (2.7) forms a system that describes the linear relation between the external source \( E^E \) and the induced current \( J \). This is the set of equations that we will use for the equivalent circuit.

### 2.1 Charge density inside the conductors

In the interior of the conductors Ohm’s law (2.12) holds. From the Maxwell equations (2.1), we know that \( \nabla \cdot \epsilon E = \rho \). Combining these two with the continuity equation (2.7), we get

\[ \frac{\partial}{\partial t} \rho = -\frac{\sigma}{\epsilon} \rho - \epsilon \left( \nabla \frac{\sigma}{\epsilon} \right) \cdot E , \]  \hspace{1cm} (2.14)

If the last term is zero, e.g. for homogeneous media, the possible solutions for the charge density are

\[ \rho(x, t) = \rho_0(x) e^{-\frac{\sigma t}{\epsilon}} . \]  \hspace{1cm} (2.15)

The only physical solution (for \( t \in [\infty, \infty] \)) is the trivial solution

\[ \rho(x, t) = 0 \quad \forall t \forall x \in \Gamma \]  \hspace{1cm} (2.16)

This shows that the charge density inside a homogeneous conductor is always zero. However, at the surface of the conductor, or at interfaces where \( \sigma \) and/or \( \epsilon \) are discontinuous, a surface charge density can be present.

It can be shown that, for a homogeneous conductor, relation (2.14) follows from the divergence of the EFIE (2.13) combined with the continuity equation (2.7). Since we will use both these equations, we should not have to enforce (2.16). However, it could be convenient to do so and reduce the number of degrees of freedom in the discretised equations. Fixing \( \rho = 0 \) on the interior
will reduce the number of unknowns and make the divergence of the EFIE (2.13) trivial. Simply fixing $\rho = 0$ on the conductor interior but using the full EFIE, will thus lead to an over-defined system. The information that is left in the EFIE should be included in a way that is complementary to the trivial divergence part. This can be achieved by taking the rotation of the EFIE.

The continuity equation can also be removed for the interior of the conductors by introducing a divergence-free current basis for the conductors interiors. Such a basis could consist of loop currents. A 2-dimensional variant of such a basis and its construction are introduced in [9] and [10, chapter 3]. A nice bonus is that a suitable discretisation of the rotation of the EFIE would again be symmetric.

Most authors will note that on the interior of the conductors the charge density should be zero, and force this by omitting the charge basis functions on the interior. However, they do not elaborate on the resulting imbalance in the number of degrees of freedom with respect to the number of equations, or the dependence that arises within the EFIE. These problems do not show, because often equivalent circuits are computed for relatively thin conducting lines. These lines are discretised along the length of the line (see Figure 1) while the variation over the cross-section of the wire is not resolved. For these discretisations, the charges are easily forced on the surface without removing any degrees of freedom.

In cases where the interior is resolved, we can just use the unchanged equations and not force the charge on the surface. By solving the full equations on the interior, the solution should have all charge on the surface. In practice, the discretisation is likely not to preserve relation (2.14) exactly, which can result in small charge densities inside the conductor. This is just one of the effects of discretisation errors.

At this stage, we will continue with the full system (2.13) and (2.7), and use these for the equivalent circuit formulation. After the discretisation, one could still eliminate the interior charge density.

### 2.2 Dielectric material

The EFIE (2.13) is only valid in a homogeneous medium. In practice, the magnetic permeability $\mu$ can often be assumed constant and equal to the vacuum magnetic permeability $\mu_0$, but the dielectric constant $\epsilon$ can vary considerably over the computational domain. This variation can be accounted for in two ways.

The first is to derive a set of new equations for $A$ and $\phi$ to replace the wave equations (2.6). One could compute a new Green function for this set of equations, and again find an integral equation formulation like the EFIE. Apart from some very simple geometries, the Green function must be approximated numerically, which is very expensive.

The second method is to move the terms that arise from the variation of $\epsilon$ to the source terms in the Maxwell equations. To do this, we split $D$ in the electric field and the polarisation field:

$$D = \epsilon_0 E + P = \epsilon_0 E + (\epsilon - \epsilon_0)E,$$

(2.17)

where $\epsilon_0$ is a constant, usually the vacuum dielectric constant. This polarisation field is moved to the source terms in the Maxwell equations,

$$\frac{1}{\mu} \nabla \times B - \epsilon_0 \frac{\partial}{\partial t} E = J + \frac{\partial}{\partial t} (\epsilon - \epsilon_0) E,$$

(2.18)

$$\epsilon_0 \nabla \cdot E = \rho - \nabla \cdot (\epsilon - \epsilon_0) E.$$  

This leads to the definition of the polarisation charges and currents

$$\rho^P = -\nabla \cdot (\epsilon - \epsilon_0)E \quad \text{and} \quad J^P = \frac{\partial}{\partial t} (\epsilon - \epsilon_0) E.$$

(2.19)

(2.20)

On the microscopic level, these polarisation charges and currents are real physical charges and currents that arise due to the polarisation of the material. They are also called the bound charges.
and currents, in contrast to the free charges $\rho$ and currents $\mathbf{J}$ in the conductors. Note that the continuity equation (2.7) also holds for the polarisation charges and currents densities. Using the total source terms

$$\rho^T = \rho + \rho^P \quad \text{and} \quad \mathbf{J}^T = \mathbf{J} + \mathbf{J}^P$$

(2.21)

in the Maxwell equations, and repeating the derivation of (2.11), we find

$$\mathbf{E}(\mathbf{x}, t) = -\mu_0 \frac{\partial}{\partial t} \int G(\mathbf{x}, \mathbf{x}', t) \mathbf{J}^T(\mathbf{x}', t') d^3\mathbf{x}' dt' - \frac{1}{\epsilon_0} \nabla \int G(\mathbf{x}, \mathbf{x}', t') \rho^T(\mathbf{x}', t') d^3\mathbf{x}' dt' + \mathbf{E}^F(\mathbf{x}, t).$$

(2.22)

In the conductors, Ohm’s law (2.12) applies to the free charge current only, so the dielectric EFIE is

$$\mu_0 \frac{\partial}{\partial t} \int G(\mathbf{x}, \mathbf{x}', t) \mathbf{J}^T(\mathbf{x}', t') d^3\mathbf{x}' dt' + \frac{1}{\epsilon_0} \nabla \int G(\mathbf{x}, \mathbf{x}', t') \rho^T(\mathbf{x}', t') d^3\mathbf{x}' dt' + \frac{\mathbf{J}^F(\mathbf{x}, t)}{\sigma(\mathbf{x})} = \mathbf{E}^F(\mathbf{x}, t) \quad \forall \mathbf{x} \in \Gamma .$$

(2.23)

Note that the source terms $\rho^T$ and $\mathbf{J}^T$ still depend on the local electric field, so the dielectric EFIE (2.23) has to be solved simultaneously with equations (2.19) through (2.22) and the continuity equation (2.7).

For now we will assuming that both $\mu$ and $\epsilon$ are constant, and continue to use the EFIE for homogeneous media (2.13). We will treat the case where different dielectric materials are present separately.

### 3 Spatial discretisation

The EFIE (2.13) and continuity equation (2.7) are spatially discretised using the Galerkin method. To this end, we introduce basis functions $\Psi_i(\mathbf{x})$ for the currents,

$$\mathbf{J}(\mathbf{x}, t) = \sum_{i=1}^{N_x} I_i(t) \Psi_i(\mathbf{x}) ,$$

(3.1)

and $\Phi_s(\mathbf{x})$ for the charges,

$$\rho(\mathbf{x}, t) = \sum_{s=1}^{N_x} Q_s(t) \Phi_s(\mathbf{x}) .$$

(3.2)

We use the current basis functions as test functions for (2.13) and the charge basis functions as test functions for (2.7). In total this gives us the equations

$$\int L(t, t') \frac{\partial}{\partial t'} I(t') dt' + \int C(t, t') Q(t') dt' + RI(t) = -V^r(t) ,$$

(3.3a)

and

$$DI(t) + \frac{\partial}{\partial t} Q(t) = 0 ,$$

(3.3b)

using definitions

$$L_{ij}(t, t') = \mu \int \int \Psi_i(\mathbf{x}) G(\mathbf{x}, \mathbf{x}', t, t') \Psi_j(\mathbf{x}') d^3\mathbf{x} d^3\mathbf{x}'$$

(3.4)

$$C_{is}(t, t') = \frac{1}{\epsilon} \int \Psi_i(\mathbf{x}) \cdot \nabla \int G(\mathbf{x}, \mathbf{x}', t, t') \Phi_s(\mathbf{x}') d^3\mathbf{x}' d^3\mathbf{x}$$

$$= -\frac{1}{\epsilon} \int \int (\nabla \cdot \Psi_i(\mathbf{x})) G(\mathbf{x}, \mathbf{x}', t, t') \Phi_s(\mathbf{x}') d^3\mathbf{x}' d^3\mathbf{x}$$

(3.5)

$$R_{ij} = \int \Psi_i(\mathbf{x}) \frac{1}{\sigma(\mathbf{x})} \Phi_j(\mathbf{x}) d^3\mathbf{x}$$

(3.6)

$$D_{sj} = \int \Phi_s(\mathbf{x}) \nabla \cdot \Psi_j(\mathbf{x}) d^3\mathbf{x}$$

(3.7)

$$E_{ij} = \int \mathbf{E}(\mathbf{x}) \frac{1}{\epsilon} \Psi_i(\mathbf{x}) \Psi_j(\mathbf{x}) d^3\mathbf{x}$$

(3.8)
and

$$-V_e^i(t) = \int \Psi_j(x) E^E(x, t) d^3x .$$

(3.9)

Notice that the spatial integral moved into the matrices, but the time integration remains in (3.3a). These time integrations, in combination with the time dependence of $L(t, t')$ and $C(t, t')$ contain the retardation effects. Each element of the matrix $L(t, t')$ is a retardation function, which is only non-zero for time differences around $t - t' = |x_i - x_j|/c$, where $x_i$ and $x_j$ denote the centre points of the support of $\Psi_i$ and $\Psi_j$. The capacitive time-delay function $C(t, t')$ behaves in the same way.

In order to simplify the notation, we introduce the time-delay operators $L^\tau$ and $C^\tau$ defined by

$$\left(L^\tau U\right)(t) = \int L(t, t') U(t') dt'$$

(3.10)

$$\left(C^\tau Q\right)(t) = \int C(t, t') Q(t') dt'$$

(3.11)

for any vector $U(t)$ of length $N_b$ and any vector $Q(t)$ of length $N_n$. The system (3.3) can now be written as

$$L^\tau \dot{I} + C^\tau \dot{Q} + RI = -V^e$$

$$D\dot{I} + \dot{Q} = 0 ,$$

(3.12a, b)

where we also abbreviated the time derivative $\frac{d}{dt} \dot{\xi}$ using the notation $\ddot{\xi}$.

If the basis functions have a small support, we might approximate the strongly peaked $L_{ij}(t, t')$ by a Dirac delta function:

$$\tilde{L}_{ij}(t, t') = \delta(t - t' - \tau_{ij}) L_{ij} \approx L_{ij}(t, t') ,$$

(3.13)

were $L_{ij}$

$$L_{ij} = \mu \int \Psi_i(x) G(x, x') \Psi_j(x') d^3x' d^3x$$

(3.14)

is the non-retarded induction, $\tau_{ij} = |x_i - x_j|/c$ is the delay time between the basis functions and $G(x, x') = 1/4\pi|x - x'|$ is the non-retarded Green function. Using this approximation, the $i$th element of the first term in (3.12a) is given by

$$\left(L^\tau I\right)_i(t) = \sum_{j=1}^{N_b} \int \tilde{L}_{ij}(t, t') \frac{\partial}{\partial t'} I_j(t') dt' = \sum_{j=1}^{N_b} L_{ij} \frac{\partial}{\partial t} I_j(t - \tau_{ij}) .$$

(3.15)

Most authors use this approximation and the notation of the last term of (3.15). This approximation will be necessary for an equivalent circuit formulation with retardation.

### 3.1 Convenient and necessary basis properties

In order to apply the above theory, we still have to pick a set of basis functions $\Phi_i$ and $\Psi_s$. In this section, we will discuss some natural properties for the basis functions, and some properties that are necessary for the equivalent circuit formulation in section 4.

Since $\rho$ and $J$ are coupled by the continuity equation (2.7), it is natural to match the current and charge basis functions, in the sense that each current basis function takes charge from one charge basis function to another, as in a finite volume discretisation. In other words,

$$\nabla \cdot \Psi_s = \Phi_{s_1(i)} - \Phi_{s_2(i)} = \sum_{s=1}^{N_n} B_{ls} \Phi_s ,$$

(3.16)
which defines a sparse $N_b \times N_n$ matrix $B$,

$$B_{ts} = \begin{cases} +1 & \text{if } \Phi_i \text{ takes charge from } \Phi_s \\ -1 & \text{if } \Phi_i \text{ puts charge in } \Phi_s \\ 0 & \text{remaining part} \end{cases}. \quad (3.17)$$

The matrix $B$ contains the connectivity information of the discretisation. Note that each row of $B$ has exactly one +1 and one −1 element, which implies that the row sums of $B$ are zero, and $B$ does not have full rank.

The matrix $B$ can be used to rewrite definition (3.6) as

$$C(t, t') = -BP(t, t'), \quad (3.18)$$

where the time-delay coefficients of potential $P(t, t')$ are defined by

$$P_{st}(t, t') = \frac{1}{\tau} \int \int \Phi_s(x) G(x, t, x', t') \Phi_t(x') d^3x' d^3x. \quad (3.19)$$

Note that, in contrast with $C$, $P$ is a symmetric matrix. We will use the notation $P^T$ for the corresponding time-delay integral operator. Definition (3.9) can also be rewritten using $B$, which leads to

$$D_{sj} = \sum_{t=1}^{N_n} B_{jt} \int \Phi_s(x) \Phi_t(x) d^3x. \quad (3.20)$$

Here we see it would be convenient to choose orthonormal charge basis functions,

$$\int \Phi_s(x) \Phi_t(x) d^3x = \delta_{st}, \quad (3.21)$$

because this will lead to $D = B^T$. In that case, $B^T$ would be a discrete version of the divergence operator, and $-B$ a discrete version of the gradient operator. Note that, in this situation, charge conservation is exact.

Using choices (3.16) and (3.21), we can rewrite the discretised EFIE (2.13) in the form

$$L^T \hat{I} - B P^T \hat{Q} + R \hat{I} = -V^e$$

$$B^T \hat{I} + \hat{Q} = 0. \quad (3.22)$$

In order to convert this system into an equivalent circuit, it is necessary that the matrix $R$ is diagonal, as we will see in section 4.1. This means that the the current basis functions must be orthogonal,

$$\int \Psi_i(x) \cdot \Psi_j(x) d^3x = 0 \quad \text{for } i \neq j. \quad (3.23)$$

In practice this is most easily achieved by using non-overlapping basis functions. However, this would mean that two current basis functions can never share (in the sense of (3.16)) a charge basis function, i.e. $B$ would have a maximum of 1 element per column. This means that it would not possible to transport charge over longer distances, which would result in an unacceptable discretisation.

In a practical PEEC context, people choose basis functions that have orthonormal basis functions (conditions (3.21) and (3.23)), but only approximately satisfy (3.16). However, they still use the form (3.22) of the equations. In practice, this just adds an extra term to the discretisation error, but poses no further problems.
Figure 1: A piece of transmission line with 3 current and 4 charge basis functions.

(a) The piecewise constant current basis function. (b) Piecewise linear current basis function.

Figure 2: A 3-dimensional representation of the 2-dimensional current basis functions. The dashed grid shows the grid for the piecewise constant charge basis functions. The current density is plotted in the vertical direction and is shown by the shaded surface.

3.2 A commonly used basis

A common choice of basis functions for the PEEC method consists of piecewise constant basis functions (block-functions) for both the currents and the charges. For example, a thin transmission line would be divided into segments as shown in Figure 1. On each segment a constant current basis function is defined. For the charge basis functions, the transmission line is again divided into segments, but shifted with respect to the current cells, such that the centers of the new charge segments correspond to the boundaries of the current cells. On the conductor surface of the charge segments, constant surface charge basis functions are defined. At the end of the transmission line, there are two half-sized charge cells. This basis does satisfy orthogonality conditions (3.21) and (3.23), but only approximately satisfies (3.16). Still, formulation (3.22) is used.

The basis described for the transmission line is essentially 1-dimensional, but it can easily be extended to 2 dimensions (see Figure 2(a)) or a 3-dimensional basis. Often, rectangular cells are used, but this can be generalised to linear deformations of rectangles [11].

3.3 A more sophisticated basis

For a more accurate discretisation, we could use a more sophisticated basis. The basis described in this section also has piecewise constant charge basis functions (block-functions), but it has piecewise linear current basis functions (tent-functions). They can be combined in such a way that relation (3.16) is satisfied exactly. A 2-dimensional example of a tent-shaped current basis function is shown in Figure 2(b)).

It is not necessary to work with rectangular cells. Suppose the conductors are subdivided in arbitrary triangles (2D) or tetrahedrons (3D). On each cell we define a constant charge basis function and for each cell boundary we define a current basis function with piecewise constant divergence. The current basis function has its support on the two connecting cells, and per unit time it takes a unit charge from the one cell to the other. This idea can be extended to cell types
with more cell boundaries, and combinations of different types, but it will be increasingly more
difficult to determine the correct current basis functions.

For this type of bases, the charge basis functions are orthonormal (3.21), but the current basis
functions overlap, so the orthogonality condition (3.23) is not met. However, the consistency
condition (3.16) is satisfied exactly. A big drawback of this basis is that the evaluation of the
matrix elements of $L$ can be even more costly than for the simpler basis described in section 3.2.

3.4 Dielectric material

In the presence of dielectric material, we have to incorporate the polarisation charge and current
densities in the model. To discretise these, we extend the current and charge basis to the dielectric
regions:

$$J^P(x, t) = \sum_{i=N_b+1}^{N_b+N_p} I_i(t) \Psi_i(x)$$

and

$$\rho^P(x, t) = \sum_{s=N_n+1}^{N_n+N_p} Q_s(t) \Phi_s(x) .$$

We will assume that for both the free and the polarisation basis functions the consistency
condition (3.16) and orthonormality condition (3.21) hold, such that we can write both the free and the
polarisation continuity equation in one matrix equation

$$B^T I + Q = 0 .$$

To simplify the notation, we will use the subscripts $f$ and $p$ to indicate the free and polarisation
parts of a vector or matrix,

$$I = \begin{pmatrix} I_f \\ I_p \end{pmatrix}, \quad Q = \begin{pmatrix} Q_f \\ Q_p \end{pmatrix}, \quad B = \begin{pmatrix} B_{ff} & 0 \\ 0 & B_{pp} \end{pmatrix} .$$

By extending the definitions for $L(t, t')$ (3.4) and $P(t, t')$ (3.19) to include the polarisation current
and charge basis functions, we can write the dielectric EFIE (2.23) as

$$L^\tau_{pp} \hat{I}_p - B_{pp} \hat{P}_{pp} Q_p - B_{pp} \hat{P}_{pp} Q_P + R_{pp} I_p = -V^e .$$

In order to discretise equation (2.22), we expand the electric field in the dielectric regions using
the $\Psi_i$ basis functions

$$E(x, t) = \sum_{i=N_b+1}^{N_b+N_p} E_i(t) \Psi_i(x) .$$

Note that the electric field is now represented by $E_f$ while $E_p$ remains undefined. We also introduce
the Galerkin discretisation of the identity operator for vector fields

$$\mathbb{I}_{ij} = \int \Psi_i(x) \Psi_j(x) d^3x .$$

The Galerkin discretised version of (2.22) can be written as

$$L^\tau_{pp} \hat{I}_p + L^\tau_{pp} \hat{I}_p - B_{pp} \hat{P}_{pp} Q_p - B_{pp} \hat{P}_{pp} Q_P + I_{pp} E_p = -V^e .$$

Also introducing the so-called excess capacitance matrix [6]

$$C^+_{ij} = \int \Psi_i(x)(\varepsilon(x) - \varepsilon_0) \Psi_j(x) d^3x ,$$

9
the discretisation of (2.20) is
\[ \|_{CV} I_t = C_{ip}^+ E_v^+ \] .
(3.33)

This completes the description of the system.

Note that, if large area’s of the domain contain dielectric material, the extra number of degrees of freedom needed to include the dielectric effects can be very large. In this case, a discretisation of the Maxwell equations itself might be more efficient since it leads to sparse matrices.

4 The equivalent circuit

In order to make an equivalent circuit formulation, we will assume that the spatially discretised system is of the form (3.22), either due to an appropriately chosen basis or by approximation.

A naming convention exists for different variants of equivalent circuit formulations. The specific model is indicated by listing the symbols that correspond to the terms that are included. \( L_p \) or \( L \) stands for partial inductances, \( P \) stands for coefficients of potential and \( R \) for series resistors. The symbol \( \tau \) is used to indicate the presence of retardation. Thus, for the full equivalent circuit corresponding to (3.22), the notation \( (L, P, R, \tau)_{PEEC} \) is used. If we neglect the time delays, we get an \( (L, P, R)_{PEEC} \) model.

For simplicity, we will first neglect time delay and describe the \( (L, P, R)_{PEEC} \) model. After that we will comment on the situation with retardation.

4.1 Without retardation: \( (L, P, R)_{PEEC} \)

In the limit case of infinite light speed, or equivalently, in the limit of very small length scales with respect to time scales, the retardation times vanish. In this limit, the time-delay operators \( L^\tau \) and \( P^\tau \) become normal matrix multiplications, and the system (3.22) can be written as
\[ R I + L \dot{I} - BPQ = -V^e \quad \text{branches} \]  
\[ B^T I + \dot{Q} = 0 \quad \text{nodes} . \]  
(4.1a)  
(4.1b)

We will show that this system can be interpreted as an electronic circuit, where each charge basis function is represented by a circuit node and each current basis function is represented by a circuit branch.

The electrostatic potential at the charge segments is given by \( V = PQ \), which will be the nodal voltages in the circuit. The charge accumulation on the charge segments is written as an extra set of currents \( I^c = \dot{Q} \). Substituting these two definitions, (4.1) becomes
\[ B^T I + I^c = 0 \quad \text{nodes} \]  
\[ R I + L \dot{I} + V^e - BV = 0 \quad \text{branches} \]  
\[ PI^c - \dot{V} = 0 \quad \text{nodes} . \]  
(4.2a)  
(4.2b)  
(4.2c)

In this formulation we can recognise a set of Kirchhoff current laws (KCR’s) in equation (4.2a), one for each charge basis function. In equation (4.2b) we can recognise a set of branch constitutive relations (BCR’s) for branches with, in series, a linear resistor \( R \), an inductor that couples to all other inductors \( L \) and a voltage source representing the external electric field \( (V^e) \). For each current basis function, there is one branch in the circuit with one BCR. The third equation, (4.2c), can be interpreted as a set BCR’s for coupled capacitances. These capacitances store the charge that is accumulated on the charge segments. However, this charge generates a potential in all the nodes. Since this type of coupled capacitance does not exist as a circuit element, they have to be rewritten. A common way of doing this is demonstrated here for the \( s \)th element of (4.2c) :
\[ \sum_{t=1}^{N} P_{st} I^c_t = V_s \quad \iff \quad I^c_s = \frac{V_s}{P_{ss}} - \sum_{t=1, t \neq s}^{N} \frac{P_{ts}}{P_{ss}} I^c_t . \]  
(4.3)
Figure 3: The PEEC equivalent circuit corresponding to one current and two charge basis functions, both for the conductors as for the dielectric regions.

The first term on the right-hand side can be interpreted as a capacitor with capacitance $1/P_{ss}$ between node $s$ and ground. The second term can be incorporated in the circuit in the form of a current controlled current source. For one branch and two accompanying nodes, the equivalent circuit is shown in Figure 3(a).

It is also possible to rewrite equation (4.2c) such that it can be interpreted as a fully connected net of capacitors. In this case each node is connected to each other node by a capacitor. The drawback is that the matrix $P$ has to be inverted in order to compute the values of these capacitances.

The PEEC can easily be combined with the real electronic circuit by connecting the circuit elements to the nodes of the PEEC. If we want to analyse the PEEC separately, we have to add current and/or voltage sources to the equivalent circuit to simulate external connections. To this end, we set up two different multi-port network formulations in section 4.3. For now, we will only introduce a set of external current sources $I_s$ at the nodes. This leads to the system

\begin{align}
B^T I + I^c &= -I^s \quad \text{(nodes)} \\
R I + L \dot{I} - BV &= -V^e \quad \text{(branches)} \\
P I^c - \dot{V} &= 0 \quad \text{(nodes)}.
\end{align}

### 4.2 Some equivalent formulations

There are several ways to write the equations of the equivalent circuit. We already showed formulation (4.1), which has $N_n + N_b$ unknowns and is closest to the original equations (2.13) and (2.7). System (4.2) is in modified-nodal formulation \[12\] and has $2N_n + N_b$ unknowns.

Eliminating $I^c$ from (4.4), and switching to a matrix notation gives

\[
\begin{pmatrix}
L & 0 & 0 & P^{-1}
\end{pmatrix}
\begin{pmatrix}
\dot{I} \\
\dot{V}
\end{pmatrix}
+ 
\begin{pmatrix}
R & -B \\
B^T & 0
\end{pmatrix}
\begin{pmatrix}
I \\
V
\end{pmatrix}
= 
\begin{pmatrix}
-V^e \\
-I^s
\end{pmatrix}.
\]

The inconvenient occurrence of $P^{-1}$ can be avoided by scaling:

\[
\begin{pmatrix}
L & 0 & 0 & MN
\end{pmatrix}
\begin{pmatrix}
\dot{I} \\
\dot{N}^{-1}V
\end{pmatrix}
+ 
\begin{pmatrix}
R & -BN \\
MPB^T & 0
\end{pmatrix}
\begin{pmatrix}
I \\
N^{-1}V
\end{pmatrix}
= 
\begin{pmatrix}
-V^e \\
-MPI^s
\end{pmatrix},
\]

where $M$ and $N$ are arbitrary invertible matrices of the same size as $P$. Choosing $N = 1$ and $M = P^{-1}$ gives equation (4.5). In this form, the second matrix has nicely balanced terms, but computing $P^{-1}$ can be very expensive. If $M = N = 1$, the second matrix can become severely ill conditioned due to very small angles between eigenvectors. Using $M \approx P^{-1}$ and $N = 1$ can resolve this problem, but $M = P^{-1}$ and $N = PO$ where $O \approx P^{-1}$ will also do the job. Note that for both cases we do not have to know $P^{-1}$. We have successfully used the inverse of the diagonal of $P$ for the approximation of $P^{-1}$. A further scaling of $M$ and $N$ should be used to reduce the difference in magnitude between the $L$ and $MN$ block of the first matrix. For simple cases, this
can be achieved by scaling both $M$ and $N$ with the square root of the ratio of the average diagonal elements of $L$ and $MN$.

The number of unknowns can be reduced by eliminating $V$, which leads to a set of second order equations

$$R \dot{I} + L \ddot{I} + BPB^T I = -\hat{V} - BPI^s \quad \text{(branches)} .$$

Conversely, we could introduce all the nodal voltages between the resistors, inductances and voltage sources and separate the capacitances and current driven current sources, but this would lead to a much larger system.

### 4.3 Multi-port networks

A multi-port network is a sub-circuit for which a (usually relatively small) number of nodes can be connected to other circuits. These nodes are called the ports. We will use a matrix $K$, to identify port numbers with node numbers.

$$K_{ij} = \begin{cases} 1 & \text{if port } j \text{ corresponds to node } i \\ 0 & \text{remaining part} \end{cases} .$$

(4.8)

In the formulations below, we assumed there is no external electric field, so $V^e = 0$. This is usually the case in the PEEC context. Furthermore, we will use a standardised notation :

$$C \ddot{x}(t) + Gx(t) = Fu(t)$$

$$y(t) = Ex(t) .$$

(4.9)

In this formulation, $u(t)$ is the driving term, which is known, $x(t)$ is the total state space vector, and $y(t)$ contains the quantities we are interested in.

In our experience it is important to use a well scaled system, so we will use the scaled system (4.6) as the basis for the multi-port formulations.

There are a lot of ways of setting up the multi-port network equations. Two standard ways are described here.

#### Impedance formulation

This formulation uses current sources $I^p$ at the ports, so $I^s = KI^p$. We find

$$C = \begin{pmatrix} L & 0 \\ 0 & MN \end{pmatrix}, \quad G = \begin{pmatrix} R & -BN \\ MPBT & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ -MPK \end{pmatrix}, \quad E = \begin{pmatrix} 0 \\ KTN \end{pmatrix}, \quad (4.10a)$$

and

$$x(t) = \begin{pmatrix} I(t) \\ N^{-1}V(t) \end{pmatrix}, \quad u(t) = I^p(t), \quad y(t) = V^p(t).$$

(4.10b)

In this formulation, the port voltages $V^p$ are computed for the given port currents $I^p$, which effectively gives us an impedance operator $Z$, $V^p = ZI^p$.

#### Admittance formulation

In the admittance formulation the port currents $I^p$ are computed for given port voltages $V^p$. This is alike an admittance operator $Y$, $I^p = YV^p$. To implement the voltage sources, we have to introduce the currents through the voltage sources as new unknowns $I^p$. This leads to

$$C^N = \begin{pmatrix} L & 0 & 0 \\ 0 & MN & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G^N = \begin{pmatrix} R & -BN & 0 \\ MPBT & 0 & MPK \\ 0 & KTN & 0 \end{pmatrix}, \quad F^N = E^{\text{yT}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, $$

(4.11a)
4.4 With retardation: \((L, P, R, \tau)\) PEEC

In order to incorporate retardation in the equivalent circuit, the approximation (3.13) must be used for \(L^r\) and \(P^r\). Taking the \((L, P, R)\) PEEC formulations we used above, and replacing all the matrices \(L\) and \(P\) by the approximated time-delay operators \(\tilde{L}^r\) and \(\tilde{P}^r\), we find the corresponding models with retardation. In effect, this means that the currents \(I\) and charges \(Q\) (or \(I^c\)) involved in the capacitive and inductive terms are evaluated at an earlier time. Apart from the evaluation times, the equivalent circuit is the same as for the case without retardation. However, most circuit simulators will not be able to handle circuits with delays.

Note that some of the formulations for the \((L, C, R)\) PEEC model contain the inverse of the matrix \(P\), for instance system (4.5). Since it is very inconvenient to formulate and work with the inverse of the time-delay operator \(\tilde{P}^r\), these system formulations can not be used for the \((L, C, R, \tau)\) PEEC model. The scaled formulation (4.6) can and should be used. For the scaling matrices, an approximation of the inverse of the no-time-delay matrix \(P\) can be used.

4.5 Dielectric material

In section 3.4, we saw that the presence of dielectric material leads to the following equations

\[
\begin{align*}
L^+_{\nu \nu} \dot{I}_x + L^+_{\nu \nu} \dot{I}_y - B_{\nu \nu} P^+_{\nu \nu} Q_{\nu} - B_{\nu \nu} P^+_{\nu \nu} Q_{\nu} + R_{\nu \nu} I_x &= -V^c \\
L^+_{\nu \nu} \dot{I}_y + L^+_{\nu \nu} \dot{I}_x - B_{\nu \nu} P^+_{\nu \nu} Q_{\nu} - B_{\nu \nu} P^+_{\nu \nu} Q_{\nu} + R_{\nu \nu} I_y &= -V^c \\
I_{\nu \nu} \dot{I}_x - C^+_{\nu \nu} \dot{E}_x &= 0 \\
B^T I + \dot{Q} &= 0.
\end{align*}
\]

Using the definitions \(V = PQ, I^c = \dot{Q}, V^+_c = \|P_{\nu \nu}\|E_{\nu}\) and \(C^+_{\nu \nu} = I_{\nu \nu}^{-1}C^+_{\nu \nu}I_{\nu \nu}^{-1}\), we can rewrite these in the form

\[
\begin{align*}
B^T I + I^c &= 0 \\
R_{\nu \nu} I_x + L^+_{\nu \nu} \dot{I}_x + L^+_{\nu \nu} \dot{I}_y - B_{\nu \nu} P^+_{\nu \nu} V_{\nu} &= -V^c \\
V^+_c + L^+_{\nu \nu} I_x + L^+_{\nu \nu} \dot{I}_x - B_{\nu \nu} P^+_{\nu \nu} V_{\nu} &= -V^c \\
I_{\nu} - C^+_{\nu \nu} \dot{V}^+_{\nu} &= 0 \\
P^T I^c - \dot{V} &= 0.
\end{align*}
\]

Assuming that \(R_{\nu \nu}\) and \(\tilde{C}^+_{\nu \nu}\) are diagonal, which is true if the \(\Psi_i\) are orthogonal, we can recognise an equivalent circuit. This is an extension of the \((L, P, R)\) PEEC circuit of section 4.1. For each polarisation charge and current basis function, a node and branch are added. Equation (4.13a) contains the Kirchhoff current laws for all the nodes. Equation (4.13b) describes the current branches for the free current basis functions in the conductor, which are the same as before. New is equation (4.13c), which describes polarisation current branches in the dielectric material. We see that these branches contain a inductor in series with a capacitor described by equation (4.13d). Equation (4.13e) describes the capacitive couplings between all the nodes, and can be represented in the circuit with ground capacitance and current driven current sources, as before. Figure 3(b) shows a dielectric current branch with two nodes. Note that, since \(B\) has two disconnected blocks (see equation (3.27)), the conductor circuit and the polarisation circuit are connected only by the capacitive and inductive couplings.
5 Laplace transformation

In circuit analysis, the Laplace transform is often used for frequency-domain analysis. The Laplace transform of a function \( f(t) \) is defined by

\[
    f(s) = \int_0^\infty e^{-st} f(t) dt \tag{5.1}
\]

Note that we use the same notation for the time-domain function and the frequency-domain function. In this document, the difference will be clear due to the use of the time variable \( t \) or the frequency variable \( s \). We will assume the basic properties of the Laplace transform are well known.

The time-delay operators \( L^\tau \) and \( P^\tau \) are convolution operators, so in the frequency-domain they are ordinary (frequency dependent) matrix operators

\[
    L^\tau_{ij}(s) = \mu \int \int \Psi_i(x) G(x - x', s) \Psi_j(x') d^3x' d^3x \tag{5.2}
\]

\[
    P^\tau_{st}(s) = \frac{1}{\varepsilon} \int \int \Phi_s(x) G(x - x', s) \Phi_t(x') d^3x' d^3x \tag{5.3}
\]

in which the frequency-domain Green function is defined by

\[
    G(x - x', s) = \frac{e^{-|x - x'|/\varepsilon}}{4\pi|x - x'|} \tag{5.4}
\]

Note that for this type of dense matrices, there are fast methods for matrix-vector multiplication without computing and storing all the matrix elements. One class of these methods are the Fast Multipole Methods (FMM) [13, 14, 15].

In the approximation (3.13), we find the Laplace transforms

\[
    \tilde{L}^\tau_{ij}(s) = L_{ij} e^{-s\tau_{ij}} \quad \text{and} \quad \tilde{P}^\tau_{st}(s) = P_{st} e^{-s\tau t}. \tag{5.5}
\]

If time delay is completely neglected, we find that \( P(s) = P \) and \( L(s) = L \) are the (frequency independent) matrices from the time domain formulation.

Assuming zero initial conditions, the Laplace transform of the system (3.22) is

\[
    sL^\tau(s) I(s) - BP^\tau(s) Q(s) + RI(s) = -V^\tau(s)
\]

\[
    B^T I(s) + sQ(s) = 0. \tag{5.6}
\]

Note that, since \( P^\tau(s) \) is a matrix operator, and it is likely to be invertible, formulations like (4.5) could again be used. However, this would not be very practical since \( P^\tau(s) \) would have to be inverted for each \( s \), which would be very costly. The Laplace transform of the impedance multi-port formulation (4.10) for time-delay systems is

\[
    sC(s) x(s) + G(s) x(s) = F(s) u(s)
\]

\[
    y(s) = E x(s), \tag{5.7}
\]

using definitions

\[
    C(s) = \begin{pmatrix} L(s) & 0 \\ 0 & M(s) N(s) \end{pmatrix}, \quad G(s) = \begin{pmatrix} R \\ M(s) P(s) B^T \\ 0 \end{pmatrix}, \tag{5.8}
\]

\[
    F(s) = \begin{pmatrix} 0 \\ -M(s) P(s) K \end{pmatrix}, \quad E(s) = \begin{pmatrix} 0 \\ K^T N(s) \end{pmatrix}, \tag{5.9}
\]

and

\[
    x(s) = \begin{pmatrix} I(s) \\ N(s)^{-1} V(s) \end{pmatrix}, \quad u(s) = I^p(s), \quad y(s) = V^p(s). \tag{5.10}
\]
In general, $M(s)$ and $N(s)$ can be chosen frequency dependent. However, for practical purposes, it is convenient to limit the number of frequency dependent terms. If necessary, the frequency dependence can be fully removed from $F$ and $E$ by using formulation (4.4). Note that if retardation is neglected, the all matrices are independent of $s$ and system (5.7) is linear in $s$.

The relation between the port currents and voltages is given by the transfer function $H(s)$,

$$x(s) = H(s)u(s),$$

$$H(s) = E(s)(G(s) + sC(s))^{-1}F(s).$$

The transfer function contains all the properties of the multi-port circuit. The frequency response of the circuit is often characterised by $H(s)$ for $s$ on the positive imaginary axis.

Using the multi-port impedance formulation (4.10), as we have done here, the transfer function is the frequency-domain impedance operator

$$V^s(s) = Z(s)I^s(s).$$

When the admittance formulation (4.11) is used, the transfer function is the frequency-domain admittance operator

$$I^s(s) = Y(s)V^s(s).$$

Using straightforward linear algebra, it can indeed be shown that the transfer function for the admittance formulation is the inverse of the transfer function for the impedance formulation,

$$Y(s) = Z(s)^{-1}.$$

### 5.1 Stability and Passivity

A linear dynamical system is said to be stable if its free-response, i.e. its solution without driving terms but with a non-zero initial condition, is bounded for $t \to \infty$. Usually, stability is a requirement for a system to make physical sense. Passivity is stronger a requirement than stability and somewhat more complicated to define. Roughly speaking, a passive system does not generate energy. The importance of passivity is that, in order to guarantee stability of a combination of coupled subsystems, all subsystems must be passive.

It can be shown that all LCR-circuits are stable and passive, so our PEEC model without retardation is stable and passive. Based on physical arguments (bounded free-response and no energy generation) we can also say that the PEEC model with retardation is stable and passive.

For non-delay systems, a stability condition can be formulated in terms of the generalised eigenvalues $\lambda$ of $G$ and $C$, $(G + \lambda C)x = 0$. The system is stable if and only if

1. All finite eigenvalues satisfy $\text{Re}(\lambda) \leq 0$.
2. All finite eigenvalues with $\text{Re}(\lambda) = 0$ are simple.

The system is passive if and only if the frequency-domain transfer function $H(s)$ is positive real, i.e. the following must be satisfied:

1. $H(s)$ is analytic for all $s \in \mathbb{C}$ with $\text{Re}(s) > 0$.
2. $H(s^*) = H(s)^*$ for all $s \in \mathbb{C}$.
3. $H(s) + H(s)^H$ is Hermitian positive semi-definite for all $s \in \mathbb{C}$ with $\text{Re}(s) > 0$.

We used the notation $^*$ to indicate the complex conjugate. For more text on stability and passivity, and more practical methods to determine if a system is stable and/or passive, see [16].
6 Model order reduction

The equivalent circuits that are generated with the PEEC method are usually incorporated in the non-linear circuit of the electronic components. The PEEC circuits are linear circuits but with dense coupling matrices and they can be quite large. Combined with the non-linear circuit, the PEEC part can severely slow down the analysis of the total circuit. For this reason, model order reduction is often used to replace the PEEC circuit with a smaller circuit with approximately the same behaviour in the desired frequency range. This often achieved by approximating the frequency-domain transfer function \( H(s) \). For most model order reduction techniques, it is not important that the system represents a circuit, and even if it does, this representation is lost during the reduced-order modelling. However, an equivalent circuit representation is still useful to facilitate the external connections to the model. Important issues in model order reduction are the preservation of stability and passivity of the model.

In absence of retardation, the transfer function is given by

\[
H(s) = E(G + sC)^{-1}F.
\]  

(6.1)

For this case, the \( s \) dependence of \( H \) is fairly clear and the behaviour of \( H(s) \) is dictated by the generalised eigenvalues and -vectors of the matrix pair \((G, C)\). Various different model order reduction techniques have been developed that are, in one way or another, based on the approximation of the generalised eigenpairs of \((G, C)\) in the frequency range of interest. An overview of some of this type of reduced-order modelling methods, with the emphasis on the Lanczos based PVL family, can be found in [16]. Other approaches can be found in [17, 18].

In the presence of retardation, reduced-order modelling is more complicated. In line with the approach for the non-retarded system, the eigenpairs of the non-linear eigenvalue problem

\[
(G(s) + sC(s))x = 0
\]

(6.2)

can be used to formulate a reduced-order model. In [19], the eigenpairs are approximated by reformulating (6.2) as a polynomial eigenvalue problem using local Taylor expansions of \( G(s) \) and \( C(s) \). The polynomial eigenvalue problem can in turn be reformulated as a larger linear eigenvalue problem.

7 Experiments

We have done some very limited experiments with the PEEC method. The main goals of these experiments are to see the method at work and to get a feeling for the influence of retardation.

The interaction matrices for these experiments where obtained with the code used in [10]. This code is restricted to thin wires and thin plates for the conductors and does not support dielectrics.

7.1 Three transmission lines

The first example consists of a set of three 1 cm long parallel wires with a radius of 2.5 µm and wire spacing 10 µm. Each wire is split in 80 charge segments. We used a two-port impedance formulation in which one port is one end of the first wire and the other port is the opposite end of the middle wire. This results in a 2 × 2 transfer function of which we plotted \( H_{12}(= Z_{12}) \) for both the case with and without retardation in figure 4(a). We see that, even at zero frequency, there is a small difference for the real part. This is the result of the finite distance between the two ports. Also notice that much larger differences start to show at the first resonance frequency (about 13 GHz). The second resonance (at 15 GHz) is virtually absent for the system with delay, while the third resonance is shifted and damped with respect to the non-retarded case. This damping of resonances can be explained by the introduction of radiation losses, which are virtually non-existent if retardation is neglected.
(a) The real and imaginary parts of $Z_{12}$ and $Z'_{12}$.

(b) The relative difference between $Z$ and $Z'$, and $Y$ and $Y'$.

Figure 4: $H(s)$ for the three thin parallel wires test case ($s = 2\pi f$).
(a) The real and imaginary parts of $Z_{12}$ and $Z_{\bar{1}2}$.

(b) The relative difference between $Z$ and $Z^\ast$, and $Y$ and $Y^\ast$.

Figure 5: $H(s)$ for the spiral inductor test case shown in figure 6 ($s = 2\pi if$).
The second test case consists of a spiral inductor as they might be used on chips. Due to the limitations of the code, the spiral is constructed using thin conducting plates. The precise geometry and discretisation can be seen in figure 6. For this problem, the ports correspond the two endpoints of the conductor and again we examine $H_{12}$ in the impedance formulation ($Z_{12}$) and the relative difference between $Z$ and $Y$ with and without retardation. The results are shown in figure 5. Notice the same type of behaviour as for the transmission line example. The 10% difference mark is now crossed at around 200 GHz, which corresponds to a wavelength of approximately 0.7 mm. This is again 10 times the largest length-scale of the geometry.

If the right port is grounded and a 5 GHz AC voltage source is connected to the left port, the resulting current and charge distribution is shown in figure 6. This shows the common effect that on the inner part of the spiral, the current is forced to the inside of the wire. This shows that the distribution of the current within the wire could be important for this problem, and should be further resolved.

The discretisation of the spiral conductor as shown in figure 6 has 1251 unknowns (468 nodes and 783 branches). The same geometry can be approximated with much less unknowns using a single conducting wire. Discretising this wire using charge segments with a length of 10 $\mu$m leads to 75 unknowns. We compared the results of this very coarse discretisation with the results of

---

**Figure 6:** Scaled currents (arrows) and charge densities (gray-scale) for the 5 GHz Fourier mode of the spiral inductor test case, with the right end grounded and an applied voltage on the left end. The charge and current densities have a $\pi/2$ phase difference.
Figure 7: $H_{12}$ for the test case of a spiral inductor for a fine and coarse discretisation.

one with 2292 unknowns (4 elements on the with of the wire). The very coarse discretisation still gives very nice results, as can be seen in figure 7.

8 Conclusions

We have seen that the PEEC formulation is nothing more than a spatial discretisation of the EFIE, which can, under some restrictions, be interpreted as an equivalent circuit. This integral equation approach leads to much fewer degrees of freedom than the differential equation approach, but it leads to full coupling, i.e. dense matrices. Still, the strong reduction of the number of degrees of freedom can lead to faster methods, especially if fast matrix multiplications like the fast multipole method are used. If the current distribution inside the conductor has to be resolved in great detail, or if there are large dielectric regions, the number of degrees of freedom can increase beyond the level where the integral equation approach can be faster than the differential equation approach. However, a simple experiment showed that very coarse discretisations of the EFIE can already give fairly accurate results.

In two small experiments, we saw that the effects of retardation become significant for wavelengths of the order of 10 times the largest distance in the geometry. For normal on-chip length scales of less than 10 μm, retardation effects can be neglected for frequencies up to 300 GHz. Note that the presence of dielectric material will lead to shorter wavelengths such that retardation will become important at lower frequencies. Retardation effects are more important for longer connections in packaging or on the printed circuit board and they are of crucial importance for radiation problems.

Model order reduction can be used to further reduce the PEEC model. For the case where retardation can be neglected, many reduced-order modelling methods are available. For the case with retardation, some methods are available, but there is still much work to do.
References


