I. INTRODUCTION

In several studies of purely two-dimensional (2-D) vortices, it was found that monopolar vortices with a shielded vorticity distribution, i.e., with a ring of opposite vorticity, can develop azimuthal instabilities. The ring then breaks up into a number of vortices that have a sense of rotation opposite to that of the core. Flierl investigated the instability of circular patches with uniform vorticity surrounded by a ring of uniform oppositely signed vorticity. He found that the instability depends on the ratio between the vorticity values in the ring and in the core, and on the relative width of the outer ring. Carton et al. introduced a useful analytical expression for a continuous vorticity distribution of a shielded vortex with zero circulation, which incorporates a steepness parameter $\alpha$. They found instability of the vortex, when the steepness of the vorticity profile exceeds a critical value. This particular vorticity profile was also used in several other studies to investigate the growth rates of small azimuthal perturbations for different wave numbers. The results of these studies are summarized in Fig. 1. For inviscid flows the azimuthal perturbation with wave number $n$ results of these studies are summarized in Fig. 1. For inviscid flows the azimuthal perturbation with wave number $n$ becomes unstable, when $a$ exceeds a critical value $a_* = 1.85$. For larger values of $a$ the growth rate of the perturbation also increases. Eventually, the $n=2$ perturbation can lead to the formation of a tripolar vortex, i.e., a vortex with an elliptical core of one sign vorticity surrounded by two satellites of opposite sign$^{6,7}$ or dipole splitting can occur (for $a \geq 3.2$). Figure 1 shows that also perturbations with wave numbers $n \geq 3$ can grow, but the steepness parameter $a$ has to become increasingly larger. The result of the growth of an instability with $n=3$ is a vortex with a triangular core surrounded by three satellite vortices of opposite vorticity.

Although the instability type is essentially two dimensional it can also arise in quasi-two-dimensional (Q2-D) flows in the laboratory. A tripole formation was observed in laboratory experiments in both rotating fluids and in density stratified fluids. Triangular vortices have also been observed in laboratory experiments, both in rotating and stratified fluids. The aim of the present study is to investigate how the essentially 3-D structure of the vortices in a stratified fluid influences their possible azimuthal instability. More specifically, the role of viscosity and of the ambient density stratification on the evolution of the vortices is studied. The results of tripole formations in laboratory experiments are shown in Sec. II, where also the differences between several observed tripoles are discussed. In Sec. III, the evolution of the 3-D vortices with a similar initial velocity and density distribution as used by Beckers et al. for $\alpha = 2$, but now for larger values of the steepness parameter, $a$, is studied by numerical simulations. The influence of the Froude number on the tripole formation is investigated and the 3-D structure of a full-grown tripole is briefly discussed. A regime study of the instability of monopolar vortices, with $500 \leq \text{Re} \leq 10,000$, $0.1 \leq F \leq 0.8$, and $2 \leq a \leq 8$, is summarized in Sec. IV. In Sec. IV we also show that both decreasing Re and increasing $F$ inhibits dipole splitting in the regime where it is usually observed for the 2-D case. Finally, a summary of the conclusions is given in Sec. V.
II. RESULTS OF LABORATORY EXPERIMENTS

The setup that was used for the laboratory experiments has been described extensively in three previous papers. Hence, only a short overview will be presented. The laboratory experiments were carried out in a nonrotating linearly stratified fluid established by applying the two-tank method. Vortices were created by applying the tangential injection method introduced by Flór and van Heijst: fluid with matched density is injected horizontally along the inner wall of a bottomless, thin-walled cylinder, which is positioned at a certain level in the stratification. An amount of fluid $\Delta V$ is injected during a period $\Delta t$ at an injection rate defined by $Q = \Delta V / \Delta t$. The injection results in a circular flow inside the cylinder, and a monopolar vortex is formed after the cylinder is carefully removed. To obtain quantitative information of the velocity field of the vortex, small polystyrene particles are added to the stratification and float at their neutrally buoyant level in the fluid. These tracer particles are illuminated from the side by a light sheet. Above the tank a video camera is mounted that records the particle motions on videotape. After the experiment the tape can be processed with a particle tracking algorithm to determine the horizontal velocity field of the fluid at the level that has been illuminated by the light sheet. From the 2-D velocity distribution the vertical vorticity component $\omega_z$ can be calculated. In all experiments the time $t=0$ is defined as the moment when the fluid injection is stopped.

In the experiments by Flór and van Heijst, a vortex was made by locally stirring the fluid with a bent rod. A monopolar vortex created in this way soon transforms into a vortex with an elliptical core surrounded by two satellites, but eventually this tripole axisymmetrizes and becomes monopolar again. In addition, they also described the emergence of a tripole, although one with very weak satellites, from a monopolar vortex created by the tangential injection method. The weak satellites of this tripole were torn into vortex filaments, which were wrapped around the core vortex, and a monopolar vortex was retrieved. We have reconsidered the vortices created by the tangential injection method and obtained vortices at higher Reynolds numbers and with different Froude numbers as in Ref. 9. In contrast to the previously reported experiments, a larger tangential injection cylinder was used, i.e., with a diameter $d = 15.0 \text{ cm}$ and height $h = 5.0 \text{ cm}$. This larger cylinder was necessary to obtain a flow with a higher Reynolds number, not only because the radius of the vortex is larger, but also because the amount of injected fluid ($\Delta V$) and the injection rate ($Q$) could be higher without spilling injected fluid over the cylinder wall. The evolution of the vortex has been studied in two ways; either by following the distribution of a passive tracer (dyed fluid), or by tracking neutrally buoyant tracer particles.

Figure 2 shows the dye-visualized evolution of an unstable vortex: almost directly after the cylinder is removed the initially circular vortex starts to deform. In Fig. 2(a), two satellite vortices are seen to be formed in the periphery of the vortex. The core vortex then transforms into an ellipse and the two satellites take positions at either side of the core: a tripole has been formed [see Fig. 2(b)]. The elliptical core rotates around its center and the satellites are corotating, so that their centers remain on one line with the core. From the moment when the tripole has been formed until the time of

![Figure 1](https://example.com/figure1.png)

**FIG. 1.** Growth rate (in arbitrary units) of three azimuthal perturbations as a function of the steepness parameter $\alpha$ [from Carnevale and Kloosterziel (Ref. 4)].

![Figure 2](https://example.com/figure2.png)

(a) $t = 87 \text{ s}$  
(b) $t = 215 \text{ s}$  
(c) $t = 1150 \text{ s}$

**FIG. 2.** Photographs of a dye visualization experiment showing the formation of a tripole. The injection parameters were $Q = 2.22 \text{ cm}^3 \text{ s}^{-1}$ and $\Delta V = 100 \text{ cm}^3$. 

the last photograph (i.e., after about 20 min), the tripole has only performed half a rotation, while it has grown substantially in size.

Dye diffuses on a much larger time scale than momentum, and the dye structure may eventually give a "fossilized" impression of the flow field of the tripolar vortex. It is therefore necessary to investigate also the evolution of the distribution of (vertical) vorticity in the tripole. Figure 3 shows contours of the vertical vorticity measured in an experiment, referred to as experiment I, performed under the same conditions as the dye visualization in Fig. 2. Continuous contours represent positive vorticity; dashed contours represent negative vorticity. In Fig. 3(a), representing the vorticity distribution at \( t = 45 \text{s} \), a core vortex can be observed, surrounded by a ring of oppositely signed vorticity. This ring appears to break up into three satellite vortices, but two of these satellites soon merge into one somewhat elongated vortex (see Fig. 3(b)]. As could also be observed in the dye visualizations, the shape of the core then becomes more elliptical, and at \( t = 165 \text{s} \) a symmetric tripole has formed. While the vortex decays in strength, mainly due to vertical diffusion, its rotation rate becomes slower, but its appearance remains tripolar. This observation contrasts with the evolution of tripoles reported by Flör and van Heijst,\ref{9} because they observed an axisymmetrization of the vortex and eventually a shielded monopole was retrieved. The present contour plots of the vertical vorticity suggest the formation of a compact tripole. The exact shape of the tripole that is eventually formed in the laboratory experiments was found to depend on the injection parameters. In another series of experiments the injection rate was doubled, but the amount of injected fluid was kept the same. Figure 4 shows the vorticity plots of such an experiment, referred to as experiment II. Two remarkable differences can be found in comparison with the two previous experiments (two realizations with a common parameter set). The core of the tripole is far less elliptical, and the satellites are wrapped around the core instead of forming separate vortices. As a result there is almost no entrainment of undyed fluid by the satellites and the tripole does not grow substantially in size. This case much more resembles the tripole described in Ref. 9, and is denoted as a weak tripolar structure.

A parameter that conveniently characterizes differences between the tripoles (apart from their size) is the (absolute) ratio \( \gamma \) between the vorticity values of the satellites and the core of the tripole. Figure 5 shows cross sections of the vertical vorticity for both tripoles. For the first tripole one finds \( \gamma \approx 0.35 \), whereas for the second \( \gamma \approx 0.20 \), and these ratios appear to remain more or less constant once the tripoles have been formed. The satellites are thus much stronger for the compact tripole than for the weak tripole, and apparently these differences also lead to very different evolutions and entrainment properties of both tripoles.

The differences between the well-developed tripoles in Figs. 2 and 3 and the weaker tripole in Fig. 4 can be explained by a close investigation of the initial monopoles. In a numerical study presented in Secs. III and IV, it will be shown how the initial conditions influence the (possible) formation of a tripole. To perform such a numerical study it is necessary to determine the Reynolds and Froude numbers of the initial vortices, and to estimate the corresponding steepness parameter \( \alpha \). In Fig. 6 the velocity \( V_m = \sqrt{u^2 + v^2} \), with \( u \) and \( v \) the velocity components in the \( x \) and \( y \) directions, of every particle in the measured flow field is plotted, as a function of the distance to the center of the vortex, for the vortices in the particle tracking experiments \( I \) and \( II \) at \( t = 30 \text{s} \). At this moment both vortices are still assumed to be circular, although the amount of scatter in the graphs indicates that this is not entirely true anymore. The maximum velocity is estimated as \( \dot{V}_1 = 0.6 \pm 0.1 \text{ cm s}^{-1} \) and \( \dot{V}_2 = 0.9 \pm 0.1 \text{ cm s}^{-1} \), yielding values for the characteristic velocity

\[ \dot{V} = 4.44 \text{ cm s}^{-1} \] and \( \Delta V = 100 \text{ cm}^3 \].
scale (the measured maximum nondimensional velocity value is $\hat{V}/V=v_{\text{max}}=1/\sqrt{16\pi e\Lambda^2}$) so that $V=3.5\hat{V}$, where the dimensionless vortex thickness $\Lambda=0.3$ was used$^{12}$ of $V_I=2.1\, \text{cm s}^{-1}$ and $V_{\Pi}=3.2\, \text{cm s}^{-1}$, respectively. Figures 6(a) and 6(b) also contain three velocity profiles, scaled to fit the data, with $\alpha=2$, 3, and 4. These profiles are based on a guess for the position where the maximum velocity occurs. This means that a large inaccuracy arises when a value for $\alpha$ is estimated for these two velocity profiles. The present fits suggest $\alpha_I\approx 3.5\pm 1.0$ for the profile in Fig. 6(a) and $\alpha_{\Pi}\approx 2.5\pm 1.0$ for Fig. 6(b). Although these values are rather inaccurate, they indicate that $\alpha_I>\alpha_{\Pi}$. The typical length scale $L$ of the flow is defined as the radius where the vorticity changes sign,$^{12}$ and it can be estimated that $L_I=6\pm 1\, \text{cm}$ and $L_{\Pi}=5\pm 1\, \text{cm}$. The buoyancy frequency is fairly easy to determine from a vertical density profile measured during the experiments: $N=1.8\, \text{rad s}^{-1}$. Together, these results yield $Re_I=V_L/L=1.3\pm 0.3\times 10^3$ and $F_I=V_L/LN=0.20\pm 0.05$ and $Re_{\Pi}=1.6\pm 0.4\times 10^3$ and $F_{\Pi}=0.35\pm 0.09$ (in both cases determined for $t=30\, \text{s}$). These numbers will be used in the numerical simulations (Sec. III) to illustrate the effects of these parameters on the formation of the tripoles.

III. NUMERICAL SIMULATIONS OF AZIMUTHALLY PERTURBED VORTICES IN A STRATIFIED FLUID

Numerical simulations of the time-dependent incompressible 3-D Navier–Stokes equations in the Boussinesq approximation have been performed with a finite-differences scheme developed by Verzicco and Orlandi.$^{15}$ The fluid density and pressure are decomposed in a part that represents the linear density profile and a part representing the perturbation with respect to the linear density profile. The buoyancy frequency $N$ is defined by $N^2=(g/\rho_0)(\partial\bar{\rho}/\partial z)$, with $\rho_0$ the constant reference density, $\bar{\rho}$ the linear density profile, and $g$ the gravity acceleration. All the equations are nondimensionalized by a typical length scale $L$ (the radius of the vortex), a characteristic velocity scale $V_*=\sqrt{2\pi\lambda\omega_{\text{max}}}$ (with $\lambda$ the vortex thickness and $\omega_{\text{max}}$ the extremum vorticity value) and a time scale $L/V$.\textsuperscript{12,13} The density perturbation $\bar{\rho}$ is scaled by...
the density difference $\Delta \rho = N^2 \rho_0 / g$. The perturbation pressure $\bar{p}$ is scaled by $\rho_0 V^2$. This yields five nondimensional equations, for the five variables $v$, $\bar{p}$, and $\bar{\rho}$,

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla \bar{p} - \frac{1}{\rho_0} \frac{1}{F^2} \bar{\rho} + \frac{1}{Re} \nabla^2 v, \tag{1}$$

$$\nabla \cdot v = 0, \tag{2}$$

$$\frac{\partial \bar{\rho}}{\partial t} + (v \nabla) \bar{\rho} - w = \frac{1}{Sc\, Re} \nabla^2 \bar{\rho}, \tag{3}$$

where $Re = VL/\nu$, with $\nu$ the kinematic viscosity of water, and $F = V/(LN)$. An additional nondimensional number can be identified here: the Schmidt number $Sc$, which is defined as $Sc = \nu / \kappa$, where $\kappa$ is the diffusivity of salt in water. A realistic value of the Schmidt number for salt stratified water equals $Sc \approx 700$. In a previous study we have shown that for the particular case of the decay of a vortex in a linearly stratified fluid $Sc = 10$ is sufficient to include all essential flow phenomena in the numerical simulations, and the present simulations rely on that observation.

In order to investigate the azimuthal instability of pancake-like vortices by means of fully 3-D numerical simulations, initial conditions for the velocity and density distribution need to be prescribed that are similar to those used in two previous papers:

$$v_{\theta,a}(r, z; t=0) = \frac{1}{\Lambda \sqrt{2\pi}} e^{-(1/2)(z/L)^2} \times \frac{1}{2} re^{-r^2}, \tag{4}$$

$$\bar{\rho}_a(r, z; t=0) = -\frac{F^2 \bar{\rho}}{4\pi \Lambda^2} \exp \left( -\frac{z^2}{\Lambda^2} \right) \times \int_0^\infty u \exp(-2u^2) du, \tag{5}$$

together with $v_r = 0$ and $v_z = 0$. This choice of cyclostrophically balanced initial conditions reduces the generation of internal waves. Moreover, it has been found that in the axisymmetric regime (before the amplitude of the unstable mode has grown substantially) the vertical and radial velocities remain an order of magnitude smaller than the azimuthal velocity (for a detailed discussion on these issues the reader is referred to Beckers et al.\cite{beckers}). The variable $\Lambda = \lambda / L$ defines the dimensionless initial thickness of the vortex. In experiments $\Lambda \approx 0.3$ and in numerical simulations $\Lambda = 0.3$; our pancake-like vortices actually represent small-aspect ratio vortices.

**A. Influence of $Re$ and $F$ on the azimuthal instability**

The effect of horizontal (or lateral) diffusion of momentum can be illustrated by considering the decay of a 2-D axisymmetric vortex. In 2-D axisymmetric flows the nondimensional vorticity equation, written for a polar coordinate system $(r, \theta)$, is given by

$$\frac{\partial \omega}{\partial t} = \frac{1}{Re} \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \omega}{\partial r} \right), \tag{6}$$

where $\omega$ is defined as $\omega = (1/r)(d/dr)(r\nu \partial \rho / \partial r)$. The vorticity equation is solved for the following initial vorticity distribution:

$$\omega_a(r, \theta; t=0) = (1 - \frac{1}{2} \alpha r^2) \exp(-r^2). \tag{7}$$

For $\alpha = 2$, a self-similar solution is found for the evolution of the vorticity in time,

$$\omega(r, t) = \frac{1}{(1 + \frac{4}{Re} t)^2} \left( 1 - \frac{r^2}{1 + \frac{4}{Re} t} \right) \times \exp \left( -\frac{r^2}{1 + \frac{4}{Re} t} \right), \tag{8}$$

and for values of $\alpha \neq 2$ a numerical evaluation of (6) is necessary. Kloosterziel\cite{kloosterziel} found that any initial vorticity distribution that has vanishing circulation will eventually attain the shape of the self-similar solution (8). For an initial vorticity distribution (7) with $\alpha |_{\theta=0}>2$ this implies a gradual decrease of the steepness parameter to $\alpha |_{t=\infty}=2$. Carnevale and Kloosterziel\cite{carnevale} performed a numerical linear stability analysis for inviscid 2-D vortices with a vorticity profile given by (7). Their results revealed how the growth rate of the azimuthal perturbations with various wave numbers depends on the value of the steepness parameter $\alpha$ (see Fig. 1). The decrease of $\alpha$ due to lateral diffusion will thus have implications for the development of azimuthal instabilities on vortices with initial values of $\alpha>2$.

The evolution of 3-D axisymmetric vortices with various initial values of $\alpha$ has been investigated, numerically focusing on the diffusion of vorticity and vortex stretching. In the present simulations an initially axisymmetric vortex was used, with an initial velocity distribution given by (4) and a density distribution given by (5), for a steepness parameter $\alpha = 4$ and an initial thickness $\Lambda = 0.3$. In order to promote the instability, a small azimuthal perturbation (with $n = 2$ and $\delta = 0.01$),

$$\tilde{u}_{\theta,a}(r, \theta; t=0) = (1 + \delta \sin 2\theta) v_{\theta,a}(r, t=0), \tag{9}$$

has been included in the initial velocity field. Although the perturbation slightly violates the incompressibility constraint, mass conservation is restored after the first integration step.

It was observed that also for a pancake-like vortex (without an azimuthal perturbation) in a stratified fluid the distribution of the vortical vorticity $\omega_a(r)$ evolves toward the $\alpha = 2$ profile for each initial value of $\alpha$. The strength of the vorticity field at the vortex symmetry plane $z=0$ is found to decrease faster in such three-dimensional (3-D) vortices due to a combination of horizontal and vertical diffusion. Vertical diffusion does not influence the stability properties of a vortex directly, because it cannot change the ratio $\gamma = \omega_{\text{ring}} / \omega_{\text{core}}$, which is the ratio between the extreme vorticity values in the ring and in the core of the vortex [note that $\gamma = \frac{1}{2} \alpha \exp(-\alpha/2) / \alpha$, thus decreasing $\gamma$ implies decreasing $\alpha$], but it causes a much faster decay of the Reynolds number of the flow at the vortex symmetry plane.
Therefore, smoothing of horizontal vorticity gradients is enhanced indirectly by vertical diffusion, and stabilizes the Q2-D flow in the symmetry plane of the vortex. The role of the Reynolds number on tripole formation can be illustrated with a comparison of the evolution of fully 3-D vortices with their 2-D counterparts with \( \text{Re} = 500, 1000, \text{and } 5000 \) (\( \alpha = 4 \)), where we have kept the Froude number small in order to separate vortex stretching from diffusion effects. For \( \text{Re} = 500 \) the tripole formation appears to cease, in contrast to the 2-D case. For \( \text{Re} = 1000 \), the core vortex of the 3-D tripole is slightly less elliptical than its 2-D counterpart, and overall the vorticity values are smaller due to vertical diffusion. The case with an initial Reynolds number of 5000 illustrates that the tripole does not split anymore into two dipoles, in contrast to the 2-D vortex, clearly indicating that the growth rate of higher-order modes is suppressed by vertical diffusion in the 3-D case.

It was also found that the value of the Froude number can have a profound influence on the evolution of the vorticity profile. Due to the cyclostrophic balance in the vortex, isopycnals are deflected toward the vortex symmetry plane. During the decay of the vortex, isopycnals return to their equilibrium position and the fluid column between the isopycnals is stretched. This results in an enhanced decrease of the vorticity ratio \( \gamma \), because vortex stretching particularly affects the (positive) core of the vortex, where the deformation of the isopycnals is largest, and not the (negative) ring. The influence of the Froude number on tripole formation is illustrated in Fig. 7, where for \( \text{Re} = 2000 \) two cases with \( F = 0.16 \) and \( F = 0.80 \) are compared. In both cases a tripole is formed eventually, but the satellites are much less pronounced and the core is less elliptical in the case with \( F = 0.80 \). Figure 8(a) compares the evolutions of \( \gamma = |\omega_{\text{satellite}}/\omega_{\text{core}}| \) for both cases. Note, however, that particularly after \( t = 20 \) the value of \( \gamma \) here characterizes the shape of the tripole by the relative strengths of the satellites and the core and it has no relation with the steepness of the vorticity profile. Clearly \( \gamma \) decreases much faster and becomes smaller for \( F = 0.80 \), indicating that the tripole is not as well developed as for the simulation with \( F = 0.16 \).

Stages as predicted by Carton and Legras for the formation of a 2-D tripole can also be distinguished during a 3-D
tripole formation. The first stage, where small azimuthal perturbations grow exponentially in strength (and perturbations with different wave numbers grow independently), takes place until \( t \approx 20 \). In the second stage, nonlinear amplification takes place: different modes start to interact and the growth of the perturbations is no longer exponential. Finally, a full-grown tripole is formed. In Fig. 8(b) the early growth of the energy \( \epsilon \) of mode \( n=2 \) is shown for \( F=0.16 \) and \( F=0.80 \) and for the corresponding 2-D case (all for \( Re=2000 \)). It indeed shows that the growth rate is always smaller for \( F=0.80 \) than for \( F=0.16 \), and that the kinetic energy contained in the mode \( n=2 \) saturates at a substantially lower level for the higher Froude number case. The data in Fig. 8(b) also indicate that the growth rate of the mode \( n=2 \) is always smaller for a 3-D vortex, compared to a 2-D vortex.

The conclusion from the previous paragraph about the growth rate of the mode \( n=2 \) should not be extrapolated to high-Reynolds number flows. The regime study to be presented in Sec. IV has revealed that for \( Re \approx 5000 \) and \( \alpha \approx 6.0 \) the early growth rate and the saturation level of the mode \( n=2 \) becomes virtually independent of the Froude number (for \( Re \leq 0.80 \)). The final stage of the evolution of these runs (with \( Re \approx 5000 \) and \( \alpha \approx 6.0 \)) appears to be strongly determined by the nonlinear amplification of the perturbation with wave number \( n=4 \) (note that only a small perturbation with wave number \( n=2 \) was added to the initial flow field). The growth rate and the saturation level of this \( n=4 \) mode depends strongly on the Froude number, and, as will be shown in Sec. IV, they determine whether or not a dipole splitting will occur.

**B. Energy balance of an unstable monopolar vortex**

An azimuthal instability can only grow if it is able to withdraw energy from some source. For vortices in a nonrotating stratified fluid, this source can be either the kinetic energy of the mean flow (barotropic instability), or the potential energy contained in the density field (baroclinic instability). This implies that azimuthal instability of flows that involve no density differences is necessarily barotropic, like, e.g., the formation of a tripolar vortex in a rotating homoge-

![Diagram](image)

**FIG. 9.** (a) Evolutions of the four terms in the energy balance (12) for the azimuthal instability of two vortices for \( Re=2000 \) and \( F=0.16 \) and (b) for \( Re=2000 \) and \( F=0.80 \).

neous fluid.\(^8\)\(^,\)\(^16\) However, for a rotating stratified fluid the instability may become predominantly baroclinic (see, e.g., Refs. 17–19).

It is possible to assess the relative contributions of the potential and kinetic energy of the mean flow to the growth of the azimuthal instability.\(^19\) This is done by a decomposition of the velocity field as \( \mathbf{v}(r, \theta, z) = \mathbf{U}(r, z) + \tilde{\mathbf{v}}(r, \theta, z) \), where \( \mathbf{U}(r, z) \) is defined as the azimuthally averaged flow velocity,

\[
\mathbf{U}(r, z) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{v}(r, \theta, z) d\theta.
\]

(10)

The decomposed velocity is substituted in the Navier–Stokes equation (1) and its azimuthally averaged part is subtracted. The resulting expression is then multiplied by \( \tilde{\mathbf{v}}(r, \theta, z) \) and integrated over the entire flow volume \( V \). Assuming that \( \tilde{\mathbf{v}}(r, \theta, z) = 0 \) at the boundary of the volume \( V \), we obtain

\[
\frac{d}{dt} \int \frac{1}{2} \tilde{\mathbf{v}}^2 dV = - \int \tilde{\mathbf{v}} \cdot \{ \tilde{\mathbf{v}} \times \nabla \} \mathbf{U} dV - \frac{1}{F^2} \int \nabla \cdot \tilde{\mathbf{v}} dV + \frac{1}{Re} \int \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}} \cdot \tilde{\mathbf{v}} dV.
\]

(11)

In short this energy balance equation can be given by

\[
\frac{dE_K}{dt} = S_M + S_P + D,
\]

(12)

and it describes how the three terms \( S_M \), \( S_P \), and \( D \) contribute to the rate of change of the kinetic energy \( E_K \) of the perturbation velocity field \( \tilde{\mathbf{v}}(r, \theta, z) \). The term \( S_M \) represents the transfer of kinetic energy from the mean flow to the azimuthal perturbation. The second term \( S_P \) represents the transformation of potential energy contained in the (azimuthally perturbed) density field into kinetic energy of the perturbation field. The quantity \( D \) represents the dissipation of the kinetic energy of the azimuthal perturbation. Figure 9 shows the evolutions of the individual terms in (12) for the tripole formations with \( F=0.16 \) and \( F=0.80 \). In the first case (\( F=0.16 \)) one observes that the rate of change of the kinetic energy of the perturbation is positive until \( t \approx 50 \), and from then on the dissipation (\( D \)) is stronger than the source term.
(\(S_M\)) and the tripole starts to decay. It can also be seen that only \(S_M\) contributes to the increase in the kinetic energy of the perturbation, and that the transformation of potential energy \(S_P\) into kinetic energy plays no role of importance on the instability. In other words, the instability of the vortices is predominantly barotropic. For the second case \((F=0.80)\) in Fig. 9(b) the contribution of \(S_P\) appears to be slightly larger, but the instability is still predominantly barotropic. In both cases it has been checked that all contributions add up to zero, so that the balance (12) is indeed satisfied.

C. Comparison with experiments

It is now possible to make a close comparison between the results from the laboratory experiments and the 3-D numerical simulations. Two numerical simulations, initialized by the analytical velocity field (4) with a small mode-2 perturbation, have been performed. In the first simulation (referred to as I) \(\text{Re}=1300, F=0.20,\) and \(\alpha=3.5,\) whereas in the second simulation (II) \(\text{Re}=1600, F=0.35,\) and \(\alpha=2.5.\) At two times, \(t_I=200\) s and \(t_{II}=150\) s, the vorticity distributions of the simulated tripoles are shown in Figs. 10(a) and 10(b) (for a comparison with the laboratory experiments; see Figs. 3 and 4). Also, the increments in the vorticity of the contours correspond to those in the vorticity contour plots of the experiments. In both cases the simulations show a tripole that is very similar to that obtained in the experiment, indicating that the different parameters (\(\text{Re}, F,\) and \(\alpha\)) indeed can explain the formation of different types of tripoles. In Fig. 10(c) the cross sections of the vorticity (scaled by the core values) are shown for the tripoles in (a) and (b), and these agree rather well with the experimentally obtained cross sections as shown in Fig. 5.

D. The 3-D structure of the tripolar vortex

Thus far, we have only studied the tripolar vortex by taking horizontal cross sections of the vertical vorticity at the symmetry plane. The present experimental techniques are not suited to measure the fully 3-D velocity field of the vortex in detail. The present numerical simulations, however, show such a close agreement with the available experimental observations, that it seems justified to investigate the 3-D structure of the tripole by using the results of numerical simulations. The discussion below is based on a simulation with \(\text{Re}=2000\) and \(F=0.16.\)

The vertical vorticity and the density perturbation give a good representation of the Q2-D flow at different levels inside the vortex. The development of the tripole is most pronounced at the vortex symmetry plane where the Reynolds number is highest. Away from the symmetry plane the core is less elliptical and also the formation of the satellites becomes progressively weaker with increasing distance from the symmetry plane. The region of the density perturbation does not have the same shape as the distribution of \(\omega_z\) of the vortex core. Instead, it is extended along a line that connects the centers of the core and the satellites. These satellites thus also induce a strong deflection of the isopycnals. Remarkably, the exact orientation of the \(\bar{\rho}\) distribution is slightly ahead in phase compared with the distribution of \(\omega_z(r, \theta)\) at the same level. This is due to the fact that the deflection of the isopycnals is mainly induced by the flow at \(z=0.\) A more detailed discussion can be found in Ref. 11.

The 3-D distribution of \(\omega_z,\) in the form of isosurfaces, provides a convenient representation of the 3-D structure of the vortex. Figure 11 shows two of these isosurfaces. The isosurface shown in Fig. 11(a) represents the vorticity inside the core for \(\omega_z=+0.02.\) The isosurfaces clearly illustrate that only the part of the core near the symmetry plane appears to be deformed. In this region one can see an ellipse with two filaments at both long ends, but somewhat further above and below the symmetry plane the core is still approximately axisymmetric. Figure 11(b) shows the isosurface for \(\omega_z=-0.02,\) i.e., the region of opposite vorticity. This structure consists of a combination of the two satellite vortices (close to the symmetry plane) and two almost axisymmetric rings of negative vorticity above and below the satellites.

IV. A REGIME STUDY OF THE INSTABILITY OF MONOPOLAR VORTICES

For the regime study a large set of numerical simulations has been performed. This investigation concerns the mapping of the instability process of shielded monopolar vortices, with a small perturbation with wave number \(n=2\) and amplitude \(\delta=0.01\) [see Eq. (9)], that we have added to the initial velocity distribution (see Sec. IV A). Additionally, a second smaller set of simulations is carried out aimed at elucidating the role of the odd wave number perturbations, that seemed to be irrelevant in the regime study based on a
mode \( n = 2 \) perturbation. In that case a small azimuthal perturbation consisting of modes with the wave numbers \( n = 2 \) and \( n = 3 \) is therefore added to the initial velocity distribution:

\[
\vec{u}_{\theta,a}(r, \theta; t=0) = \left[ 1 + \delta (\sin 2 \theta + \sin 3 \theta) \right] \vec{u}_{\theta,a}(r, t=0),
\]

(13)

with again \( \delta = 0.01 \) (see Sec. IV B). For the first set of simulations we varied the Reynolds number, the Froude number, and the steepness parameter \( \alpha \) in the following range: \( \text{Re} \in \{500, 1000, 2000, 5000, 10000\} \), \( \text{Fr} \in \{0.10, 0.20, 0.40, 0.80\} \), and \( \alpha \in \{2, 3, 4, 6, 8\} \). The second set of simulations is restricted to the following values of \( \text{Re} \), \( \text{Fr} \), and \( \alpha \): \( \text{Re} = 2000 \) (\( \text{Fr} = 0.10, 0.20, 0.40, \) and \( 0.80 \)) and \( 5000 \) (\( \text{Fr} = 0.10 \) and \( 0.20 \)) and \( \alpha = 4, 6, \) and \( 8 \).

The computations of the instability process of shielded monopolar vortices in a stratified fluid were carried out in a cylindrical container with dimensionless radius \( R = 4 \) and height \( H = 6 \) with stress-free boundary conditions on the lateral (cylindrical) boundary and periodic boundary conditions in the \( z \) direction. The initial position of the vortex is located in the center of the cylindrical container. The influence of the stress-free boundaries is negligible as long as the end products of the instability process do not collide with the boundary at \( r = 4 \). The monopole is relatively thin (\( \Lambda = 0.3 \)) and located at the half-plane \( z = 3 \), the bottom and top boundary are thus sufficiently far away. The spatial resolution \( N_r \times N_{\theta} \times N_z \) for the different runs, with only a mode \( n = 2 \) perturbation, are summarized in Table I. From Table II it can be concluded that the data obtained from the runs with \( \text{Re} = 5000 \) and \( F = 0.8 \) (\( \alpha = 8 \)), and those with \( \text{Re} = 10000 \), \( F = 0.80 \), and \( \alpha \geq 4 \) should be interpreted with care and are therefore indicative (larger Froude numbers also require an increased vertical resolution, which was impossible with the available computer resources). All the runs with \( \text{Re} = 500, 1000, \) and \( 2000 \) are well-resolved, i.e., the results are virtually independent of the grid size. The resolution and well-resolvedness for the simulations with a combined mode \( n = 2 \) and \( n = 3 \) perturbation can be deduced from Tables I and II. The dimensionless time steps used in the simulations are: \( \Delta t = 0.1 \) for the runs with a \( 65 \times 65 \times 65 \) resolution and \( \Delta t = 0.05 \) for the runs with a \( 129 \times 129 \times 65 \) resolution (and might be adapted downwards to satisfy the CFL condition). The length of the time integration for the large set of simulations was kept fixed for all runs: \( t_{\text{end}} = 100 \). The simulations with a mode \( n = 2 \) and \( n = 3 \) perturbation revealed a slow horizontal drift of the vortex out of the center of the domain. This drift constrained our simulations due to severe time-step reductions in the course of the simulation in order to satisfy the CFL condition (note that cylindrical coordinates are used). Some of these runs were therefore stopped at \( t < 100 \).

**A. Perturbation with wave number \( n = 2 \)**

The regime study of the evolution of a shielded monopolar vortex with a small perturbation with wave number \( n = 2 \) and \( \delta = 0.01 \) revealed the following evolution scenarios. The perturbation decays rapidly and no significant deformation of the monopolar vortex is observed. We denote this final state as SM (shielded monopole). All perturbed monopolar vortices with \( \alpha = 2 \) and \( \text{Re} \leq 10000 \) follow this scenario. For that reason we will not discuss these simulations.

**TABLE I.** Overview of the resolution \( N_r \times N_{\theta} \times N_z \) of the numerical experiments with azimuthally unstable monopolar vortices (\( \alpha \leq 8 \)).

<table>
<thead>
<tr>
<th>( \text{Re} )</th>
<th>( F \leq 0.80 )</th>
<th>( \alpha \leq 6 )</th>
<th>( \text{Well-resolvedness} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \leq 1000 )</td>
<td>( F \leq 0.80 )</td>
<td>( 65 \times 65 \times 65 )</td>
<td>sufficient</td>
</tr>
<tr>
<td>( 2000 )</td>
<td>( F \leq 0.40 )</td>
<td>( 65 \times 65 \times 65 )</td>
<td>sufficient</td>
</tr>
<tr>
<td>( 5000 )</td>
<td>( F \leq 0.40 )</td>
<td>( 129 \times 129 \times 65 )</td>
<td>sufficient</td>
</tr>
<tr>
<td>( \geq 5000 )</td>
<td>( F \leq 0.80 )</td>
<td>( 129 \times 129 \times 65 )</td>
<td>sufficient</td>
</tr>
</tbody>
</table>

**FIG. 11.** Three-dimensional presentation of the structure of the tripod, obtained for \( \text{Re} = 2000 \) and \( F = 0.16 \) at \( t = 80 \). The figure shows isosurfaces of \( \omega_r \) at a view at approximately 45°. Isosurfaces are drawed for the core in (a) and for the ring of opposite vorticity in (b) [rotated with respect to the isosurface shown in (a)]. Figure (a) represents the vorticity value \( \omega_r = 0.02 \) and for figure (b) \( \omega_r = -0.02 \).
in the sequel. We can distinguish three different tripolar vortices as end product of the instability (see Fig. 12): a weak tripolar vortex (WT) that tends to axisymmetrize rapidly (a similar scenario, as found in the experiments by Flör and van Heijst), a compact tripolar structure (CT), with virtually no entrainment of irrotational fluid between the slightly elliptic core and the satellites, and a noncompact tripole (NCT). The latter tripolar structure is characterized by an elliptical core and a relatively large separation distance between the satellites and the core. This relatively large separation enables the entrainment of irrotational fluid in between the core and the satellites. Note that our classification is based on the horizontal intersection of the vertical vorticity field. A similar intersection of, for example, a passive tracer field might yield a somewhat different picture: see, e.g., the dye distribution of the compact tripole shown in Fig. 2.

A fifth end product could be observed: two dipoles moving in opposite directions. This process is denoted as dipole splitting (DS), which has also been observed as a result of the instability of 2-D inviscid isolated vortices with $\chi \approx 3.2$.

Dipole splitting occurs when, due to nonlinear interactions, the azimuthal mode $n=4$ start to compete with the original perturbation with wave number $n=2$. In our simulations it predominantly occurs for large steepness parameters only. This is due to the fact that friction and vortex stretching tend to push the perturbed vortex toward the tripole regime discussed in the previous paragraph. An example, taken from the simulation with $Re=10 000$, $F=0.10$, and $\alpha=6$, is shown in Fig. 13, where one can clearly observe that nonlinear interactions predominantly amplify even wave number perturbations. Moreover, the modes $n=2$ and $n=4$ contain approximately the same amount of kinetic energy at $t=100$, and are responsible for the dipole-splitting process.

One could argue that dipole splitting is further promoted by including, together with the $n=2$ perturbation, an initial perturbation with wave number $n=4$. For that reason we have carried out several runs with the following initial perturbation: $v_{\phi,a}(r,\theta; t=0)=[1+\delta(\sin 2\theta+\sin 4\theta)] \cdot v_{\phi,a}(r, t=0)$, with $\delta=0.01$. As an illustration we consider the run with $Re=5000$, $F=0.40$ and $\alpha=6$. Although initially the mode $n=4$ is visible in the contour plots of the vertical vorticity, it decays rapidly and a noncompact tripole has been formed. This tripole is virtually indistinguishable from the one obtained with only a wave number $n=2$ perturbation. This observation is supported by inspection of the energy content $e$ of the $n=2$ and $n=4$ modes for $t \leq 20$: the kinetic energy of the azimuthal mode $n=2$ grows in the course of time in contrast with the energy content of mode $n=4$, that decreases rapidly. Additionally, a plot of the kinetic energy of the azimuthal modes reveals that no distinction can be made

FIG. 12. Characterization of the end products of the instability of a shielded monopolar vortex ($\alpha=6$, $t=80$): (a) a weak tripolar vortex (WT), (b) a compact tripole (CT), (c) a noncompact tripole (NCT) and (d) a dipole splitting (DS).
between the runs with only an $n=2$ and a combined $n=2$ and $n=4$ perturbation for $t=20$. From this numerical experiment it can be concluded that the instability is indeed dominated by the perturbation with wave number $n=2$. A few more runs have been carried out, and those with $\alpha \geq 6$ and $F=0.40$ yield the same instability scenario, as shown in Table III (NCT for Re=5000 and 10 000). However, the simulations with $\alpha=8$ show that the azimuthal mode $n=4$ promotes the formation of noncompact tripoles with a substantially larger core-to-satellite distance. Only one numerical experiment showed dipole splitting instead of the formation of a noncompact tripole: the simulation with $\alpha=8$, Re =10 000 and $F=0.20$.

In Table III we have summarized the character of the end products of the perturbed monopolar vortices for $\alpha \geq 3$ and $F \leq 0.40$ (with initially only an $n=2$ perturbation). Additionally, the instability scenario of azimuthally perturbed 2-D isolated vortices are indicated in the rows denoted with 2-D. A clear trend is visible from these data, and extrapolation toward $F=0$ is indeed in line with the 2-D computations, although the absence of vertical diffusion results in substantially less dissipation in the 2-D simulations (or phrased in another way, vertical diffusion stabilizes 3-D shielded monopoles in nonrotating linearly stratified fluids). From our data it could be conjectured that the evolution scenario for the perturbed monopolar vortex with $F=0.80$ is not that exciting. The end product usually consists of SM, WT, or an irregularly shaped tripolar structure (particularly for $\alpha \geq 4$ and Re$\geq 5000$). Due to the fact that the resolution is assumed to be fair for the simulations resulting in irregularly shaped tripolar structures (see Table II), we decided to give no further emphasis to these runs.

The ratios $\gamma'=|\omega_{\text{satellite}}/\omega_{\text{core}}|$ (for SM) and $\gamma =|\omega_{\text{satellite}}/\omega_{\text{core}}|$ (for WT, CT, and NCT) have been computed for the fully 3-D numerical computations, and the data obtained for $t=100$, which are assumed to be reliable, have been added in Table III (no data were available for the 2-D simulations). The estimated error margin is approximately 10%. These data clearly show that the final state SM is associated with $\gamma(t=100)\approx 0.16$, indicating that a rapid relaxation to a vorticity profile with $\alpha=2$ has occurred. This in-

![FIG. 13. Evolution of the kinetic energy of the azimuthal modes $n=2,4,6,8$ (dashed lines) and $n=1,3,5,7$ (drawn lines) for simulations of an unstable monopolar vortex with $\alpha=6$ (Re=10 000 and $F=0.10$).](image-url)
deed implies no instability of the vortex. A weak tripolar structure emerges when \(0.16 \leq \gamma(t=100) \leq 0.24\). The compact tripole (CT) can be associated with the following satellite-to-core vorticity ratio: \(0.20 \leq \gamma(t=100) \leq 0.45\), and the noncompact tripole (NCT) with the ratio \(0.40 \leq \gamma(t=100) \leq 0.65\). As might be expected, the satellites of the tripole become substantially stronger for WT\(\rightarrow\)CT\(\rightarrow\)NCT, and when the satellites become too strong the dipole-splitting process is initiated. The overlap between the different \(\gamma\) ranges is explained by noting that the values of \(\gamma\) are slightly biased by the initial vorticity profile, because \(\gamma(t=0) = 0.45, 0.79\) and \(1.15\) for \(\alpha=4, 6,\) and \(8\), respectively, and by the Instability \(\gamma\) (the vortex structures are still decaying).

The experimentally measured values for the ratio \(\gamma\) are consistent with the numerical data, as summarized in Table III. In our experiments we found \(\gamma=0.20\) for the weak tripolar structure, and \(\gamma=0.35\) for the compact tripole.

The following general observations can be made concerning the energy balance, as introduced in Eq. (12) (we do not discuss the simulations with \(\alpha=2\) and those in Table III with the final states indicated by SM; these runs virtually no instability is observed). It is found that in all numerical experiments with \(F \leq 0.20\), the instability is governed by the transfer of kinetic energy from the mean flow (denoted by \(S_M\)) to the azimuthal perturbation \(S_P \leq 0.03 S_M\), with \(S_P\) representing the transformation of potential energy contained in the azimuthally perturbed density field into kinetic energy of the perturbation field). For the numerical runs with \(F \geq 0.40\), the instability is still governed by \(S_M\), but \(S_P\) starts to play a more dominant role when \(Re \geq 5000\) and \(\alpha \geq 4\), with \(S_P \approx 0.1 S_M\) (for \(F = 0.40\)) and \(S_P \approx 0.25 S_M\) (for \(F = 0.80\)). It is nevertheless reasonable to conclude that the present azimuthally perturbed shielded monopolar vortices are barotropically unstable.

A numerical analysis of the Reynolds and Froude number dependence of \(S_M, S_P,\) and \(D\) shows reasonable scaling. It appears that \(S_M \propto Re^2\) (and virtually independent of the Froude number) for \(F \leq 0.40\) and \(Re \leq 5000\). Higher Reynolds number simulations revealed that \(S_M\) tends to saturate (for \(Re \geq 5000\)). Obviously, \(S_M\) increases with increasing \(\alpha\). The dissipation \(D\) scales in a similar way, although enhanced dissipation is observed for \(Re \geq 5000\) and \(F = 0.80\). This is most likely due to the emergence of 3-D small-scale structures (partly due to under-resolution of the flow in the simulation). The contribution of \(S_P\) to the energy balance equation becomes larger with increasing Reynolds number and steepness parameter. Moreover, for \(F \leq 0.40\) it is found that \(S_P \propto F^2\). This scaling can be understood by considering \(S_P = -1/2 F^2 \int \rho \delta_{\theta} dV\) and keeping in mind that in the low Froude number limit \(\delta_{\theta} \propto F^2\) and \(\delta_{\phi} \propto F^2\).

B. Perturbation with wave numbers \(n=2\) and \(n=3\)

In order to estimate the influence of the odd wave number perturbations we have explored a small Reynolds and Froude number range (\(Re=2000, F=0.40\) and \(\alpha \geq 6\); \(Re = 5000, F \leq 0.40\) and \(\alpha = 4\); \(Re = 5000, F \leq 0.20\) and \(\alpha \geq 6\)). The following general picture can be sketched: the mode \(n=3\) perturbation is amplified during the early stage of the evolution, and visible in the deformation of the outer ring of the vortex, and decays rapidly afterward. No triangular vortex is found as intermediate state with present azimuthal perturbation \((\delta=0.01\) for both \(n=2\) and \(n=3\)) for \(\alpha \leq 6\). Only the perturbation with wave number \(n=1\) is a result of nonlinear interactions between the different modes, grows substantially. It results in a slight horizontal drift of the core of the vortex. The runs with \(\alpha=8\) and \(Re=5000\) show the formation of a triangular vortex as an intermediate state (at \(t \approx 20\)), but this intermediate state is unstable. The observed end products of the instability process are in nearly all cases similar to those summarized in Table III, although the noncompact tripole are slightly asymmetric. Particularly, the core of the vortex moves closer to one of the satellites, and the vortices constituting the tripole are losing their linear arrangement. No appreciable asymmetry is observed for the compact tripole. The dipole-splitting process, as observed for the runs with \(Re=5000, F = 0.20\), and \(\alpha \geq 6\): for \(\alpha = 6\) no dipole splitting occurs, but a very asymmetric dipole has formed. It could be considered as a dipole–monopole splitting. The run with \(\alpha = 8\) revealed the formation of a rather symmetric noncompact tripole instead of a dipole-splitting behavior.

V. CONCLUSIONS

In this paper the evolution of initially axisymmetric pancake-like vortices in a nonrotating linearly stratified fluid is investigated experimentally and numerically, and compared with the evolution of similar 2-D vortices. Two-dimensional vortices are stabilized by the diffusion of vorticity, which implies that a vortex with an initially steep and therefore unstable vorticity profile will acquire a less steep profile and hence becomes stable. The value of the initial Reynolds number of the flow therefore determines how fast the stabilization of the vortex takes place and whether an unstable vortex will result in a tripole or not. The process of stabilization due to viscosity also holds for 3-D pancake-like vortices in a linearly stratified fluid, although stabilization is more effective due to vertical diffusion of momentum. However, it was found that another aspect plays a role as well. The steepness of the vorticity profile (or the ratio \(\gamma = |\omega_{\text{ring}}/\omega_{\text{core}}|\)) is also affected by the stretching mechanism associated with the disturbed shape of the isopycnals. The stretching effect becomes stronger for larger Froude numbers, and it is therefore expected that the value of \(F\) can influence the possible formation of a tripole as well. Fully 3-D numerical simulations indeed confirm this point of view, as is extensively discussed in Secs. III and IV. It is also shown that both decreasing Re and increasing \(F\) inhibits dipole splitting in the regime where it is observed for the 2-D case. Moreover, these simulations show that the formation of tripoles in a stratified fluid is essentially a barotropic, rather than a baroclinic, process.
In the numerical study of the 3-D structure of the tripole, it has been found for the case under investigation, that the instability only takes place in a thin region near the vortex symmetry plane. In that region the ring of opposite vorticity splits up in two satellite vortices, but above and below this region the vortex is still approximately axisymmetric and the ring of opposite vorticity remains almost circular.

With two sets of numerical simulations a wider range of Reynolds and Froude numbers has been explored. These simulations generally support our analysis of the role of Re, F, and \( \alpha \) on the stability of monopolar vortices in linearly stratified fluids, as briefly summarized above. Particularly, the parameter regime was extended toward experimentally nonaccessible Reynolds and Froude numbers. The results from these simulations are also compared to purely 2-D simulations of the azimuthal instability of shielded vortices, where the absence of vertical diffusion and vortex stretching modifies the instability scenario.

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